Indexes for the Degree of Departure from Models of Symmetry in Square Contingency Tables

Thesis submitted in partial fulfilment of the requirements
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by

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Under the Guidance of

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Dedicated to my beloved parents and to my lovely partner.

Certificate

This is to certify that the thesis entitled **"Indexes for the Degree of Departure from Models of Symmetry in Square Contingency Tables"**, submitted by **Tomotaka Momozaki** to the Tokyo University of Science, for the award of the degree of **Doctor of Science** in Faculty of Science and Technology, is a record of the original, bona fide research work carried out by him under our supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations related to the award of the degree.

The results contained in this thesis have not been submitted in part or in full to any other University or Institute for the award of any degree or diploma to the best of our knowledge.

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Declaration

I declare that this written submission represents my ideas in my own words. Where others' ideas and words have been included, I have adequately cited and referenced the original source. I declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated, or falsified any idea/data/fact/source in my submission. I understand that any violation of the above will cause disciplinary action by the Institute and can also evoke penal action from the source which has thus not been properly cited or from whom proper permission has not been taken when needed.

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Tomotaka Momozaki
Date: October, 12, 2023

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Abstract

In contingency table analysis, we are interested in dependency structures among categorical variables. The independence model, where the cell probability (joint distribution of categorical variables) is expressed as a product of the marginal cell probabilities (marginal distributions), is one of the well known models for the dependent structure among variables, and we investigate whether the data generating process is the independent model using Pearson's chi-squared test. Meanwhile, we are interested in different probabilistic structures in square contingency tables, where the row and column categorical variables have the same categories, since in most cases the independent model does not hold. Examples are the Symmetry model that represent the equality of cell probabilities at symmetric positions with respect to the main diagonals of the contingency table and its special cases such as the marginal homogeneity model, some extended Symmetry models and asymmetry models. Besides statistical hypothesis testings, there are other methods for inferring the probabilistic structure in contingency tables using indexes for the degree of departure from some model instead of the Pearson's correlation coefficient for independence between continuous variables. This thesis develops indexes for symmetry in two-way square contingency tables.

Chapter 1 briefly explains backgrounds and motivations of the works given in the subsequent chapters. Chapter 2 proposes the index which is a two-dimensional vector with two indexes as its elements, represents the degree of departure from the Symmetry model from a different perspective from the conventional indexes, and allows for easier interpretation in the data analysis. Chapter 3 proposes a directional index that can distinguish between two types of asymmetry structures, expresses the degree of departure from them, and is easier to interpret in data analysis than existing ones. Chapter 4 provides some discussion and concluding remarks.

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Chapter 1

Introduction

1.1 Background

This thesis highlights contingency tables and methodologies for their analysis, which are of great importance in a wide variety of fields such as medicine, psychology, social sciences, education, sports, and so on. The contingency table is a common data representation when all observations are categorical data, and the chi-square test of goodness-of-fit test (Pearson, 1900), which is still very commonly used today, was introduced more than one century ago. For a history of contingency table analysis before that, see Stigler (2002). Tables formed from two binary variables (e.g., smoker or not and hypertension or not) and having four attributes ((smoker and hypertension), (smoker and not hypertension), (nonsmoker and hypertension), and (nonsmoker and not hypertension)) are referred to as 2×2 contingency tables, and in the case of categorical variables with multiple categories (with r and c) instead of binary variables, they are referred to as $r \times c$ contingency tables, they are collectively referred to as two-way contingency tables. When they are formed by more than two categorical variables, they are termed multi-way contingency tables. This thesis focuses on two-way contingency tables and does not deal with multi-way contingency tables in the following sections, but readers who are interested in the methodology for the analysis of multi-way contingency tables are referred to Bishop et al. (2007) and Agresti (2010). Categorical variables that consist of contingency tables are classified into those that have a natural ordering between categories and those that do not, and are called nominal and ordinal categorical variables, respectively. Examples of ordinal categorical variables are disease progression (stage 1, 2, 3, 4) and opinion on a certain policy (disagree, neutral, agree), and the distance between successive categories is unknown. On the other hand, nominal categorical variables have no order between categories, such as brand of a certain product and nationality. Binary variables are treated as nominal or ordinal categorical variables. The type of categorical variable of interest, i.e., nominal or ordinal, also determines the analysis method to be used. The method is appropriate for the analysis of nominal categorical contingency tables if the results are invariant to arbitrary permutations of the categories. The method for nominal categorical variables can also be used for ordinal categorical variables unless it does not use information on the order between categories. On the other hand, the method for ordinal categorical variables is inappropriate for the analysis of nominal categorical variables since it presents result that are not invariant to arbitrary permutations of categories.

