「学位論文」

Safety Assist Control via Zeroing Control Barrier Function

(零化制御バリア関数を用いた安全アシスト制御に関する研究)

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Abstract

The demand for safety assurance of human-operated systems has been increasing as Advanced Emergency Braking (AEB) systems become mandatory for all new cars. Technologies as typified by AEB systems belong to a class of human assist control, as they try to assist human-operated systems so that safety constraints are always satisfied. However, the design of a human assist controller is challenging because it requires not only safety assurance but also continuity and optimality. The latter properties, continuity and optimality, lead to increasing operability and passenger comfort, as they indicate that a controller does not suddenly intervene and prevents violation of safety constraints with minimal modification of human operator inputs.

Control Barrier Functions (CBFs) attract attention in recent nonlinear control theory as they can establish forward invariant sets. If a prescribed set is forward invariant, the states of systems starting from the set stay there eternally. A CBF-based controller renders a safe set where the states of systems should stay forward invariant; that is, it produces theoretical safety assurance of control systems. There are two general types of CBFs that have contrasting properties: Reciprocal CBFs (RCBFs) that blow up and Zeroing CBFs (ZCBFs) that vanish on the boundary of a safe set. Although RCBFs are more suitable for some applications, ZCBFs are preferable as they are well defined outside of a safe set. As a matter of fact, they are mainly utilized for the design of a safety filter due to their robustness properties. However, the ZCBFs employed in many previous studies have some mathematical problems. For example, they cannot yield the theoretical safety of nonautonomous systems where human-operated systems are categorized. Moreover, a ZCBF-based safety filter fails to be continuous without a relative degree condition, which is not ideal for human assist control.

In this dissertation, the author proposes a ZCBF-based safety (human) assist controller that guarantees the theoretical safety of nonautonomous systems. The results of this dissertation mainly consist of two parts: a safety assist controller for time-invariant state constraint problems such as collision avoidance with stationary obstacles and a safety assist controller for time-dependent state constraint problems such as collision avoidance with moving obstacles.

For time-invariant safety constraints, the author provides a strict version of ZCBFs, Strict Zeroing Control Barrier Functions (S-ZCBFs), that solve the problems of the conventional ZCBF definition. Then, the author proposes an S-ZCBF-based safety (human) assist controller that renders a safe set forward invariant, indicating that it guarantees the theoretical safety of nonautonomous systems. It is worth emphasizing that the proposed controller prevents solutions to nonautonomous systems from having a finite escape time; that is, solutions are global and result in forward invariance. Moreover, the author shows that the proposed controller is continuous and minimally invasive, which is ideal for human assist control. Lastly, considering that almost all real control systems have input constraints,

the author proposes a safety assist controller that satisfies both state and input constraints.

For time-dependent safety constraints, the author first extends S-ZCBF to Time-varying Zeroing Control Barrier Functions (Tv-ZCBFs). The author then proposes a Tv-ZCBF-based safety assist controller that fulfills time-dependent state constraints while inheriting the ideal properties of the S-ZCBF-based controller. Noticing that the velocity of time-varying obstacles is unknown to control systems, the author also proposes a safety assist controller for control systems in unknown environments or under input disturbances. The proposed controller satisfies the safety constraints of input-disturbed systems, although the conventional method only ensures the boundedness of the system's states.

In Chapter 1, the author emphasizes the importance and difficulty of human assist control. Then, the author summarizes the recent studies on CBFs and points out the problems of conventional methods.

In Chapter 2, the author introduces some basic definitions and mathematical foundations that will be used in this dissertation.

In Chapter 3, the author discusses a time-invariant state constraint problem. In the first part of this chapter, the author proposes an S-ZCBF-based safety assist controller that guarantees the theoretical safety of human-operated systems. The second part of this chapter demonstrates the design of CBFs for input-constrained systems and proposes a safety assist controller that satisfies both state and input constraints. The effectiveness of the proposed methods is confirmed by computer simulation.

In Chapter 4, the author discusses a time-dependent state constraint problem. In the first part of this chapter, the author proposes a Tv-ZCBF-based safety assist controller that is applicable to time-dependent safety constraints, having the properties of the S-ZCBF-based controller. The second part of this chapter proposes a safety assist controller for control systems under input disturbances, motivated by unknown time-varying obstacle velocity. The last part of this chapter confirms the effectiveness of the proposed methods by computer simulation and experiments on an electric wheelchair.

In Chapter 5, the author summarizes this study and refers to open issues of the control methods proposed in this dissertation.

Keywords:

Nonlinear Control, Control Barrier Function, Human Assist control.

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Chapter 1

Introduction

1.1 Background

Automatic Emergency Braking (AEB) systems have been mandatory on all new passenger vehicles introduced in Japan since 2021 and in the European Union since 2022. More recently, the U.S. Department of Transportation's National Highway Traffic Safety Administration (NHTSA) has advocated for all new passenger cars and light trucks sold in the United States to be equipped with AEB systems. AEB is a safety technology designed to assist drivers in mitigating or avoiding collisions. The growing demand for AEB systems is mainly due to traffic accidents caused by human errors. According to NHTSA, the main cause of 94 % critical pre-crash events is attributed to drivers [42, 60]. On the other hand, previous studies [21, 24] showed that vehicles equipped with AEB systems experienced approximately 38 % fewer rear-end crashes, and AEB pedestrian detection systems reduced pedestrian fatalities by approximately 27 %, indicating the effectiveness of AEB technology. This way of thinking about safety-critical control expands not only to cars but also to robotic systems [51], quadrotors [9], and so on [23].

Recent research on nonlinear control systems tends to prioritize safety beyond stability as safety-critical systems are deployed into increasingly complex real-world environments due to the rapid development of sensing, communication, and computation technology. Technologies such as AEB and Lane Keeping Assistance (LKA) systems are related to safety-critical control because they assist human operators in meeting preliminarily defined safety constraints, e.g., collision avoidance with obstacles. However, under the premise of the existence of operators or passengers, meeting safety constraints may not be sufficient from the point of view of innocuity [31]. For example, a discontinuous safety assist controller interferes with the operability of human-operated systems or decreases passenger comfort. According to the National Consumer Affairs Center of Japan, a sudden AEB intervention that potentially comes from discontinuity arouses fears of passengers [52]. Moreover, it might cause unexpected behaviors of control systems that affect the environment and result in another risky event [42]. Therefore, theoretical safety of human-operated systems should be guaranteed by a continuous and minimally invasive safety assist controller.

1.1.1 Control Barrier Function

In recent nonlinear control theory, Control Barrier Functions (CBFs) have rapidly become of huge interest as they can yield theoretical safety assurance with the notion of forward invariance [4, 31]. When we say that a prescribed set is forward invariant, then the system's states starting from the set stay there for all the future time [16]. CBFs are known to be a powerful tool for establishing forward invariant sets. Concretely, a controller produced from CBFs, often referred to as a safety filter, renders a safe set, whose outside indicates violation of safety constraints, forward invariant [7]. The basic idea of safety filters is to modify a nominal controller that works for the primary purpose of the application, such as equilibrium stabilization, trajectory tracking, or open-loop forcing of the system, only when a safety violation is imminent [45]. The effectiveness of CBF-based safety filters has been successfully confirmed through various fields and control strategies, e.g., autonomous vehicles [2, 32], robotic systems [58], quadrotors [26, 37, 41], and multi-robot systems [28, 62].

CBFs were first defined in [69] and later refined and popularized in [7]. Currently, there are two general types of CBF that have contrasting properties [5, 74]: a Reciprocal CBF (RCBF) and a Zeroing CBF (ZCBF). RCBFs tend to infinity when the system's state approaches the boundary of a safe set, and are undefined outside the safe set, an unsafe set. The term "barrier" describes this blow-up property and originates in optimization theory [18, 65]. While RCBFs are useful for some applications, e.g., multiple safety constraints [33], their use makes control systems sensitive to noise, since a large control signal is required when the system's state is close to the boundary of a safe set [70]. On the other hand, ZCBFs first proposed in [74] can be defined in an unsafe set, since they vanish on the set boundary. The robustness property of ZCBFs enhances their employment for real applications considering modeling errors or disturbances [36]. Recent vigorous research on safety-critical systems has produced different forms of CBFs, including Input-to-State Safe CBF [2, 3, 44], High-order CBF [65, 70, 73], Adaptive CBF [71, 72], and Integral CBF [6]; significantly, many of them are based on a ZCBF. However, contrary to active studies on ZCBFs, mathematical problems related to safety assurance and the theoretical limitation of the applicable range remain.

Safety Assurance Problem

In the ZCBF-based approach, Nagumo's theorem [16] plays an essential role in theoretical safety assurance of control systems, since it gives necessary and sufficient conditions for forward invariance. Again, when a safe set is forward invariant, the trajectory of systems eternally stays there; that is, for a given dynamical autonomous system $\dot{x} = f(x)$ with a state $x \in \mathbb{R}^n$ and a safe set $\mathcal{S} \subset \mathbb{R}^n$, all solutions starting from $\mathcal{S}$ satisfy $x(t) \in \mathcal{S}$ for $\forall t \geq 0$. Therefore, all solutions at least need to be forward complete to apply Nagumo's theorem. Generally speaking, however, solutions to nonlinear systems are not always forward complete due to their nonlinearity. That is, the states of nonlinear systems can go to infinity in finite time, known as finite escape time [40]. As the research [70] where Brezis's theorem [54] is used points out this problem, many of the previous studies in the CBF community ensure safety only for the maximal time interval of existence of solutions to control systems; in other words, they ensure forward pre-invariance of a safe set [48]. Moreover, the uniqueness of solutions is also required for the use of Nagumo's theorem, which results in investigating whether the right-hand side of $\dot{x} = f(x, u(x))$ and a safety filter $u(x)$ are (locally) Lipschitz continuous in x.

Another problem of Nagumo's theorem is that it cannot ensure safety of a nonautonomous system, which is given by $\dot{x} = f(x, t)$ where the right-hand side depends not only on a state x but also on time $t \in \mathbb{R}$. Therefore, it is not suitable for theoretical discussion of a safety assist control such as AEB technology, since human-operated systems are categorized into nonautonomous systems.

CBF for Input-constrained Systems

Automobiles cannot stop immediately due to maximum deceleration; in other words, a braking distance never becomes 0, unlike a thinking distance. Control theory refers to such systems as input-constrained systems because admissible inputs are limited; in the above case, we cannot use deceleration inputs beyond the maximum deceleration value. Since almost all real control systems have input constraints, the design of a safety filter that satisfies both state (safety) and input constraints is crucial.

Studies on CBFs considering input-constrained systems have recently emerged [32]– [27]. For a system with a dynamically defined controller, the authors in [32] introduced Control-Dependent CBFs (CD-CBFs) by taking a controller as a new state. Related to [32], Integral CBFs (I-CBFs) were defined in [6]. The controller based on the CBFs above succeeded in simultaneously satisfying state and input constraints. However, these frameworks assume the differentiability of a controller and hence are not suitable for human assist control because requiring a human operator input to be differentiable excessively limits the scope of applications. A controller within input constraints might render only a subset of a safe set forward invariant; hence the authors in [1] introduced the notion of an inner safe set and defined Input-Constrained CBFs (IC-CBFs) to satisfy both state and input constraints. However, finding an inner safe set is difficult because multiple IC-CBF candidates need to be constructed. Moreover, these frameworks are not applicable to a nonautonomous system since Nagumo's theorem guarantees the system's safety.

CBF for Input-disturbed Systems

Uncertainties such as unknown disturbances and unmodeled dynamics pose risks to safety assurance in real-world implementations. Several studies on CBFs considering disturbances have begun to be carried out in recent years. For example, a robust CBF-based controller is known to be an effective technique to ensure safety of control systems under external disturbances [36, 51]. For a control system with input disturbances, the authors in [44] proposed Input-to-State Safety CBFs (ISSf-CBFs) and showed that an ISSf-CBF-based controller ensured the boundedness of the system state. The notion of ISSf is an extension of Input-to-State Stability (ISS), which characterizes stability of nonlinear systems under input disturbances [61]. More recently, Tunable Input-to-State Safety CBFs (TISSf-CBFs) were proposed in [3] with the aim of a less conservative controller design. However, ISSf-CBF-based controllers ensure the forward invariance of a larger set dependent on the magnitude of input disturbances, not an original safe set. That is, the controller does not eliminate the risk of safety violation, since the system's state runs off the original safe set if a large input disturbance is added to the control system.

CBF for Time-dependent State Constraint Problem

Many of studies on CBFs deal with time-invariant state constraint problems such as stationary obstacle avoidance or moving obstacle avoidance by considering systems given as relative coordinates; that is, a CBF candidate only depends on the state of control systems. Several studies on CBFs deal with a time-dependent state constraint problem. Considering nonautonomous systems, the author in [73] defined high-order Time-varying ZCBFs (Tv-ZCBFs) by generalizing first-order ZCBFs proposed in [74]. However, this research only deals with single-input systems. Similarly to [73], the research [47] provided the formal definition of first-order Tv-ZCBFs for autonomous systems. They also introduced the notion of candidate Tv-ZCBFs so that a time-varying safe set is not empty. However, the formal definition of first-order Tv-ZCBFs for nonautonomous systems is not derived. To the best of the author's knowledge, there are few CBF studies dealing with time-dependent state constraint problems for nonautonomous systems; solving this problem by CBFs seems to be in an immature phase.

The difficulty of a time-dependent state constraint problem is that a safety filter needs to handle unknown information. For example, the velocity of moving obstacles is unknown to control systems. To design a safety filter, we need complete information about unknown environments or estimate unknown parameters. However, in real applications, it is difficult to estimate the true value of unknown parameters, e.g., obstacle velocities, due to the measurement noise. Therefore, a CBF-based controller should ensure safety of control systems even if there exist estimation errors of unknown parameters.

1.1.2 Related Work to Safety-Critical Control

In exiting AEB systems, time to collision (TTC) is often used as a trigger for entering warning or emergency braking phases [38, 46, 59]. TTC is the time left before rear-end collisions occur if the current velocity and course are maintained constant. A high TTC threshold can warn drivers or automatically provide a braking action, which implies safer but conservative behaviors. On the other hand, a low TTC threshold can decrease conservatism but increase the risk of safety violation. Furthermore, TTC-based AEB systems decelerate sharply without driver involvement to avoid a potential collision or mitigate damage from safety violations. This indicates that the existing methods produce neither a theoretical safety guarantee nor a continuous safety assist controller.

In nonlinear control theory, Model Predictive Control (MPC) is as popular as CBF-based methods for safety-critical control [13, 17]. The basic concept of MPC is to use a dynamic model of a control system to forecast system behaviors and optimize the forecast to produce the best decision at each time step [35]. Thanks to the recent development of computation technology, an MPC-based controller attracts attention in various control fields. However, its high computational cost stays still as a problem due to, for example, the performance limit of a control computer mountable on an automobile. Generally speaking, moreover, an MPC-based controller is not explicitly obtained since a

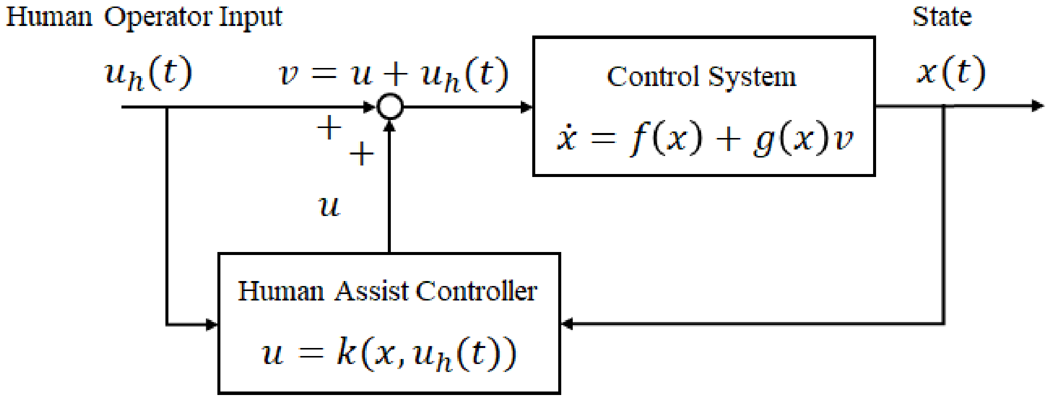

Figure 1.1: Human Assist Control.

mathematical programming problem yielding the controller is solved online and numerically.

As a similar concept to CBFs, Barrier-Lyapunov Functions (BLFs) is known as tools for the design of a feedback controller that simlutaneously achieve asymptotic stability of an equilibrium point while avoiding an unsafe set [55, 66]. The drawback of BLF-based methods is about the feasibility; that is, if the stabilization and safety objectives are in conflict, no feedback controller can be obtained. In addition, since control systems with a BLF-based controller are actively repelled from the safe set boundary, it produces a more conservative behavior compared with a CBF-based controller. Furthermore, neither MPC nor BLFs entail the notion of a nominal control [45]. In other words, their approach is not suitable for modifying a controller designed for prescribed control objectives or assisting a human operator input, which do not always generate safe actions of control systems.

1.1.3 Human Assist Control

A human assist controller needs to meet not only safety constraints but also optimality and continuity. According to [25], the human assist control problem that provides the ideal properties of a human assist controller is introduced as follows:

1. A human assist controller renders a safe set forward invariant for any human operator inputs.

2. A human assist controller is optimal in the sense of a preliminarily defined cost function or functional.

The first condition is related to theoretical safety assurance of human-operated systems. As mentioned previously, human-operated systems are classified into nonautonomous systems because they are affected by a human operator input, which is time-varying and cannot be derived in advance. Therefore, some other mathematical tools need to be employed for safety assurance of human-operated systems.

The second condition is about the relationship between a human operator input and a human assist controller. Similarly to previous studies on CBFs, a human assist controller should intervene only when safety constraints are at risk of violation. This condition eliminates, for example, a controller that fully cancels an acceleration input given by a driver and results in a stationary automobile.

To this end, the research [50] considered the control scheme shown in Fig. 1.1 and proposed human assist control using a relaxed CBF for a human-operated system. This human assist control enables humans to operate a system freely while guaranteeing forward invariance of a safe set. It is worth highlighting that the human assist controller proposed in [50] is continuous without a relative degree condition, which cannot be achieved by the CBFs defined in [7, 74]. The continuity of a controller is another essential property for human assist control because it increases operability and passenger comfort. However, since a relaxed CBF is based on an RCBF that blows up on the boundary, it is still sensitive to noise in control systems [65] and restricts the scope of applications. Moreover, using a relaxed CBF may cause numerical calculation problems due to its blowing-up property.

1.2 Contribution

In this dissertation, the author proposes a ZCBF-based safety (human) assist controller that guarantees theoretical safety of nonautonomous systems. Importantly, the controllers proposed in this dissertation are continuous and guarantee safety with minimal intervention, indicating that it is an ideal human assist controller. As mentioned above, the employment of ZCBFs will expand the scope of control applications compared to an RCBF-based controller and a relaxed CBF-based human assist controller.

The results of this dissertation consist of two parts. The first contribution, given in Chapter 3, is the establishment of safety assist control for time-invariant state constraint problems. The second contribution, given in Chapter 4, is the establishment of safety assist control for time-dependent state constraint problems.

1.2.1 Safety Assist Control for Time-invariant State Constraint Problem

In the first part of Chapter 3, the author considers a generalized nonautonomous control-affine system. The biggest contribution of this dissertation, a ZCBF-based safety assist controller, is available here. The author firstly defines the "strict" version of a ZCBF, a Strict Zeroing Control Barrier Function (S-ZCBF). Using the strict conditions for a ZCBF, the author then provides a global condition for an S-ZCBF leading to the existence of global solutions, forward complete solutions, to nonautonomous systems. Then, the author proposes a human assist controller using an S-ZCBF and proves that the proposed controller ensures forward invariance of a safe set by employing Gronwall's lemma that is applicable regardless of the system's nonautonomy. The author simultaneously proves that the proposed controller is continuous and that any solution to a system can be arbitrarily extended, i.e., every solution is forward complete. The author also proves that the proposed controller is minimally invasive. The author finally studies a mathematical example, a state constraint problem for a non-Lipschitz system, and confirms the effectiveness of the proposed controller by computer simulation.

In the second part of Chapter 3, the author considers an input-constrained control system. In real applications, almost all control systems have input constraints, e.g., a limit of actuators. To meet both state and input constraints, the notion of viability kernels advocated in viability theory becomes of interest [10]. If the state of a control system starts from a viability kernel, which is a subset of a safe set, at least one solution remains in the safe set. In other words, we can always choose a controller within input constraints that renders the viability kernel controlled forward invariant [19, 31]. Motivated by this, the author modifies the definition of an S-ZCBF by introducing a CBF for viability kernels, since the objective is rephrased as guaranteeing the forward invariance of a viability kernel. Then, the author proposes a CBF-based human assist controller that renders a viability kernel forward invariant by restricting the scope of control systems. The author studies a double-integrator system as an example and demonstrates how to construct a CBF for viability kernels. The effectiveness of the proposed controller is confirmed by computer simulation.

1.2.2 Safety Assist Control for Time-dependent State Constraint Problem

The methods proposed in Chapter 3 cannot ensure safety of nonautonomous systems when a safe set is time-varying, i.e., a state constraint is time-dependent. In the first part of Chapter 4, the author proposes a safety assist controller that is applicable to time-dependent state constraint problems. Importantly, the proposed controller inherits the rich properties of an S-ZCBF-based controller, continuity and optimality. The notion of forward invariance is only used for a time-invariant (fixed) safe set. Therefore, the author considers a graph space, i.e., a subset of a product space consisting of a system's state and a time variable, as a safe region; in this setting, forward invariance of a graph space implies that time-dependent safety constraints are satisfied. The author also provides assumptions on set-valued maps that prevent a time-varying safe set from being empty. Then, the author defines a Time-varying ZCBF (Tv-ZCBF) for nonautonomous systems by extending an S-ZCBF. Using the conditions for a Tv-ZCBF, the author proves the compactness regarding a graph space. After that, the author proposes a Tv-ZCBF-based human assist controller that ensures the forward invariance of a graph space. The author simultaneously proves that the proposed controller ensures the forward completeness of solutions. Lastly, the author confirms the effectiveness of the proposed controller by considering a mathematical example and conducting computer simulation.

In the latter part of Chapter 4, the author considers an input-disturbed control system. Motivation comes from the difficulty of applying the Tv-ZCBF-based controller in real experiments. Concretely, the velocity of moving obstacles is generally unknown to control systems and needs to be estimated. However, the estimate of the obstacle velocity contains estimation errors, and it is difficult to obtain its true value. Estimation errors might lead to violation of state constraints as the Tv-ZCBF-based controller allows a control system to approach considerably close to the safe set boundary. Therefore, the author aims to design a human assist controller that renders a safe set forward invariant even if the estimation error exists. Here, the author introduces an Input-to-State Constrained Safety ZCBF (ISCSf-ZCBF) that ensures safety of control systems under input disturbances. The author lastly designs the proposed controller for a time-dependent state constraint problem of an electric wheelchair. The effectiveness of the proposed controller is confirmed by computer simulation and experiment.

Chapter 2

Preliminaries

This chapter introduces some basic definitions and mathematical foundations that will be used in this dissertation.

2.1 Notation

The sets of positive and non-negative real numbers are denoted by $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$. The empty set is denoted by $\emptyset$. The boundary, the interior, and the complement of a set $\mathcal{S} \subset \mathbb{R}^n$ are denoted by $\partial\mathcal{S}$, $\text{int}(\mathcal{S})$ and $\mathcal{S}^c$, respectively.

A C^1 continuously differentiable function has continuous derivatives. For a C^1 continosly differentiable function $h(x)$ and a vector function $f(x)$, define the Lie derivative notation $L_f h(x) := (\partial h/\partial x)(x) f(x)$.

2.2 Comparison Functions

The use of comparison functions has become standard in control theory. In this section, the author introduces some comparison functions used in this dissertation.

Definition 2.1 (Class $\mathscr{K}$ function [40]) *A continuous function $\alpha : [0, a) \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathscr{K}$ ($\alpha \in \mathscr{K}$) if it is strictly increasing and $\alpha(0) = 0$.*

Definition 2.2 (Class $\mathscr{K}_\infty$ function [40]) *A continuous function $\alpha : [0, a) \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathscr{K}_\infty$ ($\alpha \in \mathscr{K}_\infty$) if it belongs to class $\mathscr{K}$, $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$.*

Definition 2.3 (Extended class $\mathscr{K}$ function [74]) *A continuous function $\alpha : (-b, a) \to \mathbb{R}$ for some $a, b > 0$ is said to belong to extended class $\mathscr{K}$ ($\alpha \in \mathscr{K}_e$) if it is strictly increasing and $\alpha(0) = 0$.*

Definition 2.4 (class $\mathcal{L}$ function [39]) *A continuous function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ is said to belong to class $\mathcal{L}$ ($\rho \in \mathcal{L}$) if it is strictly decreasing and $\rho(r) \to 0$ as $r \to \infty$.*

2.3 Ordinary Differential Equations

In control theory, control systems are generally modeled by a first-order ordinary differential equation (ODE).

In this section, we consider the following ODE:

$$\dot{x} = f(x, t), \tag{2.1}$$

where $x \in \mathbb{R}^n$ denotes a state, $t \in \mathbb{R}$ a time variable, and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ a mapping, respectively. A special case of (2.1) arises when the mapping f only depends on x, i.e.,

$$\dot{x} = f(x). \tag{2.2}$$

We call the system modeled by (2.1) a nonautonomous system and the system modeled by (2.2) an autonomous system, respectively.

A classical solution $x : \mathcal{I} \to \mathbb{R}^n$ to the ODE (2.1) with the initial value $x(t_0) = x_0$ is a mapping such that the following two conditions hold [12].

1. $x(t)$ is of class C^1.
2. $\dot{x}(t) = f(x(t), t), \quad \forall t \in \mathcal{I}$.

Here, the set $\mathcal{I}$ is an interval of $\mathbb{R}$ such that $t_0 \in \mathcal{I} \subset \mathbb{R}$. If $\mathcal{I} = \mathbb{R}$, a solution $x(t)$ is said to be global. The qualifier "classical" is used to distinguish other concepts of ODE solutions, e.g., a Carathéodory solution for discontinuous differential equations [20]. In this dissertation, however, we are only interested in a classical solution; therefore, we sometimes call it a solution to the ODE (2.1) for brevity.

2.3.1 Cauchy-Peano Existence Theorem [20]

The existence of solutions to the ODE (2.1) depends on the property of a mapping $f(x, t)$. In this subsection, the author introduces the Cauchy-Peano existence theorem (Peano's theorem) that states the existence of a classical solution.

Consider the following rectangle $\mathcal{R}$ about the point (x_0, t_0):

$$\mathcal{R} : \ |t - t_0| \leq a, \ \ |x - x_0| \leq b \ \ (a, b > 0). \tag{2.3}$$

If a mapping $f(x, t)$ of the ODE (2.1) is continuous, it is bounded on $\mathcal{R}$. Therefore, there exists a positive constant $M \in \mathbb{R}_{>0}$ such that

$$M = \max \ |f(x, t)| \ \ ((x, t) \in \mathcal{R}). \tag{2.4}$$

Let

$$\alpha = \min \left(a, \frac{b}{M}\right). \tag{2.5}$$

Then, the following theorem is well known as the Cauchy-Peano existence theorem.

Theorem 2.1 *If a mapping $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is continuous on the rectangle $\mathcal{R}$, then there exists a classical solution $x(t)$ to the ODE* (2.1) *on $|t - t_0| \leq \alpha$ for which $x(t_0) = x_0$.*

Importantly, Peano's theorem only assures the existence of a local solution $x(t)$, i.e., solutions are not generally defined for $\forall t \in (-\infty, \infty)$. If a solution $x(t)$ to the ODE (2.1) is defined for $\forall t \geq 0$, it is called forward complete [8].

2.3.2 Gronwall's Lemma [30]

In this dissertation, the author adopts the following Gronwall lemma ([30, 40]) to ensure forward invariance of a safe set, which is introduced later.

Theorem 2.2 *Let $\lambda \in \mathbb{R}$ be a constant, and $v : [t_0, t_1] \to \mathbb{R}_{\geq 0}$ be a non-negative continuous function. If an absolute continuous non-negative function $z : [t_0, t_1] \to \mathbb{R}_{\geq 0}$ satisfies*

$$z(t) \leq \lambda + \int_{t_0}^{t} v(s)z(s)ds, \tag{2.6}$$

for $\forall t \in [t_0, t_1]$, then on the same interval, the following inequality holds:

$$z(t) \leq \lambda \exp\left(\int_{t_0}^{t} v(s)ds\right). \tag{2.7}$$

We can obtain the following corollary under the assumption that $z(t)$ is a C^1 continuously differentiable function.

Corollary 2.1 *Let $v : [t_0, t_1] \to \mathbb{R}_{\geq 0}$ be a non-negative continuous function. If a non-negative C^1 continuously differentiable function $z : [t_0, t_1] \to \mathbb{R}_{\geq 0}$ satisfies*

$$\dot{z}(t) \geq -v(t)z(t), \tag{2.8}$$

for $\forall t \in [t_0, t_1]$, then on the same interval, the following inequality holds:

$$z(t) \geq z(t_0) \exp\left(-\int_{t_0}^{t} v(s)ds\right). \tag{2.9}$$

In particular, if $v(t) = \gamma \in \mathbb{R}_{\geq 0}$ is a non-negative constant, then the following inequality holds for $\forall t \in [t_0, t_1]$:

$$z(t) \geq z(t_0) \exp\left(-\gamma(t - t_0)\right). \tag{2.10}$$

2.3.3 Lipschitz Continuity

Peano's theorem ensures that at least one local solution to the ODE (2.1) exists. In other words, the ODE (2.1) with a given initial condition might have several solutions. For example, the following ODE

$$\dot{x} = 2x^{1/2}, \quad x(0) = 0 \tag{2.11}$$

has a solution $x(t) = t^2$. On the other hand, another solution $x(t) = 0$ to this equation exists. This example implies that the continuity condition on $f(x,t)$ of (2.1) is insufficient to ensure the uniqueness of solutions.

