Note on the Representation of an Arbitrary Function in Mathematical Physics,

BY

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Dr. Haar has given some general theorems for the non-convergency and the summability of the series of orthogonal functions, and he has applied these theorems to the series of Sturm-Liouville's functions, etc. (1) Subsequently he has obtained the theorems analogous to Riemann's, Cantor's and Du Bois-Reymond's for the series of Sturm-Liouville's functions. (2)

The main object of this note is to treat (i) the series of orthogonal functions occurring in theory of cooling of a sphere and (ii) that in the theory of lateral vibration of a bar, by applying Haar's method.

I.

1. The solutions of the differential equation

$$\frac{d^2\varphi}{dx^2} + \rho^2\varphi = 0$$

with the boundary conditions

$$\varphi\,|^0 = 0, \qquad \frac{d\varphi}{dx} + H\varphi\,|^1 = 0 \qquad (H > 0),$$

are given by the orthogonal functions

$$\varphi_n(x) = \frac{\sqrt{2}}{\sqrt{1 - \dfrac{\sin\rho_n\cos\rho_n}{\rho_n}}}\sin\rho_n x \qquad (n = 0, 1, 2, \ldots.),$$

$\rho_0, \rho_1, \rho_2, \ldots.$ being the positive roots of the equation

$$\rho\cos\rho + H\sin\rho = 0.$$

Since it is known that

$$\rho_n = \frac{2n+1}{2}\pi + \frac{A_n}{n}$$

where A_n remains finite as n approaches to infinity, we have

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(1) Haar, Math. Ann., 69 (1910), p. 331.

(2) „ „ „ , 71 (1911), p. 38.

$$\sin \rho_n x = \sin \frac{2n+1}{2}\pi x + \frac{A'_n(x)}{n}, \quad \sin \rho_n = \pm 1 + \frac{A'}{n},$$

$$\cos \rho_n x = \cos \frac{2n+1}{2}\pi x + \frac{A''_n(x)}{n}, \quad \cos \rho_n = \frac{A''}{n},$$

and
$$\varphi_n(x) = \sqrt{2}\sin \frac{2n+1}{2}\pi x + \frac{A_n(x)}{n},$$

where $|A'|$, $|A''|$, $|A_n(x)|$, $|A'_n(x)|$, $|A''_n(x)|$ are smaller than a finite number independent of n and x.

The representation of an arbitrary function $f(x)$ by the series $\Sigma a_n \varphi_n(x)$ $(0<x<1)$ was treated by Prof. Fujisawa,[1] Prof. Kneser[2] and others. But their investigations were restricted to the convergence problem of Fourier's series with respect to $\varphi_n(x)$

$$\sum_{n=0}^{\infty} \sin \rho_n x \frac{2\int_0^1 f(u) \sin \rho_n u \, du}{1 - \frac{\sin \rho_n \cos \rho_n}{\rho_n}} \text{ [3]}.$$

2. In this article I will prove the following theorem which may be called the non-convergence theorem of Fourier's series with respect to $\varphi_n(x)$.

There exists Fourier's series with respect to $\varphi_n(x)$, corresponding to a continuous function, which fails to converge at a certain point.

If we put

$$K_n(a, t) = \varphi_1(a)\varphi_1(t) + \varphi_2(a)\varphi_2(t) + \dots + \varphi_n(a)\varphi_n(t),$$

then

$$K_n(a, t) = 2\sum_{p=1}^{n} \sin \frac{2p+1}{2}\pi a \cdot \sin \frac{2p+1}{2}\pi t + \Phi_n(a, t)$$

(1) Fujisawa, Ueber eine in der Wärmeleitungstheorie auftretende nach den Wurzeln einer transcendenten Gleichung fortschreitende Reihe, Strassburger Dissertation (1886); Journal of the College of Science, Tôkyô Imperial University, 2 (1888), p. 1.

(2) Kneser, Math. Ann., 58 (1904), p. 101.