One of the interests in the analysis of $r \times c$ contingency tables is the independence (non-association) between the two variables. The Pearson's chi-square statistic (Pearson, 1900) is often used for this purpose. There are also the log-likelihood ratio statistic (Wilks, 1935), Freeman-Tukey statistic (Freeman and Tukey, 1950), Neyman's modified chi-square statistic (Neyman, 1949), modified log-likelihood ratio statistic (Kullback, 1959), and the family of power-divergence statistics (Cressie and Read, 1984) that contain them as special cases. For theoretical properties and other details on these statistics, see Read and Cressie (2012). See also Pardo (2018) for the family of ϕ-divergence (f-divergence) statistics containing the family of power-divergence statistics as a special case. These statistics are used for testing independence under large samples asymptotically following a chi-square distribution with $(r-1)(c-1)$ degrees of freedom.

In the analysis of two-way contingency tables, besides the independence between two variables, the association between them is of interest. The test using the above mentioned statistic does not reveal the association between the variables, and when we are interested in the association, other methods of analysis are used. As one of them, there have been numerous studies on indexes measuring the strength of the association (the degree of departure from independence) between variables in two-way contingency tables. One of the well-known indexes of the association is the Pearson's phi-squared (Pearson, 1904), which is defined based on the Pearson's chi-square statistic. Other well-known indexes include the Tschuprow's coefficient (Tschuprow, 1939) and the Cramér's coefficient (Cramér, 1946). Tomizawa et al. (2004) proposed a family of indexes for the association including these indexes via the power-divergence. Urasaki et al. (2023) also proposed a family of indexes for the association via the f-divergence containing the power-divergence. The most popular index of the association may be the odds ratio. In the analysis of 2×2 contingency tables, the odds ratio is highly used in medicine and other fields and of great importance. The "local" odds ratio is defined for a subtable formed by four cells in adjacent rows and adjacent columns of a $r \times c$ contingency table, and is a index of "local" association. All local odds ratios in the subtables can be used to represent the association of the contingency table (Goodman, 1969, 1979b). Models of uniform association, row (or column) effect association, row by column effect association, and linear by linear association are the extended association model via the odds ratios (Goodman, 1979b, 1985, 1986; Agresti, 1983b; Liu and Agresti, 2005). Other analytical methods for the association between variables include the correspondence analysis to visualize the association, see Greenacre (1984) and Beh and Lombardo (2014) for more details.

The association in two-way contingency tables described so far is referred to as the "symmetric" association, which is characterized by the fact that categorical variables are not distinguished into the predictor and the response variable. In two-way contingency tables formed by the predictor and response variable, we are more interested in the "asymmetric" association than in the "symmetric" association (D'Ambra and Lauro, 1992; Beh et al., 2007; Beh and Lombardo, 2014). That is, the predictability of the response variable given the predictor is of interest to us. The asymmetric association is either "directional" or "nondirectional" (Wei and Kim, 2017; Wei et al., 2022). The directional association is when one categorical variable is the response variable and the others are predictors, and the nondirectional association is when each variable affects each other but in different degrees. One of the analyses for the asymmetric association is the use of the index of the proportional reduction in variation (PRV) from the marginal distribution to the conditional distribution of the response variable (Agresti, 2010). The Goodman-Kruskal tau index (Goodman and Kruskal, 1954) and the Theil's uncertainty coefficient (Theil, 1970) are PRV indexes defined with the Gini concentration (Gini, 1912) and the Shannon entropy (Shannon, 1948), respectively, as the variation measure of the response variable. Tomizawa et al. (1997) proposed a PRV index via the diversity index (Patil and Taillie, 1982) including the Gini concentration and the Shannon entropy. Momozaki et al. (2023) proposed a PRV index via a family of functions (termed the f-function in the literature) including the diversity index. Moreover to the PRV index, other indexes for the asymmetric association have been studied in recent years, such as the functional chi-square statistics (Zhang and Song, 2013) and the subcopula-based index (Wei and Kim, 2017), for example. The subcopula-based index of the asymmetric association is the index based on the subcopula-based regression, Wei and Kim (2017) derived its theoretical properties and Wei et al. (2022) also proposed a Bayesian inference of the index. In the analysis of the asymmetric association in contingency tables, there are other methods such as the non-symmetrical correspondence analysis (D'Ambra and Lauro, 1992) than using above mentioned indexes. For details of the non-symmetrical correspondence analysis and other analysis methods for the asymmetric association, see Beh and Lombardo (2014).