The following sufficient condition for uniqueness of solutions to (2.1),

$$\|f(x,t) - f(y,t)\| \leq L\|x - y\| \tag{2.12}$$

for all (x,t) and (y,t) in some neighborhood of (x_0, t_0), is known as the Lipschitz condition with a Lipschitz constant $L \in \mathbb{R}_{>0}$. Here, the author introduces an existence and uniqueness theorem for local solutions to (2.1) where the local Lipschitz condition is employed [40].

Theorem 2.3 *Consider the ODE* (2.1) *with the initial value* $x(t_0) = x_0$. *Let* $f(x,t)$ *be continuous in* t *and satisfies the local Lipschitz condition:*

$$\|f(x,t) - f(y,t)\| \leq L\|x - y\|, \quad \forall x, y \in B := \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq r\}, \ \forall t \in [t_0, t_1]. \tag{2.13}$$

Then, there exists some $\delta \in \mathbb{R}_{>0}$ *such that a unique solution* $x(t)$ *to* (2.1) *exists over* $[t_0, t_0 + \delta]$.

Importantly, nonlinear systems of the form (2.1) may generally have a finite escape time due to its nonlinearity (See Example 3.1). That is, a state of nonlinear systems can go to infinity in finite time. The following theorem finds a unique global solution $x(t)$ to (2.1) with the initial value $x(t_0) = x_0$, i.e., $x(t)$ is defined over $[t_0, t_1]$ where t_1 may be arbitrarily large [40].

Theorem 2.4 *Consider the ODE* (2.1) *with the initial value* $x(t_0) = x_0$. *Let* $f(x,t)$ *be continuous in* t *and satisfies the global Lipschitz condition:*

$$\|f(x,t) - f(y,t)\| \leq L\|x - y\|, \quad \forall x, y \in \mathbb{R}^n, \ \forall t \in [t_0, t_1]. \tag{2.14}$$

Then, there exists a unique solution $x(t)$ *to* (2.1) *over* $[t_0, t_1]$.

2.4 Set-Valued Maps [11]

In this dissertation, a time-dependent constrained control problem, e.g., collision avoidance with moving obstacles, will be discussed in Section 4 using set-valued maps. Therefore, in this section, the author introduces some basic properties of set-valued maps based on the literature [11].

A set-valued map is literally a map that associates a point with a set defined as follows.

Definition 2.5 (Set-valued map and its graph) *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces. A set-valued map $\mathcal{F} : \mathcal{X} \rightsquigarrow \mathcal{Y}$ is characterized by its graph $\mathcal{G}(\mathcal{F})$:*

$$\mathcal{G}(\mathcal{F}) := \{(x, y) \in \mathcal{X} \times \mathcal{Y} \mid y \in \mathcal{F}(x)\}, \tag{2.15}$$

which is the subset of the product space $\mathcal{X} \times \mathcal{Y}$. The domain of $\mathcal{F}$ is the subset of elements $x \in \mathcal{X}$ such that $\mathcal{F}(x)$ is not empty:

$$Dom(\mathcal{F}) := \{x \in \mathcal{X} \mid \mathcal{F}(x) \neq \emptyset\}. \tag{2.16}$$

It is trivial from Definition 2.5 that the following holds for any $x \in \mathcal{X}$:

$$\mathcal{F}(x) = \{y \in \mathcal{Y} \mid (x, y) \in \mathcal{G}(\mathcal{F})\}. \tag{2.17}$$

Definition 2.6 (Upper semi-continuity) *A set-valued map $\mathcal{F} : \mathcal{X} \rightsquigarrow \mathcal{Y}$ is upper semi-continuous at $x_0 \in Dom(\mathcal{F})$ if and only if for any neighborhood $\mathcal{U}$ of $\mathcal{F}(x_0)$,*

$$\exists \eta > 0 \;\; \textit{such that} \;\; \forall x \in B_{\mathcal{X}}(x_0, \eta) := \{x \in \mathcal{X} \mid \|x - x_0\| \leq \eta\}, \; \mathcal{F}(x) \subset \mathcal{U}. \tag{2.18}$$

It is said to be upper semi-continuous if it is upper semi-continuous at every point $x_0 \in Dom(\mathcal{F})$.

Definition 2.7 (Lower semi-continuity) *A set-valued map $\mathcal{F} : \mathcal{X} \rightsquigarrow \mathcal{Y}$ is lower semi-continuous at $x_0 \in Dom(\mathcal{F})$ if and only if for any $y \in \mathcal{F}(x_0)$ and for any sequence of elements $x_n \in Dom(\mathcal{F})$ such that $x_n \to x_0$ as $n \to \infty$, there exists a sequence of elements $y_n \in \mathcal{F}(x_n)$ such that $y_n \to y$ as $n \to \infty$.*

It is said to be lower semi-continuous if it is lower semi-continuous at every point $x_0 \in Dom(\mathcal{F})$.

Proposition 2.1 *The graph of an upper semi-continuous set-valued map $\mathcal{F} : \mathcal{X} \rightsquigarrow \mathcal{Y}$ with closed domain and closed values is closed.*

The converse is true under the assumption that $\mathcal{Y}$ is compact.

2.5 Set Invariance

In recent nonlinear control theory, theoretical safety assurance agrees with set invariance. In this section, the author introduces the notion of forward invariance and Nagumo's theorem, which is a necessary and sufficient condition for forward invariance regarding autonomous systems.

2.5.1 Forward Invariance

According to the literature [15, 16], forward invariance with respect to an autonomous system given by the ODE (2.2) is defined as follows. Note that in the literature forward invariance is introduced as positive invariance, and the role of the word "forward" (also "positive") implies that the property regards the future.

Definition 2.8 *(Forward invariance [16]) A set $\mathcal{S} \subset \mathbb{R}^n$ is forward invariant with respect to* (2.2) *if every solution to* (2.2) *with the initial condition $x(0) = x_0 \in \mathcal{S}$ is globally defined and such that $x(t) \in \mathcal{S}$ for $\forall t \geq 0$.*

Remark 2.1 *Strictly speaking, the notion of forward invariance for nonautonomous systems is different from that regarding autonomous systems since the system* (2.1) *is not time-invariant [16, Definition 4.10]. However, in this study, we fix an initial time t_0 to $t_0 = 0$ so that the notion accords to the case of autonomous systems, which is given by the definition above. Note that initializing to $t_0 = 0$ is a common assumption in control engineering although it leads to loss of generality.*

In Definition 2.8, the existence of global solutions is assumed. As already mentioned, however, nonlinear systems sometimes have a finite escape time. This implies that a solution to (2.2) starting from $x(0) \in \mathcal{S}$ may satisfy $x(t) \in \mathcal{S}$ for $\forall t \in Dom(x)$ where $Dom(x) \subset [0, \infty)$ is a domain of a mapping $x : \mathbb{R} \to \mathbb{R}^n$, but not over $[0, \infty)$. The notion of forward pre-invariance points out this problem peculiar to nonlinear systems.

Definition 2.9 *(Forward pre-invariance [48]) A set $\mathcal{S} \subset \mathbb{R}^n$ is forward pre-invariant with respect to* (2.2) *if every solution $x(t)$ to* (2.2) *with initial condition $x(0) = x_0 \in \mathcal{S}$ satisfies $x(t) \in \mathcal{S}$ for $\forall t \in Dom(x) \subset [0, \infty)$.*

2.5.2 Nagumo's Theorem [16, 49]

Nagumo's theorem is of fundamental importance for establishing forward invariant sets as it provides a necessary and sufficient condition for forward invariance of closed sets. The notion of a tangent cone is useful to state the theorem. Before introducing these, we firstly define the notion of a distance from a set.

Definition 2.10 *(Distance from a set [16]) Consider a nonempty set $\mathcal{S} \subset \mathbb{R}^n$ and a point $y \in \mathbb{R}^n$. Then, the distance from a set is defined as follows:*

$$dist(y, \mathcal{S}) = \inf_{w \in \mathcal{S}} \|y - w\|. \tag{2.19}$$

Then, the following notion of Bouligand's tangent cone plays an important role in Nagumo's theorem.

Definition 2.11 *(Bouligand's tangent cone [16]) Consider a closed set $\mathcal{S} \subset \mathbb{R}^n$. Then, the tangent cone to $\mathcal{S}$ at $x \in \mathbb{R}^n$ is defined as follows:*

$$\mathcal{T}_{\mathcal{S}}(x) = \left\{ z \in \mathbb{R}^n \;\middle|\; \liminf_{\tau \to 0} \frac{dist(x + \tau z, \mathcal{S})}{\tau} = 0 \right\}. \tag{2.20}$$

From the definition of tangent cone, it is easy to see that

1. if $x \in \mathrm{int}(\mathcal{S})$, then $\mathcal{T}_{\mathcal{S}}(x) = \mathbb{R}^n$.
2. if $x \notin \mathcal{S}$, then $\mathcal{T}_{\mathcal{S}}(x) = \emptyset$.

That is, the tangent cone $\mathcal{T}_{\mathcal{S}}(x)$ is non-trivial only on the boundary of $\mathcal{S}$. It is worth stressing that the boundary $\partial\mathcal{S}$ satisfies $\partial\mathcal{S} \subset \mathcal{S}$ since the set $\mathcal{S}$ is closed.

A special case of a tangent cone can be derived when a target set is given as a practical set [16, Definition 4.9]; if the closed set $\mathcal{S}$ is given as the following 0-superlevel set of a C^1 continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$:

$$\begin{aligned} \mathcal{S} &= \{x \in \mathbb{R}^n \mid h(x) \geq 0\}, \\ \partial\mathcal{S} &= \{x \in \mathbb{R}^n \mid h(x) = 0\}, \\ \mathrm{Int}(\mathcal{S}) &= \{x \in \mathbb{R}^n \mid h(x) > 0\}, \end{aligned} \tag{2.21}$$

then the tangent cone to $\mathcal{S}$ at x is given as follows [16, 63]:

$$\mathcal{T}_{\mathcal{S}}(x) = \left\{ z \in \mathbb{R}^n \;\middle|\; \frac{\partial h}{\partial x}(x) z \geq 0, \quad \forall x \in \partial\mathcal{S} \right\}. \tag{2.22}$$

Now, we can introduce Nagumo's theorem that states necessity and sufficiency for forward invariance, which was first discovered by Mitio Nagumo [49].

Theorem 2.5 *(Nagumo's theorem [16]) Consider the autonomous system* (2.2)*. Assume that for each initial condition $x(0)$ in an open set $\mathcal{O} \subset \mathbb{R}^n$, it admits a unique solution defined for $\forall t \geq 0$. Let $\mathcal{S} \subset \mathcal{O}$ and $\partial\mathcal{S} \subset \mathcal{S}$ be a closed set and its boundary. Then, $\mathcal{S}$ is forward invariant with respect to* (2.2) *if and only if the following condition holds:*

$$f(x) \in \mathcal{T}_{\mathcal{S}}(x), \quad \forall x \in \partial\mathcal{S}. \tag{2.23}$$

The theorem assumes that a solution is unique. If the assumption fails to be satisfied, Nagumo's theorem only ensures weak forward invariance [16, Definition 4.2], i.e., there exists at least one global solution $x(t)$ to (2.2) with $x(0) \in \mathcal{S}$ such that $x(t) \in \mathcal{S}$ for $\forall t \geq 0$.

2.6 Convex Optimization

In control engineering, a controller needs to be optimal somehow in many cases. In this section, we consider the following inequality-constrained optimization problem:

$$\text{minimize } f(u) \tag{2.24}$$
$$\text{subject to } g_1(u) \le 0, \ldots, g_r(u) \le 0, \tag{2.25}$$

where mappings $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$ are C^1 continuously differentiable convex functions. Since both a cost function $f(u)$ and a constrained function $g_i(u)$ are convex, this is classified as an inequality-constrained convex optimization problem (IC-COP). Accordingly, a local minimum for the IC-COP (2.24) (2.25) corresponds to its global minimum due to the convexity of the problem.

Let $Act(u)$ be the set of active inequality constraints:

$$Act(u) = \{i \mid g_i(u) = 0\}. \tag{2.26}$$

Define the Lagrangian function for the IC-COP (2.24) (2.25) as follows:

$$L(x, \lambda) = f(u) + \sum_{i=1}^{r} \lambda_i g_i(u). \tag{2.27}$$

Then, the following proposition is known as the Karush-Kuhn-Tucker (KKT) condition that is necessary for optimal solutions. Note that the KKT condition also becomes sufficient if the optimization problem is convex.

Proposition 2.2 *(Karush-Kuhn-Tucker (KKT) condition [14]) Let u^* be a local minimum of the IC-COP* (2.24) (2.25). *Assume that there exists a feasible vector $\bar{u}$ satisfying*

$$g_i(\bar{u}) < 0, \quad \forall j \in Act(u^*). \tag{2.28}$$

Then, there exists a unique Lagrange multiplier vector $\lambda^* = (\lambda_1^*, \ldots, \lambda_r^*)$ such that

$$\nabla_u L(u^*, \lambda^*) = 0, \tag{2.29}$$
$$\lambda_i^* \ge 0, \quad i = 1, \ldots, r, \tag{2.30}$$
$$\lambda_i^* = 0, \quad \forall i \notin Act(u^*). \tag{2.31}$$

Here, the assumption (2.28) is called the Slater constraint qualification, which guarantees the existence of Lagrange multipliers, which is standard for convex inequalities [14, Proposition 3.3.9].

Chapter 3

Safety Assist Control via Zeroing Control Barrier Function

In this chapter, the author proposes a safety assist controller (human assist controller) for nonautonomous control systems. The state constraint problem considered in this chapter is time-invariant, i.e., a safe set where we wish the system's states to stay is fixed and does not shrink or grow depending on time. The safety assurance will be achieved by designing a safety assist controller that renders a safe set forward invariant.

This chapter starts with an autonomous control system and introduces a Zeroing Control Barrier Function (ZCBF) proposed in [7, 74]. Then, the author points out the technical problems of the conventional ZCBF definition when considering nonautonomous systems. Concretely, Nagumo's theorem employed for theoretical safety assurance in the literature generates mathematical problems; Section 3.2 will motivate the first objective of this chapter. Before designing a controller, the author defines what an "ideal" human assist controller is in Section 3.3. In Section 3.4, the author provides a formal definition of a ZCBF that can ensure the safety of nonautonomous systems. Then, the author proposes a safety assist controller that solves the problems of the conventional method, which is the biggest contribution of this research. The author will show that the proposed controller not only ensures the safety of nonautonomous systems but is also continuous and optimal, indicating that it is ideal for human assist control. The effectiveness of the proposed controller will be confirmed by considering a mathematical example.

The second part of this chapter considers input-constrained control systems. Here, the author aims to design a safety assist controller that ensures both state and input constraints. To achieve the objective, the notion of viability kernels will become of interest. The viability kernel is a subset of a safe set, and we can find a controller that ensures the safety of control systems starting from there within input constraints. In Section 3.5, therefore, a ZCBF will be defined for viability kernels when considering input-constrained systems. Moreover, the author proposes a safety assist controller that renders a viability kernel forward invariant, indicating that state constraints are satisfied while meeting input constraints. The author lastly illustrates how to construct a ZCBF for viability kernel by considering double-integrator systems and conducts computer simulations for effectiveness verification.

3.1 Zeroing Control Barrier Function for Autonomous System [7,74]

In this section, we consider the following nonlinear autonomous control system:

$$\dot{x} = f(x) + g(x)u, \tag{3.1}$$

where $x \in \mathcal{D} \subset \mathbb{R}^n$ denotes a state, $u \in \mathbb{R}^m$ a control input, respectively. Assume that mappings $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are locally Lipschitz continuous. If a state feedback controller $u = k(x)$ is (locally) Lipschitz continuous, there exists a maximal time interval $\mathcal{I}(x_0) \subset \mathbb{R}$ for the initial state $x_0 = x(0)$ such that the unique solution $x(t)$ to (3.1) exists for $\forall t \in \mathcal{I}(x_0)$. Here, we assume that the unique solution $x(t)$ is forward complete, i.e., $\mathcal{I}(x_0) = [0, \infty)$ to simplify the discussion.

Similarly to Subsection 2.5.2, we define the following closed set $\mathcal{S}_a \subset \mathcal{D}$ as the 0-superlevel set of a continuously differentiable function $h_a : \mathcal{D} \to \mathbb{R}$:

$$\begin{aligned} \mathcal{S}_a &= \{x \in \mathcal{D} \mid h_a(x) \geq 0\}, \\ \partial\mathcal{S}_a &= \{x \in \mathcal{D} \mid h_a(x) = 0\}, \\ \mathrm{Int}(\mathcal{S}_a) &= \{x \in \mathcal{D} \mid h_a(x) > 0\}, \end{aligned} \tag{3.2}$$

where $\partial\mathcal{S}_a$ and $\mathrm{Int}(\mathcal{S}_a)$ are the boundary and the interior of $\mathcal{S}_a$, respectively. If $x \in \mathcal{S}_a$, the system (3.2) is safe, and we refer to the set $\mathcal{S}_a$ as a safe set in this section.

To establish the safety of autonomous systems, Xu et al. introduce the following Zeroing Control Barrier Function (ZCBF).

Definition 3.1 *(Zeroing Control Barrier Function [74]) Consider the system* (3.1) *and the safe set $\mathcal{S}_a$ defined by* (3.2)*. Then, a function $h_a : \mathcal{D} \to \mathbb{R}$ is a Zeroing Control Barrier Function (ZCBF) if there exists an extended class $\mathscr{K}$ function $\bar{\alpha} \in \mathscr{K}_e$ such that the following inequality holds:*

$$\begin{aligned} \sup_{u\in\mathbb{R}^m} \dot{h}_a(x,u) &= \sup_{u\in\mathbb{R}^m}[L_f h_a(x) + L_g h_a(x)u] \\ &\geq -\bar{\alpha}\left(h_a(x)\right), \ \forall x \in \mathcal{D}. \end{aligned} \tag{3.3}$$

Given a ZCBF, we can define the set:

$$\mathcal{K}(x) = \{u \in \mathbb{R}^m \mid L_f h_a + L_g h_a u + \bar{\alpha}\left(h_a(x)\right) \geq 0\}, \tag{3.4}$$

which is the admissible control set for $\forall x \in \mathcal{D}$. Then, the forward invariance of $\mathcal{S}_a$ is ensured by the following theorem.

Theorem 3.1 *([7, 74]) Consider the system* (3.1) *and the safe set $\mathcal{S}_a \subset \mathcal{D}$ defined by* (3.2)*. If a function $h_a : \mathcal{D} \to \mathbb{R}$ is a ZCBF, then any Lipschitz continuous controller $u = k(x) \in \mathcal{K}(x)$ ensures the forward invariance of $\mathcal{S}_a$.*

Since the safe set $\mathcal{S}_a$ is defined as a practical set, it is easy to see that Nagumo's theorem ensures the forward invariance of $\mathcal{S}_a$. That is, for any $x \in \partial\mathcal{S}_a$, $\dot{h}_a(x) \geq -\bar{\alpha}\left(h_a(x)\right) = 0$ with a Lipschitz controller $u = k(x) \in \mathcal{K}(x)$ as the function $\bar{\alpha}\left(h(x)\right)$ belongs to extended class $\mathscr{K}$.

3.2 Motivation

The result given in [7, 74] cannot be applied directly to nonautonomous systems. This section points out the problems of the conventional method.

For the sake of simplicity, in the preceding section, we assumed that a solution to the system (3.1) is forward complete. Importantly, Nagumo's theorem does not guarantee forward invariance without the forward completeness of solutions. However, as mentioned in Section 2.3.3, the states of nonlinear systems can go to infinity in finite time, called finite escape time [40, 48]. The following example illustrates the case of forward pre-invariance.

Example 3.1 *Consider the following system:*

$$\dot{x} = -x^2 + u \tag{3.5}$$

with the initial state $x_0 = x(0) = -1$. Assume that a safe set $\mathcal{S}_a$ is given by

$$\mathcal{S}_a = \{x \in \mathbb{R} \mid x \leq 0\}. \tag{3.6}$$

Since the right-hand side of (3.5) *is locally Lipschitz continuous in x, the unique solution to* (3.5)

$$x(t) = \frac{1}{t-1} \tag{3.7}$$

exists over $\mathcal{I}(x_0) = [0, 1)$ and satisfies $x(t) \in \mathcal{S}_a$ for $\forall t \in \mathcal{I}(x_0)$ with $u \equiv 0$. Although $\mathcal{S}_a$ is forward pre-invariant, it is not forward invariant since the solution has a finite escape time $T = 1$ due to the nonlinearity of (3.5).

Nagumo's theorem also requires the uniqueness of solutions for forward invariance. This imposes the (local) Lipschitz continuity on mappings of the system (3.1) and a controller $u \in \mathcal{K}(x)$, which might restrict the scope of applications.

Moreover, Nagumo's theorem only ensures forward invariance of a safe set for autonomous systems. The following example illustrates that it is not applicable to nonautonomous systems [63].

Example 3.2 *Consider the following system:*

$$\dot{x} = t^3, \tag{3.8}$$

with the initial state $x_0 = x(0) = 0$. Assume that a safe set $\mathcal{S}_a$ is given by $\mathcal{S}_a = \{0\}$. At $t = 0$, $f(x, t) = t^3 \in \mathcal{T}_{\mathcal{S}_a}(x)$ for each $x \in \partial\mathcal{S}_a$, indicating that Nagumo's condition (2.23) *holds. However, the solution to* (3.8)

$$x(t) = \frac{1}{4}t^4, \tag{3.9}$$

implying that $x(t) \notin \mathcal{S}_a$ for $t \in (0, \infty)$. Therefore, $\mathcal{S}_a$ is not forward invariant.

In addition, using the ZCBF condition given by Definition 3.1, a ZCBF-based controller might not be continuous without a relative degree condition, as clarified in the following section (see Remark 3.3 and Example 3.3).

3.3 Problem Statement

Consider the following nonlinear time-varying (nonautonomous) control system with a human operator input $u_h(t)$:

$$\dot{x} = f(x, t) + g(x, t) [u + u_h(t)], \tag{3.10}$$

where $x \in \mathcal{D} \subset \mathbb{R}^n$ denotes a state, $u \in \mathbb{R}^m$ a control input, respectively. Assume simply that a mapping $u_h : \mathbb{R} \to \mathbb{R}^m$ is continuous and that mappings $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n \times m}$ are continuous in $(x, t) \in \mathbb{R}^n \times \mathbb{R}$.

Then, divide a state space $\mathcal{D}$ into a safe set $\mathcal{S}$ and an unsafe set $\mathcal{S}^c$, i.e., $\mathcal{S} \cup \mathcal{S}^c = \mathcal{D}$. Assume that the safe set $\mathcal{S}$, namely where the system state should stay, is open and time-invariant. Moreover, assume that the safe set $\mathcal{S}$ is defined as the strict 0-superlevel set of a C^1 continuously differentiable function $h : \mathcal{D} \to \mathbb{R}$:

$$\begin{aligned} \mathcal{S} &= \{x \in \mathcal{D} \mid h(x) > 0\}, \\ \partial\mathcal{S} &= \{x \in \mathcal{D} \mid h(x) = 0\}, \end{aligned} \tag{3.11}$$

where $\partial\mathcal{S}$ is the boundary of $\mathcal{S}$ (Fig. 3.1). Note that the open safe set implies $\mathcal{S} = \mathrm{Int}(\mathcal{S})$ and $x \in \mathcal{S}$ denotes state constraints.

To clarify what conditions the ideal human assist controller should fulfill, the definition of the human assist control problem is introduced as follows.

Definition 3.2 *(Human Assist Control (Based on [25])) Consider the system* (3.10) *and the safe set* $\mathcal{S}$ *defined by* (3.11). *Then, a controller* $u = k(x, t)$ *is a human assist controller if the following conditions hold:*

(P1) For any human operator input $u_h : \mathbb{R} \to \mathbb{R}^m$, *the controller* $u = k(x, t)$ *renders* $\mathcal{S}$ *forward invariant in the sense of Definition 2.8, i.e., if the initial state* $x_0 := x(0)$ *satisfies* $x_0 \in \mathcal{S}$, *then* $x(t) \in \mathcal{S}$ *for* $\forall t \geq 0$.

(P2) The controller $u = k(x, t)$ *is optimal in the sense of a preliminarily defined cost function, i.e., the controller* u *renders* $\mathcal{S}$ *forward invariant with minimal intervention.*

(P3) The controller $u = k(x, t)$ *is continuous in* $(x, t) \in \mathbb{R}^n \times \mathbb{R}$.

The condition *(P1)* is related to the theoretical safety assurance. As mentioned in Remark 2.1, the notion of forward invariance with respect to nonautonomous systems is not the same as the one with respect to autonomous systems. Again, since we fix an initial time t to $t = 0$, forward invariance given by Definition 2.8 is employed in the human assist control problem. The condition *(P2)* concerns the optimality. Due to this condition, a controller that fully cancels a human operator input is not a human assist controller; e.g., an automobile at rest while a human operator steps on an accelerator is safe but not optimal. The condition *(P3)* is unique to this study, which leads to increasing operability and passenger comfort.

Problem 3.1 *The first objective of this chapter is to design a human assist controller* $u = k(x, t)$ *for the system* (3.10) *satisfying the conditions (P1), (P2) and (P3).*

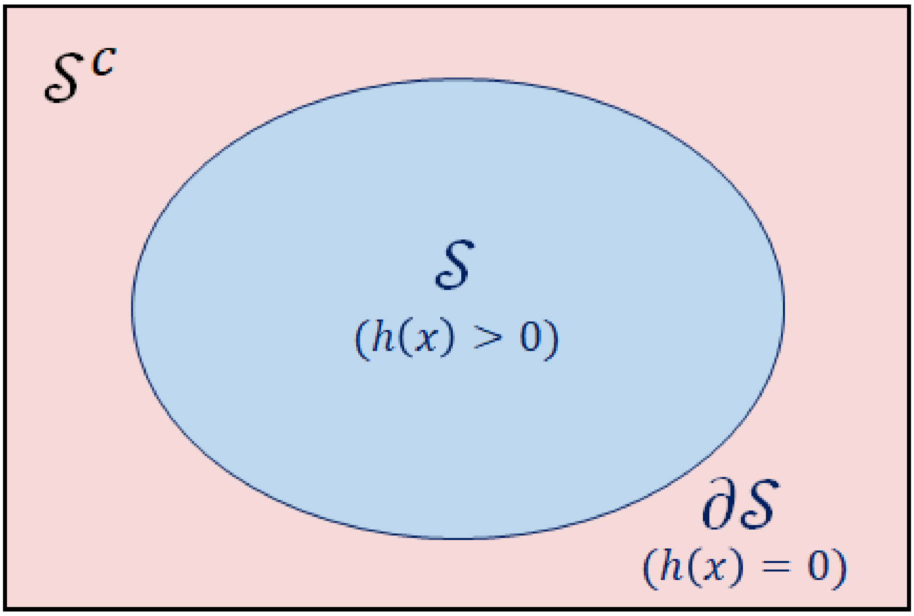

Figure 3.1: Safe Set.

3.4 Strict Zeroing Control Barrier Function for Continuous Safety Assist Control

In this section, the author proposes a human assist controller for nonautonomous systems using a ZCBF.

3.4.1 Strict Zeroing Control Barrier Function

In this subsection, the author defines the "strict" version of a ZCBF to solve the problems of the conventional ZCBF given by Definition 3.1.

Definition 3.3 *(Strict Zeroing Control Barrier Function) Consider the system* (3.10) *and the safe set* $\mathcal{S}$ *defined by* (3.11). *Then, a* C^1 *continuously differentiable function* $h : \mathcal{D} \to \mathbb{R}$ *is a Strict Zeroing Control Barrier Function (S-ZCBF) if the following conditions hold.*

(A1) $h(x)$ *is a proper function for* $\forall x \in \mathcal{S}$*; for any positive constant* $L \in \mathbb{R}_{>0}$*, the superlevel set* $\{x | h(x) \geq L\}$ *is compact.*

(A2) There exists a locally Lipschitz continuous extended class $\mathcal{K}$ *function* α *such that the following inequality holds:*

$$\sup_{u \in \mathbb{R}^m} \dot{h}(x, t, u, u_h(t)) > -\alpha\left(h(x)\right), \ \forall x \in \mathcal{D}. \tag{3.12}$$

Remark 3.1 *The significant property obtained from the condition (A1) is that the solution* $x(t)$ *to the system* (3.10) *does not have a finite escape time, which will be clarified in the next subsection. Moreover, the invariance of the CBF's properties through coordinate transformation will hold if the condition (A1) is satisfied [64].*

Remark 3.2 *The condition (A2) requires that an extended class* $\mathcal{K}$ *function* α *is locally Lipschitz continuous in h, whereas a ZCBF for autonomous systems given in Definition 3.5 simply requires continuity [47]. The easiest way to confirm that S-ZCBF candidates satisfy the condition (A2) is to verify whether the following strict inequality holds:*

$$L_g h(x) = 0 \;\Rightarrow\; \dot{h} = L_f h(x) > -\alpha(h(x)). \tag{3.13}$$

Using the strict conditions for an S-ZCBF, the following theorem that provides a global condition leading to a global solution to (3.10) can be obtained.

Theorem 3.2 *Consider the system* (3.10) *and the safe set* $\mathcal{S}$ *defined by* (3.11). *Then, a* C^1 *continuously differentiable function* $h : \mathcal{D} \to \mathbb{R}$ *is an S-ZCBF if and only if the condition (A1) and the following condition hold.*

(A2') There exists a positive constant $\gamma \in \mathbb{R}_{>0}$ *such that the following inequality holds:*

$$\sup_{u \in \mathbb{R}^m} \dot{h}(x, t, u, u_h(t)) > -\gamma h(x), \;\forall x \in \mathcal{S}. \tag{3.14}$$

To prove Theorem 3.2, the author first introduces the following lemma.

