With the Author's Compliments

K. OGURA,

Note on Stewart's and Luchterhandt's Theorems.

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M. FUJIWARA, J. ISHIWARA, T. KUBOTA, and K. OGURA.

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Note on Stewart's and Luchterhandt's Theorems,

BY

K. Ogura in Sendai.

1. In Crelle's Journal 23, Luchterhandt proved the following two theorems by analytic geometry: (i) If A, B, C, D be any four points on a circle, O be any point in the plane of the circle, and (BCD), (CDA), (DAB), (ABC) be the areas of the four triangles BCD, CDA, DAB, ABC respectively, then

$$OA^2.\ (BCD)-OB^2.\ (CDA)+OC^2.\ (DAB)-OD^2.\ (ABC)=0$$ [1];

and (ii) the similar relation holds good for any five points on a sphere and any one point in space [2].

Joachimsthal [3] proved these theorems by the theory of determinants, and Möbius [4] by his barycentric calculus. Mr. Lachlan [5] has derived the former from the well-known theorem du to Stewart [6]; his method of proof is quite elementary, but is not easily applicable to the proof of the latter.

In this note I will give another proof of Luchterhandt's and Stewart's theorems, and also an extension of Stewart's theorem in space. The following method is not only simple, systematic and elementary, but is of interest as it shows us some close relations between Stewart's and Luchterhandt's theorems.

2. If A, B, C, D be any four points in the same plane, then

(1) $$(ABC)-(BCD)+(CDA)-(DAB)=0,$$

$(ABC),\ldots.$ being the area of the triangle $ABC,\ldots.$. Particularly, when these four points are on a circle K whose radius is R,

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(1) Reproduced in Salmon, Conic sections, 6th ed., p. 87.

(2) We see in Baltzer's Theorie und Anwendungen der Determinanten, 4. Aufl., p. 251, the latter was already obtained by Feuerbach in his Untersuchung der dreieckigen Pyramide, p. 15.

(3) Joachimsthal, Crelle J. 40.

(4) Möbius, Crelle J. 26 = Ges. Werke, I.

(5) Lachlan, Modern pure geometry, p. 21.

(6) Stewart, Some general theorems of considerable use in the higher parts of mathematics. For the history of this theorem, see Chasles, Aperçu historique p. 175.

$$4\,R\cdot(ABC) = AB\cdot BC\cdot CA, \ldots\ldots .$$

Hence

(2) $AB\cdot BC\cdot CA - BC\cdot CD\cdot DB + CD\cdot DA\cdot AC - DA\cdot AB\cdot BD = 0.$

Now if we invert this figure with respect to a point O which is in the plane of the circle K, but not on its circumference, and denote the inverses of the points A, B, C, D by A', B', C', D' respectively, then the points A', B', C', D' lie on a circle K' and the equation (2) is transformed into

(3) $OD'^2\cdot A'B'\cdot B'C'\cdot C'A' - OA'^2\cdot B'C'\cdot C'D'\cdot D'B' + OB'^2\cdot C'D'\cdot D'A'\cdot A'C' - OC'^2\cdot D'A'\cdot A'B'\cdot B'D' = 0.$

But if R' be the radius of the circle,

$$4R'\cdot(A'B'C') = A'B'\cdot B'C'\cdot C'A', \ldots\ldots .$$

Therefore we obtain the following equation

(4) $OD'^2\cdot(A'B'C') - OA'^2\cdot(B'C'D') + OB'^2\cdot(C'D'A') - OC'^2\cdot(D'A'B') = 0,$

which is Luchterhandt's theorem in plane.

Again if we invert the figure thus obtained with respect to the point D' and denote the inverses of the points O, A', B', C' by O'', A'', B'', C'' respectively, then the points A'', B'', C'' lie on a straight line K'' and the equation (3) is transformed into

(5) $A''B''\cdot B''C''\cdot C''A'' + O''A''^2\cdot B''C'' + O''B''^2\cdot C''A'' + O''C''^2\cdot A''B'' = 0,$

which is nothing but Stewart's theorem.

Remark I. If we invert (2) with respect to the point D, we get also Stewart's theorem.

Remark II. In the particular case where O coincides with D', (3) takes the form

$$A'D'^2\cdot B'C'\cdot C'D'\cdot D'B' - B'D'^2\cdot A'C'\cdot C'A'\cdot A'D' + C'D'^2\cdot A'B'\cdot B'D'\cdot D'A' = 0.$$

Hence

$$A'D'\cdot B'C' - B'D'\cdot A'C' + C'D'\cdot A'B' = 0$$

which shows the well-known theorem due to Ptolemy.

Therefore Ptolemy's theorem may be considered as a particular case of Luchterhandt's theorem.