(3) The functions $\varphi_n(x)$ are characteristic functions of the integral equation

$$\varphi_n(x) = \rho_n^2 \int_0^1 K(x, \xi)\varphi_n(\xi)\, d\xi$$

having the continuous and symmetrical kernel

$$K(x, \xi) = x\frac{1 + H(1-\xi)}{1+H} \qquad \xi > x,$$
$$= \xi \frac{1 + H(1-x)}{1+H} \qquad \xi < x.$$

Hence an arbitrary "quellenmässige" function $f(x)$ may be represented as Fourier's series with respect to $\varphi_n(x)$ which converges uniformly and absolutely.

where

$$\Phi_n(a, t) = \sqrt{2}\sum_{p=1}^{n} \frac{A_p(t)}{p} \sin \frac{2p+1}{2}\pi a$$
$$+ \sqrt{2}\sum_{p=1}^{n} \frac{A_p(a)}{p} \sin \frac{2p+1}{2}\pi t + \sum_{p=1}^{n} \frac{A_p(a) \cdot A_p(t)}{p^2}.$$

Since
$$|A_p(x)| < G,$$

G being a positive finite number independent of p and x, from the convergency of $\sum_{p=1}^{\infty} \frac{1}{p} \sin \frac{2p+1}{2}\pi x$, it follows that $|\Phi_n(a, t)|$ is smaller than a finite number independent of n, a and t.

Next in

$$2\sum_{p=1}^{n} \sin \frac{2p+1}{2}\pi a \cdot \sin \frac{2p+1}{2}\pi t$$
$$= \frac{\sin(n+1)\pi(a-t) - \sin\pi(a-t)}{2 \sin \frac{\pi(a-t)}{2}} - \frac{\sin(n+1)\pi(a+t) - \sin\pi(a+t)}{2\sin\frac{\pi(a+t)}{2}},$$

$$\left| \frac{\sin(n+1)\pi(a+t) - \sin\pi(a+t)}{2\sin\frac{\pi(a+t)}{2}} \right| \text{ and}$$

$$\left| \frac{\sin \pi(a-t)}{2 \sin \frac{\pi(a-t)}{2}} \right| = \left| \cos \frac{\pi(a-t)}{2} \right| \quad \begin{pmatrix} 0 < a < 1 \\ 0 \leqq t \leqq 1 \end{pmatrix}$$

are smaller than a finite number independent of n, a, t.

But
$$\int_0^1 \left| \frac{\sin(n+1)\pi(t-a)}{\sin\frac{\pi(t-a)}{2}} \right| dt = \frac{2}{\pi}\int_{-\frac{a}{2}\pi}^{\frac{1-a}{2}\pi} \left| \frac{\sin(2n+2)\vartheta}{\sin\vartheta} \right| d\vartheta$$

$$> \frac{2}{\pi}\int_0^{\frac{1-a}{2}\pi} \left| \frac{\sin(2n+2)\vartheta}{\sin\vartheta} \right| d\vartheta > \frac{2}{\pi}\int_0^{\frac{1-a}{2}\pi} \left| \frac{\sin(2n+2)\vartheta}{\vartheta} \right| d\vartheta.$$

Now since

$$|\sin(2n+2)\vartheta| > \sin\frac{\pi}{8} = \mu$$

in the interval

$$\frac{\frac{1}{8}+p}{2n+2}\pi < \vartheta < \frac{\frac{7}{8}+p}{2n+2}\pi \qquad (p = 0, 1, \dots)$$

we obtain

$$\int_0^{\frac{1-a}{2}\pi} \left| \frac{\sin(2n+2)\vartheta}{\vartheta} \right| d\vartheta \geqq \mu \sum_{p=1}^{\nu} \log\left(1 + \frac{6}{8p+1}\right),$$

ν being the greatest positive integer such that

$$\frac{\frac{7}{8}+\nu}{2n+2} \leqq \frac{1-a}{2}.$$

But since ν diverges to infinity with n, and since the series $\sum_{p=1}^{\infty} \log\left(1+\frac{6}{8p+1}\right)$ diverges, we can choose n so that

$$\int_0^1 \left| \frac{\sin(n+1)\pi(t-a)}{\sin\frac{\pi(t-a)}{2}} \right| dt$$

is greater than any assigned positive value.(1)

Hence $$\lim_{n=\infty} \int_0^1 |K_n(a,t)|\, dt = \infty.$$

Therefore, by Haar's theorem, (2) it follows that there exists Fourier's series with respect to $\varphi_n(x)$, corresponding to a continuous function $f(x)$, which does not converge at the point $x=a$.