Lemma 3.1 *Consider the system* (3.10) *and the safe set* $\mathcal{S}$ *defined by* (3.11). *Assume that* $x \in \mathcal{S}$. *If a* C^1 *continuously differentiable function* $h : \mathcal{D} \to \mathbb{R}$ *is an S-ZCBF, i.e.,* $h(x)$ *is a proper function and satisfies the inequality* (3.12), *then there exists a positive constant* $\gamma \in \mathbb{R}_{>0}$ *such that*

$$\gamma \geq \frac{\alpha\left(h(x)\right)}{h(x)}, \;\forall x \in \mathcal{S}. \tag{3.15}$$

Proof Note that $x \in \mathcal{S}$ implies $h(x) > 0$. Since a function α is locally Lipschitz continuous, α can be represented as follows:

$$\alpha\left(h(x)\right) = \gamma_1 h(x) + \delta(h(x)), \tag{3.16}$$

where $\gamma_1 \in \mathbb{R}_{>0}$ is a positive constant and $\delta : \mathbb{R} \to \mathbb{R}$ is a higher-order term of $h(x)$. Then consider the following limit:

$$\lim_{h \to 0} \frac{1}{h}\left(\frac{1}{2}\gamma_1 h - \delta(h)\right) = \frac{1}{2}\gamma_1 > 0. \tag{3.17}$$

Hence there exists a positive constant $L \in \mathbb{R}_{>0}$ such that

$$\frac{1}{2}\gamma_1 h(x) \geq \delta(h(x)), \tag{3.18}$$

for $\forall x \in \mathcal{W} := \{x|\ h(x) < L\} \subset \mathcal{S}$. (3.16) and (3.18) imply that there exists a positive constant $\gamma\ (> \gamma_1) \in \mathbb{R}_{>0}$ such that

$$\gamma \geq \frac{\alpha\,(h(x_a))}{h(x_a)},\ \forall x_a \in \mathcal{W} \subset \mathcal{S}. \tag{3.19}$$

Thus, we now consider the case of $h(x) \geq L$, i.e., $x \in \mathcal{S}\backslash\mathcal{W}$. Since $h(x)$ is a proper function for $\forall x \in \mathcal{S}$, a superlevel set $\mathcal{S}\backslash\mathcal{W} = \{x|\ h(x) \geq L\}$ is compact. Hence there exists a positive constant $\gamma \in \mathbb{R}_{>0}$ such that

$$\gamma \geq \frac{\alpha\,(h(x_b))}{h(x_b)},\ \forall x_b \in \mathcal{S}\backslash\mathcal{W} \subset \mathcal{S}, \tag{3.20}$$

due to the extreme value theorem [53].

Therefore, according to (3.19) and (3.20), there exists a positive constant $\gamma \in \mathbb{R}_{>0}$ such that

$$\gamma \geq \frac{\alpha\,(h(x))}{h(x)},\ \forall x \in \mathcal{S}, \tag{3.21}$$

and the above discussion completes the proof. □

Using Lemma 3.1, we can now prove Theorem 3.2.

Proof Note that $x \in \mathcal{S}$ implies $h(x) > 0$. According to Lemma 3.1, in the case of $x \in \mathcal{S}$, *(A2)* can be rewritten as follows:

$$\begin{aligned}\sup_{u \in \mathbb{R}^m} \dot{h}(x, t, u, u_h(t)) > -\alpha\,(h(x)) &= -\frac{\alpha\,(h(x))}{h(x)}h(x) \\ &\geq -\gamma h(x),\ \forall x \in \mathcal{S}.\end{aligned} \tag{3.22}$$

Thus, *(A2)* is equivalent to the global condition *(A2')*. Therefore, $h(x)$ is an S-ZCBF if and only if the conditions *(A1)* and *(A2')* hold. □

Theorem 3.2 provides the global Lipschitz condition for an S-ZCBF, which is the powerful result for the indefinite extension of the solution $x(t)$ to (3.10). This will be discussed in the following subsection.

3.4.2 Human Assist Control using Strict Zeroing Control Barrier Function

In the preceding subsection, the author provided the strict conditions for a ZCBF. Using an S-ZCBF, the author proposes a human assist control law for the nonautonomous system (3.10) in this subsection.

Theorem 3.3 *Consider the system* (3.10)*, the safe set $\mathcal{S}$ defined by* (3.11) *and an S-ZCBF $h : \mathcal{D} \to \mathbb{R}$ satisfying the conditions (A1) and (A2'). Then, for any continuous mapping $u_h : \mathbb{R} \to \mathbb{R}^m$, the following human assist controller $u = k(x, t)$ ensures the forward invariance of $\mathcal{S}$.*

$$u = k(x,t) = \begin{cases} -\dfrac{I(x,t,u_h(t)) - J(x)}{\|L_g h(x,t)\|^2}(L_g h(x,t))^T \\ \qquad\qquad \text{if } I(x,t,u_h(t)) < J(x) \\ 0 \qquad\quad \text{if } I(x,t,u_h(t)) \geq J(x), \end{cases} \tag{3.23}$$

where functions $I : \mathcal{D} \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ and $J : \mathcal{D} \to \mathbb{R}$ are defined by

$$\begin{aligned} I(x,t,u_h) &= L_f h(x,t) + L_g h(x,t) \cdot u_h(t), \\ J(x) &= -\gamma h(x). \end{aligned} \tag{3.24}$$

Proof Denote the closed-loop system (3.10) with the initial state $x_0 = x(0) \in \mathcal{S}$ and the proposed human assist controller (3.23) as follows:

$$\begin{aligned} \dot{x} &= f_{cl}(x,t,k(x,t),u_h(t)) \\ &= f(x,t) + g(x,t)\left[k(x,t) + u_h(t)\right]. \end{aligned} \tag{3.25}$$

The first step is to prove the continuity of the proposed controller (3.23) to ensure that there exists a local solution $x(t)$ to (3.25).

Note that $L_f h(x,t), L_g h(x,t)$ and $u_h(t)$ are all continuous mappings, and hence functions $I(x,t,u_h)$ and $J(x)$ are continuous. Moreover, $u \to 0$ as $I \to J$ uniformly when $L_g h(x,t) \neq 0$. Hence (3.23) is continuous when $L_g h(x,t) \neq 0$. Thus, we need to prove the continuity of (3.23) at $L_g h(x,t) = 0$. According to the condition *(A2')* for an S-ZCBF, for $\forall t_0 \geq 0$ such that $L_g h(t_0) = L_g h(x(t_0), t_0) = 0$, the following inequality holds:

$$\dot{h} = L_f h(t_0) > -\gamma h(t_0). \tag{3.26}$$

This implies that there exists $\varepsilon \in \mathbb{R}_{>0}$ such that

$$\dot{h} = L_f h(t_0) - \varepsilon \geq -\gamma h(t_0). \tag{3.27}$$

Since $L_g h(x,t) \cdot u_h(t)$ is a continuous mapping, there exists a neighborhood $\mathcal{V} \subset \mathcal{D} \times \mathbb{R}$ of $(x(t_0), t_0)$ such that

$$\|L_g h(x,t) \cdot u_h(t)\| \leq \varepsilon, \ \forall (x,t) \in \mathcal{V}. \tag{3.28}$$

Thus, the following inequality holds for any $(x,t) \in \mathcal{V}$:

$$L_f h(x,t) + L_g h(x,t) \cdot u_h(t) \geq -\gamma h(x). \tag{3.29}$$

Accordingly, this implies that $I(x, t, u_h) \geq J(x)$. Hence $u = k(x, t) = 0$ for any $(x, t) \in \mathcal{V}$, and this implies that there exists a neighborhood $\mathcal{V}$ of $L_g h(x, t) = 0$ such that $u = k(x, t) = 0$ for $\forall (x, t) \in \mathcal{V}$. Thus, $u = k(x, t)$ is continuous for $\forall (x, t) \in \mathcal{D} \times \mathbb{R}$. Therefore, for any continuous mapping $u_h(t)$, the mapping f_{cl} is continuous in (x, t) and hence there exists a local solution $x(t)$ to (3.25).

The second step is to show that the proposed controller (3.23) ensures the forward invariance of the safe set $\mathcal{S}$.

The time-derivative of an S-ZCBF h is calculated as follows:

$$\begin{aligned} \dot{h} &= \frac{dh}{dx}\frac{dx}{dt} \\ &= L_f h + L_g h \cdot k(x, t) + L_g h \cdot u_h(t). \end{aligned} \tag{3.30}$$

Case 1: $I \geq J$.
Since $u = 0$, the following inequality holds:

$$\dot{h} = I(x, t, u_h) = L_f h + L_g h \cdot u_h(t) \geq -\gamma h(x). \tag{3.31}$$

Case 2: $I < J$.
Substituting (3.23) into (3.30) yields

$$\dot{h} = J(x) = -\gamma h(x). \tag{3.32}$$

Hence in both cases 1 and 2, the following inequality holds:

$$\dot{h} \geq -\gamma h(x). \tag{3.33}$$

Since the right-hand side of (3.33) is globally Lipschitz continuous in h for $\forall x \in \mathcal{S}$, there exists a unique lower bounded function $\underline{h}(t)$ over $[0, t_1]$, where $t_1 \in \mathbb{R}_{>0}$ can be arbitrarily extended by Theorem 2.4. Here, the open safe set implies that $h(x_0) > 0$. Thus, the following inequality holds according to Gronwall's lemma 2.1:

$$\begin{aligned} h(x(t)) \geq \underline{h}(t) &= h(x_0) \exp(-\gamma t) \\ &\geq h(x_0) \exp(-\gamma t_1) > 0, \forall t \in [0, t_1], \end{aligned} \tag{3.34}$$

for any local solution $x(t)$ starting from $x_0 \in \mathcal{S}$. Since $h(x)$ is a proper function for $\forall x \in \mathcal{S}$, the set $\mathcal{M} := \{x |\, h(x) \geq h(x_0) \exp(-\gamma t_1)\} \subset \mathcal{S}$ is compact, and hence there exists a positive constant $r \in \mathbb{R}_{>0}$ such that $\{x |\, \|x - x_0\| \leq r\} \supset \mathcal{M}$. Then consider the following rectangle $\mathcal{R}$:

$$\mathcal{R}: \quad 0 \leq t \leq t_1,\ \|x - x_0\| \leq r, \tag{3.35}$$

where r is sufficiently large. Since the mapping f_{cl} is continuous in (x, t) on the rectangle $\mathcal{R}$, there exists at least one solution $x(t)$ to (3.25) defined on $[0, t_1]$ by Theorem 2.1, and $t_1 \in \mathbb{R}_{>0}$ can be extended indefinitely by (3.34); for any constant $t_1 \in \mathbb{R}_{>0}$, there exists a solution $x : [0, t_1] \to \mathcal{S}$ for any initial state $x_0 \in \mathcal{S}$. Therefore, the proposed controller (3.23) ensures the forward invariance of $\mathcal{S}$, i.e., $x(t) \in \mathcal{S}$ for $\forall t \geq 0$. □

Remark 3.3 *The first part of the proof indicates the continuity of the proposed human assist controller* (3.23). *In [74], the continuity of a controller is ensured only in the case that the relative degree one condition, $L_g h_a(x) \neq 0$ for $\forall x \in \mathcal{D}$, is satisfied. On the other hand, the proposed controller* (3.23) *is continuous for $\forall (x, t) \in \mathcal{D} \times \mathbb{R}$ even in the case of $L_g h(x, t) = 0$, i.e., the continuity of the mapping f_{cl} in (x, t) is guaranteed regardless of the relative degree condition. Note that if the inequality in the condition (A2') (or (A2)) is not strict, the proposed controller might fail to be continuous at $L_g h = 0$.*

The following example illustrates the case that a controller fails to be continuous when the inequality (3.12) (or (3.14)) is not strict.

Example 3.3 *Consider the following system:*

$$\dot{x}_1 = \frac{K x_1^4 - x_1}{2K x_1^3 + 1} + x_2, \tag{3.36}$$

$$\dot{x}_2 = u + u_h(t), \tag{3.37}$$

where $K \in \mathbb{R}_{>0}$ is a positive constant. Assume that a safe set is given by

$$\mathcal{S} = \{x \in \mathbb{R}^2 \mid x_1 < 0\}. \tag{3.38}$$

Here, define the following S-ZCBF candidate:

$$h(x) = \frac{x_1}{K x_1^3 + x_1 x_2^2 - 1}, \tag{3.39}$$

so that it satisfies the condition (A1) regarding the properness of functions. In this case, choosing $\gamma = 1$ yields $\dot{h} = -\gamma h(x)$ at $x_2 = 0$, indicating that $h(x)$ does not satisfy the strict condition (A2') and results in the discontinuity of u at $L_g h = 0$.

Figures 3.2, 3.3 illustrate computer simulation results for $x_0 = (-5, -2)$, $u_h = 3$, $K = 0.001$, and $\gamma = 1$. From Fig. 3.2, we can confirm that the proposed controller (3.23) *satisfies the state constraint $x(t) \in \mathcal{S}$ as $x_1(t) < 0$ for $\forall t \geq 0$. However, from Fig. 3.3, we can also confirm that the controller is discontinuous at $x_2 = 0$ due to the discussion above.*

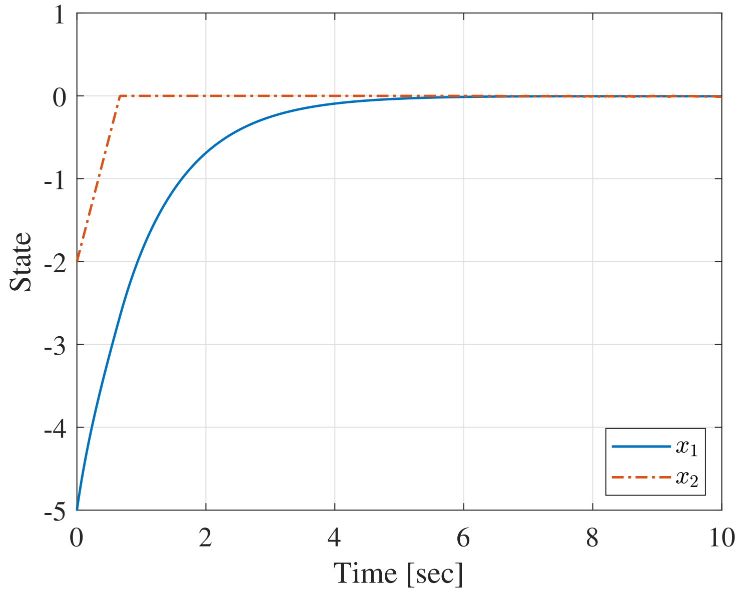

Figure 3.2: Computer Simulation: State (Example 3.3).

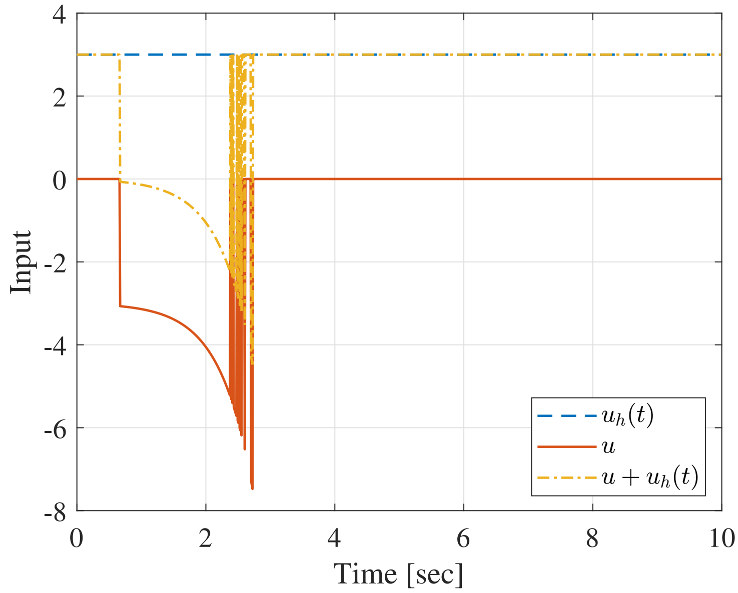

Figure 3.3: Computer Simulation: Input (Example 3.3).

Remark 3.4 *The second part of the proof indicates the forward invariance of* $\mathcal{S}$*. Using the condition (A1), the proposed human assist controller can ensure the forward completeness of solutions to the closed-loop system* (3.25)*. Note that* $\mathcal{S}$ *is not forward invariant if solutions have a finite escape time.*

If a ZCBF candidate is not proper, i.e., the condition *(A1)* does not hold, solutions to (3.25) might not be forward complete. That is, a safe set $\mathcal{S}$ is not forward invariant. The following example illustrates the importance of the condition *(A1)*.

Example 3.4 *Reconsider the modified Example 3.1, i.e.,*

$$\dot{x} = -x^2 + u + u_h(t), \tag{3.40}$$

with the initial state $x_0 = x(0) = -1$ *and* $u_h(t) \equiv 0$*. Assume that a safe set* $\mathcal{S}$ *is given by*

$$\mathcal{S} = \{x \in \mathbb{R} \mid x < 0\}. \tag{3.41}$$

Then, a ZCBF candidate $h_a(x)$ *and an S-ZCBF* $h(x)$ *candidate are constructed as follows:*

$$h_a(x) = -x, \tag{3.42}$$

$$h(x) = -\frac{x}{1 - Kx^3}, \tag{3.43}$$

where $K \in \mathbb{R}_{>0}$ *is a positive constant. Note that these candidates are valid as both* $L_g h_a(x) \neq 0$ *and* $L_g h(x) \neq 0$ *are always satisfied.*

Figures 3.4, 3.5 illustrate computer simulation results for $x_0 = -1$*,* $u_h = 0$*,* $K = 0.01$*, and* $\gamma = 1$*. From Fig. 3.4, we can confirm that the proposed controller* (3.23) *with the S-ZCBF* (3.43) *extends the solution and consequently ensures the forward invariance of* $\mathcal{S}$*. On the other hand, from Fig. 3.5, the controller with the ZCBF* (3.42) *does not work although the solution has a finite escape time* $T = 1$*. Accordingly, the conventional ZCBF-based controller does not render* $\mathcal{S}$ *forward invariant.*

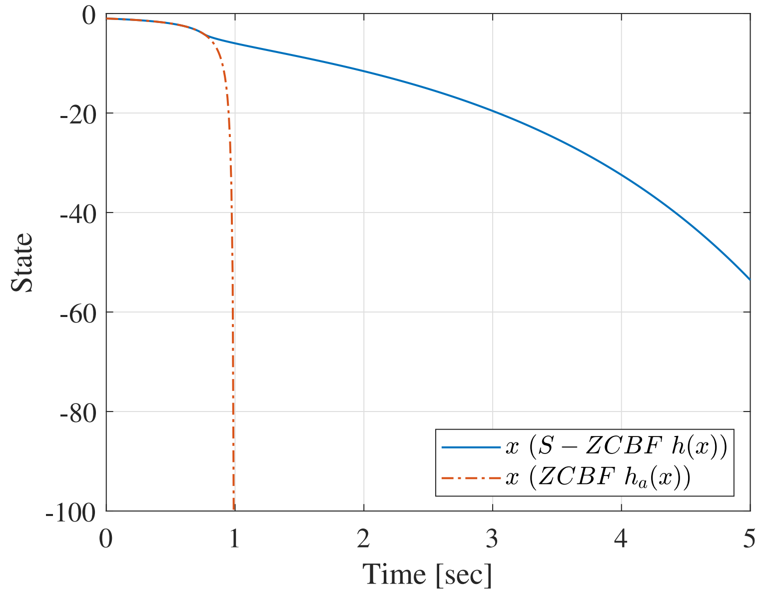

Figure 3.4: Computer Simulation: State (Example 3.4).

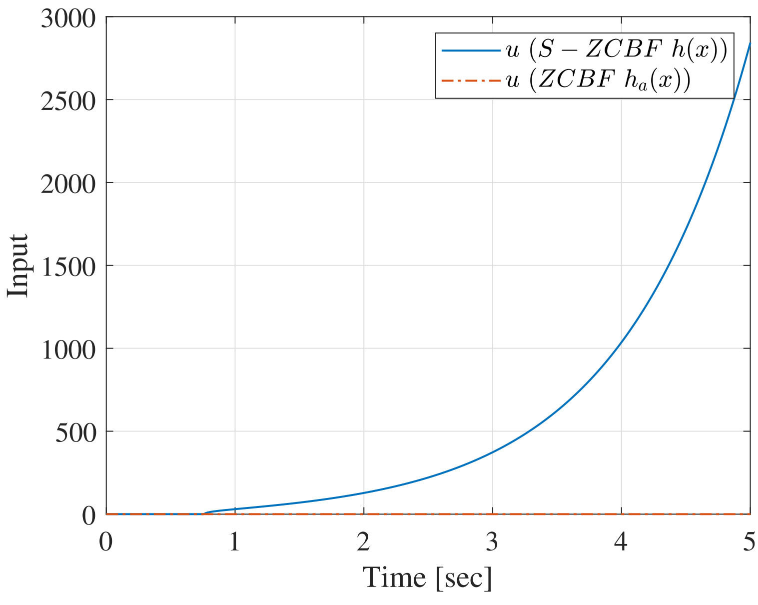

Figure 3.5: Computer Simulation: Input (Example 3.4).

Remark 3.5 *Importantly, Gronwall's lemma is valid for safety assurance regardless of whether control systems are autonomous or nonautonomous, whereas Nagumo's theorem is only applicable to autonomous systems. The alternative approach to ensure the forward invariance of $\mathcal{S}$ can be seen in [28,47], where comparison lemma is used. In [57], Gronwall's lemma is applied under the assumption that there exists $\gamma \in \mathbb{R}_{>0}$ such that* (3.14) *holds. Note that an S-ZCBF satisfying the conditions (A1) and (A2) ensures the existence of such γ, which yields the global condition (A2').*

The author next proves that the human assist controller (3.23) is minimally invasive. The proof of this property implies the optimality of (3.23) described in the second condition of Definition 3.2. This property improves the operability of a human-operated system and increases passenger comfort when considering, for instance, a human-operated automobile.

Lemma 3.2 *Consider the system* (3.10)*, the safe set $\mathcal{S}$ defined by* (3.11) *and an S-ZCBF $h : \mathcal{D} \to \mathbb{R}$. Then, the human assist controller* (3.23) *minimizes the Euclidian norm $\|u\|$ for $\forall (x,t) \in \mathcal{D} \times \mathbb{R}_{\geq 0}$ such that the following inequality holds:*

$$I(x,t,u_h) + L_g h(x,t)u \geq J(x). \tag{3.44}$$

Proof When $I(x,t,u_h) \geq J(x)$, $u = 0$ by (3.23) and $\|u\| = 0$; this implies that (3.23) minimizes the Euclidian norm. Therefore, we consider the case of $I(x,t,u_h) < J(x)$. Finding the controller that minimizes the Euclidian norm $\|u\|$ satisfying (3.44) is equivalent to solving the following optimization problem:

$$\begin{aligned} &\text{Minimize } \|u\|^2 \\ &\text{s. t. } L_f h(x,t) + L_g h(x,t)u + L_g h(x,t)u_h(t) \geq -\gamma h(x). \end{aligned} \tag{3.45}$$

The KKT optimality conditions given by Proposition 2.2 for (3.45) imply there exists Lagrange multiplier $\lambda \geq 0$ such that

$$\begin{aligned} &2u^T - \lambda L_g h(x,t) = 0, \\ &\lambda \left(-I(x,t,u_h) - L_g h(x,t)u + J(x)\right) = 0. \end{aligned} \tag{3.46}$$

When $\lambda = 0$, $u = 0$ and this implies $I(x,t,u_h) \geq J(x)$. Hence, we consider the case of $\lambda \neq 0$ and $I(x,t,u_h) < J(x)$. Then, Lagrange multiplier λ is uniquely determined by (3.46) as follows:

$$\lambda = -2\frac{I(x,t,u_h) - J(x)}{\|L_g h(x,t)\|^2}. \tag{3.47}$$

Note that the KKT optimality conditions for a nonlinear optimization problem are necessary but not sufficient; however, it is sufficient because the optimization problem considered here is convex. Thus, the unique Lagrange multiplier λ given by (3.47) minimizes the Euclidian norm $\|u\|$.

Therefore, the following controller u is the solution to the optimization problem (3.45):

$$u = -\frac{I(x, t, u_h) - J(x)}{||L_g h(x, t)||^2}(L_g h(x, t)),^T \tag{3.48}$$

when $I(x, t, u_h) < J(x)$. This is equivalent to (3.23) in the case of $I(x, t, u_h) < J(x)$, and the above discussion completes the proof. □

3.4.3 Mathematical Example: Non-Lipschitz Control System

In this section, we study a mathematical example, a system whose $g(x, t)$ is continuous in (x, t) but not Lipschitz continuous in x, to confirm the effectiveness of the proposed human assist controller.

Consider the following control system:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= g_2(x, t)\,(u + u_h(t))\,, \end{aligned} \tag{3.49}$$

where $x = [x_1, x_2]^T \in \mathbb{R}^2$ denotes a state, $u \in \mathbb{R}$ a control input, and $u_h : \mathbb{R} \to \mathbb{R}$ a human operator input, respectively. Moreover, we assume that a mapping $g_2(x, t)$ is of the form:

$$g_2(x, t) = 3 + \sum_{n=0}^{\infty} a^n \cos(b^n \pi x_1) + 0.5 \sin t, \tag{3.50}$$

where $a \in (0, 1)$, $b \in \mathbb{R}_{>0}$ is a positive odd integer, and $ab > 1 + 3\pi/2$. The second term of (3.50) is known as the Weierstrass function, which is continuous everywhere, but differentiable nowhere and not Lipschitz continuous in x_1 [68]. Note that the first term of (3.50) avoids $g_2(x, t) = 0$. Note also that substituting $f(x) = [x_2\ 0]^T$, $g(x) = [0\ g_2(x, t)]^T$ into (3.10) yields the system (3.49).

A safe set is given by $\mathcal{S} = (-\infty, 0) \times \mathbb{R} \subset \mathbb{R}^2$, whereas an unsafe set $\mathcal{S}^c = \mathbb{R}^2 \backslash \mathcal{S} \subset \mathbb{R}^2$. We assume that the initial state $x_0 = x(0)$ is safe, i.e., $x_0 \in \mathcal{S}$.

Controller Design

Firstly, in a similar fashion to [50], we construct a relaxed CBF candidate $B : \mathcal{S} \to \mathbb{R}$ as follows:

$$B(x) = -\frac{1}{x_1} + L_1 x_1^2 + L_2 x_2^2, \tag{3.51}$$

where $L_1, L_2 \in \mathbb{R}_{>0}$ are positive constants. Since a relaxed CBF is based on a reciprocal CBF, the relationship between a reciprocal CBF and a ZCBF according to [7] yields the following S-ZCBF candidate illustrated in Fig. 3.6 that satisfies the condition *(A1)*:

$$h(x) = \frac{1}{B(x)} = \frac{x_1}{L_1 x_1^3 + L_2 x_1 x_2^2 - 1}. \tag{3.52}$$

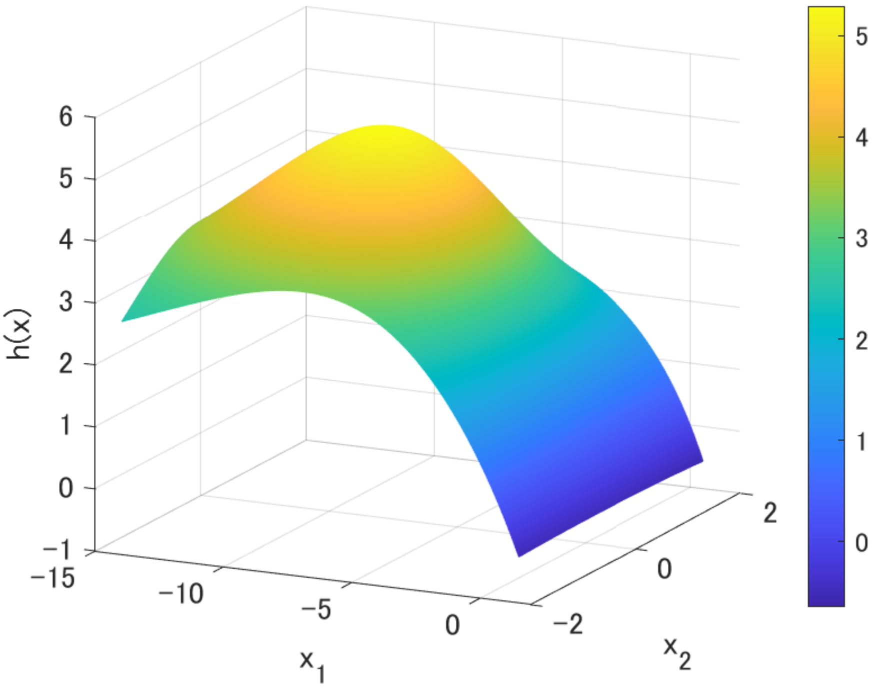

Figure 3.6: Mathematical Example: S-ZCBF (3.52) for $L_1 = 0.001$ and $L_2 = 0.05$.

As can be seen in Fig. 3.6, the condition *(A1)* generates an S-ZCBF whose shape resembles a mountain in the corresponding safe set.