As mentioned above, we are interested in the association between categorical variables in the analysis of two-way contingency tables, but this interest changes in "square" contingency tables formed by commensurable, i.e., categorical variables with the same classification of row and column variables. Square contingency tables are often presented in various fields such as medicine, social sciences, psychology, education, sports, and so on, especially in social mobility surveys and panel surveys in which data are repeatedly observed at different points in time. In square contingency tables obtained in such surveys, there is often the association between the variables. Moreover, since the observations are concentrated in the cells on the main diagonal, more attention is paid to the off-diagonal cells. Therefore, in the analysis of square contingency tables, there have been extensive studies about models of symmetry and marginal homogeneity. Consider an $r \times r$ square contingency table $\boldsymbol{N}$ where the (i, j)-th cell observation is denoted by n_{ij} for $i, j = 1, 2, \ldots, r$ and its probability distribution $\boldsymbol{\Pi}$ where π_{ij} denotes the probability that an observation falls into the (i, j)-th cell. The first test for the hypothesis $\pi_{1\cdot} = \pi_{\cdot 1} \Leftrightarrow \pi_{12} = \pi_{21}$ where $\pi_{i\cdot} = \sum_{t=1}^{2} \pi_{it}$ and $\pi_{\cdot i} = \sum_{s=1}^{2} \pi_{si}$ in 2×2 square contingency tables was proposed by

McNemar (1947), and then Bowker (1948) proposed a test for the symmetry model

$$\pi_{ij} = \pi_{ji} \quad \text{for } i < j$$

as a generalization of McNemar (1947)'s test for $r \times r$ square contingency tables. When the symmetry model holds, $\pi_{i\cdot} = \pi_{\cdot i}$, that is, the marginal distributions of categorical variables are equivalent, this is referred to as the marginal homogeneity model. Note that even if the marginal homogeneity model holds, it does not mean that the symmetry model holds. Stuart (1955) proposed a test for the marginal homogeneity model. Models of symmetry and marginal homogeneity are linked through the quasi-symmetry model originally introduced by Caussinus (1965). The quasi-symmetry model implies that the odds ratios in the square contingency table are symmetric with respect to the main diagonal (Goodman, 1979b). The quasi-symmetry model has been studied extensively since then. The reader is referred to Tahata (2022) for a review of the history of the quasi-symmetry model, related studies, and recent progress. Since the quasi-symmetry model deals with categorical variables as nominal, when the variables are ordinal, other models are considered, such as the models of conditional symmetry (McCullagh, 1978, known also as triangular symmetry in Goodman, 1979a) and diagonals-parameter symmetry (Goodman, 1979a). Many other models for square contingency tables of nominal and ordinal categorical variables have been studied; see Tahata and Tomizawa (2014) and Tahata (2020) for more details.

Although the model of symmetry is parsimonious and has good interpretability, there are relatively few square contingency tables with the probability structure of the symmetry model practically common. Hence, there has been a great deal of research on the above mentioned less restrictive models and their goodness-of-fit tests as alternatives to the symmetry model. When the symmetry model does not hold, a detailed analysis is conducted through the decomposition of the model in order to investigate the cause of the failure (Tomizawa and Tahata, 2007). It is also of interest to measure the degree of departure from the symmetry model in certain directions. For example, for nominal square contingency tables Becker (1990) treated the quasi-symmetry model as a model that measures the degree of departure from the symmetry model in the direction of marginal inhomogeneity, and introduced parameters that by reparameterizing it represent the degree. For ordinal square contingency tables, Goodman (1979a) showed that the models of conditional symmetry and diagonals-parameter symmetry can be interpreted as measuring the degree of departure from the symmetry model in certain directions.