3. Next I will prove the analogue of Fejér's theorem.

Fourier's series with respect to $\varphi_n(x)$, corresponding to a function which lies in the domain of the functions $\varphi_n(x)$, is always summable. Particularly, Fourier's series with respect to $\varphi_n(x)$, corresponding to a continuous functions $f(x)$, is always summalle, and its sum is equal to $f(x)$.

As in the case of Sturm-Liouville's function, in order to prove this theorem, we are only to shew that

$$\int_0^1 \left| \frac{K_1(a,t)+K_2(a,t)+\ldots+K_n(a,t)}{n} \right| dt \quad (0<a<1)$$

is smaller than a finite number independent of n and a (3).

Now we have

(1) This may be proved also by Fejér's method. See Fejér, Journ. f. Math., 138 (1910), p. 28.
(2) Haar, Math. Ann., 69, p. 335.
(3) Haar, loc. cit. p. 357.

$$\frac{1}{n}\sum_{p=1}^{n} K_p(a,t) = \frac{2}{n}\sum_{p=1}^{n}\sum_{r=1}^{p} \sin\frac{2r+1}{2}\pi a \cdot \sin\frac{2r+1}{2}\pi t + \frac{1}{n}\sum_{p=1}^{n}\Phi_p(a,t)$$

$$= \frac{\sin\frac{2n+3}{2}\pi(a-t)\cdot\sin\frac{n}{2}\pi(a-t)}{2n\left\{\sin\frac{\pi}{2}(a-t)\right\}^2} - \cos\frac{\pi}{2}(a-t)$$

$$+\cos\frac{\pi}{2}(a+t) + \frac{\Psi_n(a,t)}{n} + \frac{1}{n}\sum_{p=1}^{n}\Phi_p(a,t)$$

$$= \frac{1}{2n}\left\{\cos\frac{\pi}{4}(a-t)\frac{\sin\frac{2n+1}{2}\pi(a-t)}{\sin\frac{\pi}{2}(a-t)}\right\}^2$$

$$+\frac{1}{n}\frac{\sin(n+1)\pi(a-t)}{\sin\frac{\pi}{2}(a-t)} + \cos\frac{\pi}{2}(a+t)$$

$$-\cos\frac{\pi}{2}(a-t) + \frac{\Psi_n(a,t)}{n} + \frac{1}{n}\sum_{p=1}^{n}\Phi_p(a,t)$$

where $|\Psi_n(a,t)|$ and $\left|\frac{1}{n}\sum_{p=1}^{n}\Phi_p(a,t)\right|$ are smaller than a finite number independent of n, a, t.

But

$$\frac{1}{n}\int_0^1 \left| \frac{\sin(n+1)\pi(a-t)}{\sin\frac{\pi}{2}(a-t)} \right| dt = \frac{2}{n\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}(1-a)} \left| \frac{\sin(2n+2)\vartheta}{\sin\vartheta} \right| d\vartheta$$

$$< \frac{4}{n\pi}\int_0^{\frac{\pi}{2}} \left| \frac{\sin(2n+2)\vartheta}{\vartheta} \right| d\vartheta < \frac{4}{n\pi}\cdot\frac{\pi}{2}(2n+2).$$

Similarly $$\frac{1}{n}\int_0^1 \left| \cos\frac{\pi}{4}(a-t)\frac{\sin\frac{2n+1}{2}\pi(a-t)}{\sin\frac{\pi}{2}(a-t)} \right|^2 dt$$

is smaller than a finite number independent of a and n. Thus our theorem is proved.

4. If the series

$$f(x) = \sum a_n\varphi_n(x) \qquad (\lim_{n=\infty} a_n = 0)$$

converge at the point $x=a$, then it is easily seen that the series

$$F(x) = -\sum \frac{a_n\varphi_n(x)}{\rho_n^2}$$

converges at the point $x = a$ uniformly and absolutely.

Here I will prove that

$$\lim_{\delta=0}\frac{F(a+\delta)-2F(a)+F(a-\delta)}{\delta^2}=f(a);$$

that is, *Rieman's theorem of trigonometrical series holds good in the series* $\Sigma a_n\varphi_n(x)$.