The Lie derivative of h is calculated as follows:

$$\begin{aligned} L_f h(x) &= -\frac{2L_1 x_1^3 + 1}{\left(L_1 x_1^3 + L_2 x_1 x_2^2 - 1\right)^2}\, x_2, \\ L_g h(x,t) &= -\frac{2L_2 x_1^2 x_2}{\left(L_1 x_1^3 + L_2 x_1 x_2^2 - 1\right)^2}\, g_2(x,t). \end{aligned} \tag{3.53}$$

From (3.53), we can confirm that there exists $x \in \mathcal{S}$ such that $L_g h = 0$, where $x_2 = 0$ in this case, as previously mentioned in Remark 3.3. However, we can also see that $L_f h = 0$ when $L_g h = 0$. Therefore, the following strict inequality holds at $x_2 = 0$ for any locally Lipschitz function $\alpha : \mathbb{R} \to \mathbb{R}$:

$$\dot{h} = 0 > -\alpha(h(x)), \quad \forall x \in \mathcal{S}, \tag{3.54}$$

indicating that the S-ZCBF candidate (3.52) satisfies the condition *(A2)*.

Consequently, we can design a human assist controller for the system (3.49) using (3.23).

Computer Simulation

Figures 3.7–3.9 illustrate computer simulation results for $x_0 = (-5, -1)$, $u_h(t) = 1 - 0.5\cos(\pi t/20)$, $L_1 = 0.001$, $L_2 = 0.05$, $\gamma = 0.3$, the summation n from 0 to 100 in (3.50) with $a = 0.5$, $b = 15$. Figures 3.7–3.9 denote the time series of $g_2(x,t)$, the system's state, and the input time series, respectively.

As shown in Fig. 3.8, the state x_2 decreases as the state x_1 approaches the unsafe set $\mathcal{S}^c$. Consequently, we can confirm that the system state x starting from the safe set $\mathcal{S}$ remains in $\mathcal{S}$. From Fig. 3.9, the human assist controller u works properly to keep the system state in $\mathcal{S}$, whereas $u_h(t)$ drives the system into $\mathcal{S}^c$. We can also confirm that the controller u is continuous at $x_2 = 0$, which is the case of $L_g h = 0$, because the constructed ZCBF (3.52) satisfies the strict inequality in the condition *(A2')*.

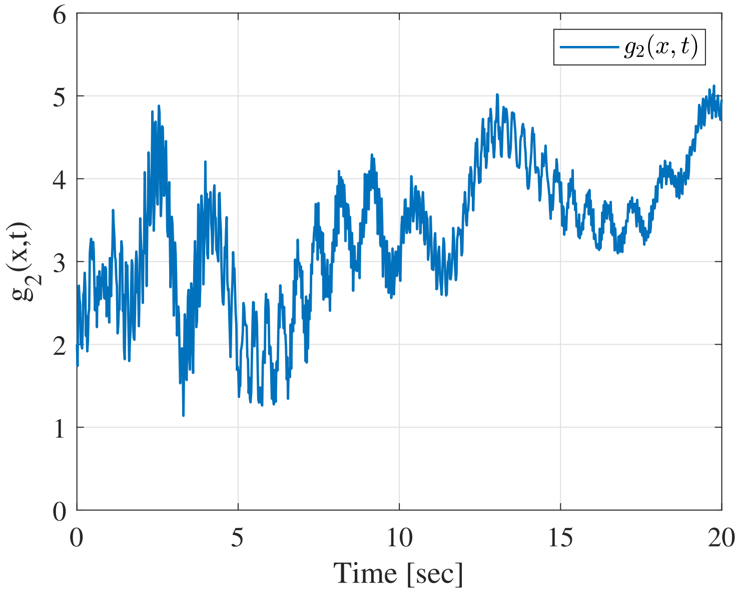

Figure 3.7: Computer Simulation: Function $g_2(x,t)$ (Mathematical Example).

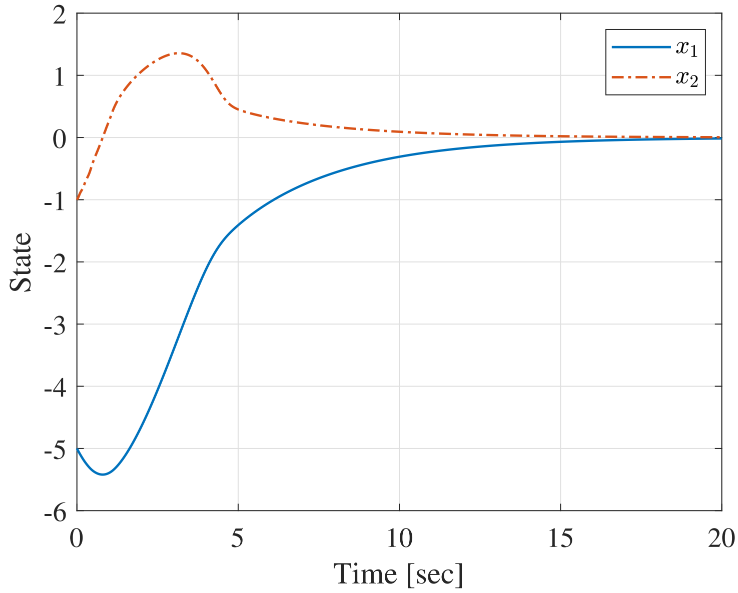

Figure 3.8: Computer Simulation: State (Mathematical Example).

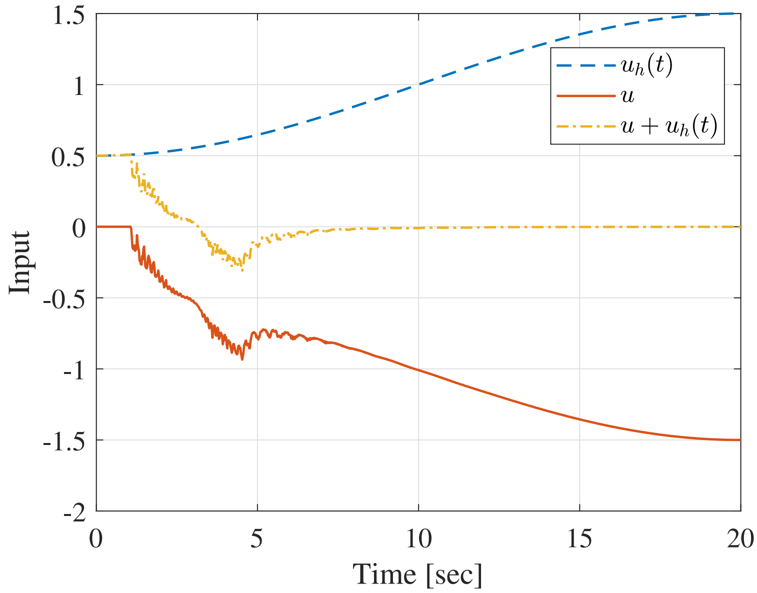

Figure 3.9: Computer Simulation: Input (Mathematical Example).

3.5 Input-constrained Safety Assist Control

In real applications, control systems generally have input constraints. For example, human-operated automobiles cannot instantly stop due to a stopping distance, which consists of a thinking distance and a braking distance. A thinking distance is unique to human-operated automobiles, and it might be shortened or ideally eliminated by constructing a human assist controller. However, it is impossible to eliminate a braking distance because maximum deceleration values are bounded. In this case, the maximum deceleration indicates the existence of input constraints.

While the S-ZCBF-based safety assist controller proposed in Section 3.4 ensures the safety of human-operated systems, it cannot consider input-constrained systems. The following example motivates the problem.

Example 3.5 *Consider the following double-integrator automobile system with an input constraint* $u + u_h \in [-a, a], a > 0$*:*

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= u + u_h(t), \end{aligned} \tag{3.55}$$

where $x_1 \in \mathbb{R}$ *denotes a position,* $x_2 \in \mathbb{R}$ *a velocity, respectively. Let the safe set* $\mathcal{S}$ *be*

$$\mathcal{S} = \{x \in \mathbb{R}^2 \mid x_1 < 0\}. \tag{3.56}$$

In this example, another approach is taken to design an S-ZCBF; that is, the following S-ZCBF is designed by multiplying a proper function $p : \mathbb{R}^2 \to \mathbb{R}_{>0}$*:*

$$h(x) = -x_1 p(x) = -\frac{x_1}{1 + K(x_1^2 + x_2^2)}, \tag{3.57}$$

where $K \in \mathbb{R}_{>0}$ *is a positive constant. Then, the Lie derivative of* h *is calculated as follows:*

$$L_f h(x) = -\frac{1 - K(x_1^2 - x_2^2)}{\left(1 + K\left(x_1^2 + x_2^2\right)\right)^2} x_2, \quad L_g h(x) = \frac{2K x_1 x_2}{\left(1 + K\left(x_1^2 + x_2^2\right)\right)^2}. \tag{3.58}$$

Accordingly, we can design a human assist controller for the system (3.55) *using* (3.23)*. Note that we choose a locally Lipschitz function* $\alpha : \mathbb{R} \to \mathbb{R}$ *as follows:*

$$J(x) = -\alpha(h(x)) = -\gamma_1 h(x) - \gamma_2 h^2(x), \tag{3.59}$$

where $\gamma_1, \gamma_2 \in \mathbb{R}_{>0}$ *are positive constants.*

Figures 3.10, 3.11 illustrate the results of computer simulation for $x_0 = (-10, 10)$*,* $u_h(t) = 1.5 - 0.5\cos(\pi t/5)$*,* $K = 0.001$*,* $\gamma_1 = \gamma_2 = 1.0$*,* $a = 10$*. Figure 3.10 denotes the state of the system* (3.55)*, and Fig. 3.11 the control input, respectively. From Fig. 3.10, we can confirm that the state constraint* $x(t) \in \mathcal{S}$ *is satisfied as* $x_1(t) < 0$ *for* $\forall t \geq 0$*. However, from Fig. 3.11, we can also confirm the violation of the input constraint* $u + u_h(t) \in [-a, a]$*. This seems to be caused by the neglect of input constraints when designing an S-ZCBF.*

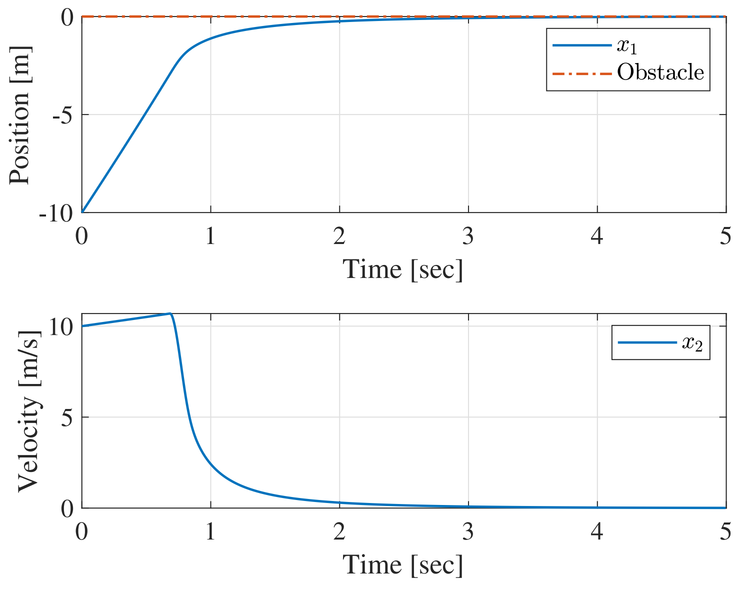

Figure 3.10: Computer Simulation: State (Motivation).

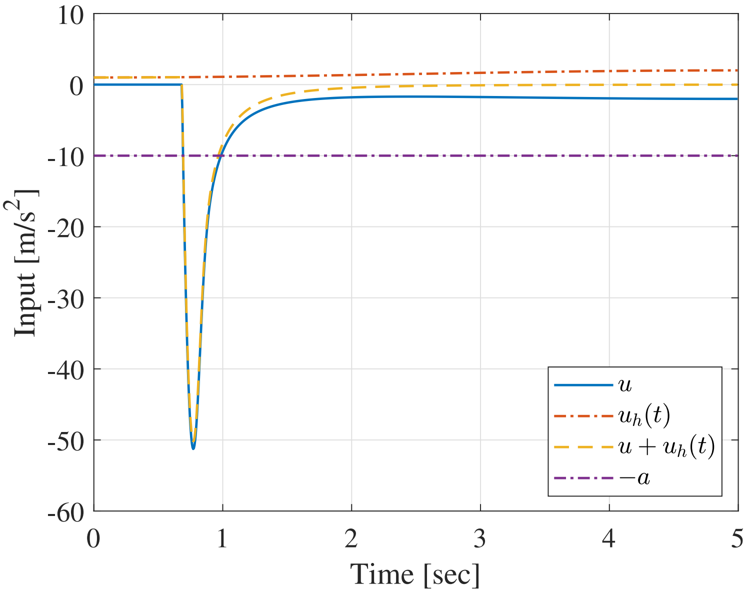

Figure 3.11: Computer Simulation: Input (Motivation).

As almost all real control systems have input constraints, a safety assist controller needs to satisfy not only state constraints but also input constraints.

3.5.1 Problem Setup

In this section, consider the following nonlinear control system with a constrained human operator input $u_h : \mathbb{R} \to \mathcal{U}_s$:

$$\dot{x} = f(x) + g(x)[u + u_h(t)], \tag{3.60}$$

where $x \in \mathbb{R}^n$ denotes a state, $u \in \mathbb{R}^m$ a control input, and $\mathcal{U}_s \subset \mathbb{R}^m$ a bounded input constraint on $u_h(t)$. Assume that the total control input $u + u_h$ has the same bounded input constraint as $u_h(t)$, i.e., $u+u_h \in \mathcal{U}_s$. Accordingly, assume that there exists an input constraint $u \in \mathcal{U} := \{u \in \mathbb{R}^m \mid u+u_h \in \mathcal{U}_s\}$ where $\mathcal{U}$ is a convex bounded subset of $\mathbb{R}^m$. Assume also that a mapping $u_h : \mathbb{R} \to \mathcal{U}_s$ is continuous in t, and mappings $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are continuous in x.

As in Section 3.3, a state space $\mathbb{R}^n$ is divided into a safe set $\mathcal{S}$ and an unsafe set $\mathcal{S}^c$, i.e., $\mathcal{S} \cup \mathcal{S}^c = \mathbb{R}^n$.

To design a safety assist controller that satisfies both state and input constraints, we employ the notion of a viability kernel advocated in viability theory [10]. The motivation for its employment comes from the definition of forward invariance. Specifically, when we say that the safe set $\mathcal{S}$ is forward invariant, "all" the initial states contained in $\mathcal{S}$ must satisfy $x(t) \in \mathcal{S}$ for $\forall t \geq 0$. However, it is impossible to guarantee the forward invariance of $\mathcal{S}$ for some $x_0 \in \mathcal{S}$ when considering input

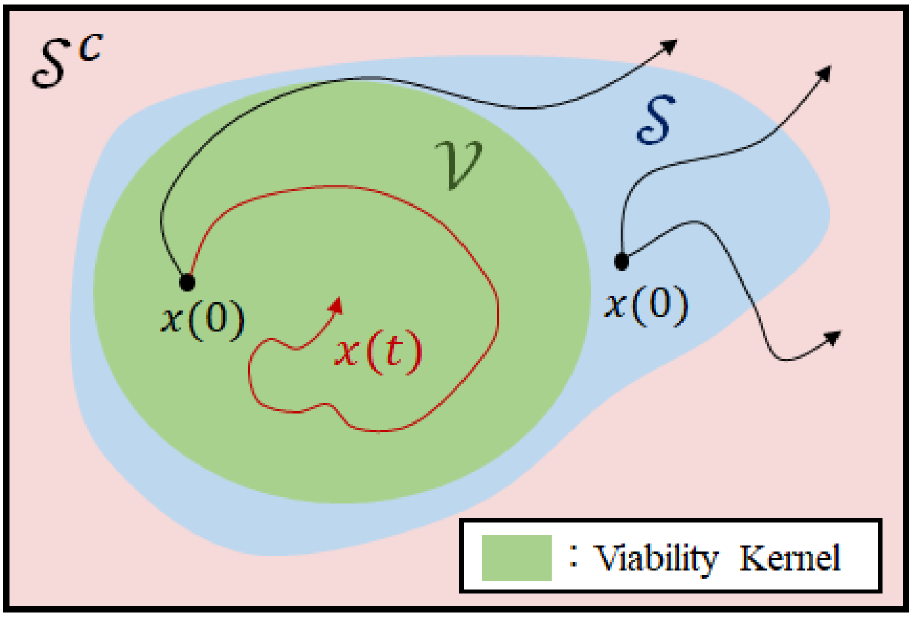

Figure 3.12: Viability Kernel.

constraints, i.e., only a subset of a safe set might be forward invariant within input constraints. Therefore, forward invariance of a safe set is not always a correct statement on input-constrained systems. Motivated by this, we consider a viability kernel for a safe set [10]. Originally, the subset of a safe set is a viability kernel if at least one solution starting from there remains in the safe set (Fig. 3.12). For systems with control inputs, a viability kernel is defined as follows [19].

Definition 3.4 *(Viability Kernel [10, 19]) The viability kernel is a subset of a safe set, i.e., $\mathcal{V}_\mathcal{S} := Viab(\mathcal{S}) \subseteq \mathcal{S}$, defined as follows:*

$$\mathcal{V}_\mathcal{S} = \{x \in \mathcal{S} \mid \exists u \in \mathcal{U} \text{ such that } x(t) \in \mathcal{S},\ \forall t \geq 0\}. \tag{3.61}$$

In other words, as long as the initial state satisfies $x_0 \in \mathcal{V}_\mathcal{S}$, there exists a controller $u \in \mathcal{U}$ that ensures both the state constraint $x(t) \in \mathcal{S}$ and the input constraint $u + u_h(t) \in \mathcal{U}_s$ for $\forall t \geq 0$. Accordingly, to satisfy both state and input constraints, a human assist controller at least renders a viability kernel forward invariant. Importantly, the viability kernel $\mathcal{V}_\mathcal{S}$ is a subset of its safe set, the forward invariance of $\mathcal{V}_\mathcal{S}$ implies that $x(t) \in \mathcal{S}$ for $\forall t \geq 0$.

Problem 3.2 *The second objective of this chapter is to design a human assist controller $u = k(x, t)$ for the input-constrained system* (3.60) *satisfying the conditions (P2), (P3) and the following condition:*

(P1') For any human operator input $u_h : \mathbb{R} \to \mathbb{R}^m$, the controller $u = k(x, t)$ renders the viability kernel $\mathcal{V}_\mathcal{S}$ forward invariant, i.e., if the initial state $x_0 := x(0)$ satisfies $x_0 \in \mathcal{V}_\mathcal{S}$, then $x(t) \in \mathcal{V}_\mathcal{S} \subset \mathcal{S}$ for $\forall t \geq 0$.

3.5.2 Zeroing Control Barrier Function for Viability Kernel

In Section 3.4 (or the literature [7]), ZCBF candidates are defined as positive (or non-negative) functions in a safe set as defined in (3.11). Outside the viability kernel $\mathcal{V}_{\mathcal{S}}^c = \mathbb{R}^n \backslash \mathcal{V}_{\mathcal{S}}$, however, any controllers within the input constraints cannot render the safe set $\mathcal{S}$ forward invariant. That is, there is no valid ZCBF enforcing itself if $x_0 \in \mathcal{V}_{\mathcal{S}}^c$ because the corresponding solution to the closed-loop system of (3.60) starting from $x_0 \in \mathcal{V}_{\mathcal{S}}^c$ has to leave $\mathcal{S}$ for $\forall u \in \mathcal{U}$. Therefore, a CBF should be defined for a viability kernel, not for the safe set itself.

Definition 3.5 *Consider the system* (3.60)*, and a viability kernel $\mathcal{V}_{\mathcal{S}}$ for the safe set $\mathcal{S}$. Then, a C^1 continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ is a (zeroing) control barrier function (for a viability kernel) if the following conditions hold.*

(B1) The viability kernel $\mathcal{V}_{\mathcal{S}}$ for the safe set $\mathcal{S}$ is defined as the strict 0-superlevel set of $h(x)$:

$$\mathcal{V}_{\mathcal{S}} = \{x \in \mathbb{R}^n \mid h(x) > 0\}, \tag{3.62}$$
$$\partial\mathcal{V}_{\mathcal{S}} = \{x \in \mathbb{R}^n \mid h(x) = 0\}, \tag{3.63}$$
$$\mathcal{V}_{\mathcal{S}}^c = \{x \in \mathbb{R}^n \mid h(x) \leq 0\}. \tag{3.64}$$

(B2) $h(x)$ is proper for $\forall x \in \mathcal{V}_{\mathcal{S}}$; for any positive constant $L \in \mathbb{R}_{>0}$, the superlevel set $\{x | h(x) \geq L\}$ is compact.

(B3) For any continuous mapping $u_h : \mathbb{R} \to \mathcal{U}_s$, there exists a locally Lipschitz continuous function $\alpha \in \mathcal{K}_e$ such that the following inequality holds:

$$\sup_{u \in \mathcal{U}} \dot{h}\,(x, u, u_h(t)) > -\alpha(h(x)), \ \forall x \in \mathcal{V}_{\mathcal{S}}. \tag{3.65}$$

The following result ensures the forward invariance of the viability kernel defined by (3.62).

Theorem 3.4 *Consider the system* (3.60)*, the viability kernel $\mathcal{V}_{\mathcal{S}}$ defined by* (3.62) *and a ZCBF $h : \mathbb{R}^n \to \mathbb{R}$. Then, any continuous controller $u = k(x, t)$ such that $\dot{h} \geq -\alpha\,(h(x))$ renders the viability kernel $\mathcal{V}_{\mathcal{S}}$ forward invariant and accordingly ensures $x(t) \in \mathcal{S}$ for $\forall t \geq 0$.*

Proof Note that the continuity of $u = k(x, t)$ implies that there exists a local solution to (3.60). Note also that $x_0 \in \mathcal{V}_{\mathcal{S}}$ implies $h(x_0) > 0$. Then, the time-derivative of a CBF satisfies $\dot{h} \geq -\alpha\,(h(x))$ by the assumption. The extension of Lemma 3.1 implies that there exists a positive constant $\eta \in \mathbb{R}_{>0}$ such that

$$\eta h(x) \geq \alpha\,(h(x))\,, \ \ \forall x \in \mathcal{V}_{\mathcal{S}}, \tag{3.66}$$

and hence

$$\dot{h} \geq -\alpha\,(h(x)) \geq -\eta h(x), \ \ \forall x \in \mathcal{V}_{\mathcal{S}}. \tag{3.67}$$

Thus, the following inequality holds according to Gronwall's lemma 2.1:

$$h(t) \geq h(x_0)\exp(-\eta t) > 0,\ \forall t \in [0, t_1], \tag{3.68}$$

where $t_1 \in \mathbb{R}_{>0}$ can be arbitrarily extended by Theorem 2.4. Since $h(x)$ is a proper function for $\forall x \in \mathcal{V}_\mathcal{S}$, the level set $\{x | h(x) \geq h(x_0)\exp(-\eta t_1)\} \subset \mathcal{V}_\mathcal{S}$ is compact; therefore, any continuous controller $u = k(x, t)$ such that $\dot{h} \geq -\alpha(h(x))$ renders the viability kernel $\mathcal{V}_\mathcal{S}$ forward invariant, and the inclusion $\mathcal{V}_\mathcal{S} \subseteq \mathcal{S}$ implies that $x(t) \in \mathcal{S}$ for $\forall t \geq 0$. □

To find a minimally invasive controller, we seem to re-solve the optimization problem considering input constraints. However, if we restrict the scope of control systems, the S-ZCBF-based controller can be directly applied to an input-constrained human assist control problem.

Lemma 3.3 *Consider the following single-input system ($m = 1$) with the norm input constraint:*

$$\dot{x} = f(x) + g(x)[u + u_h(t)], \tag{3.69}$$

$$u \in \mathcal{U} = \{u \in \mathbb{R} \mid |u + u_h(t)| \leq a,\ \ a > 0\}, \tag{3.70}$$

and a CBF $h : \mathbb{R}^n \to \mathbb{R}$ for the viability kernel $\mathcal{V}_\mathcal{S}$ defined by (3.62)*. Then, for any continuous mapping $u_h : \mathbb{R} \to Int(\mathcal{U}_s) = \{u_h \in \mathbb{R} \mid |u_h| < a,\ a > 0\}$, the following human assist controller $u = k(x, t)$ renders $\mathcal{V}_\mathcal{S}$ forward invariant while always satisfying the input constraint* (3.70)*:*

$$k(x,t) = \begin{cases} -\dfrac{1}{L_g h(x)}\left(I(x, u_h(t)) + \alpha(h(x))\right) \\ \qquad \text{if } I(x, u_h(t)) < -\alpha(h(x)) \\ 0 \qquad \text{if } I(x, u_h(t)) \geq -\alpha(h(x)), \end{cases} \tag{3.71}$$

where $\alpha : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous extended class $\mathcal{K}$ function and a function $I : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is defined by

$$I(x, u_h(t)) = L_f h(x) + L_g h(x) u_h(t). \tag{3.72}$$

Proof The proposed controller (3.71) ensures the forward invariance of $\mathcal{V}_\mathcal{S}$ by Theorem 3.4 because applying (3.71) to the time-derivative of a CBF h yields $\dot{h} \geq -\alpha(h(x))$ in both cases $I(x, u_h(t)) < -\alpha(h(x))$ and $I(x, u_h(t)) \geq -\alpha(h(x))$.

When $I(x, u_h(t)) \geq -\alpha(h(x))$, i.e., $k(x,t) = 0$, the input constraint (3.70) is evidently satisfied due to the assumption $u_h \in (-a, a)$. Hence, we consider the case of $I(x, u_h(t)) < -\alpha(h(x))$. If $L_g h \geq 0$, the control input $u_M = -u_h(t) + a$ maximizes $\dot{h}$ and satisfies

$$\max_{u \in \mathcal{U}} \dot{h}(x, u, u_h) = L_f h + L_g h\, a > -\alpha(h(x)), \tag{3.73}$$

by the condition *(B3)* for a CBF. Hence, the following inequality holds:

$$-\frac{1}{L_g h}\left(L_f h + \alpha(h(x))\right) < a. \tag{3.74}$$

When $I(x, u_h(t)) < -\alpha(h(x))$, (3.71) is rewritten as follows:

$$k(x,t) + u_h(t) = -\frac{1}{L_g h}\left(L_f h + \alpha(h(x))\right). \tag{3.75}$$

Accordingly, $k(x,t) + u_h(t) < a$ is satisfied. If $L_g h < 0$, the control input $u_m = -u_h(t) - a$ maximizes $\dot{h}$ and $k(x,t) + u_h(t) > -a$ is satisfied in a similar fashion to the case of $L_g h \geq 0$. Therefore, the proposed controller (3.71) satisfies the input constraint (3.70). □

Here, the proposed controller (3.71) is a single-input form of the S-ZCBF-based controller (3.23) in Theorem 3.3. Moreover, Lemma 3.3 implies that the controller (3.71) does not render the input constraint (3.70) active. Therefore, by the continuity of the controller (3.23), the following theorem is immediately obtained.

Theorem 3.5 *Consider the single-input system* (3.69)*, the input constraint* (3.70) *and a CBF* $h : \mathbb{R}^n \to \mathbb{R}$ *for the viability kernel* $\mathcal{V}_S$ *defined by* (3.62)*. Then, for any continuous mapping* $u_h : \mathbb{R} \to (-a, a)$*, the human assist controller* $u = k(x,t)$ *defined by* (3.71) *is continuous in* $(x,t) \in \mathbb{R}^n \times \mathbb{R}$.

3.5.3 Example: Double-Integrator System

In this subsection, we reconsider Example 3.5 to confirm the effectiveness of the proposed controller (3.71). Consider again the following double-integrator automobile system with an input constraint $u + u_h \in [-a, a], a > 0$:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= u + u_h(t), \end{aligned} \tag{3.76}$$

where $x_1 \in \mathbb{R}$ denotes a position, $x_2 \in \mathbb{R}$ a velocity, respectively. Let the safe set $\mathcal{S}$ be

$$\mathcal{S} = \{x \in \mathbb{R}^2 \mid x_1 < 0\}. \tag{3.77}$$

Derivation of Viability Kernel

Note that we handle the case of $x_2 \geq 0$ and $u + u_h(t) \geq -a$, because the system moves away from the unsafe set $\mathcal{S}^c$ in the case of $x_2 < 0$. Using the maximum deceleration value $a > 0$, the minimum braking distance d_{min} is calculated as follows:

$$\begin{aligned} d_{min} &= x_2\tau - \frac{1}{2}a\tau^2 \\ &= x_2\frac{x_2}{a} - \frac{1}{2}a\left(\frac{x_2}{a}\right)^2 = \frac{x_2^2}{2a}, \end{aligned} \tag{3.78}$$

where $\tau \in \mathbb{R}_{\geq 0}$ is the time taken to stop with $-a < 0$. Hence we derive the viability kernel of the safe set $\mathcal{S} = (-\infty, 0) \times \mathbb{R} \subset \mathbb{R}^2$ as follows:

$$\mathcal{V}_\mathcal{S} = \left\{ x \in \mathbb{R}^2 \;\middle|\; -x_1 - \frac{x_2^2}{2a} > 0 \right\}. \tag{3.79}$$

If $x_0 \in \mathcal{V}_\mathcal{S}$, there exists a controller $u \in \mathcal{U}$ that renders $x(t) \in \mathcal{S}$ for $\forall t \geq 0$.