Just as there are measures of the degree of departure from independence as described above, there have been many studies on indexes of the degree of departure from the symmetry model in the analysis of square contingency tables. Tomizawa (1994) proposed two indexes to measure the degree of departure from the symmetry model in nominal square contingency tables. One of them is constructed using the Kullback-Leibler divergence (or the weighted average of the conditional Shannon entropy) and the other one using the Pearson's chi-square type discrepancy (or the weighted average of conditional Gauss discrepancy). Tomizawa et al. (1998) proposed an index for the symmetry model using the power-divergence including those divergences (or the weighted average of the diversity index including the Shannon entropy and the Gauss discrepancy). Tomizawa et al. (2001) proposed an index to measure the degree of departure from the symmetry model in ordinal square contingency tables based on the power-divergence or the diversity index by using certain cumulative probabilities defined only for ordinal categorical variables. Iki and Tomizawa (2018) proposed other index for the symmetry model in ordinal square contingency tables using different definitions of cumulative probabilities. There are also indexes that measure the degree of departure from models other than the symmetry model. Tomizawa (1995) proposed two indexes, one based on the Kullback-Leibler divergence (or the weighted average of the conditional Shannon entropy) and the other one using the Pearson's chi-square type discrepancy (or the weighted average of conditional Gauss discrepancy) to measure the degree of departure from marginal homogeneity in nominal square contingency tables. Tomizawa and Makii (2001) proposed an index for marginal homogeneity based on the power-divergence (or the weighted average of the diversity index). In ordinal square contingency tables, indexes for marginal homogeneity have been proposed for Tomizawa et al. (2003), Tahata et al. (2008), Iki et al. (2012), and Nakagawa et al. (2020), using various definitions of cumulative probabilities. Tomizawa (1992) and Tahata et al. (2004) proposed indexes to measure the degree of departure from the quasi-symmetry model, Tomizawa and Saitoh (1999b), Tomizawa and Saitoh (1999a), and Saigusa et al. (2021) proposed indexes to measure the degree of departure from the conditional symmetry model, Tomizawa and Kato (2003), Tomizawa et al. (2005), Miyamoto et al. (2010), and Kurakami et al. (2013) proposed indexes to measure the degree of departure from the diagonals-parameter symmetry model.

The indexes for models of symmetry and marginal homogeneity described above may represent two probability structures at the maximum degree of departure, and they cannot be distinguished. As the Pearson's correlation coefficient indicates positive correlation when the value is positive and negative correlation when the value is negative, it is sometimes desirable to distinguish the probability structures of the maximum degree of departure in the index for symmetry and marginal homogeneity as well. Tahata et al. (2009) proposed an index that can distinguish between the two probability structures, complete-upper-asymmetry and complete-lower-asymmetry, represented by the maximum value of the indexes of Tomizawa (1994), Tomizawa et al. (1998), and Tomizawa et al. (2001). Tahata et al. (2010) also proposed an index that can distinguish between complete-upper-asymmetry and complete-lower-asymmetry using the cumulative probabilities. Yamamoto et al. (2011) and Iki et al. (2019) proposed an index that can distinguish between two marginal inhomogeneity represented by the maximum value of the indexes of Tomizawa et al. (2003) and Iki et al. (2012), respectively.

While the indexes for models of symmetry and asymmetry are suitable for representing the degree of departure from the model of interest, vector type indexes that use these indexes as elements have been studied in recent years. Vector type indexes inherit the properties of the indexes that include them as elements but also provide new capabilities to interests that could not be addressed in the analysis of square contingency tables. Ando et al. (2017) developed a vector type index that can visually represent which direction the probability structure is deviated from the symmetry model, complete-upper-asymmetry or complete-lower-asymmetry, and the degree of departure, using Tomizawa et al. (2001) and Tahata et al. (2010). Ando (2019), Ando et al. (2019), and Ando et al. (2021) proposed vector type indexes that can visually represent in which direction the probability structure deviates from the marginal homogeneity to the two marginal inhomogeneities and the degree of departure. Ando (2021a) proposed a vector type index that can simultaneously visually represent the degree of departure from the quasi-symmetry model and its direction. Other vector type indicators are proposed by Ando et al. (2019), Ando (2020), Ando et al. (2021), Ando (2021b), Ando (2021c), and Ando (2022).

1.2 Overview of the thesis

This thesis proposes two indexes for symmetry. The first one is a two-dimensional vector type index that measures the degree of departure from the symmetry model in nominal square contingency tables. The vector type indexes can be combined with two or more other indexes to provide additional usefulness while preserving the properties of those indexes. Tomizawa et al. (1998) proposed an index to represent the degree of departure from the symmetry model to a defined maximum asymmetry structure, where both edges of its range $[0, 1]$ are trivial to interpret (0 and 1 indicate symmetry and its maximum asymmetry structure, respectively), but not for values in between. To address this issue, we propose a two-dimensional vector type index that can be combined with two kinds of indexes to give useful interpretations of the degree of departure from the symmetry model, while retaining the interpretability of the symmetry model and the maximum asymmetry structure in Tomizawa et al. (1998).