If we put $\Delta^2F(a,\delta)=F(a+\delta)-2F(a)+F(a-\delta)$, then

$$\frac{\Delta^2F(a,\delta)}{\delta^2}=\Sigma a_n\sqrt{2}\sin\frac{2n+1}{2}\pi a\cdot\left(\frac{\frac{2n+1}{2}\pi}{\rho_n}\right)^2\left\{\frac{\sin\frac{2n+1}{2}\pi\frac{\delta}{2}}{\frac{2n+1}{2}\pi\frac{\delta}{2}}\right\}^2$$

$$+\Sigma\frac{a_n}{\rho_n^2}\frac{\Delta^2A_n(a,\delta)}{\delta^2}.$$

From the equation

$$\frac{d^2\varphi_n(x)}{dx^2}+\rho_n^2\varphi_n(x)=0,$$

we get

$$\frac{1}{n}\frac{d^2A_n(x)}{dx^2}+\frac{A_n(x)}{n}+\sqrt{2}\sin\frac{2n+1}{2}\pi x\left\{\rho_n^2-\left(\frac{2n+1}{2}\pi\right)^2\right\}=0.$$

Again if we put

$$\frac{\Delta^2A_n(a,\delta)}{\delta^2}=\frac{d^2A_n(a)}{da^2}+\varepsilon_n(a),$$

we can take δ so that $|\varepsilon_n(a)|<\varepsilon$, ε being an arbitrary small quantity independent of n, a, and $\lim_{\delta=0}\varepsilon=0$.

Hence

$$\frac{\Delta^2F(a,\delta)}{\delta^2}=\Sigma a_n\left(\frac{\sqrt{2}\sin\frac{2n+1}{2}\pi a}{\rho_n^2}\left[\left\{\left(\frac{\sin\frac{2n+1}{2}\pi\frac{\delta}{2}}{\frac{2n+1}{2}\pi\frac{\delta}{2}}\right)^2-1\right\}\left(\frac{2n+1}{2}\pi\right)^2-\rho_n^2\right]+\frac{A_n(a)}{n}-\frac{\varepsilon_n(a)}{n\rho_n^2}\right).$$

But since $\Sigma\frac{a_n}{n\rho_n^2}$ and $\Sigma a_n\frac{\varepsilon_n(a)}{n\rho_n^2}$ converge uniformly and absolutely, and since

$$\Sigma\left|\frac{a_n\varepsilon_n(a)}{n\rho_n^2}\right|<\varepsilon\,\Sigma\left|\frac{a_n}{n\rho_n^2}\right|,$$

we get

$$\lim_{\delta=0}\Sigma\left|\frac{a_n\varepsilon_n(a)}{n\rho_n^2}\right|=0.$$

Hence it follows that

$$\lim_{\delta=0}\Sigma\frac{a_n\varepsilon_n(a)}{n\rho_n^2}=0.$$

Now in

$$\Sigma a_n\left(\frac{\frac{2n+1}{2}\pi}{\rho_n}\right)^2\cdot\sqrt{2}\cdot\sin\frac{2n+1}{2}\pi a\left\{\frac{\sin\frac{2n+1}{2}\pi\frac{\delta}{2}}{\frac{2n+1}{2}\pi\frac{\delta}{2}}\right\}^2,$$

if we put

$$a_n'=\sqrt{2}\,a_n\left(\frac{\frac{2n+1}{2}\pi}{\rho_n}\right)^2,$$

then $\lim_{n=\infty}a'_n=0$;

and $\Sigma a'_n\sin\frac{2n+1}{2}\pi a$ is covergent.

Hence, by a method similar to the proof of Riemann's theorem,(1) we can see that

$$\lim_{\delta=0}\Sigma a'_n\sin\frac{2n+1}{2}\pi a\left\{\frac{\sin\frac{2n+1}{2}\pi\frac{\delta}{2}}{\frac{2n+1}{2}\pi\frac{\delta}{2}}\right\}^2=\Sigma a'_n\sin\frac{2n+1}{2}\pi a.$$

Hence we have the final result

$$\lim_{\delta=0}\frac{\Delta^2E(a,\delta)}{\delta^2}=\Sigma a_n\left\{\sqrt{2}\sin\frac{2n+1}{2}\pi a+\frac{A_n(a)}{n}\right\}$$

$$=f(a).$$

From this result, by the repetition of Haar's method of proof, we can prove the analogues of Cantor's and Du Bois-Reymond's theorems. Hence we shall not go on further.

II.