Remark 3.6 *Considering the case of* $x_2 < 0$*, the viability kernel* $\mathcal{V}_\mathcal{S}$ *can be formally derived as follows:*

$$\mathcal{V}_\mathcal{S} = \left\{ x \in \mathbb{R}^2 \;\middle|\; -x_1 - \frac{x_2^2 + |x_2|x_2}{4a_{max}} > 0 \right\}. \tag{3.80}$$

As mentioned above, in the case of $x_2 < 0$*, the system* (3.76) *does not approach* $\mathcal{S}^c$*. This implies that* $\mathcal{V}_\mathcal{S} = \mathcal{S} = \{x \in \mathbb{R}^2 \mid x_1 < 0\}$ *as confirmed by* (3.80)*.*

Controller Design

Modifying the viability kernel (3.80) yields the following ZCBF $h : \mathbb{R}^2 \to \mathbb{R}$ illustrated in Fig. 3.13:

$$\begin{aligned} h(x) &= -\left(x_1 + \frac{x_2^2 + |x_2|x_2}{4a} \right) p(x) \\ &= -\left(x_1 + \frac{x_2^2 + |x_2|x_2}{4a} \right) \frac{1}{1 + K(x_1^2 + x_2^2)}, \end{aligned} \tag{3.81}$$

where $K \in \mathbb{R}_{>0}$ is a positive constant. Note that the function $p : \mathbb{R}^2 \to \mathbb{R}_{>0}$ is required to satisfy the condition *(B2)* maintaining the conditions for a CBF w.r.t. the viability kernel (3.80). Then, the Lie derivative of h for $x_2 \geq 0$ is calculated as follows:

$$\begin{aligned} L_f h(x) &= -\frac{a - K(ax_1^2 - ax_2^2 + x_1 x_2^2)}{a\left(1 + K(x_1^2 + x_2^2)\right)^2} x_2, \\ L_g h(x) &= -\frac{1 + K(x_1 - 2a)x_1}{a\left(1 + K(x_1^2 + x_2^2)\right)^2} x_2. \end{aligned} \tag{3.82}$$

Note that the ZCBF (3.81) is differentiable at $x_2 = 0$. Consequently, we can obtain a human assist controller for the system (3.76) by (3.71). Note that we design the function α as follows: for non-negative constants $\eta_1, \eta_2 \in \mathbb{R}_{\geq 0}$,

$$\alpha\left(h(x)\right) = \eta_1 h(x) + \eta_2 h^2(x). \tag{3.83}$$

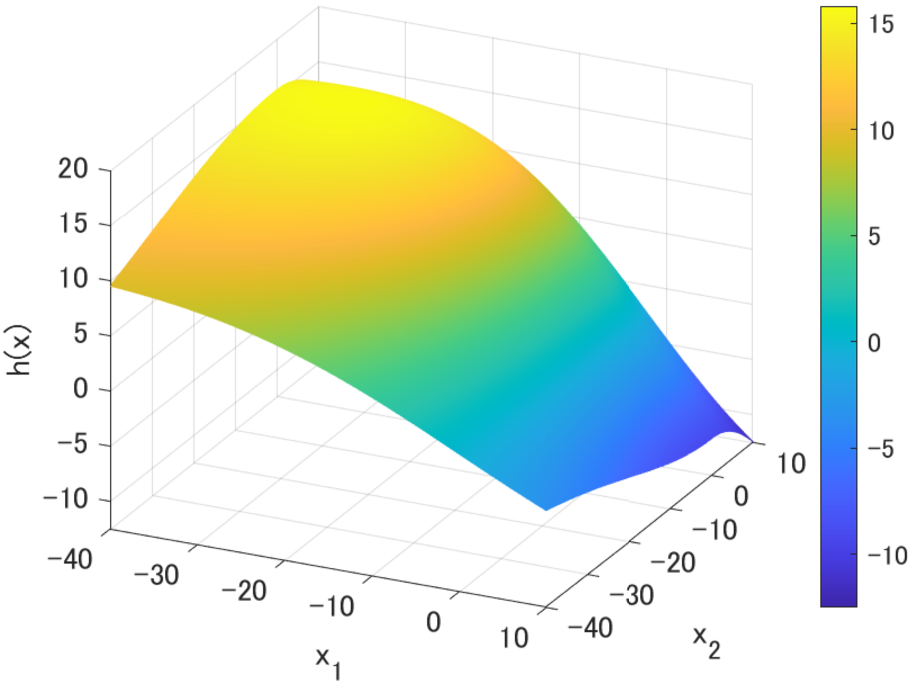

Figure 3.13: Control Example: ZCBF (3.81) for $K = 0.001$.

Computer Simulation: Time-varying Human Operator Input

Figures 3.14, 3.15 illustrate computer simulation results for $x_0 = (-10, 10)$, $a = 10$, $u_h(t) = 1.5 - 0.5\cos(\pi t/5)$, $K = 0.001$, $\eta_1 = \eta_2 = 1$. Figure 3.14 denotes the system state, and Fig. 3.15 the control input, respectively. Note that the simulation condition is the same as in Example 3.5.

As seen in Fig. 3.14, the system (3.76) smoothly decelerates and stays in the safe set $\mathcal{S}$. In addition, Fig. 3.15 shows that the proposed controller does not violate the input constraint $u+u_h \geq -a$; this is because the initial state $x_0 = (-10, 10)$ is contained in the viability kernel (3.80).

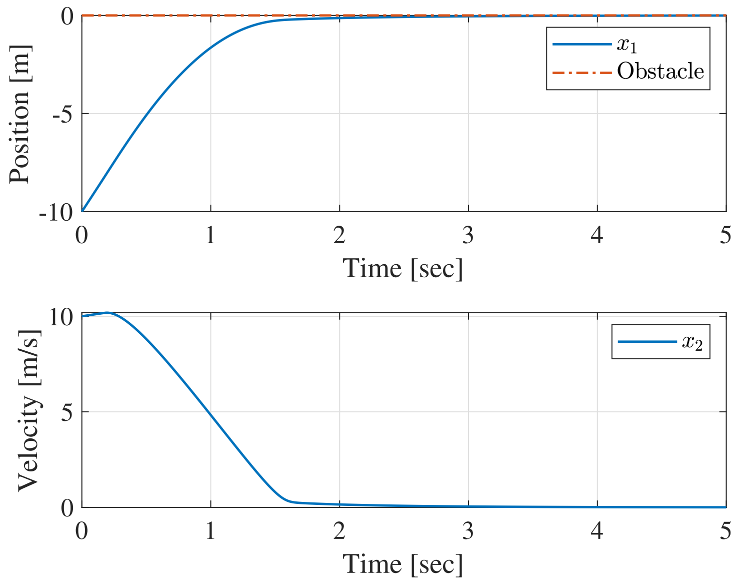

Figure 3.14: Computer Simulation: State (Time-varying Human Operator Input).

Computer Simulation: Parametric Properties

To confirm the parametric properties of the controller (3.71), the parameter $\eta_1 \in \mathbb{R}_{\geq 0}$ is chosen for some cases. Figures 3.16–3.18 illustrate the results of computer simulation for $x_0 = (-10, 10)$, $a = 10$, $u_h = 0$, $K = 0.001$, $\eta_2 = 1$ and $\eta_1 = 0.1, 10, 1000$. Figure 3.16 denotes the position x_1, Fig. 3.17 the velocity x_2, and Fig. 3.18 the control input, respectively.

Figure 3.16 shows that the controller (3.71) keeps the system state in $\mathcal{S}$ for each $\eta_1 \in \mathbb{R}_{\geq 0}$. Furthermore, Fig. 3.18 shows that the controller (3.71) satisfies the input constraint $u + u_h \geq -a$ for each $\eta_1 \in \mathbb{R}_{\geq 0}$ and the larger η_1 yields the steep property of the controller within the input constraint.

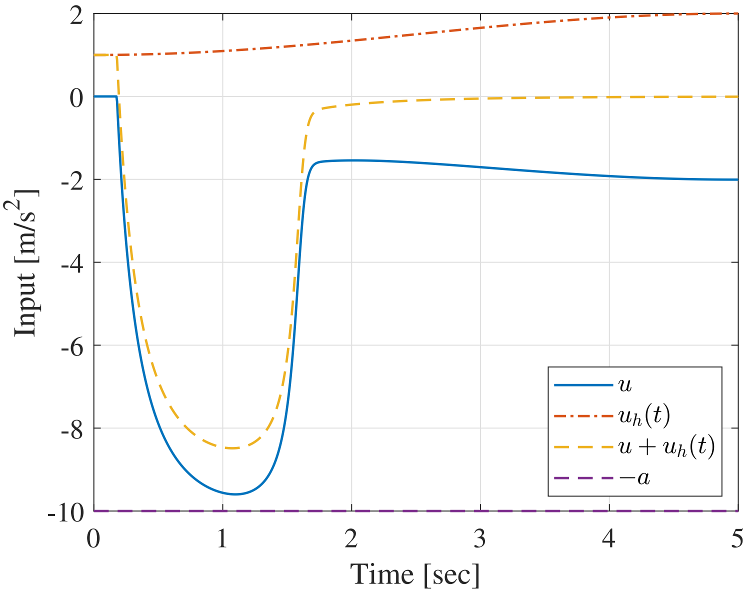

Figure 3.15: Computer Simulation: Input (Time-varying Human Operator Input).

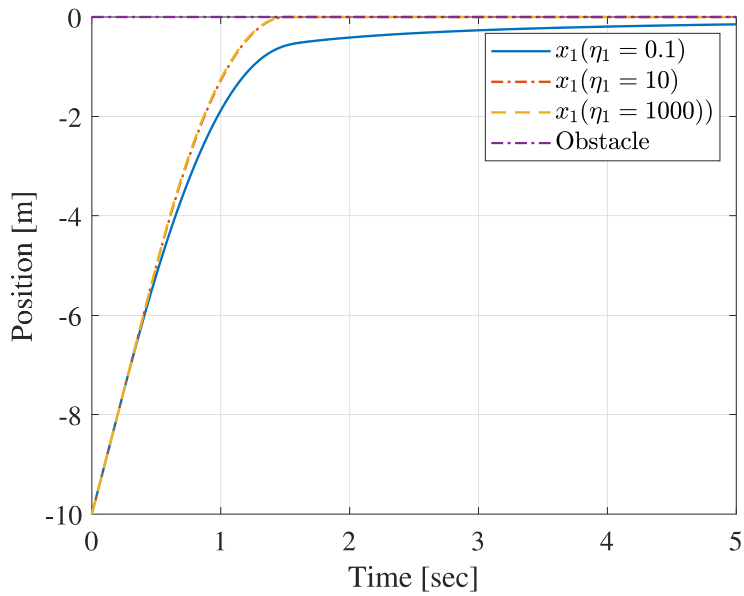

Figure 3.16: Computer Simulation: Position (Parametric Properties).

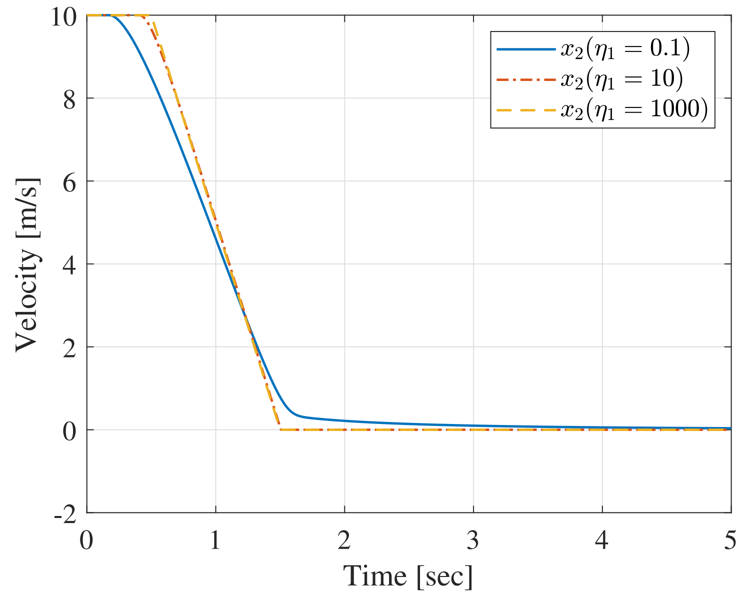

Figure 3.17: Computer Simulation: Velocity (Parametric Properties).

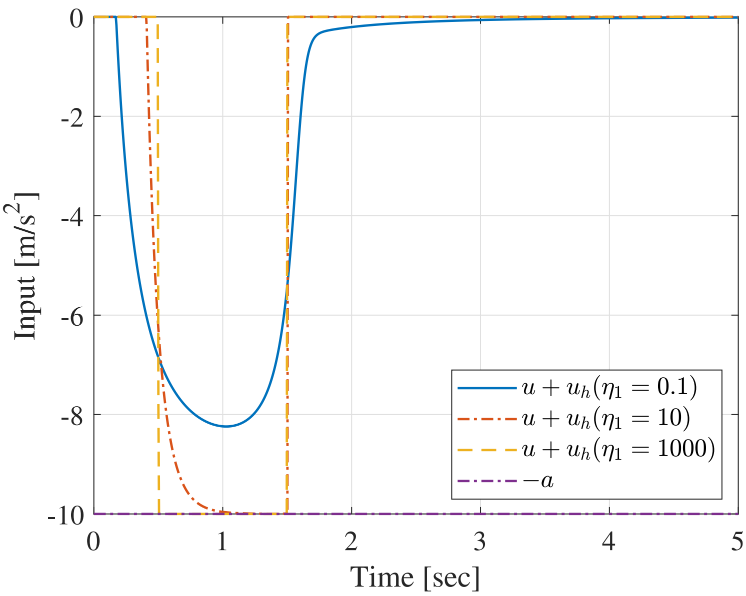

Figure 3.18: Computer Simulation: Input (Parametric Properties).

Computer Simulation: Boundary of Viability Kernel

As confirmed previously, the system (3.76) stays in $\mathcal{S}$ by the controller (3.71) while satisfying the input constraint $u + u_h \geq -a$ if $x_0 \in \mathcal{V}_\mathcal{S}$. In the case of $x_2(0) = 10$, according to the viability kernel (3.80) and the CBF (3.81), the boundary of $\mathcal{V}_\mathcal{S}$ is determined as $x = (-5, 10) \in \partial\mathcal{V}_\mathcal{S}$. To confirm the controller property on the boundary $\partial\mathcal{V}_\mathcal{S}$, we show the following results of computer simulation for $x_2(0) = 10$.

Figures 3.19–3.21 illustrate computer simulation results for $x_1(0) = -9, -7, -5$, $x_2(0) = 10$, $a = 10$, $u_h = 0$, $K = 0.000001$, $\eta_1 = \eta_2 = 1$. Figure 3.19 denotes the position x_1, Fig. 3.20 the velocity x_2, and Fig. 3.21 the control input, respectively.

Figure 3.19 shows that the controller (3.71) starts to work at an earlier time as $x_1(0)$ approaches the boundary $\partial\mathcal{V}_\mathcal{S}$. Since the safe set $\mathcal{S}$ and the corresponding viability kernel $\mathcal{V}_\mathcal{S}$ are open, in the case of $x_0 = (-5, 10) \in \partial\mathcal{V}_\mathcal{S}$, the system reaches the boundary $\partial\mathcal{S}$, which implies violation of state constraints; however, the input constraint is satisfied as seen in Fig. 3.21.

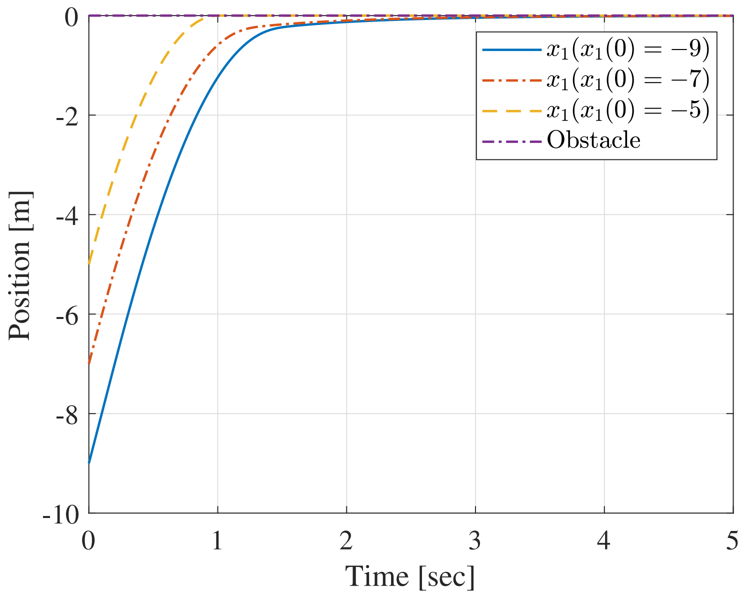

Figure 3.19: Computer Simulation: Position (Boundary of Viability Kernel).

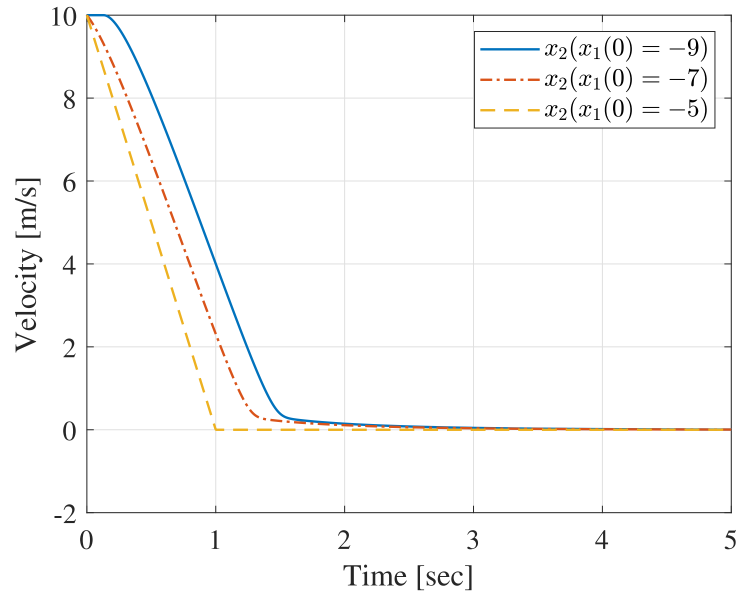

Figure 3.20: Computer Simulation: Velocity (Boundary of Viability Kernel).

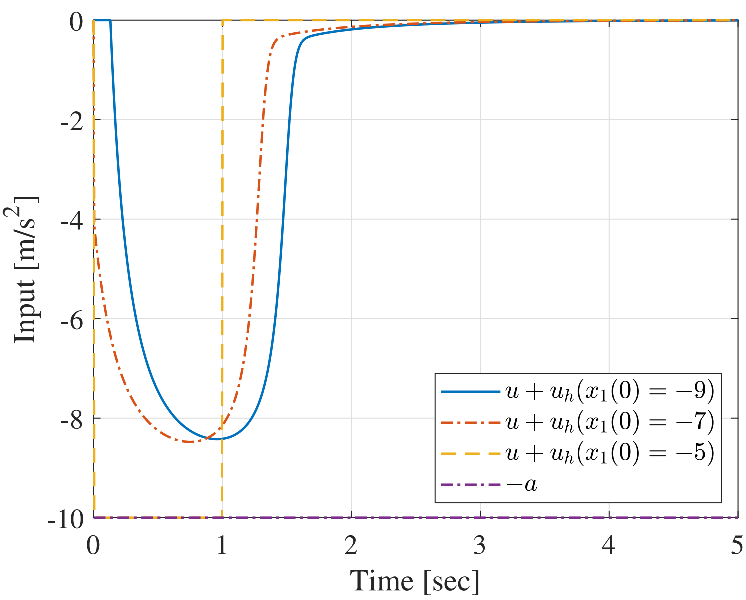

Figure 3.21: Computer Simulation: Input (Boundary of Viability Kernel).

Chapter 4

Safety Assist Control via Time-varying Zeroing Control Barrier Function

In this chapter, the author deals with a time-dependent state constraint problem such as collision avoidance of moving obstacles, and proposes a safety assist controller for a nonautonomous control system. The safety assurance mentioned here will be achieved by rendering a graph space forward invariance. The motivation for the employment of a graph space comes from the notion of forward invariance. Concretely, the notion is only applicable to time-invariant sets; therefore, it cannot be directly used for time-dependent sets. Considering this problem, the author will focus on a subset of product space $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ and apply the notion of forward invariance for time-dependent sets by regarding its subset, a graph space, as time-invariant.

This chapter starts with a generalized nonautonomous control-affine system, which was considered in the first part of Chapter 3. In Section 4.1, the author first introduces the notion of a graph space and then provides a definition of Time-varying ZCBF (Tv-ZCBF) for nonautonomous systems by extending an S-ZCBF. Second, the author proposes a safety assist controller that renders a graph space forward invariant, which indicates the satisfaction of time-dependent safety constraints. The author also shows that the proposed controller is continuous and optimal as in the case of a time-invariant state constraint problem. The effectiveness of the proposed controller will be confirmed by considering a mathematical example.

The method proposed in the first part of this chapter assumes that a human assist controller contains complete information on the motion of environments. However, generally speaking, the velocity vector of moving obstacles is unknown to control systems and needs to be estimated. In real applications, the existence of noise or disturbance makes it difficult to obtain the true value of the obstacle velocity. This problem specific to time-dependent safety constraints motivates the goal of the second part of this chapter. In Section 4.2, the author proposes a safety assist controller for control systems under input disturbances with the new type of ZCBFs, Input-to-State Constrained Safety ZCBF (ISCSf-ZCBF). The proposed safety assist controller can ensure the safety of input-disturbed systems for arbitrary unknown input disturbances while satisfying continuity and optimality.

In the last part of this chapter, Section 4.3, we return to the time-dependent state constraint problem under the assumption that the obstacle velocity is unknown. The author proposes an ISCSf-ZCBF-based safety assist controller that renders a graph space forward invariant by considering the estimation error of the obstacle velocity as input disturbances. That is, the proposed controller meets time-dependent safety constraints without knowledge of the motion of environments. The effectiveness of the proposed controller will be confirmed by conducting experiments on an electric wheelchair.

4.1 Time-dependent State Constraint Problem

In this chapter, we firstly consider the following time-varying nonlinear control-affine system with a human operator input $u_h(t)$:

$$\dot{x} = f(x,t) + g(x,t)[u + u_h(t)], \tag{4.1}$$

where $x \in \mathbb{R}^n$ denotes a state, $u \in \mathbb{R}^m$ a control input. Assume that a mapping $u_h : \mathbb{R} \to \mathbb{R}^m$ is continuous in t, and mappings $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n\times m}$ are continuous in (x,t). Note that the uniqueness of a solution $x : [0,t_1] \to \mathbb{R}^n$ to (4.1) with the initial state $x_0 := x(0)$ is not guaranteed as the Lipschitz continuity in x to the mappings $f(x,t)$ and $g(x,t)$ is not required.

In this chapter, the author aims to design a human assist controller that satisfies $x \in \mathcal{S}(t)$ for $\forall t \geq 0$. To this end, first divide a state space $\mathbb{R}^n$ into time-dependent subsets: a safe set $\mathcal{S}(t) \subset \mathbb{R}^n$ and an unsafe set $\mathcal{S}^c(t) = \mathbb{R}^n \backslash \mathcal{S}(t) \subset \mathbb{R}^n$. We assume that the safe set $\mathcal{S}(t)$ is open and defined as the strict 0-superlevel set of a C^1 continuously differentiable function $\mathfrak{h} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$:

$$\begin{aligned} \mathcal{S}(t) &= \{x \in \mathbb{R}^n \mid \mathfrak{h}(x,t) > 0\}, \\ \partial\mathcal{S}(t) &= \{x \in \mathbb{R}^n \mid \mathfrak{h}(x,t) = 0\}. \end{aligned} \tag{4.2}$$

The method proposed in Section 3.4 cannot ensure the safety of nonautonomous systems when a safe set $\mathcal{S} \subset \mathbb{R}^n$ is time-dependent, i.e., when $x \in \mathcal{S}(t)$ denotes the state constraint. Moreover, the notion of forward invariance cannot be directly used for time-dependent sets. Therefore, consider the following time-invariant graph space $\mathcal{G} \subset \mathbb{R}^{n+1}$ illustrated in Fig. 4.1:

$$\mathcal{G} = \{(x,t) \in \mathbb{R}^{n+1} \mid x(t) \in \mathcal{S}(t)\}, \tag{4.3}$$

which is a fixed subset of the product space $\mathbb{R}^{n+1}$ [11, 34]. It is worth stressing that $(x,t) \in \mathcal{G}$ if $x(t) \in \mathcal{S}(t)$. This implies that the forward invariance of $\mathcal{G}$ corresponds to $x(t) \in \mathcal{S}(t)$ for $\forall t \geq 0$. Assume that for any $(x(t_0), t_0) \in \mathcal{G}$, there exists an open neighborhood $\mathcal{V} \subset \mathcal{G}$ of $(x(t_0), t_0)$ to prevent $\exists t_e \in \mathbb{R}_{>0}$ such that $\mathcal{S}(t_e) = \emptyset$.

Problem 4.1 *The first objective of this chapter is to design a human assist controller $u = k(x,t)$ for the nonautonomous system* (4.1) *satisfying the conditions (P2), (P3) given in Definition 3.2 and the following condition:*

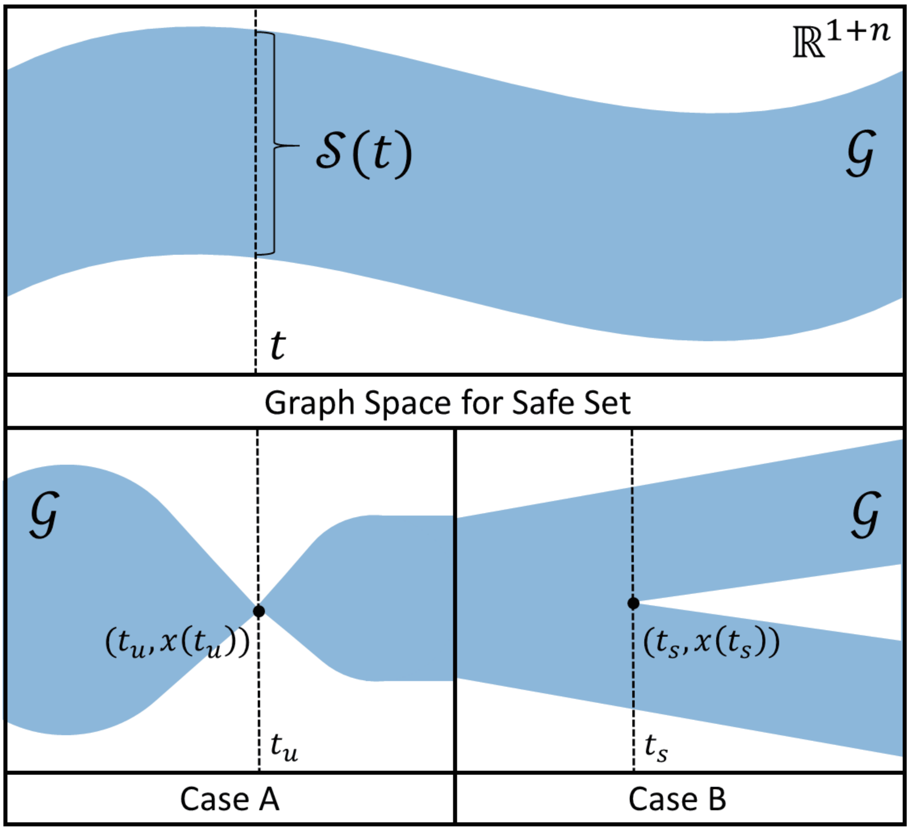

Figure 4.1: Graph Space.

(P1") *For any human operator input* $u_h : \mathbb{R} \to \mathbb{R}^m$*, the controller* $u = k(x, t)$ *renders the graph space* $\mathcal{G}$ *forward invariant,*

$$\begin{aligned} &(x(0), 0) \in \mathcal{G} \\ &\qquad \Rightarrow \{(x(t), t) \in \mathbb{R}^{n+1} \mid \forall t \geq 0\} \subset \mathcal{G}, \end{aligned} \tag{4.4}$$

indicating that $x(t) \in \mathcal{S}(t)$ *for* $\forall t \geq 0$.

4.1.1 Time-varying Safe Set

When considering a time-dependent state constraint problem, we need to derive the conditions ensuring $\mathcal{S}(t) \neq \emptyset$ for $\forall t \geq 0$. To this end, the literature [47] introduced a candidate Tv-ZCBF, which is a conservative assumption as a solution $x(t) \in \mathcal{S}(t)$ needs to exist.

In this chapter, as mentioned in the preceding section, we set the assumption regarding a graph space $\mathcal{G}$, i.e., for any $(x(t_0), t_0) \in \mathcal{G}$, there exists an open neighborhood $\mathcal{V} \subset \mathcal{G}$ of $(x(t_0), t_0)$. This assumption evidently ensures $\mathcal{S}(t) \neq \emptyset$ for $\forall t \geq 0$. Importantly, this assumption also avoids $\exists t_u \in \mathbb{R}_{>0}$ such that $\mathcal{S}(t_u) = \{x(t_u)\}$ where $\{x(t_u)\} \subset \mathbb{R}^n$ is a singleton, i.e., we cannot choose the open neighborhood of $(x(t_u), t_u) \in \mathcal{G}$ because a singleton is a closed set. In other words, Case A illustrated in Fig. 4.1 will never happen under the assumption because states of the system are always on a safe set boundary when a safe set is a singleton, which implies that the safety constraint is not satisfied. Accordingly, the assumption is useful to ensure that $\mathcal{S}(t)$ is open for $\forall t \geq 0$.