The second is an (one-dimensional) index that can simultaneously represent the degree and direction of asymmetry in ordinal square contingency tables. Iki and Tomizawa (2018) defined two types of asymmetry structures as probability structures of the maximum departure from the symmetry model. However, the index proposed by Iki and Tomizawa (2018) cannot distinguish in which direction the degree of departure is. For the purposes of applications, it may be required to interpret the results of distinguishing two asymmetry structures. Therefore, we propose an index that can distinguish these asymmetries and represent the degree of departure in each direction by using the arc-cosine function.

This thesis is organized as follows. Chapter 1 introduces the background and previous works on the analysis of contingency tables, especially two-way contingency tables and square contingency tables, and provides an overview of each chapter in this thesis.

The content in Chapter 2 comes from Momozaki et al. (2021). In the analysis of two-way contingency tables, the degree of departure from independence is measured using measures of association between row and column variables (e.g., Yule's coefficients of association and of colligation, Cramér's coefficient, and Goodman and Kruskal's coefficient). On the other hand, in the analysis of square contingency tables with the same row and column classifications, we are interested in measuring the degree of departure from symmetry rather than independence. Over past years, many studies have proposed various types of indexes based on their power divergence (or diversity index) to represent the degree of departure from symmetry. This study proposes a two-dimensional index to measure the degree of departure from symmetry in terms of the log odds of each symmetric cell with respect to the main diagonal of the table. By measuring the degree of departure from symmetry in terms of the log odds of each symmetric cell, the analysis results are easier to interpret than existing indexes. Numerical experiments show the utility of the proposed two-dimensional index. We show the usefulness of the proposed two-dimensional index by using real data.

The content in Chapter 3 comes from Momozaki et al. (2023). For square contingency tables with ordered categories, an index based on Cressie and Read (1984)'s power-divergence (or Patil and Taillie (1982)'s diversity-index) has been proposed in order to measure the degree of departure from symmetry. Although there are two types of maximum asymmetry (i.e., whether (1) all the observations concentrate in the top-right cell in the table, or (2) they concentrate in the bottom-left cell) represented by the maximum value of Iki and Tomizawa (2018)'s index, the existing index cannot distinguish the two directions of maximum asymmetry. This paper proposes a directional index based on the arc-cosine function in order to simultaneously represent the degree and directionality of asymmetry. The proposed index would be useful for comparing degrees of asymmetry for several square contingency tables. Numerical examples show the utility of the proposed index using some datasets. We evaluate the usefulness of the proposed index by applying it to real data of the clinical study. The proposed index provides analysis results that are easier to interpret than the existing index.

Chapter 4 provides some discussion and concluding remarks.

Chapter 2

Two-Dimensional Index of Departure from the Symmetry Model for Square Contingency Tables with Nominal Categories

2.1 Introduction

For two-way contingency tables, an analysis is generally performed to see whether the independence between the row and column classifications holds. Meanwhile, for the analysis of square contingency tables with the same row and column classifications, there are many issues related to symmetry rather than independence. This is because, in square contingency tables, there is a strong association between the row and column classifications. Consider an $r \times r$ square contingency table. Let π_{ij} denote the probability that an observation will fall in the ith row and jth column of the table ($i, j = 1, \ldots, r$). Bowker (1948) proposed the symmetry model defined by

$$\pi_{ij} = \pi_{ji} \quad \text{for all } i < j.$$

This symmetry model, however, often does not hold when applied to real data. When the symmetry model fits real data poorly, other symmetry (e.g., quasi symmetry (Caussinus, 1965) and partial symmetry (Saigusa et al., 2016)) models, or asymmetry (e.g., conditional symmetry (McCullagh, 1978), linear diagonals-parameter symmetry (Agresti, 1983a), and conditional difference asymmetry (Tomizawa et al., 2004)) models are applied to these real data.

In the analysis of two-way contingency tables, the degree of departure from independence is assessed by using measures of association between the row and column variables. Measures of association include, for example, Yule's coefficients of association and of colligation (Yule, 1900, 1912), Cramér's coefficient (Cramér, 1946), and Goodman and Kruskal's coefficient (Goodman and Kruskal, 1954). For details, see Bishop et al. (2007) and Agresti (2013).