5. The solutions of the differential equation

(1) Riemann, Gesammelte Werke, 2. Aufl., p. 246.

$$\frac{d^4\varphi}{dx^4} - \lambda^4\varphi = 0$$

with the boundary conditions

$$\varphi\,|^0 = \varphi\,|^1 = 0, \qquad \frac{d\varphi}{dx}\Big|^0 = \frac{d\varphi}{dx}\Big|^1 = 0,$$

are given by the orthogonal functions

$$\varphi_n(x) = \cos\lambda_n x - \operatorname{ch}\lambda_n x - (\sin\lambda_n x - \operatorname{sh}\lambda_n x)\,\frac{\cos\lambda_n - \operatorname{ch}\lambda_n}{\sin\lambda_n - \operatorname{sh}\lambda_n},$$

$\lambda_0, \lambda_1, \lambda_2, \ldots$ being the positive roots of the equation

$$\cos\lambda \cdot \operatorname{ch}\lambda = 1.$$

It is known that

$$\lambda_n = \frac{2n+1}{2}\pi - B_n e^{-\frac{2n+1}{2}\pi}$$

and

$$\varphi_n(x) = \sqrt{2}\cos\left(\frac{2n+1}{2}\pi x + \frac{\pi}{4}\right) + (-1)^n e^{-\frac{2n+1}{2}\pi(1-x)} - e^{-\frac{2n+1}{2}\pi x} + B_n(x)\cdot e^{-\frac{2n+1}{2}\pi},$$

where $|B_n|$ and $|B_n(x)|$ are smaller than a finite number independent of n and x.

Recently Profs. Stekloff and Tamarkine (1) have obtained some convergence theorems and also a summation theorem of Fourier's series with respect to $\varphi_n(x)$. Here I will give another proof of the summation theorem, and also the non-convergence theorem, the analogues of Riemann's and Cantor's theorems. In the proof of the last theorem, it will be found the application of Dr. M. Riesz's extension (2) of Schwarz's theorem.

6. First of all I will prove the non-convergence theorem:

There exists Fourier's series with respect to $\varphi_n(x)$, corresponding to a continuous function, which fails to converge at the point $x = a$ $(0 < a < 1)$.

If we put

(1) Stekloff et Tamarkine. Problème des vibrations transversales d'une verge élastique homogene, Rendiconti del Circolo mat. di Palermo, 31 (1911), p. 341.

(2) M. Riesz, Math. Ann., 71 (1911), p. 61-63.

$$K_n(a, t) = \varphi_1(a)\,\varphi_1(t) + \varphi_2(a)\,\varphi_2(t) + \ldots + \varphi_n(a)\,\varphi_n(t),$$

then

$$K_n(a, t) = 2\sum_{p=1}^{n}\cos\left(\frac{2p+1}{2}\pi a + \frac{\pi}{4}\right)\cos\left(\frac{2p+1}{2}\pi t + \frac{\pi}{4}\right) + \phi_n(a, t),$$

where

$$\phi_n(a, t) = \sqrt{2}\sum_{p=1}^{n}(-1)^p e^{-\frac{2p+1}{2}\pi(1-t)}\cdot\cos\left(\frac{2p+1}{2}\pi a + \frac{\pi}{4}\right)$$

$$-\sqrt{2}\sum_{p=1}^{n} e^{-\frac{2p+1}{2}\pi t}\cdot\cos\left(\frac{2p+1}{2}\pi a + \frac{\pi}{4}\right) + \sum_{p=1}^{n}\omega_p(a, t),$$

the series $\sum_{p=1}^{\infty}\omega_p(a, t)$ $(0 \leqq t \leqq 1)$ being convergent.

Since

$$\int_0^1 |K_n(a, t)|\,dt > \int_\eta^{1-\eta} |K_n(a, t)|\,dt \quad (0 < a < 1)$$

where η is a positive number such that $1 - a\eta > 0$, we are only to prove

$$\lim_{n=\infty}\int_\eta^{1-\eta} |K_n(a, t)|\,dt = \infty.$$

Now $\displaystyle\sum_{p=1}^{\infty}(-1)^p e^{-\frac{2p+1}{2}\pi(1-t)}\cdot\cos\left(\frac{2p+1}{2}\pi a + \frac{\pi}{4}\right)$

is convergent if $\quad 0 \leqq t \leqq 1 - \eta < 1$,

and also $\displaystyle\sum_{p=1} e^{-\frac{2p+1}{2}\pi t}\cdot\cos\left(\frac{2p+1}{2}\pi a + \frac{\pi}{4}\right)$

is convergent if $\quad 0 < \eta \leqq t \leqq 1$.