Remark 4.1 *The assumptions include the case that there exists $t_s \in \mathbb{R}_{>0}$ such that the separation of a safe set occurs. For example, Case B illustrated in Fig. 4.1 satisfies that for any $(x(t_s), t_s) \in \mathcal{G}$, $\exists \mathcal{V} \in \mathcal{G}$ of $(x(t_s), t_s)$, and that the safe set $\mathcal{S}(t)$ is open because $\{x(t_s)\} \not\subset \mathcal{S}(t_s)$ is a singleton. However, in this case, the time-varying function $\mathfrak{h} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ for the safe set $\mathcal{S}(t)$ defined by* (4.2) *might fail to be differentiable at t_s. Therefore, we remove the separation cases in this dissertation.*

Remark 4.2 *The time-varying safe set $\mathcal{S}(t)$ can be regarded as a set-valued map $\mathcal{S} : t \rightsquigarrow \mathbb{R}^n$ and the continuity of $\mathcal{S}$ seems to prevent $\mathcal{S}(t) \neq \emptyset$ for $\forall t \geq 0$. However, the continuity of a set-valued map, in particular the lower semicontinuity, is generally defined on the following domain [11]:*

$$Dom(\mathcal{S}) := \{t \in \mathbb{R} \mid \mathcal{S}(t) \neq \emptyset\}. \tag{4.5}$$

Accordingly, the following set-valued map $\mathcal{S} : t \rightsquigarrow \mathbb{R}$ is continuous in t:

$$\mathcal{S}(t) = \begin{cases} (0,1) & \text{if } t \in [0, t_e] \\ \emptyset & \text{if } t \in (t_e, \infty). \end{cases} \tag{4.6}$$

Therefore, the continuity of a set-valued map is insufficient for ensuring $\mathcal{S}(t) \neq \emptyset$ for $\forall t \geq 0$.

4.1.2 Time-varying Zeroing Control Barrier Function

In this subsection, the author defines a Tv-ZCBF for the nonautonomous system (4.1).

Definition 4.1 *Consider the system* (4.1) *and the graph space $\mathcal{G}$ for the safe set $\mathcal{S}(t)$ defined by* (4.2). *Then, a C^1 continuously differentiable function $\mathfrak{h} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a time-varying zeroing control barrier function (Tv-ZCBF) if the following conditions hold.*

(C1) For any fixed $t_0 \in \mathbb{R}$, $\mathfrak{h}(x, t_0)$ is a proper function w.r.t. $\forall x \in \mathcal{S}(t_0)$; the superlevel set $\{x | \mathfrak{h}(x, t_0) \geq L\}$ is compact for any positive constant $L \in \mathbb{R}_{>0}$.

(C2) There exists a locally Lipschitz continuous function $\alpha \in \mathcal{K}_e$ such that the following inequality holds:

$$\sup_{u \in \mathbb{R}^m} \dot{\mathfrak{h}}(x, t, u, u_h(t)) > -\alpha\left(\mathfrak{h}(x, t)\right), \tag{4.7}$$

where

$$\dot{\mathfrak{h}} = L_f \mathfrak{h}(x, t) + L_g \mathfrak{h}(x, t) \cdot (u + u_h(t)) + \frac{\partial \mathfrak{h}}{\partial t}(x, t).$$

Using the condition *(C1)* for a Tv-ZCBF, we can ensure the compactness of a graph space by the following lemma.

Lemma 4.1 *Consider the system* (4.1) *and the graph space* $\mathcal{G}$ *for the safe set* $\mathcal{S}(t)$ *defined by* (4.2). *Let* $t_1 \in \mathbb{R}_{>0}$, $L \in \mathbb{R}_{>0}$ *be positive constants. If a* C^1 *continuously differentiable function* $\mathfrak{h} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ *is a Tv-ZCBF, the following graph* $\beta_1(L, t_1)$ *is compact for any* t_1 *and satisfies* $\beta_1(L, t_1) \subset \mathcal{G}$.

$$\beta_1(L, t_1) = \{(x, t) \in \mathbb{R}^{n+1} \mid \mathfrak{h}(x, t) \geq L, t \in [0, t_1]\}. \tag{4.8}$$

Proof Consider the following set-valued map $\beta : t \rightsquigarrow \mathbb{R}^n$:

$$\beta(t) = \{x \in \mathbb{R}^n \mid \mathfrak{h}(x, t) \geq L\}. \tag{4.9}$$

Note that $\beta(t)$ is upper semicontinuous since $\mathfrak{h}(x, t)$ is a C^1 continuously differentiable function. Here, the graph $\beta_1(L, t_1)$ defined by (4.8) can be rewritten as follows:

$$\beta_1 = \{(x, t) \in \mathbb{R}^{n+1} \mid x \in \beta(t), t \in [0, t_1]\}. \tag{4.10}$$

Accordingly, for any fixed $t_1 \in \mathbb{R}_{\geq 0}$, the graph $\beta_1(L, t_1)$ is closed as $\beta(t)$ with the closed domain $[0, t_1]$ is upper semicontinuous by Proposition 2.1.

Due to the compactness of a Tv-ZCBF given by the condition *(C1)*, there exists $M(t_0) \in \mathbb{R}_{>0}$ such that

$$M(t_0) \geq \|x(t_0)\|, \quad \forall x \in \beta(t_0), \tag{4.11}$$

for any fixed $t_0 \in \mathbb{R}_{>0}$. Hence there exists a positive constant $M \in \mathbb{R}_{>0}$ such that the following inequality holds:

$$M \geq M(t_0) \geq \|x(t_0)\|, \quad \forall x \in \beta(t_0), \tag{4.12}$$

for any $t \in [0, t_1]$. Therefore, the graph $\beta_1(L, t_1)$ is bounded because the following inequality holds for any $t \in [0, t_1]$:

$$\|(x, t)\| \leq \sqrt{M^2 + t_1^2}. \tag{4.13}$$

Thus, the graph $\beta_1(L, t_1)$ is compact. Moreover, since $L \in \mathbb{R}_{>0}$ is a positive constant, $x \in \beta(t) \subset \mathcal{S}(t)$ and hence $(x, t) \in \beta_1(L, t_1) \subset \mathcal{G}$ for any $t \in [0, t_1]$; the above discussion completes the proof. □

Then, the following result will be useful to ensure the forward completeness of a solution $x(t)$ to (4.1).

Theorem 4.1 *Consider the system* (4.1) *and the graph space* $\mathcal{G}$ *for the safe set* $\mathcal{S}(t)$ *defined by* (4.2). *Assume that* $(x, t) \in \mathcal{H} := \{(x, t) | x \in \mathcal{S}(t), t \in [0, t_1]\}$ *for any positive constant* $t_1 \in \mathbb{R}_{>0}$. *If a* C^1 *continuously differentiable function* $\mathfrak{h} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ *is a Tv-ZCBF, there exists a positive constant* $\eta \in \mathbb{R}_{>0}$ *such that the following inequality holds:*

$$\eta \mathfrak{h}(x, t) \geq \alpha (\mathfrak{h}(x, t)), \quad \forall (x, t) \in \mathcal{H}. \tag{4.14}$$

Proof Note that $(x,t) \in \mathcal{H} \subset \mathcal{G}$ implies that $\mathfrak{h}(x,t) > 0$ for $\forall t \in [0, t_1]$. Since a function $\alpha \in \mathcal{K}_e$ is locally Lipschitz continuous, it can be rewritten as follows:

$$\alpha(\mathfrak{h}(x,t)) = \eta_1 \mathfrak{h}(x,t) + \delta(\mathfrak{h}(x,t)), \tag{4.15}$$

where $\eta_1 \in \mathbb{R}_{>0}$ is a positive constant and $\delta : \mathbb{R} \to \mathbb{R}$ is a higher-order term of $\mathfrak{h}(x,t)$. Accordingly, there exists a positive constant $L \in \mathbb{R}_{>0}$ such that

$$\frac{1}{2}\eta_1 \mathfrak{h}(x,t) \geq \delta(\mathfrak{h}(x,t)), \tag{4.16}$$

for $\forall(x,t) \in \beta_0 := \{(x,t) | 0 < \mathfrak{h}(x,t) < L, t \in [0,t_1]\} \subset \mathcal{H}$. Therefore, there exists a positive constant $\eta \in \mathbb{R}_{>0}$ such that the following inequality holds:

$$\eta \mathfrak{h}(x,t) \geq \alpha(\mathfrak{h}(x,t)), \quad \forall(x,t) \in \beta_0. \tag{4.17}$$

Next, consider the case that $(x,t) \in \beta_1 = \{(x,t) | \mathfrak{h}(x,t) \geq L, t \in [0,t_1]\} \subset \mathcal{H}$. According to Lemma 4.1, the graph β_1 is compact. Therefore, there exists a positive constant $\eta \in \mathbb{R}_{>0}$ such that the following inequality holds:

$$\eta \geq \frac{\alpha(\mathfrak{h}(x,t))}{\mathfrak{h}(x,t)}, \quad \forall(x,t) \in \beta_1, \tag{4.18}$$

due to the extreme value theorem [53]. Thus, the inequality (4.14) holds by (4.17) and (4.18) as $\beta_0 \cup \beta_1 = \mathcal{H}$. □

4.1.3 Safety Assist Control via Time-varying Zeroing Control Barrier Function

The author proposes a human assist controller that ensures the safety of the nonautonomous system (4.1) using a Tv-ZCBF, which is the main result of this section.

Theorem 4.2 *Consider the system* (4.1)*, the graph space $\mathcal{G}$ for the safe set $\mathcal{S}(t)$ defined by* (4.2)*, a Tv-ZCBF $\mathfrak{h} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and the following control input:*

$$u = k(x,t) = \begin{cases} -\dfrac{I(x,t,u_h(t)) + \alpha(\mathfrak{h}(x,t))}{\|L_g \mathfrak{h}(x,t)\|^2} (L_g \mathfrak{h}(x,t))^T \\ \qquad \text{if } I(x,t,u_h(t)) < -\alpha(\mathfrak{h}(x,t)) \\ 0 \qquad \text{if } I(x,t,u_h(t)) \geq -\alpha(\mathfrak{h}(x,t)), \end{cases} \tag{4.19}$$

where a function $\alpha \in \mathcal{K}_e$ is locally Lipschitz continuous and a function $I : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ is defined by

$$I(x,t,u_h(t)) = L_f \mathfrak{h}(x,t) + L_g \mathfrak{h}(x,t) \cdot u_h(t) + \frac{\partial \mathfrak{h}}{\partial t}. \tag{4.20}$$

Then, the control input defined by (4.19) *is a human assist controller that ensures the forward invariance of $\mathcal{G}$, i.e., $x(t) \in \mathcal{S}(t)$ for $\forall t \geq 0$.*

Proof Consider the following closed-loop system:

$$\begin{aligned}\dot{x} &= f_{cl}(x, t, k(x, t), u_h(t)) \\ &= f(x, t) + g(x, t)\left[k(x, t) + u_h(t)\right],\end{aligned} \tag{4.21}$$

which is the system (4.1) with the proposed human assist controller (4.19).

Firstly, we will prove the continuity of the proposed controller (4.19) to ensure the existence of a local solution $x(t)$ to (4.21). Note that $L_f\mathfrak{h}(x, t)$, $L_g\mathfrak{h}(x, t)$ are continuous in (x, t), and $u_h(t)$ is continuous in t; hence the function $I(x, t, u_h(t))$ is also continuous in (x, t). Moreover, uniformly $k(x, t) \to 0$ as $I(x, t, u_h(t)) \to -\alpha\left(\mathfrak{h}(x, t)\right)$ when $L_g\mathfrak{h}(x, t) \neq 0$. Theorefore, (4.19) is continous in (x, t) when $L_g\mathfrak{h}(x, t) \neq 0$. Thus, we need to prove the continuity of (4.19) at (x, t) where $L_g\mathfrak{h}(x, t) = 0$. Since $\mathfrak{h}(x, t)$ is a Tv-ZCBF, the following inequality hold:

$$\dot{\mathfrak{h}} = L_f\mathfrak{h}(t_0) + \frac{\partial\mathfrak{h}}{\partial t}(t_0) > -\alpha\left(\mathfrak{h}(t_0)\right), \tag{4.22}$$

for $\forall t_0 \geq 0$ such that $L_g\mathfrak{h}(t_0) = L_g\mathfrak{h}\left(x(t_0), t_0\right) = 0$. This implies that there exists $\varepsilon \in \mathbb{R}_{>0}$ such that

$$\dot{\mathfrak{h}} = L_f\mathfrak{h}(t_0) + \frac{\partial\mathfrak{h}}{\partial t}(t_0) - \varepsilon \geq -\alpha\left(\mathfrak{h}(t_0)\right). \tag{4.23}$$

Since a mapping $L_g\mathfrak{h}(x, t) \cdot u_h(t)$ is continuous, there exists a neighborhood $\mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}$ of $(x(t_0), t_0)$ such that

$$\|L_g\mathfrak{h}(x, t) \cdot u_h(t)\| \leq \varepsilon, \quad \forall (x, t) \in \mathcal{U}. \tag{4.24}$$

Therefore, the following inequality holds for any $(x, t) \in \mathcal{U}$:

$$L_f\mathfrak{h}(x, t) + L_g\mathfrak{h}(x, t) \cdot u_h(t) + \frac{\partial\mathfrak{h}}{\partial t}(x, t) \geq -\alpha\left(\mathfrak{h}(x, t)\right). \tag{4.25}$$

This implies that $I(x, t, u_h(t)) \geq -\alpha\left(\mathfrak{h}(x, t)\right)$, and accordingly $k(x, t) = 0$ for any $(x, t) \in \mathcal{U}$. Therefore, in the neighborhood of $L_g\mathfrak{h}(x, t) = 0$, $k(x, t) = 0$ is always satisfied; (4.19) is continuous for $\forall (x, t) \in \mathbb{R}^n \times \mathbb{R}$. Thus, there exists a local solution $x(t)$ to (4.21) as the mapping f_{cl} is continuous in $(x, t) \in \mathbb{R}^n \times \mathbb{R}$.

Secondly, we will prove that the proposed controller (4.19) ensures the forward invariance of the graph space $\mathcal{G}$. Note that $x_0 \in \mathcal{S}(0)$ implies $\mathfrak{h}(x_0, 0) > 0$, and $S(t) \neq \emptyset$ for $\forall t \geq 0$ due to the assumptions on $\mathcal{G}$. By applying (4.19) to the time-derivative of a Tv-ZCBF $\mathfrak{h}$, i.e.,

$$\dot{\mathfrak{h}} = L_f\mathfrak{h} + L_g\mathfrak{h} \cdot k(x, t) + L_g\mathfrak{h} \cdot u_h(t) + \frac{\partial\mathfrak{h}}{\partial t}, \tag{4.26}$$

the following inequality holds:

$$\dot{\mathfrak{h}} \geq -\alpha\left(\mathfrak{h}(x, t)\right), \tag{4.27}$$

in both $I(x, t, u_h(t)) < -\alpha\,(\mathfrak{h}(x, t))$ and $I(x, t, u_h(t)) \geq -\alpha\,(\mathfrak{h}(x, t))$. According to *Theorem 4.1*, there exists a positive constant $\eta \in \mathbb{R}_{>0}$ such that the following inequality holds:

$$\dot{\mathfrak{h}} \geq -\alpha\,(\mathfrak{h}(x, t)) \geq -\eta\mathfrak{h}(x, t), \tag{4.28}$$

for $\forall(x, t) \in \mathcal{H} = \{(x, t) | x \in \mathcal{S}(t), t \in [0, t_1]\} \subset \mathcal{G}$. Since the right-hand side of (4.28) is globally Lipschitz continuous in $\mathfrak{h}$, there exists the unique lower bounded function $\underline{\mathfrak{h}}(t)$ over $[0, t_1]$ where $t_1 \in \mathbb{R}_{>0}$ can be arbitrarily extended by Theorem 2.4 Moreover, the following inequality holds by Gronwall's lemma 2.2:

$$\begin{aligned} \mathfrak{h}(x(t), t) \geq \underline{\mathfrak{h}}(t) &= \mathfrak{h}(x_0, 0) \exp(-\gamma t) \\ &\geq \mathfrak{h}(x_0, 0) \exp(-\gamma t_1) := L > 0, \end{aligned} \tag{4.29}$$

for $\forall t \in [0, t_1]$ due to the existence of a local solution $x(t)$ to (4.21). Therefore, $(x(t), t) \in \beta_1(L, t_1) \subset \mathcal{G}$ for $\forall t \in [0, t_1]$, according to Lemma 4.1. Thus, $x(t) \in \mathcal{S}(t)$ is ensured over $[0, t_1]$. Since t_1 can be arbitrarily chosen, the proposed controller (4.19) ensures the forward invariance of $\mathcal{G}$, i.e., $x(t) \in \mathcal{S}(t)$ for $\forall t \geq 0$. □

Remark 4.3 *The proposed human assist controller* (4.19) *is continuous in* (x, t) *regardless of the relative degree condition. Note that its continuity is guaranteed by requiring the inequality* (4.7) *in the condition (C2) to be strict. Moreover, a solution to* (4.21) *having a finite escape time might exist without the properness of a Tv-ZCBF.*

The proposed human assist controller (4.19) is minimally invasive, which is the desired property of a human assist control. The proof of the following lemma is done exactly the same way as in Lemma 3.2.

Lemma 4.2 *Consider the system* (4.1)*, the graph space* $\mathcal{G}$ *for the safe set* $\mathcal{S}(t)$ *defined by* (4.2) *and a Tv-ZCBF* $\mathfrak{h} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$*. Then, the human assist controller* (4.19) *minimizes the Euclidian norm* $||u||$ *for* $\forall(x, t) \in \mathbb{R}^n \times \mathbb{R}$ *while satisfying the following inequality:*

$$I(x, t, u_h(t)) + L_g\mathfrak{h}(x, t) \cdot u \geq -\alpha(\mathfrak{h}(x, t)). \tag{4.30}$$

4.1.4 Mathematical Example

In this section, we consider a system whose $g(x, t)$ is not Lipschitz continuous in x because the requirement on mappings f and g of the system (4.1) is simply to be continuous in (x, t) as in Section 3.3. Then, we confirm the effectiveness of the proposed human assist controller by computer simulation.

Consider the following control system:

$$\begin{aligned} \dot{x}_1 &= f_1(x, t), \\ \dot{x}_2 &= g_2(x, t)\,(u + u_h(t))\,, \end{aligned} \tag{4.31}$$

where $x = [x_1, x_2]^T \in \mathbb{R}^2$ denotes a state, $u \in \mathbb{R}$ a control input, $u_h : \mathbb{R} \to \mathbb{R}$ a human operator input, and mappings $f_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are of the form:

$$f_1(x,t) = x_2 - 0.1t, \tag{4.32}$$

$$g_2(x,t) = 3 + \sum_{n=0}^{\infty} a^n \cos(b^n \pi x_1) + 0.5 \sin t, \tag{4.33}$$

where $a \in (0,1)$, $b \in \mathbb{R}_{>0}$ is a positive odd integer, and $ab > 1 + 3\pi/2$. Note that the second term of (4.33) is the Weierstrass function that is not Lipschitz continuous in x_1, and the first term avoids $g_2(x,t) = 0$.

Assume that the state x_1 has a time-dependent constraint $x_1 \in \mathcal{S}_1(t) = (-\infty, x_o(t))$, and a safe set is defined by $\mathcal{S}(t) = \mathcal{S}_1(t) \times \mathbb{R}$, where $x_o : \mathbb{R} \to \mathbb{R}$ is a smooth time-varying function. Note that an unsafe set is denoted by $\mathcal{S}^c(t) = \mathbb{R} \backslash \mathcal{S}_1(t) \times \mathbb{R}$. Then, the graph space for the safe set $\mathcal{S}(t)$ is denoted by $\mathcal{G} = \{(x,t) \in \mathbb{R}^2 | x_1 < x_o(t)\}$.

Controller Design

Define the following Tv-ZCBF candidate:

$$\begin{aligned} \mathfrak{h}(x,t) &= -(x_1 - x_o(t))p(x,t) \\ &= -\frac{x_1 - x_o(t)}{1 + L_1(x_1 - x_o(t))^2 + L_2(x_2 - \dot{x}_o(t))^2}, \end{aligned} \tag{4.34}$$

where $L_1, L_2 \in \mathbb{R}_{>0}$ are positive constants. We can easily confirm that the function (4.34) satisfies the conditions for a Tv-ZCBF. Firstly, for any fixed $t_0 \in \mathbb{R}$, $\mathfrak{h}(x,t_0) \to 0$ as $x \to -\infty$ or $x \to \partial\mathcal{S}(t_0)$, which implies the function (4.34) satisfies the condition *(C1)*. Secondly, $L_f\mathfrak{h}$, $L_g\mathfrak{h}$ and $\partial\mathfrak{h}/\partial t$ can be calculated as follows:

$$\begin{aligned} L_f\mathfrak{h} &= \left[L_1(x_1 - x_o)^2 - L_2(x_2 - \dot{x}_o)^2 - 1\right] p^2(x,t) f_1(x,t), \\ L_g\mathfrak{h} &= 2L_2(x_1 - x_o)(x_2 - \dot{x}_o)p^2(x,t)g_2(x,t), \\ \frac{\partial\mathfrak{h}}{\partial t} &= -\left[L_1(x_1 - x_o)^2 - L_2(x_2 - \dot{x}_o)^2 - 1\right] p^2(x,t)\dot{x}_o \\ &\quad - 2L_2(x_1 - x_o)(x_2 - \dot{x}_o)p^2(x,t)\ddot{x}_o. \end{aligned} \tag{4.35}$$

Considering $x_1 < x_o(t)$ as $x \in \mathcal{S}(t)$, we can confirm that there exists $x \in \mathcal{S}(t)$ such that $L_g\mathfrak{h} = 0$, i,e., $x_2 = \dot{x}_o(t)$. Then, in the neighborhood of $\partial\mathcal{S}(t)$ with $L_1 \ll 1$, the following inequality holds:

$$\begin{aligned} L_f\mathfrak{h} + \frac{\partial\mathfrak{h}}{\partial t} &= -0.1t\left[L_1(x_1 - x_o)^2 - 1\right] p^2(x,t) > 0 \\ &> -\alpha(\mathfrak{h}(x,t)), \end{aligned} \tag{4.36}$$

for any locally Lipschitz continuous function $\alpha \in \mathcal{K}_e$, which implies that the function (4.34) satisfies the condition *(C2)*. Therefore, the function $\mathfrak{h}(x,t)$ defined by (4.34) is a Tv-ZCBF. Accordingly, we can design a human assist controller $u = k(x,t)$ for the system (4.31) by using (4.19).

Computer simulation

Figures 4.2–4.5 show computer simulation results for $x_0 = (0, -3)$, $u_h(t) = 1 - 0.5\cos(\pi t/20)$, $(x_o(0), \dot{x}_o(0)) = (5, 0)$, $\ddot{x}_o = -0.01t$, $L_1 = 0.0001$, $L_2 = 0.05$, the summation n from 0 to 100 in (4.33) with $a = 0.5$, $b = 15$. Note that we chose $\alpha(\mathfrak{h}(x,t)) = \eta\mathfrak{h}(x,t)$ where $\eta = 0.3$. Figures 4.2–4.5 denote the time series of $g_2(x,t)$, the state x_1, the state x_2, and the control input, respectively.

From Figs. 4.3–4.5, the state x_1 satisfies $x_1 \in \mathcal{S}_1(t)$, and hence $x(t) \in \mathcal{S}(t)$ is satisfied by the proposed controller $u = k(x,t)$. From Figs. 4.4–4.5, we can also confirm that the proposed controller $u = k(x,t)$ is continuous at $x_2 = \dot{x}_o$ where $L_g b = 0$. As we mentioned in Remark 4.3, this continuity is ensured because the constructed Tv-ZCBF (4.34) satisfies the condition *(C2)*.

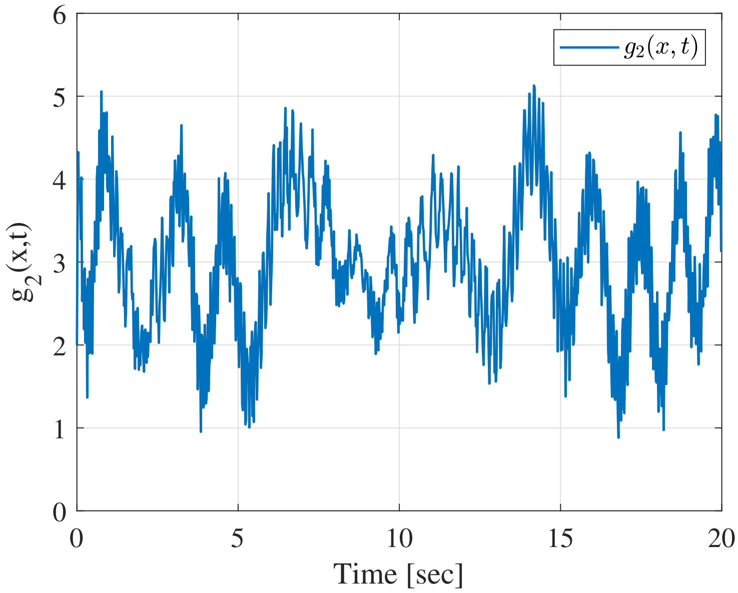

Figure 4.2: Computer Simulation: Function $g_2(x,t)$ (Mathematical Example).

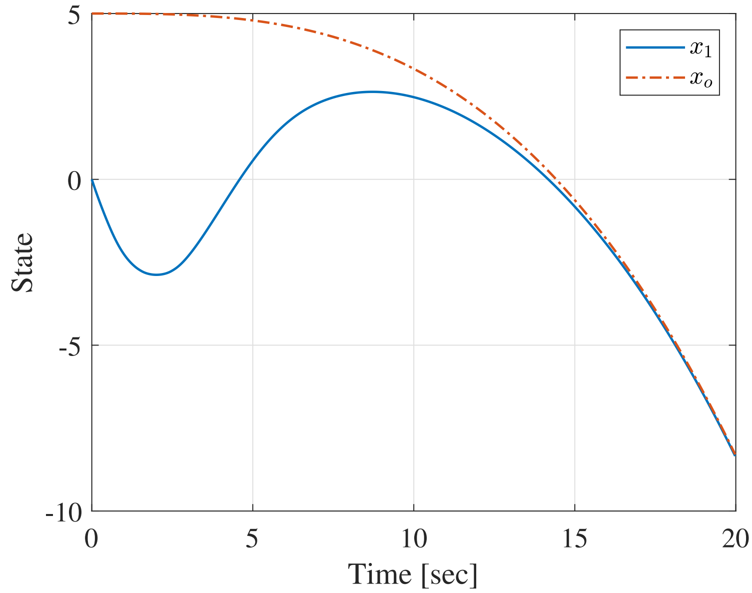

Figure 4.3: Computer Simulation: State x_1 (Mathematical Example).

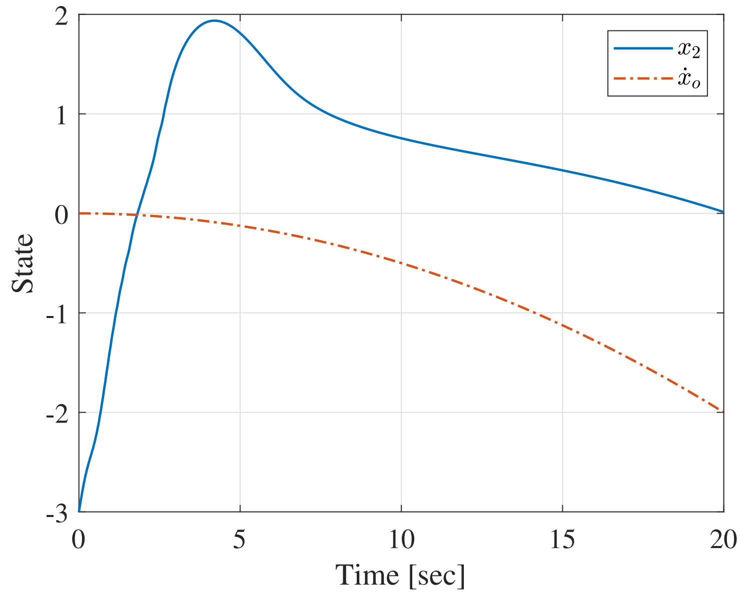

Figure 4.4: Computer Simulation: State x_2 (Mathematical Example).

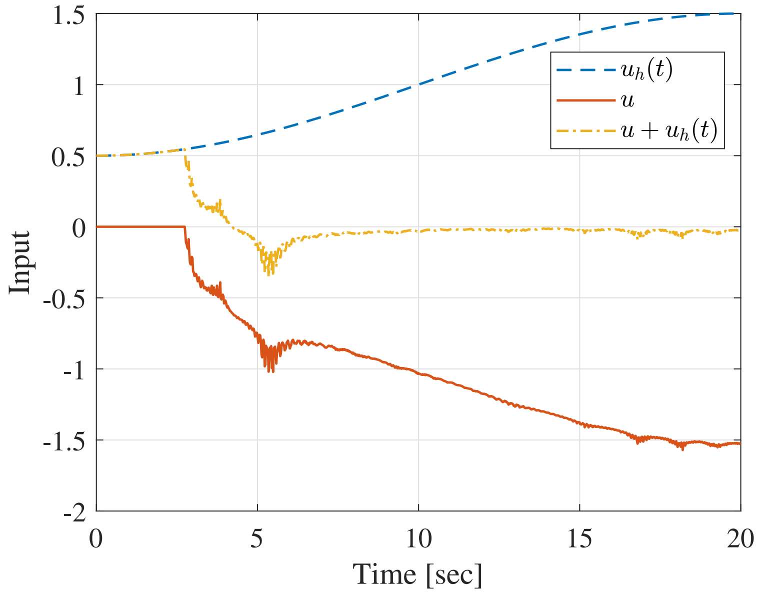

Figure 4.5: Computer Simulation: Input (Mathematical Example).

4.2 Input-disturbed Safety Assist Control

In this chapter, we secondly consider the following nonlinear control-affine system with a human operator input $u_h(t)$ and an input disturbance $d(t)$:

$$\dot{x} = f(x) + g(x)\left[u + u_h(t) + d(t)\right], \tag{4.37}$$

where $x \in \mathbb{R}^n$ denotes a state, $u \in \mathbb{R}^m$ a control input. Assume that mappings $u_h : \mathbb{R} \to \mathbb{R}^m$, $d : \mathbb{R} \to \mathbb{R}^m$ are continuous, and mappings $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are continuous. Assume moreover that an input disturbance $d(t)$ is unknown but bounded.