3. Next I will treat analogous theorems in space.

If A, B, C, D, E be any five points in space, then

(1) $(ABCD) - (BCDE) + (CDEA) - (DEAB) + (EABC) = 0,$

$(ABCD), \ldots\ldots$ being the volume of the tetrahedron $ABCD, \ldots\ldots$. Particularly, when these five points are on a sphere K whose radius is R, we have by the well-known theorem (1)

(1) Jungius (Biographie von Guhrauer 1850); Carnot, Mém. sur la relation qui existe entre les distances de cinq points etc.; Crelle, Math. Aufsätze I; v. Staudt, Crelle J. 57.

$$24R.\,(ABCD) = p_{ABCD}, \ldots\ldots,$$

where $p_{ABCD}, \ldots\ldots$ stand for

$$\sqrt{(AB\cdot CD + BC\cdot AD + AC\cdot BD)(-AB\cdot CD + BC\cdot AD + AC\cdot BD)(AB\cdot CD - BC\cdot AD + AC\cdot BD)(AB\cdot CD + BC\cdot AD - AC\cdot BD)}, \ldots$$

Hence

(2) $p_{ABCD} - p_{BCDE} + p_{CDEA} - p_{DEAB} + p_{EABC} = 0.$

Now if we invert this figure with respect to a point O which is not on the sphere K, and denote the inverses of A, B, C, D, E by A', B', C', D', E' respectively, then the points A', B', C', D', E' lie on a sphere K' and the equation (2) is transformed into

(3) $OE'^2\cdot p_{A'B'C'D'} - OA'^2\cdot p_{B'C'D'E'} + OB'^2\cdot p_{C'D'E'A'} - OC'^2\cdot p_{D'E'A'B'} + OD'^2\cdot p_{E'A'B'C'} = 0.$

But if R' be the radius of the sphere K',

$$24R'\cdot(A'B'C'D') = p_{A'B'C'D'}, \ldots\ldots .$$

Therefore we obtain the following equation

(4) $OE'^2\cdot(A'B'C'D') - OA'^2\cdot(B'C'D'E') + OB'^2\cdot(C'D'E'A) - OC'^2\cdot(D'E'A'B') + OD'^2\cdot(E'A'B'C') = 0,$

which shows Luchterhandt's theorem in space.

Again if we invert the figure thus obtained with respect to the point E' and denote the inverses of the points O', A', B', C', D' by O'', A'', B'', C'', D'' respectively, then the points A'', B'', C'', D'' lie on a plane K'' and the equation (3) is transformed into

(5) $p_{A''B''C''D''} - O''A''^2\cdot q_{B''C''D''} + O''B''^2\cdot q_{C''D''A''} - O''C''^2\cdot q_{D''A''B''} + O''D''^2\cdot q_{A''B''C''} = 0,$

where $q_{B''C''D''}, \ldots$ stand for

$$\sqrt{(B''C'' + C''D'' + D''B'')(-B''C'' + C''D'' + D''B)(B''C'' - C''D'' + D''B'')(B''C'' + C''D'' - D''B'')}, \ldots .$$

But by Hero's formula

$$q_{B''C''D''} = 4(B''C''D''), \ldots\ldots ,$$

we obtain the following equation

(6) $\frac{1}{4}p_{A''B''C''D''} - O''A''^2\cdot(B''C''D'') + O''B''^2\cdot(C''D''A'') - O''C''^2\cdot(D''A''B'') + O''D''^2\cdot(A''B''C'') = 0,$

which may be considered as an extension of Stewart's theorem. (1)

Remark I. If we invert (2) with respect to the point E, we get also this extension of Stewart's theorem.

Remark II. In the particular case where O coincides with E', (3) gives an extension of Ptolemy's theorem

$$E'A'^2\cdot p_{B'C'D'E'} - E'B'^2\cdot p_{C'D'E'A'} + E'C'^2\cdot p_{D'E'A'B'} - E'D'^2\cdot p_{E'A'B'C'} = 0.$$

(1) In his Leçons de géométrie élémentaire II, p. 315, Prof. Hadamard has given this extension of Stewart's theorem by barycentric calculus.

Remark III. In the particular case where A'', B'', C'', D'' are on a circle, (6) will take, by Ptolemy's theorem, the form

$$O''A''^2 \cdot (B''C''D'') - O''B''^2 \cdot (C''D''A'') + O''C''^2 \cdot (D''A''B'') - O''D''^2 \cdot (A''B''C'') = 0.$$

Hence in Luchterhand's theorem in plane, the point O need not lie in the plane of the circle. [1] Therefore it follows that Luchterhandt's theorem in plane may be considered as a particular case of the extension of Stewart's theorem in space.

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(1) This remark was mentioned already by Baltzer in his Determinanten, 4. Aufl., p. 252.

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