And we have

$$2\sum_{p=1}^{n}\cos\left(\frac{2p+1}{2}\pi a + \frac{\pi}{4}\right)\cdot\cos\left(\frac{2p+1}{2}\pi t + \frac{\pi}{4}\right)$$

$$= \frac{\cos(n+1)\pi(a+t) - \cos\pi(a+t)}{2\sin\frac{\pi(a+t)}{2}}$$

$$+ \frac{\sin(n+1)\pi(a-t) - \sin\pi(a-t)}{2\sin\frac{\pi(a-t)}{2}}.$$

But as in Art. 2, we see that

$$\int_{\eta}^{1-\eta}\left|\frac{\sin(n+1)\pi(t-a)}{\sin\dfrac{\pi(t-a)}{2}}\right|dt>\mu\sum_{p=1}^{\nu'}\log\left(1+\frac{6}{8p+1}\right)$$

where ν' is the greatest positive integer such that

$$\frac{\dfrac{7}{8}+\nu'}{2n+2}\leqq\frac{1-a-\eta}{2}.$$

Hence

$$\lim_{n=\infty}\int_{\eta}^{1-\eta}|K_n(a,t)|\,dt=\infty.$$

Thus our theorem is proved.

7. In this article, by Haar's method, I will prove the following summation theorem due to Stekloff and Tamarkine.

Fourier's series with respect to $\varphi_n(x)$, corresponding to a continuous function $f(x)$, is always summable and its sum is equal to $f(x)$. This theorem may be generalised as in Art. 3.

If we put

$$s_p=\sum_{r=1}^{p}\sin(2r+1)\frac{\pi a}{2},\qquad s'_p=\sum_{r=1}^{p}(-1)^r\sin(2r+1)\frac{\pi a}{2},$$

$$c_p=\sum_{r=1}^{p}\cos(2r+1)\frac{\pi a}{2},\qquad c'_p=\sum_{r=1}^{p}(-1)^r\cos(2r+1)\frac{\pi a}{2},$$

then it is well-known that

$$\lim_{n=\infty}\frac{1}{n}\sum_{p=1}^{n}s_p=\frac{1}{2}\operatorname{cosec}\frac{\pi a}{2},\quad \lim_{n=\infty}\frac{1}{n}\sum^{n}s_p'=0,$$

$$\lim_{n=\infty}\frac{1}{n}\sum_{p=1}^{n}c_p=0,\quad \lim_{n=\infty}\frac{1}{n}\sum_{p=1}^{n}c_p'=-\frac{1}{2}\sec\frac{\pi a}{2}.\ (^1)\qquad (0<a<1),$$

From these results, it follows that

$$\left|\frac{1}{n}\sum_{p=1}^{n}\phi_p(a,t)\right|\qquad(0<a<1,\ 0\leqq t\leqq 1)$$

is smaller than a finite number independent of a and t.

But we have

$$\frac{2}{n}\sum_{p=1}^{n}\sum_{r=1}^{p}\cos\left(\frac{2r+1}{2}\pi a+\frac{\pi}{4}\right)\cos\left(\frac{2r+1}{2}\pi t+\frac{\pi}{4}\right)$$

$$=\left\{\frac{\sin(n+3)\vartheta\cdot\sin n\vartheta}{2n\sin^2\vartheta}-\cos\vartheta\right\}+\left\{\frac{\cos(n+3)\vartheta'\cdot\sin n\vartheta'}{2n\sin^2\vartheta'}-\cos\vartheta'\right\}$$

(1) Bromwich, Infinite series (1908), p. 276.

where $\qquad \vartheta=\dfrac{\pi(a-t)}{2},\qquad \vartheta'=\dfrac{\pi(a+t)}{2}.$

Hence the method used in Art. 3 can be applied at once to shew that

$$\int_0^1\left|\frac{1}{n}\sum_{p=1}^{n}K_p(a,t)\right|dt\qquad(0<a<1)$$

is smaller than a finite number independent of a and n.

Thus our theorem is proved.