Consider the following time-invariant open set $\mathcal{S}$:

$$\begin{aligned} \mathcal{S} &= \{x \in \mathbb{R}^n \mid h(x) > 0\}, \\ \partial\mathcal{S} &= \{x \in \mathbb{R}^n \mid h(x) = 0\}, \end{aligned} \tag{4.38}$$

where a function $h : \mathbb{R}^n \to \mathbb{R}$ is a C^1 continuously differentiable function. Here we assume that a solution $x(t)$ to the system (4.37) is forward complete to simplify the discussion. Then, the set $\mathcal{S}$ is forward invariant if for every initial state $x_0 := x(0) \in \mathcal{S}$, $x(t) \in \mathcal{S}$ for $\forall t \geq 0$, and we call $\mathcal{S}$ a safe set where the state of the system (4.37) should stay. Accordingly, the state constraint is denoted by $x \in \mathcal{S}$.

4.2.1 Input-to-State Safety [3, 44]

This subsection introduces the notion of input-to-state safety (ISSf) and input-to-state safety control barrier functions (ISSf-CBFs) firstly proposed by [44].

Consider the following control system with an input disturbance $d(t)$:

$$\dot{x} = f(x) + g(x)\left[u + d(t)\right], \tag{4.39}$$

where $x \in \mathbb{R}^n$ denotes a state, $u \in \mathbb{R}^m$ a control input. Assume that a continuous mapping $d : \mathbb{R} \to \mathbb{R}^m$ is bounded, i.e., $\exists \bar{\lambda} \in \mathbb{R}_{>0}$ such that $\|d\|_\infty := \sup_{t \geq 0} \|d(t)\| \leq \bar{\lambda}$, and mappings $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are locally Lipschitz continuous. A safe set $\bar{\mathcal{S}}$ for the system (4.39) is defined as a closure of (4.38), i.e., $\bar{\mathcal{S}} = \mathrm{cl}(\mathcal{S})$ and hence $x \in \partial\mathcal{S}$ is safe in this subsection.

Then, the notion of ISSf and ISSf-CBFs are defined as follows [3].

Definition 4.2 *Consider the system* (4.39) *with a bounded input disturbance $d(t)$ satisfying $\|d\|_\infty \leq \bar{\lambda}$, the safe set $\bar{\mathcal{S}}$ and a C^1 continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$. Then, the system* (4.39) *is input-to-state safe (ISSf) if there exists a function $\kappa \in \mathcal{K}_\infty$ such that the following set $\bar{\mathcal{S}}_{\bar{\lambda}} \supseteq \bar{\mathcal{S}}$ is forward invariant.*

$$\begin{aligned} \bar{\mathcal{S}}_{\bar{\lambda}} &:= \{x \in \mathbb{R}^n \mid h(x) + \kappa(\bar{\lambda}) \geq 0\}, \\ \partial\bar{\mathcal{S}}_{\bar{\lambda}} &:= \{x \in \mathbb{R}^n \mid h(x) + \kappa(\bar{\lambda}) = 0\}, \\ int(\bar{\mathcal{S}}_{\bar{\lambda}}) &:= \{x \in \mathbb{R}^n \mid h(x) + \kappa(\bar{\lambda}) > 0\}. \end{aligned} \tag{4.40}$$

In this case, the safe set $\bar{\mathcal{S}}$ is said to be an input-to-state safe set (ISSf set).

Definition 4.3 *Consider the system* (4.39) *with a bounded input disturbance $d(t)$ satisfying $\|d\|_\infty \leq \bar{\lambda}$, the safe set $\bar{\mathcal{S}}$ and a C^1 continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ with* $(\partial h/\partial x)(x) \neq 0$ *when $h(x) = 0$. Then, h is an input-to-state safe control barrier function (ISSf-CBF) on $\bar{\mathcal{S}}$ if there exists a function $\alpha \in \mathcal{K}_{\infty,e}$ and a positive constant $\delta \in \mathbb{R}_{>0}$ such that the following inequality holds:*

$$\sup_{u \in \mathbb{R}^m} \dot{h}(x, u) \geq -\alpha\left(h(x)\right) + \frac{\|L_g h(x)\|^2}{\delta}. \tag{4.41}$$

Designing any Lipschitz continuous controller satisfying $u \in K_{\mathrm{ISSf}}(x)$ where

$$K_{\mathrm{ISSf}}(x) = \left\{ u \in \mathbb{R}^m \;\middle|\; \dot{h} \geq -\alpha\left(h(x)\right) + \frac{\|L_g h(x)\|^2}{\delta} \right\}, \tag{4.42}$$

ensures the forward invariance of $\bar{\mathcal{S}}_{\bar{\lambda}}$, and consequently renders the safe set $\bar{\mathcal{S}}$ ISSf [3, Theorem 2].

4.2.2 Problem Statement

An ISSf-CBF-based controller $u \in K_{\text{ISSf}}(x)$ renders the system state bounded for any bounded input disturbance. For designing more flexible controllers, [3] extended an ISSf-CBF to a tunable ISSf-CBF (TISSf-CBF) using the value of a function $h(x)$. Concretely, parametrizing a function $\kappa(h(x), \delta)$ in (4.40) (or equivalently a function $\delta(h(x))$ in (4.41)) where $\kappa(r, \cdot) \in \mathcal{K}_\infty$ for $\forall r \in \mathbb{R}$ results in less invasive controllers compared to $u \in K_{\text{ISSf}}(x)$. However, these frameworks ensure the forward invariance of a larger set $\bar{\mathcal{S}}_{\bar{\lambda}} \supseteq \bar{\mathcal{S}}$, not the original safe set $\bar{\mathcal{S}}$; they are effective only for small input disturbances since the set $\bar{\mathcal{S}}_{\bar{\lambda}}$ grows as $\bar{\lambda}$ becomes larger. Moreover, modifying the safe set $\bar{\mathcal{S}}$ by the set $\bar{\mathcal{S}}' \subseteq \bar{\mathcal{S}}$ so as to $\bar{\mathcal{S}}'_{\bar{\lambda}} \subseteq \bar{\mathcal{S}}$ might also increase conservatism.

The S-ZCBF-based human assist controller proposed in Section 3.4 ensures the forward invariance of the original safe set $\mathcal{S}$ only when $d(t) \equiv 0$, not allowing $\exists t_d \geq 0$ such that $d(t_d) \neq 0$ for $\forall t \geq t_d$ since it maintains a human operator input to a maximal degree.

Problem 4.2 *The second objective of this chapter is to design a human assist controller $u = k(x, t)$ for the input-disturbed system* (4.37) *satisfying the conditions (P1), (P2) and (P3). Importantly, satisfying the condition (P1) inidicates the forward invariance of the original safe set $\mathcal{S}$.*

4.2.3 Input-to-State Constrained Safety Zeroing Control Barrier Function

In this subsection, we introduce an ISCSf-ZCBF as follows.

Definition 4.4 *Consider the system* (4.37) *and the safe set $\mathcal{S}$ defined by* (4.38)*. Then, a C^1 continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ is an input-to-state constrained safety zeroing control barrier function (ISCSf-ZCBF) if the following conditions hold.*

(D1) *$h(x)$ is a proper function for $\forall x \in \mathcal{S}$; the superlevel set $\{x | h(x) \geq L\}$ is compact for any positive constant $L \in \mathbb{R}_{>0}$.*

(D2) *There exists a locally Lipschitz continuous function $\alpha \in \mathcal{K}_e$ and a function $\rho \in \mathcal{L}$ such that the following inequality holds:*

$$\begin{aligned} & h(x) \leq \rho(\|d(t)\|) \\ & \Rightarrow \sup_{u \in \mathbb{R}^m} \dot{h}(x, u, u_h(t), d(t)) > -\alpha(h(x)). \end{aligned} \tag{4.43}$$

Since the time-derivative of an ISCSf-ZCBF contains an unknown input disturbance $d(t)$, it is difficult to construct a human assist controller directly using the condition *(D2)*. Thus, we introduce the following theorem that provides the equivalent condition to *(D2)* and enables us to construct a controller without using or estimating the unknown input disturbance $d(t)$.

Theorem 4.3 *Consider the system* (4.37) *and the safe set $\mathcal{S}$ defined by* (4.38)*. Then, a C^1 continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ is an ISCSf-ZCBF if the condition (D1) and the following condition holds.*

(D2') There exists a locally Lipschitz continuous function $\alpha \in \mathcal{K}_e$ and a function $\rho \in \mathcal{L}$ such that the following inequality holds:

$$\begin{aligned} &h(x) \le \rho(\|d(t)\|) \\ &\quad \Rightarrow \; L_f h(x) + L_g h(x) \cdot (u + u_h(t)) \\ &\qquad > -\alpha(h(x)) + \left[\rho^{-1}\left(h(x)\right)\right]^2. \end{aligned} \tag{4.44}$$

To prove Theorem 4.3, the following lemma is introduced.

Lemma 4.3 *Consider the system* (4.37) *and the safe set $\mathcal{S}$ defined by* (4.38)*. Then, for any continuous mappings $u_h : \mathbb{R} \to \mathbb{R}^m$ and $d : \mathbb{R} \to \mathbb{R}^m$, there exists a control input $u \in \mathbb{R}^m$ such that*

$$\begin{aligned} \dot{h}(x, u, u_h(t), d(t)) > &- \alpha(h(x)) \\ &+ \left[\rho^{-1}\left(h(x)\right)\right]^2 + L_g h(x) \cdot d(t), \end{aligned} \tag{4.45}$$

if and only if there exists a control input $u \in \mathbb{R}^m$ such that

$$\begin{aligned} \dot{h}(x, u, u_h(t), d(t)) > &-\alpha(h(x)) \\ &+ \left[\rho^{-1}\left(h(x)\right)\right]^2 + \frac{\|L_g h(x)\|^2}{4} + L_g h(x) \cdot d(t). \end{aligned} \tag{4.46}$$

Proof The time-derivative of h is calculated as follows:

$$\dot{h} = L_f h(x) + L_g h(x) \cdot (u + u_h(t)) + L_g h(x) \cdot d(t). \tag{4.47}$$

Assume that there exists a control input $u_a \in \mathbb{R}^m$ such that the inequality (4.45) holds, i.e., we suppose

$$\begin{aligned} &L_f h(x) + L_g h(x) \cdot (u_a + u_h(t)) + L_g h(x) \cdot d(t) \\ &> -\alpha(h(x)) + \left[\rho^{-1}\left(h(x)\right)\right]^2 + L_g h(x) \cdot d(t). \end{aligned} \tag{4.48}$$

Then, choosing $u = u_a + (L_g h(x))^T/4$ yields

$$\begin{aligned} &L_f h(x) + L_g h(x) \cdot (u + u_h(t)) + L_g h(x) \cdot d(t) \\ &> -\alpha(h(x)) + \left[\rho^{-1}\left(h(x)\right)\right]^2 + \frac{\|L_g h(x)\|^2}{4} + L_g h(x) \cdot d(t). \end{aligned} \tag{4.49}$$

The sufficiency is proved vice versa; the above discussion completes the proof. □

Now, we can prove Theorem 4.3.

Proof Note that $x \in \mathcal{S}$ implies $h(x) > 0$. The boundedness of an unknown input disturbance $d(t)$ implies that there exists a positive constant $\lambda \in \mathbb{R}_{>0}$ such that $\|d(t)\| \le \lambda$ for $\forall t \ge 0$. Owning to $\rho \in \mathcal{L}$, the inequality $\rho(\|d(t)\|) \ge \rho(\lambda)$ holds. Then, in the case of $h(x) > \rho(\|d(t)\|)$, the inequality

$h(x(t)) > \rho(\lambda)$ holds. Hence $x(t) \in \mathcal{S}$ is evidently ensured for $\forall t \geq 0$ as long as $d(t)$ is bounded since $h(x)$ is a proper function; the level set $\{y|h(y) \geq \rho(\lambda)\}$ is compact and hence $\{y|h(y) \geq \rho(\lambda)\} \subset \mathcal{S}$.

Thus, we consider the case that there exists $t_1 \geq 0$ such that $h(x) \leq \rho(\lambda)$ holds. Then, the following inequality holds:

$$h(x(t)) \leq \rho(\lambda) \leq \rho(\|d(t)\|), \tag{4.50}$$

for $\forall t \geq t_1$ due to $\rho \in \mathcal{L}$. Then, substituting (4.44) into the time-derivative of h yields

$$\begin{aligned} \dot{h} =& L_f h(x) + L_g h(x) \cdot (u + u_h(t)) + L_g h(x) \cdot d(t) \\ &> -\alpha(h(x)) + \left[\rho^{-1}(h(x))\right]^2 + L_g h(x) \cdot d(t). \end{aligned} \tag{4.51}$$

According to Lemma 4.3, there exists a control input $u \in \mathbb{R}^m$ such that the inequality (4.49) holds. Hence by the Cauchy–Schwarz inequality,

$$\frac{\|L_g h(x)\|^2}{4} + L_g h(x) \cdot d(t) + \|d(t)\|^2 \geq 0, \tag{4.52}$$

the following inequality holds:

$$\begin{aligned} \dot{h} &> -\alpha(h(x)) + \left[\rho^{-1}(h(x))\right]^2 + \frac{\|L_g h(x)\|^2}{4} + L_g h(x) \cdot d(t) \\ &\geq -\alpha(h(x)) + \left[\rho^{-1}(h(x))\right]^2 - \|d(t)\|^2, \\ &\geq -\alpha(h(x)) + \left[\rho^{-1}(h(x))\right]^2 - \lambda^2. \end{aligned} \tag{4.53}$$

Here considering $\rho \in \mathcal{L}$, an inverse function ρ^{-1} is also strictly decreasing. Hence the following inequality holds:

$$\rho^{-1}(h(x(t))) \geq \lambda \geq \|d(t)\|, \tag{4.54}$$

according to (4.50). Thus, substituting the square of (4.54) into (4.53) yields $\dot{h} > -\alpha(h(x))$; therefore, the condition (D2') is equivalent to (D2), and the above discussion completes the proof. □

4.2.4 Safety Assist Control via ISCSf-ZCBF

In this subsection, the author proposes an ISCSf-ZCBF-based human assist controller that ensures the forward invariance of $\mathcal{S}$.

Theorem 4.4 *Consider the system* (4.37)*, the safe set* $\mathcal{S}$ *defined by* (4.38) *and an ISCSf-ZCBF* $h : \mathbb{R}^n \to \mathbb{R}$ *satisfying the conditions (D1) and (D2). Then, for any continuous mappings* $u_h : \mathbb{R} \to \mathbb{R}^m$

and $d : \mathbb{R} \to \mathbb{R}^m$*, the following human assist controller* $u = k(x, t)$ *ensures the forward invariance of* $\mathcal{S}$.

$$u = k(x,t) = \begin{cases} -\dfrac{I(x,u_h(t)) - J(x)}{||L_g h(x)||^2}(L_g h(x))^T \\ \qquad \text{if } I(x,u_h(t)) < J(x) \\ 0 \qquad \text{if } I(x,u_h(t)) \geq J(x), \end{cases} \tag{4.55}$$

where functions $I : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$*,* $J : \mathbb{R}^n \to \mathbb{R}$ *and* $\beta : \mathbb{R} \to \mathbb{R}_{\geq 0}$ *are defined by*

$$\begin{aligned} I(x,u_h(t)) &= L_f h(x) + L_g h(x) \cdot u_h(t), \\ J(x) &= -\alpha(h(x)) + \beta(h(x)) + \frac{||L_g h(x)||^2}{4}, \\ \beta(h(x)) &\geq \left[\rho^{-1}(h(x))\right]^2. \end{aligned} \tag{4.56}$$

Proof We denote the closed-loop system (4.37) with the proposed human assist controller (4.55) as follows:

$$\begin{aligned} \dot{x} &= f_{cl}(x, k(x,t), u_h(t), d(t)) \\ &= f(x) + g(x)(k(x,t) + u_h(t)) + g(x)d(t). \end{aligned} \tag{4.57}$$

Note that $x \in \mathcal{S}$ implies $h(x) > 0$, and the time-derivative of an ISCSf-ZCBF h is denoted by (4.47).

Similarly to the first part of the proof of Theorem 4.3, the state constraint $x(t) \in \mathcal{S}$ is satisfied for any $\forall t \geq t_0 \geq 0$ if $h(x) > \rho(||d(t)||)$ holds for $\forall t \geq t_0$ due to the boundedness of $d(t)$ and the properness of $h(x)$.

Thus, we consider the case that there exists $t_1 \geq 0$ such that $h(x) \leq \rho(||d(t)||)$, i.e., the inequality (4.50) holds for $\forall t \geq t_1$.

Case 1: $I(x, u_h(t)) < J(x)$.

Substituting the proposed controller (4.55) into the time-derivative (4.47) and applying the Cauchy–Schwarz inequality (4.52) yield

$$\begin{aligned} \dot{h} &= J(x) + L_g h(x) \cdot d(t) \\ &\geq J(x) - \frac{||L_g h(x)||^2}{4} - ||d(t)||^2 \\ &\geq J(x) - \frac{||L_g h(x)||^2}{4} - \lambda^2 \\ &\geq J(x) - \frac{||L_g h(x)||^2}{4} - \left[\rho^{-1}(h(x))\right]^2, \end{aligned} \tag{4.58}$$

since a function ρ belongs to class $\mathcal{L}$. Hence by the definition of a function β, the following inequality holds:

$$\dot{h} \geq J(x) - \frac{||L_g h(x)||^2}{4} - \beta((h(x))) = -\alpha(h(x)). \tag{4.59}$$

Case 2: $I(x, u_h(t)) \geq J(x)$.

In a similar fashion to the case of $I(x, u_h(t)) < J(x)$, substituting $k(x, t) = 0$ into (4.47) yields

$$\begin{aligned}\dot{h} &= I(x, u_h(t)) + L_g h(x) \cdot d(t) \\ &\geq I(x, u_h(t)) - \frac{\|L_g h(x)\|^2}{4} - \beta(h(x)) \\ &\geq J(x) - \frac{\|L_g h(x)\|^2}{4} - \beta(h(x)) = -\alpha(h(x)). \end{aligned} \tag{4.60}$$

Accordingly, the following inequality holds in both cases $I(x, u_h(t)) < J(x)$ and $I(x, u_h(t)) \geq J(x)$ by the proposed controller (4.55) when the inequality (4.50) is satisfied:

$$\dot{h} \geq -\alpha(h(x)). \tag{4.61}$$

Since an ISCSf-ZCBF h is proper, there exists a positive constant $\gamma \in \mathbb{R}_{>0}$ that satisfies the inequality (3.15) according to Lemma 3.1, i.e., the following inequality holds for $\forall x \in \mathcal{S}$:

$$\dot{h} \geq -\alpha(h(x)) \geq -\gamma h(x). \tag{4.62}$$

Hence the following inequality holds according to Gronwall's lemma 2.2 for $\forall t \in [t_1, t_2]$:

$$\begin{aligned} h(t) &\geq h(x_0(t_1)) \exp(-\gamma(t - t_1)) \\ &\geq h(x_0(t_1)) \exp(-\gamma(t_2 - t_1)) > 0. \end{aligned} \tag{4.63}$$

where $t_2 \geq t_1 \geq 0$, and t_2 can be arbitrarily extended by Theorem 2.4. Similarly to the case of $h(x) > \rho(\|d(t)\|)$, the properness of an ISCSf-ZCBF h implies that the level set $\{y | h(y) \geq h(x(t))\}$ is compact and hence $\{y | h(y) \geq h(x(t))\} \subset \mathcal{S}$. Thus, the state constraint $x(t) \in \mathcal{S}$ is satisfied for $\forall t \geq t_1$ in the case of $h(x) \leq \rho(\|d(t)\|)$.

Therefore, the closed-loop system (4.57) satisfies $x(t) \in \mathcal{S}$ for $\forall t \geq 0$, i.e., the proposed controller (4.55) ensures the forward invariance of $\mathcal{S}$. □

The proposed human assist controller (4.55) has the other outstanding properties described by the following lemma, which indicates that it satisfies the condition ($P2$) and ($P3$).

Lemma 4.4 *The human assist controller* (4.55) *is continuous in* $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ *regardless of the relative degree condition* $L_g h(x) \neq 0$ *for* $\forall x \in \mathbb{R}$.

Lemma 4.5 *The human assist controller* (4.55) *minimizes the Euclidian norm* $\|u\|$ *for* $\forall (x, t) \in \mathbb{R}^n \times \mathbb{R}$ *such that the following inequality holds:*

$$\begin{aligned} &L_f h(x) + L_g h(x) \cdot u_h(t) + L_g h(x) \cdot u \\ &\qquad \geq -\alpha(h(x)) + \beta(h(x)) + \frac{\|L_g h(x)\|^2}{4}. \end{aligned} \tag{4.64}$$

4.3 Time-varying Obstacle Avoidance for Electric Wheelchair

In this section, we aim to design a human assist controller to avoid collisions with a time-varying obstacle $x_o(t) \in \mathbb{R}^l$ whose velocity $\dot{x}_o(t)$ is unknown by using the methods proposed in Section 4.1 and 4.2.

Consider the following control system:

$$\dot{x} = f(x) + g(x)\left[u + u_h(t)\right], \tag{4.65}$$

where $x \in \mathbb{R}^n$ denotes a state, $u \in \mathbb{R}^m$ a control input. Assume that mappings $u_h : \mathbb{R} \to \mathbb{R}^m$ is continuous, and mappings $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are continuous.

We also consider the following time-dependent open safe set $\mathcal{S}(x_o(t))$:

$$\begin{aligned} \mathcal{S}(x_o(t)) &= \{x \in \mathbb{R}^n \mid \mathfrak{h}(x, x_o(t)) > 0\}, \\ \partial\mathcal{S}(x_o(t)) &= \{x \in \mathbb{R}^n \mid \mathfrak{h}(x, x_o(t)) = 0\}, \end{aligned} \tag{4.66}$$

where a function $\mathfrak{h} : \mathbb{R}^{n+1} \to \mathbb{R}$ is a C^1 continuously differentiable function. Then, we consider the pair $(x, t) \in \mathbb{R}^{n+1}$ such that

$$\mathcal{G} = \{(x, t) \in \mathbb{R}^{n+1} \mid x(t) \in \mathcal{S}(x_o(t))\}, \tag{4.67}$$

since the notion of forward invariance is only applicable for a time-invariant safe set, which was discussed in Section 4.1. Again, we call a time-invariant set $\mathcal{G}$ a graph space.

Problem 4.3 *The last objective of this chapter is to design a human assist controller $u = k(x, t)$ for the system* (4.65) *satisfying the conditions (P1"), the forward invariance of the graph space* (4.67)*, (P2) and (P3) given in Definition 3.2, under the assumption that the obstacle velocity vector $\dot{x}_o(t)$ is unknown.*

4.3.1 ISCSf-ZCBF for Time-varying Obstacle Avoidance

As the safe set $\mathcal{S}(x_o(t))$ is time-dependent, the time-derivative of h along a solution to the system (4.65) is calculated as follows:

$$\dot{\mathfrak{h}} = L_f\mathfrak{h} + L_g\mathfrak{h}(u + u_h(t)) + \frac{\partial \mathfrak{h}}{\partial x_o}\dot{x}_o. \tag{4.68}$$

Here, we need to estimate the unknown $\dot{x}_o(t)$ to design a human assist controller $u = k(x, t)$. Hence we employ the following high-gain observer based first-order differentiator proposed by [67]:

$$\begin{aligned} \dot{\hat{z}}_1 &= \hat{z}_2 + (\alpha_1/\varepsilon)(x_o(t) - \hat{z}_1), \\ \dot{\hat{z}}_2 &= (\alpha_2/\varepsilon^2)(x_o(t) - \hat{z}_1), \\ y &= \hat{z}_2 = \hat{\dot{x}}_o(t), \end{aligned} \tag{4.69}$$

where the polynomial $s^2 + \alpha_1 s + \alpha_2$ is Hurwitz, and $\varepsilon \in \mathbb{R}_{>0}$ is a positive constant. Note that the observer gain K_o is given by

$$K_o = \left[\frac{\alpha_1}{\varepsilon}, \ \frac{\alpha_2}{\varepsilon^2}\right]^T. \tag{4.70}$$

Ideally, the estimation error $d(t) := \hat{\dot{x}}_o(t) - \dot{x}_o(t)$ shrinks to zero as the observer gain K_o grows to infinity, i.e., $\varepsilon \to 0$. However, a large gain magnifies the measurement noise, and hence we need to choose the suitable bounded gain; as a result, the estimation error $d(t)$ necessarily remains. Therefore, the direct substitution of $\hat{\dot{x}}_o(t)$ into $\dot{x}_o(t)$ leads to the failure of theoretical safety assurance.

To this end, the author proposes the following ISCSf-ZCBF for a time-dependent state constraint problem by regarding the estimation error $d(t)$ as input disturbance.

Definition 4.5 *Consider the system* (4.65)*, the safe set* (4.66) *with the graph space* (4.67) *and the high-gain observer* (4.69)*. Then, a C^1 continuously differentiable function $\mathfrak{h} : \mathbb{R}^{n+1} \to \mathbb{R}$ is an input-to-state constrained safety zeroing control barrier function (ISCSf-ZCBF) if the following conditions hold.*

(E1) For any fixed $t_0 \in \mathbb{R}$, $\mathfrak{h}(x, x_o(t_0))$ is a proper function w.r.t. $\forall x \in \mathcal{S}(x_o(t_0))$; the superlevel set $\{x |\ \mathfrak{h}(x, x_o(t_0)) \geq L\}$ is compact for any positive constant $L \in \mathbb{R}_{>0}$.

(E2) There exists a locally Lipschitz continuous function $\alpha \in \mathcal{K}_e$ and a function $\rho \in \mathcal{L}$ such that the following inequality holds:

$$\begin{aligned} &\mathfrak{h}(x, x_o(t)) \leq \rho(\|d(t)\|) \\ &\Rightarrow \sup_{u \in \mathbb{R}^m} \dot{\mathfrak{h}}(x, x_o(t), u, u_h(t), d(t)) > -\alpha(\mathfrak{h}(x, x_o(t))), \end{aligned} \tag{4.71}$$

where

$$\dot{\mathfrak{h}} = L_f \mathfrak{h} + L_g \mathfrak{h}(u + u_h(t)) + \frac{\partial \mathfrak{h}}{\partial x_o}(\hat{\dot{x}}_o(t) - d(t)).$$

According to Theorem 4.3 and 4.4, the following corollaries are immediately obtained.

Corollary 4.1 *Consider the system* (4.65)*, the safe set* (4.66) *with the graph space* (4.67) *and the high-gain observer* (4.69)*. Then, a C^1 continuously differentiable function $\mathfrak{h} : \mathbb{R}^{n+1} \to \mathbb{R}$ is an ISCSf-ZCBF if the condition (E1) and the following condition hold.*

(E2') There exists a locally Lipschitz continuous function $\alpha \in \mathcal{K}_e$ and a function $\rho \in \mathcal{L}$ such that the following inequality holds:

$$\begin{aligned} &\mathfrak{h}(x, x_o(t)) \leq \rho(\|d(t)\|) \\ &\quad \Rightarrow \ L_f \mathfrak{h} + L_g \mathfrak{h} \cdot (u + u_h(t)) + \frac{\partial \mathfrak{h}}{\partial x_o} \hat{\dot{x}}_o(t) \\ &\qquad > -\alpha(\mathfrak{h}(x, x_o(t))) + \left[\rho^{-1}\left(\mathfrak{h}(x, x_o(t))\right)\right]^2. \end{aligned} \tag{4.72}$$

Corollary 4.2 *Consider the system* (4.65)*, the safe set* (4.66) *with the graph space* (4.67)*, and an ISCSf-ZCBF* $\mathfrak{h} : \mathbb{R}^{n+1} \to \mathbb{R}$ *with the following high-gain observer:*

$$\begin{aligned} \dot{\hat{z}}_1 &= \hat{z}_2 + (\alpha_1/\varepsilon)(x_o(t) - \hat{z}_1), \\ \dot{\hat{z}}_2 &= (\alpha_2/\varepsilon^2)(x_o(t) - \hat{z}_1), \\ y &= \hat{z}_2 = \hat{\dot{x}}_o(t). \end{aligned} \tag{4.73}$$

Then, for any continuous mappings $u_h : \mathbb{R} \to \mathbb{R}^m$ *and the estimation error* $d : \mathbb{R} \to \mathbb{R}^m$*, the following human assist controller* $u = k(x, t)$ *ensures the forward invariance of* $\mathcal{G}$*.*

$$u = k(x,t) = \begin{cases} -\dfrac{I(x, x_o(t), u_h(t)) - J(x, x_o(t))}{\|L_g\mathfrak{h}(x, x_o(t))\|^2}(L_g\mathfrak{h}(x, x_o(t)))^T \\ \qquad\qquad\qquad \text{if } I(x, x_o(t), u_h(t)) < J(x, x_o(t)) \\ 0 \qquad\qquad\qquad \text{if } I(x, x_o(t), u_h(t)) \geq J(x, x_o(t)), \end{cases} \tag{4.74}$$

where functions $I : \mathbb{R}^{n+1} \times \mathbb{R}^m \to \mathbb{R}$*,* $J : \mathbb{R}^{n+1} \to \mathbb{R}$ *and* $\beta : \mathbb{R} \to \mathbb{R}_{\geq 0}$ *are defined by*

$$\begin{aligned} I(x, x_o(t), u_h(t)) &= L_f\mathfrak{h}(x, x_o(t)) + L_g\mathfrak{h}(x, x_o(t)) \cdot u_h(t) + \frac{\partial \mathfrak{h}}{\partial x_o}\hat{\dot{x}}_o(t), \\ J(x, x_o(t)) &= -\alpha(\mathfrak{h}(x, x_o(t))) + \beta(\mathfrak{h}(x, x_o(t))) + \frac{1}{4}\left\|\frac{\partial \mathfrak{h}}{\partial x_o}\right\|^2, \\ \beta(\mathfrak{h}(x, x_o(t))) &\geq \left[\rho^{-1}(\mathfrak{h}(x, x_o(t)))\right]^2. \end{aligned} \tag{4.75}$$

4.3.2 Application to Electric Wheelchair

In this subsection, we experimentally confirm the effectiveness of the proposed human assist controller by using an electric wheelchair WHILL model CR (Fig. 4.6) whose model is described as follows:

$$\dot{x} = \frac{1}{80}(u + u_h(t)), \tag{4.76}$$

where $x[m] \in \mathbb{R}$ denotes a position, $u \in \mathbb{R}$ a control input, and $u_h \in \mathbb{R}$ a human operator input. Note that the input to the system (4.76) is given within $|u + u_h(t)| \in [0, 100][\%]$, and the electric wheelchair moves forward at $0.125[m/s]$ when $u + u_h(t) = 10[\%]$. Note also that the function $g(x) = 1/80$ was identified through the experiment with the input $u + u_h(t) = 50[\%]$.