8. If the series

$$f(x)=\Sigma\,a_n\varphi_n(x)\qquad\left(\lim_{n=\infty}a_n=0\right)$$

converge at the point $x=a$, then it is easily seen that the series

$$F(x)=\Sigma\frac{a_n}{\lambda_n^4}\varphi_n(x)$$

converge at the point $x=a$ uniformly and absolutely. Here I will prove the analogues of Riemann's theorem:

If we put

$$\Delta^4F(a,\delta)=F(a+2\delta)-4F(a+\delta)+6F(a)-4F(a-\delta)+F(a-2\delta),$$

then $$\lim_{\delta=0}\frac{\Delta^4F(a,\delta)}{\delta^4}=f(a).$$

As in Art. 4, by using the equation

$$\frac{d^4\varphi_n(x)}{dx^4}-\lambda_n^4\varphi_u(x)=0$$

we get

$$\frac{\Delta^4F(a,\delta)}{\delta^4}=\Sigma a_n\sqrt{2}\cos\left(\frac{2n+1}{2}\pi a+\frac{\pi}{4}\right)\left(\frac{\dfrac{2n+1}{2}\pi}{\lambda_n}\right)^4\left\{\frac{\sin\dfrac{2n+1}{2}\pi\dfrac{\delta}{2}}{\dfrac{2n+1}{2}\pi\dfrac{\delta}{2}}\right\}^4$$

$$+\Sigma a_n\left\{\left[1-\left(\frac{\dfrac{2n+1}{2}\pi}{\lambda_n}\right)^4\right]\sqrt{2}\cos\left(\frac{2n+1}{2}\pi a+\frac{\pi}{4}\right)+v_n(a)\right\}+\Sigma\frac{a_n}{\lambda_n^4}\varepsilon_n(a),$$

where

$$v_n(a)=\varphi_n(a)-\sqrt{2}\cos\left(\frac{2n+1}{2}\pi a+\frac{\pi}{4}\right)$$

$$=(-1)^n e^{-\frac{2n+1}{2}\pi(1-a)}-e^{-\frac{2n+1}{2}\pi a}+B_n(a)\,e^{-\frac{2n+1}{2}\pi},$$

and $$|\varepsilon_n(a)|<\varepsilon$$

ε being an arbitrary small number independent of n, a, and $\lim_{\delta=0}\varepsilon=0$.

Hence

$$\lim_{\delta=0}\frac{\Delta^4 F(a,\delta)}{\delta^4} = \lim_{\delta=0}\Sigma a_n\sqrt{2}\cos\left(\frac{2n+1}{2}\pi a+\frac{\pi}{4}\right).$$

$$\cdot\left(\frac{\frac{2n+1}{2}\pi}{\lambda_n}\right)^4\cdot\left\{\frac{\sin\frac{2n+1}{2}\pi\frac{\delta}{2}}{\frac{2n+1}{2}\pi\frac{\delta}{2}}\right\}^4$$

$$+\Sigma a_n\left\{\left[1-\left(\frac{\frac{2n+1}{2}\pi}{\lambda_n}\right)^4\right]\sqrt{2}\cos\left(\frac{2n+1}{2}\pi a+\frac{\pi}{4}\right)+v_n(a)\right\}.$$

If we put $$a'_n = a_n\left(\frac{\frac{2n+1}{2}\pi}{\lambda_n}\right)^4,$$

then $$\lim_{n=\infty} a'_n = 0;$$

and

$$\Sigma a'_n\left(\cos\frac{2n+1}{2}\pi a-\sin\frac{2n+1}{2}\pi a\right)=\Sigma\left(\frac{\frac{2n+1}{2}\pi}{\lambda_n}\right)^4\{a_n\varphi_n(a)-a_n v_n(a)\}$$

is convergent.

Hence, be Fejer's theorem,(1) we can see that

$$\lim_{\delta=0}\Sigma a'_n\left(\cos\frac{2n+1}{2}\pi a-\sin\frac{2n+1}{2}\pi a\right)\left\{\frac{\sin\frac{2n+1}{2}\pi\frac{\delta}{2}}{\frac{2n+1}{2}\pi\frac{\delta}{2}}\right\}^4$$

$$=\Sigma a'_n\left(\cos\frac{2n+1}{2}\pi a-\sin\frac{2n+1}{2}\pi a\right).$$

Therefore we have the final result

$$\lim_{\delta=0}\frac{\Delta^4 F(a,\delta)}{\delta^4} = \Sigma a_n\left\{\sqrt{2}\cos\left(\frac{2n+1}{2}\pi a+\frac{\pi}{4}\right)+v_n(a)\right\}$$

$$=f(a).$$

9. Lastly I will prove the analogue of Cantor's theorem:

If $$\sum_{n=0}^{\infty} a_n\varphi_n(x) = 0$$

for all points of the interval $0<x<1$, *then*

$$a_n = 0 \qquad (n=0,1,2,3\ \ldots).$$

(1) Fejér, Math. Ann., 58 (1904), p. 69.

Since $\Sigma a_n\varphi_n(x)$ is convergent,

$$F(x) = \Sigma\frac{a_n}{\lambda_n^4}\varphi_n(x) \qquad \left(\lambda_n = \frac{2n+1}{2}\pi + B_n e^{-\frac{2n+1}{2}\pi}\right)$$

is uniformly convergent.

But from the equation

$$\frac{d^4\varphi_n(x)}{dx^4} = \lambda_n^4\psi_n(x)$$

we have $$\frac{d^2\varphi_n(x)}{dx^2} = \lambda_n^4\int^x\!\!\int^x\varphi_n(x)\,dx\,dx$$

$$=\frac{\lambda_n^4}{\left(\frac{2n+1}{2}\pi\right)^2}\varphi_n(x)\,c_n(x)$$

where $|c_n(x)|$ is smaller than a finite number independent of n and x.

Hence

$$\Sigma\frac{a_n}{\left(\frac{2n+1}{2}\pi\right)^2}\varphi_n(x)\,c_n(x) \quad\text{and}\quad \Sigma\frac{a_n}{\lambda_n^4}\frac{d^2\varphi_n(x)}{dx^2}$$

are uniformly convergent. Therefore it follows that $F(x)$ and $\frac{d^2F(x)}{dx^2}$ are continuous function of x $(0<x<1)$.

And by the last article we have

$$\lim_{\delta=0}\frac{\Delta^4 F(a,\delta)}{\delta^4} = 0.$$

Hence by M. Riesz's theorem which is an extention of Schwarz's theorem, we see that $F(x)$ is a polynomial of the third degree. So we have $$\frac{d^4F(x)}{dx^4} = 0.$$

Now since

$$F(x) = \Sigma\frac{a_n}{\lambda_n^4}\varphi_n(x)$$

is uniformly convergent and

$$\int_0^1\{\varphi_n(x)\}^2\,dx = 1, \quad \int_0^1\varphi_m(x)\,\varphi_n(x)\,dx = 0 \quad (m \gtrless n),$$

we get

$$\frac{a_n}{\lambda_n^4} = \int_0^1 F(x)\,\varphi_n(x)\,dx.$$

But

$$\frac{d^4 \varphi_n(x)}{d x^4} = \lambda_n^4 \varphi_n(x)$$

and

$$\frac{d^4 F(x)}{d x^4} = 0 .$$

Hence

$$a_n = \int_0^1 \left\{ \frac{d^4 \varphi_n(x)}{d x^4} F(x) - \varphi_n(x) \frac{d^4 F(x)}{d x^4} \right\} d x$$

$$= \left[\frac{d^3 \varphi_n(x)}{d x^3} F(x) - \frac{d^2 \varphi_n(x)}{d x^2} \frac{d F(x)}{d x} + \frac{d \varphi_n(x)}{d x} \frac{d^2 F(x)}{d x^2} - \varphi_n(x) \frac{d^3 F(x)}{d x^3} \right]_0^1 .$$

Since

$$F(x) = \Sigma \frac{a_n}{\lambda_n^4} \varphi_n(x) \qquad \text{and} \qquad \frac{d F(x)}{d x} = \Sigma \frac{a_n}{\lambda_n^4} \frac{d \varphi_n(x)}{d x}$$

are uniformly convergent, from the boundary conditions

$$\varphi(x) \Big|^0 = \varphi(x) \Big|^1 = \frac{d \varphi(x)}{d x} \Big|^0 = \frac{d \varphi(x)}{d x} \Big|^1 = 0 ,$$

we have

$$F(x) \Big|^0 = F(x) \Big|^1 = \frac{d F(x)}{d x} \Big|^0 = \frac{d F(x)}{d x} \Big|^1 = 0 .$$

Therefore we obtain $\quad a_n = 0 .$

We shall not give the proof of the analogue of Du Bois-Reymond's theorem, for it is easily obtained by Haar's method.

Sendai, November 1911.