Consider the time-varying state constraint problem illustrated in Fig. 4.8 where $x_o(t) \in \mathbb{R}$ denotes an obstacle and its velocity $\dot{x}_o(t)$ is unknown. We employ Intel Real Sense Depth Camera D435i (shortly, D435i) whose technical specification is given in Appendix A to get a distance $x_o(t) - x$ (Fig. 4.7). Then, we set a time-varying safe set $\mathcal{S}(x_o(t)) = \{x \in \mathbb{R} \mid x + 0.3 < x_o(t)\}$ considering the ideal range of D435i. Accordingly, a graph space $\mathcal{G}$ for the safe set $\mathcal{S}(x_o(t))$ is obtained as follows:

$$\mathcal{G} = \{(x, t) \in \mathbb{R}^2 \mid x + 0.3 < x_o(t)\}. \tag{4.77}$$

As the velocity $\dot{x}_o(t)$ is unknown, we employ the following high-gain observer-based differentiator:

$$\begin{aligned}
\dot{\hat{z}}_1 &= \hat{z}_2 + (\alpha_1/\varepsilon)(x_o(t) - \hat{z}_1),\\
\dot{\hat{z}}_2 &= (\alpha_2/\varepsilon^2)(x_o(t) - \hat{z}_1),\\
y &= \hat{z}_2 = \hat{\dot{x}}_o(t).
\end{aligned} \tag{4.78}$$

Figure 4.6: Electric Wheelchair WHILL model CR.

Figure 4.7: Depth Sensor D435i.

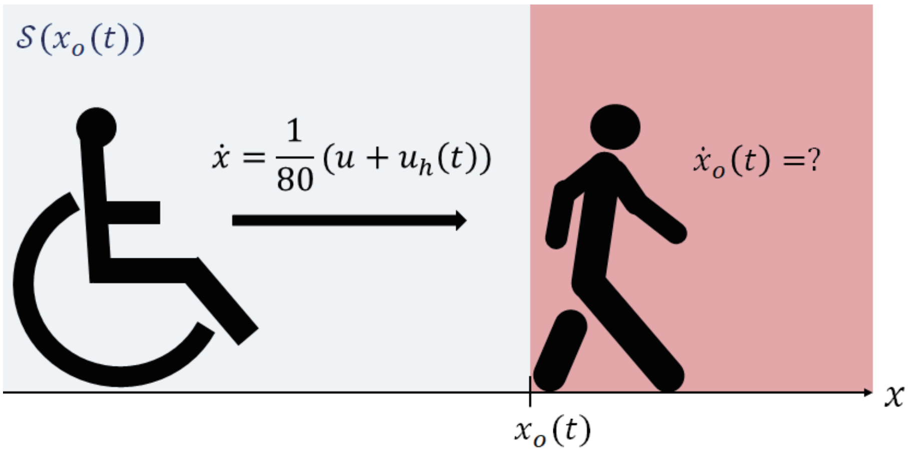

Figure 4.8: Control Model.

Controller Design

For the graph space (4.77) and the high-gain observer based differentiator (4.78), we construct the following ISCSf-ZCBF:

$$\begin{aligned} \mathfrak{h}(x, x_o(t)) &= (x_o(t) - x - 0.3)p(x, x_o(t)) \\ &= \frac{x_o(t) - x - 0.3}{1 + L(x - x_o(t))^2}, \end{aligned} \tag{4.79}$$

where $L \in \mathbb{R}_{>0}$ is a positive constant. Here, the function $p : \mathbb{R}^2 \to \mathbb{R}_{>0}$ is required to satisfy the condition *(E1)*, but we choose L sufficiently small to approximate $p(x, x_o) = 1$ in the controller design below; note that this function is needed to ensure the forward completeness of a solution $x(t)$ to (4.76). Accordingly, we have

$$L_f \mathfrak{h} = 0, \;\; L_g \mathfrak{h} = -\frac{1}{80}, \;\; \frac{\partial \mathfrak{h}}{\partial x_o} \hat{\dot{x}}_o(t) = \hat{\dot{x}}_o(t). \tag{4.80}$$

Moreover, we design a class $\mathcal{L}$ function ρ and a function β as follows:

$$\rho(\mathfrak{h}(x, x_o(t))) = \frac{\gamma}{\mathfrak{h}^2(x, x_o(t))}, \tag{4.81}$$

$$\beta(\mathfrak{h}(x, x_o(t))) = \left[\rho^{-1}\left(\mathfrak{h}(x, x_o(t))\right)\right]^2 = \frac{\gamma}{\mathfrak{h}(x, x_o(t))}, \tag{4.82}$$

where $\gamma \in \mathbb{R}_{>0}$ is a positive constant. Finally, we design the following human assist controller $u = k(x, t)$:

$$k(x,t) = \begin{cases} 80\left(I(x, x_o(t), u_h(t)) - J(x, x_o(t))\right) \\ \qquad \text{if } I(x, x_o(t), u_h(t)) < J(x, x_o(t)) \\ 0 \quad \text{if } I(x, x_o(t), u_h(t)) \geq J(x, x_o(t)), \end{cases} \tag{4.83}$$

where functions I and J with positive constants $\eta_1, \eta_2 \in \mathbb{R}_{>0}$ are defined by

$$I(x, x_o(t), u_h(t)) = -u_h(t)/80 + \hat{\dot{x}}_o(t), \tag{4.84}$$

$$J(x, x_o(t)) = -\eta_1 \mathfrak{h}(x, x_o(t)) - \eta_2 \mathfrak{h}^2(x, x_o(t)) + \frac{\gamma}{\mathfrak{h}(x, x_o(t))} + \frac{1}{4}. \tag{4.85}$$

Simulation Result

Figures 4.9–4.11 illustrate computer simulation results for $x(0) = 0$, $u_h = 15$, $\eta_1 = 0.5$, $\eta_2 = 0$, $\gamma = 0.0001$, $(z_1(0), z_2(0)) = (2, 0)$, $\alpha_1 = 3$, $\alpha_2 = 2$, $\varepsilon = 0.5$. Note that the position of the time-varying obstacle $x_o(t)$ is given by

$$x_o(t) = \begin{cases} 2.0 & \text{if } t < 3 \\ 1.25 + 0.75 \cos \frac{\pi}{6}(t-3) & \text{if } t \geq 3. \end{cases}$$

Figure 4.9 denotes the position, Fig. 4.10 the output of the high-gain observer-based differentiator (4.78) and Fig. 4.11 the control input.

From those figures, we can confirm that the wheelchair position satisfies $x \in \mathcal{S}(x_o(t))$ with the proposed controller (4.83) while the high-gain observer-based differentiator with $\varepsilon = 0.5$ remains the estimation error.

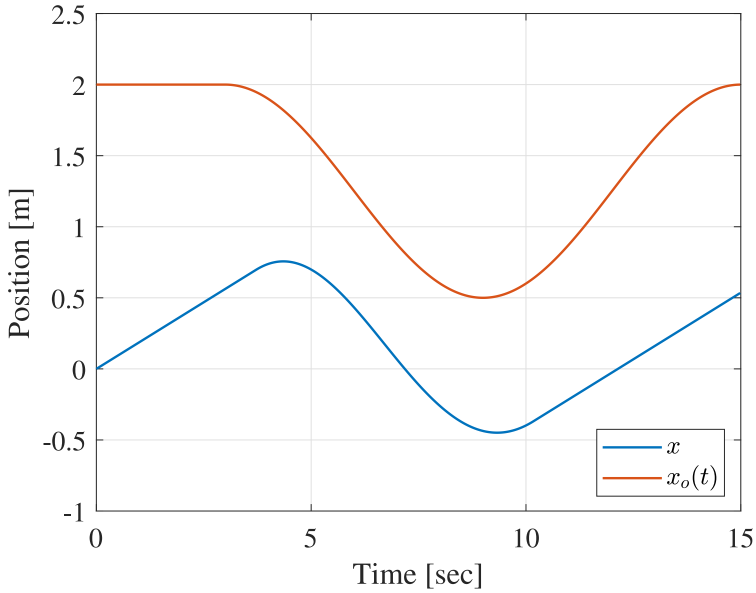

Figure 4.9: Computer Simulation: Wheelchair Position.

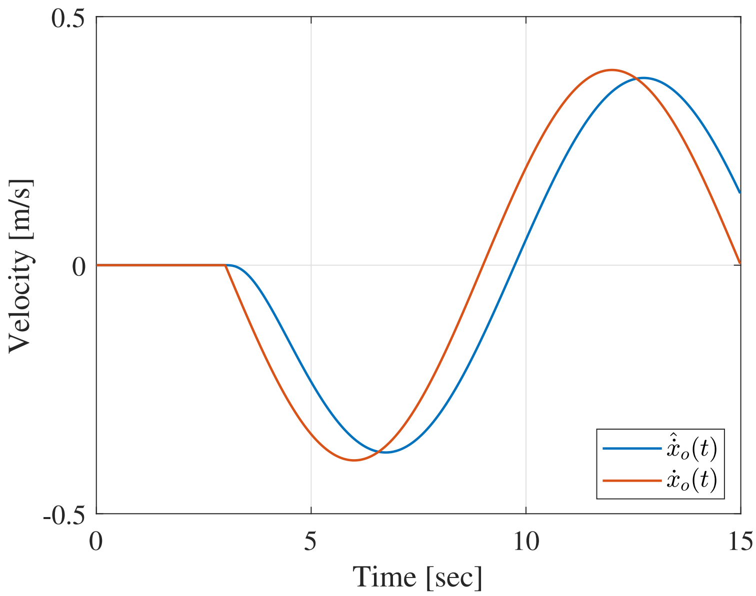

Figure 4.10: Computer Simulation: Estimate of $\dot{x}_o(t)$.

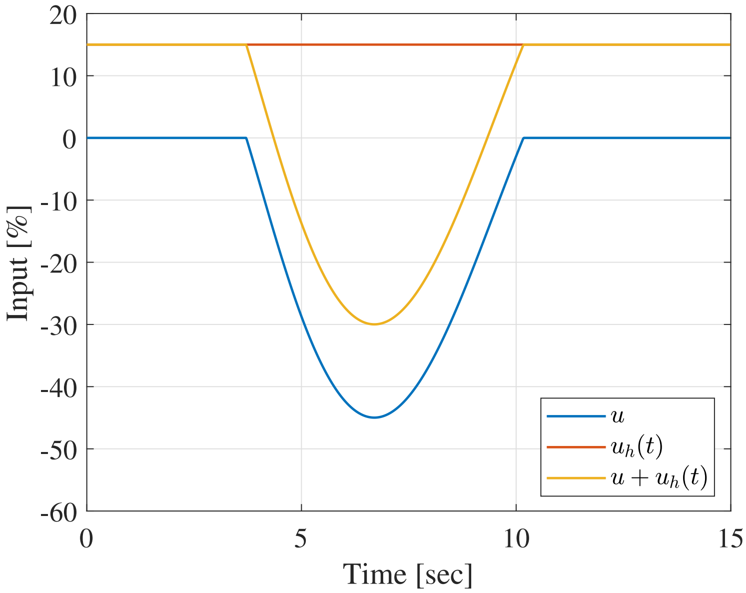

Figure 4.11: Computer Simulation: Velocity Input.

Experimental Result 1: Comparison to Computer Simulation

Figures 4.12–4.14 illustrate experimental results for $x(0) = 0$, $u_h = 15$, $\eta_1 = 0.5$, $\eta_2 = 0$, $\gamma = 0.0001$, $(z_1(0), z_2(0)) = (2, 0)$, $\alpha_1 = 3$, $\alpha_2 = 2$, $\varepsilon = 0.5$, which is the same condition as *Simulation Result*. Figure 4.12 denotes the position, Fig. 4.13 the output of the high-gain observer-based differentiator (4.78) and Fig. 4.14 the control input.

From Fig. 4.12, we can confirm that the time-varying state constraint $x \in \mathcal{S}(x_o(t))$ is satisfied. From Fig. 4.13, we can confirm that the high-gain observer-based differentiator (4.78) properly estimates the obstacle velocity $\dot{x}_o(t)$ compared to Fig. 4.10. From Fig. 4.14, we can confirm that the proposed controller (4.83) works when the obstacle $x_o(t)$ approaches the wheelchair position. We can also confirm that the proposed controller (4.83) vanishes when the obstacle $x_o(t)$ is far from the wheelchair position.

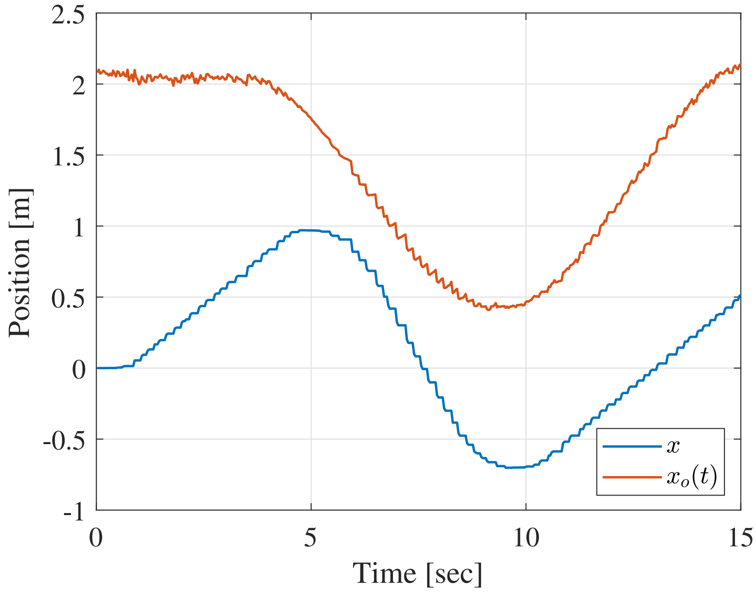

Figure 4.12: Experimental Result 1: Wheelchair Position.

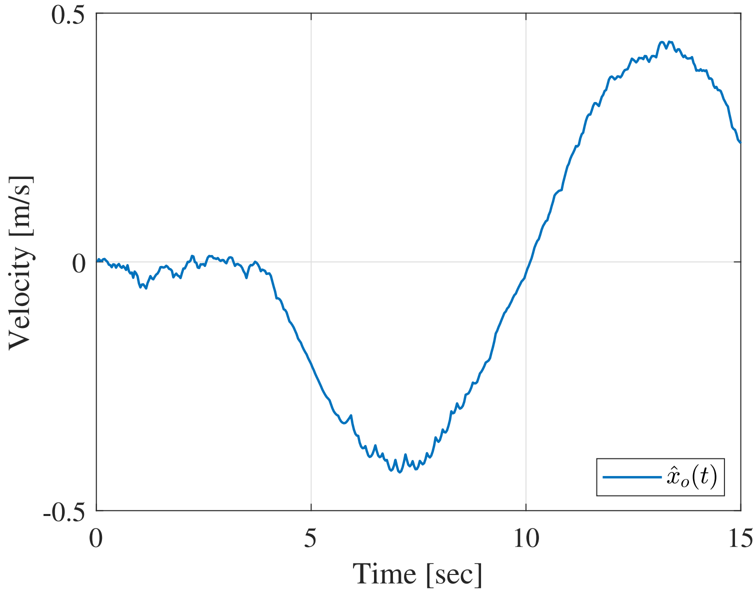

Figure 4.13: Experimental Result 1: Estimate of $\dot{x}_o(t)$.

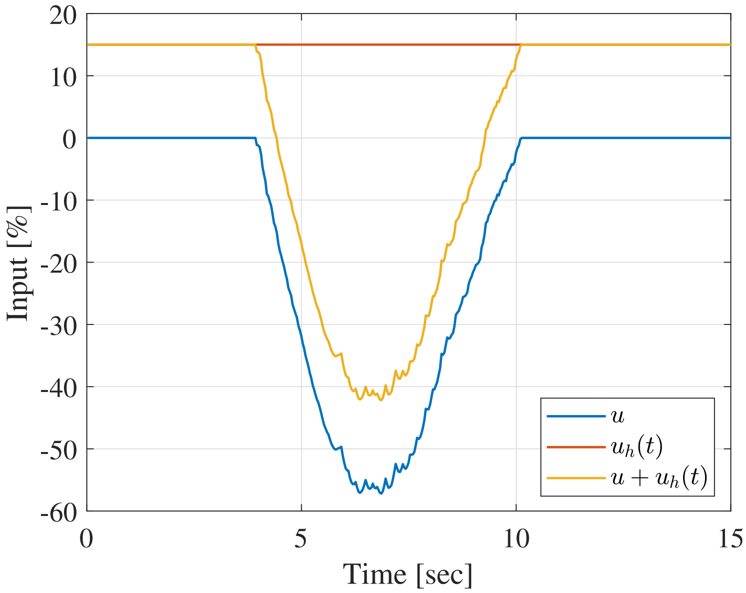

Figure 4.14: Experimental Result 1: Velocity Input.

Experimental Result 2: Time-varying Human Operator Input

Figures 4.15–4.17 illustrate experimental results for $x(0) = 0$, $\eta_1 = 0.5$, $\eta_2 = 0$, $\gamma = 0.0001$, $(z_1(0), z_2(0)) = (0, 0)$, $\alpha_1 = 3$, $\alpha_2 = 2$, $\varepsilon = 0.5$. In this experiment, a human operator provides a time-varying control input $u_h(t)$. Figure 4.15 denotes the position, Fig. 4.16 the output of the high-gain observer-based differentiator (4.78) and Fig. 4.17 the control input.

From Fig. 4.15, we can confirm that the time-varying state constraint $x \in \mathcal{S}(x_o(t))$ is satisfied. From Fig. 4.16, we can confirm that the high-gain observer-based differentiator (4.78) estimates the obstacle velocity $\dot{x}_o(t)$. From Fig. 4.17, we can confirm that the proposed controller (4.83) works when the obstacle $x_o(t)$ approaches the wheelchair position by the time-varying human operator input. Specifically, the human operator is meant to avoid collisions when the obstacle approaches. When it is not enough to do so, the human assist controller works to satisfy the state constraint.

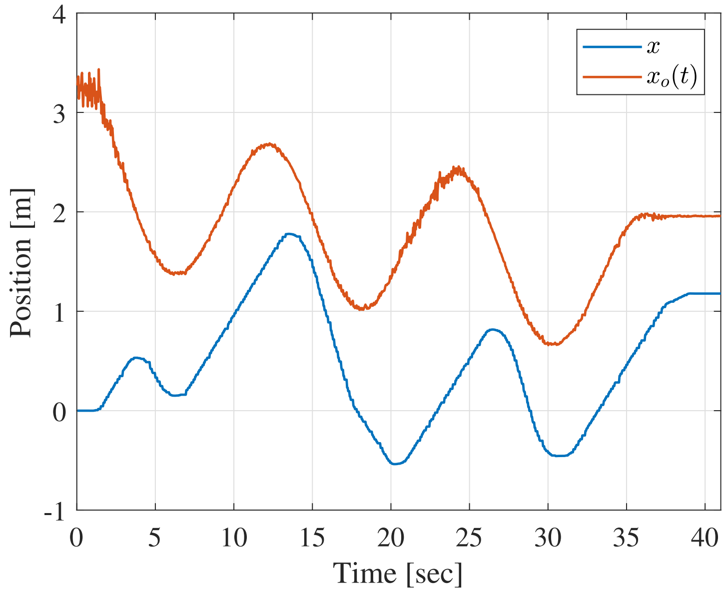

Figure 4.15: Experimental Result 2: Wheelchair Position.

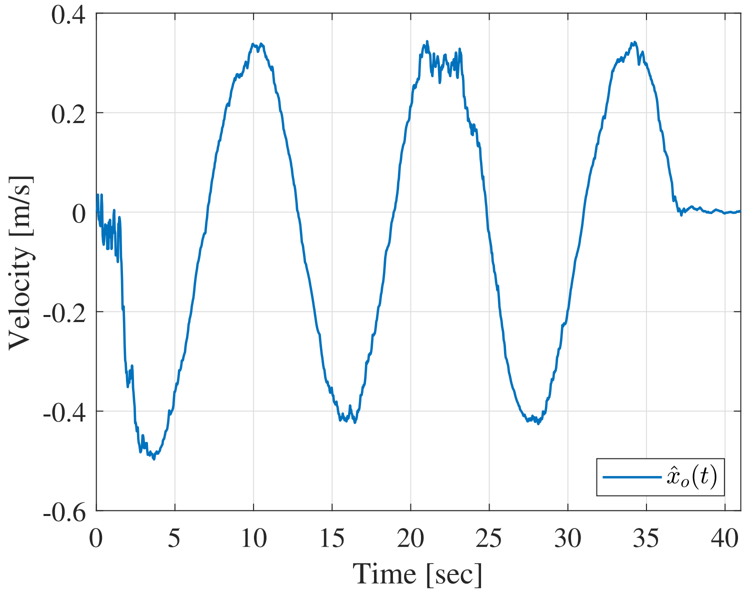

Figure 4.16: Experimental Result 2: Estimate of $\dot{x}_o(t)$.

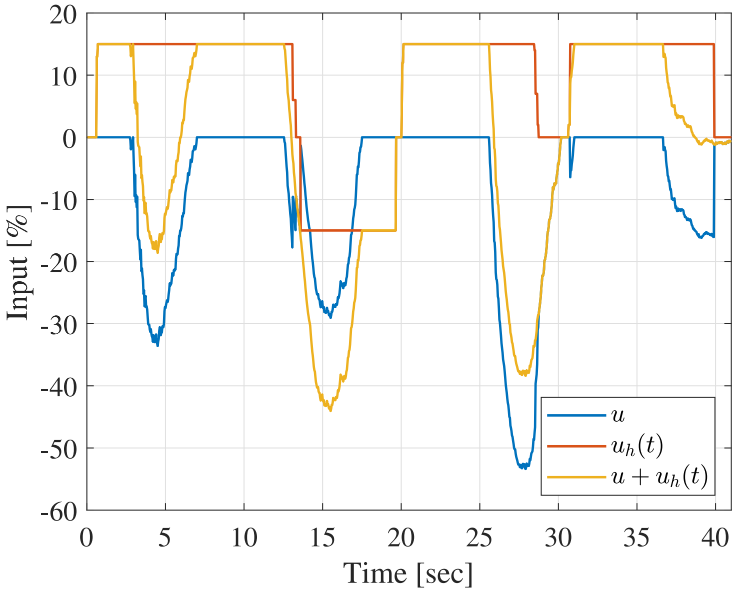

Figure 4.17: Experimental Result 2: Velocity Input.

Chapter 5

Conclusion

5.1 Summary

In this dissertation, the author proposed a safety assist controller that ensured safety of nonautonomous systems. The author pointed out the problems of the conventional method related to Zeroing Control Barrier Functions (ZCBFs) for autonomous systems [7].

The author firstly considered a time-invariant state constraint problem. The author provided the formal definition of ZCBFs, Strict ZCBFs (S-ZCBFs). Then, the author proposed an S-ZCBF-based safety assist controller that guaranteed safety of nonautonomous systems. The author simultaneously proved that the proposed controller was continuous and optimal. Moreover, the author showed that the proposed controller avoided solutions to nonautonomous systems from having a finite escape time, indicating the forward completeness of solutions. Next, the author considered input-constrained control systems and proposed a safety assist controller that met both state and input constraints employing the notion of viability kernels. The effectiveness of the proposed controllers was confirmed by computer simulation.

The author next considered a time-dependent state constraint problem. The author provided the definition of Time-varying ZCBFs (Tv-ZCBFs) with the notion of graph space. Then, the author proposed a Tv-ZCBF-based safety assist controller that rendered graph space forward invariant, which implied that the time-dependent safety constraint was satisfied. Importantly, the proposed controller inherited the properties of the S-ZCBF-based controller. Then, the author proposed a safety assist controller for input-disturbed systems. Here, an Input-to-State Constrained Safety ZCBF (ISCSf-ZCBF) was introduced. Using an ISCSf-ZCBF, the proposed controller ensured the safety of input-disturbed systems. The author lastly applied an ISCSf-ZCBF-based controller to time-varying obstacle avoidance of an electric wheelchair. The effectiveness of the proposed controller was confirmed by computer simulation and experiment.

5.2 Future Work

In this section, the author refers to open issues of the methods proposed in this dissertation.

5.2.1 Advantages of the Use of Zeroing Control Barrier Function

Here, the author lists the future advantages of using ZCBFs instead of reciprocal CBFs. Importantly, these advantages will be due to the property of ZCBFs that can be defined outside of a safe set or a viability kernel.

Damage Mitigation Controller

When considering input-constrained systems, there might be a case such that violation of safety constraints is inevitable. This implies that the system's states start from outside of viability kernels. Moreover, when considering time-dependent state constraint problems, a safe set might be empty at some point; e.g., moving obstacles approaching control systems from various directions narrow a safe set and eventually make it empty. In these cases, a controller must be designed so that the damage of safety violation is minimized.

Robust Controller

In practical problems, control systems generally have modeling errors that make their states leave a safe set in finite time. The author believes that there are mainly two approaches for this problem. The first method is to design a CBF for a subset of viability kernels considering the worst case of disturbances or response delays. This method will increase conservatism, since it takes a margin between a control system and an obstacle, but does not give up the forward invariance of an original safe set. The second method is to keep the state of control systems as close to a safe set or a viability kernel as possible even after the violation of safety constraints occurs. This method might also be achieved by using the ZCBF property similarly to the case of a damage mitigation controller.

5.2.2 Open Issues

Theoretical Issues

In this dissertation, the continuity of mappings in control system models given as an ODE plays an essential role for theoretical safety assurance. However, control systems or controllers are not always continuous in general problems. For example, a CBF-based decision making controller is known as a discontinuous controller because in this case a CBF fails to be differentiable [56]. While there are some studies on nonsmooth ZCBFs, their methods does not produce a decision making controller as Filippov's operator is applied for dealing with a differential inclusion [28, 29]. Therefore, it is still challenging to ensure the safety of discontinuous control systems using ZCBFs.

In Section 3.5, the author restricted control systems to single-input systems with norm input constraints. This is because the continuity of solutions to optimization problems is not ensured when considering multi-input systems even if input constraints are affine [7]. Moreover, explicit solutions to an optimization problem for a minimally invasive controller will be difficult to derive and need to be solved numerically. Therefore, guaranteeing the continuity and the explicitness of a safety assist controller for input-constrained multi-input systems remains future works.

Practical Issues

In this dissertation, the author mainly focused on the theoretical aspect of ZCBFs and only dealt with simple control systems for the effectiveness validation. Therefore, future work needs to establish ZCBF design methods for more complex control systems. For example, a more detailed automobile model contains an actuator time delay, a rolling resistance, a self aligning torque, a friction circle constraint, and so on. Moreover, for the improvement of passenger comfort, considering a jerk as an input constraint and deriving a viability kernel for a jerk-constrained automobile model will improve passenger comfort. This practical factors should be taken into account for the ZCBF design. In other words, once a ZCBF can be obtained, it is easy to design a safety assist controller proposed in this dissertation.

The consideration of a shape of control targets or obstacles should also be studied. The research on reciprocal CBFs considering this problem is progressing compared with ZCBFs, and the literature [25, 75] will be useful for the ZCBF design.

Appendix A

Intel RealSense Depth Camera D435i [22]

In the experiment shown in Subsection 4.3.2, Intel RealSense Depth Camera D435i (Fig.4.7) is employed to measure the distance between an electirc wheelchair and an obstacle. Table A.1 lists the technical specification of D435i [22, 43]. The reason of its employment is that it is cheaper, more easily available than Laser Range Finder [43]. Most importantly, it has a short measurement period that leads to a smooth intervention of a safety assist controller.

Table A.1: Technical Specification of D435i

Depth Technology	Stereoscopic
Measurement Period [sec]	0.0111
Ideal Range [m]	0.3 to 3
Minimum Depth Distance at Max Resolution [m]	0.28
Horizontal Field of View [°]	87 ± 3
Vertical Field of View [°]	58 ± 1
Depth Stream Output Frame Rate [fps]	90

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Achievements

Peer-reviewed Journal

1. Issei Tezuka and Hisakazu Nakamura. Safety Assist Control for Nonautonomous Control-affine Systems via Time-varying Control Barrier Function, *IFAC-PapersOnline*, Vol. 56, No. 1, pp. 187-192, 2023.
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2. Issei Tezuka, Taisuke Kuramoto, and Hisakazu Nakamura. Input-to-State Constrained Safety Zeroing Control Barrier Function and Its Application to Time-varying Obstacle Avoidance for Electric Wheelchair, *IFAC-PapersOnline*, Vol. 55, No. 41, pp. 44-51, 2022.
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