In addition, in the analysis of square contingency tables with the same row and column classifications, we are interested in measuring the degree of departure from the symmetry model. Over the past few years, many studies have proposed indexes to represent the degree of departure from the symmetry model. Tomizawa et al. (1998) and Tomizawa et al. (2001) proposed the various types of indexes based on power divergence (or the diversity index) to represent the degree of departure from the symmetry model. Ando et al. (2019) and Ando (2020) proposed two-dimensional indexes to represent the degree of departure from symmetry. A two-dimensional index allows us to visually compare the degrees of departure from symmetry in multiple data sets by using confidence regions and allows us to easily interpret the results of data analysis.

This study proposes a two-dimensional index to measure the degree of departure from the symmetry model in terms of the log odds of each symmetric cell with respect to the main diagonal of the table. By measuring the degree of departure from symmetry in terms of the log odds of each symmetric cell, the analysis results are easier to interpret than existing indexes. This paper is organized as follows. Section 2.2 introduces the proposed index and shows the properties of the proposed index. Section 2.3 derives the confidence region of the proposed index. Section 2.4 shows the usefulness of the proposed index by applying it to real data. Section 2.5 discusses properties of the proposed index by using several asymmetry models. Section 2.6 describes the concluding remarks.

2.2 Two-Dimensional Index and Its Properties

This section proposes a two-dimensional index to measure the degree of departure from the symmetry model in terms of the log odds of each symmetric cell with respect to the main diagonal of the table. By using the weighted geometric mean indexes of the diversity index as the elements of the proposed two-dimensional index, the proposed two-dimensional index has more useful properties than the index proposed by Tomizawa et al. (1998), which measures the degree of departure from the symmetry model. Section 2.2.1 describes two univariate indexes of weighted geometric mean type that are elements of the proposed two-dimensional index and their characteristics. Section 2.2.2 shows the relationship between the elements of the proposed two-dimensional index and describes the properties of the proposed two-dimensional index.

2.2.1 Univariate Index of Weighted Geometric Mean Type

For an $r \times r$ square contingency table with nominal categories, Saigusa et al. (2016) proposed the weighted geometric mean of the diversity index as follows. Assuming that $\pi_{ij} + \pi_{ji} > 0$ for all $i < j$, for $\lambda > -1$,

$$\tau^{(\lambda)} = \prod_{i<j} \left[\phi_{ij}^{(\lambda)}\right]^{(\pi_{ij}^* + \pi_{ji}^*)},$$

where

$$\phi_{ij}^{(\lambda)} = 1 - \frac{\lambda 2^\lambda}{2^\lambda - 1} H_{ij}^{(\lambda)}, \quad H_{ij}^{(\lambda)} = \frac{1}{\lambda}\left[1 - (\pi_{ij}^c)^{\lambda+1} - (\pi_{ji}^c)^{\lambda+1}\right],$$

$$\pi_{ij}^c = \frac{\pi_{ij}}{\pi_{ij} + \pi_{ji}}, \quad \pi_{ij}^* = \frac{\pi_{ij}}{\delta}, \quad \delta = \sum_{i \neq j} \pi_{ij}.$$

The values at $\lambda = 0$ are taken to be the continuous limit as $\lambda \to 0$. Note that $H_{ij}^{(\lambda)}$ is Patil and Taillie (1982)'s diversity index of degree λ including the Shannon entropy ($\lambda = 0$), and the real number λ is chosen by the user. The index $\tau^{(\lambda)}$ has the following characteristics: (i) $0 \leq \tau^{(\lambda)} \leq 1$; (ii) $\tau^{(\lambda)} = 0$ if and only if $\pi_{ij} = \pi_{ji}$ for at least one $i < j$; and (iii) $\tau^{(\lambda)} = 1$ if and only if the degree of asymmetry is maximum in the sense that $\pi_{ij} = 0$ (then $\pi_{ji} > 0$) or $\pi_{ji} = 0$ (then $\pi_{ij} > 0$) for all $i < j$.

For an $r \times r$ square contingency table with nominal categories, we define a weighted geometric mean univariate index of the diversity index, which has a different formula from index $\tau^{(\lambda)}$, as follows. Assuming that $\pi_{ij} + \pi_{ji} > 0$ for all $i < j$, for $\lambda > -1$,

$$\Phi^{(\lambda)} = 1 - \prod_{i<j} \left[1 - \phi_{ij}^{(\lambda)}\right]^{(\pi_{ij}^* + \pi_{ji}^*)}.$$

The values at $\lambda = 0$ are taken to be the continuous limit as $\lambda \to 0$. Note that

$H_{ij}^{(\lambda)}$ is Patil and Taillie (1982)'s diversity index of degree λ including the Shannon entropy ($\lambda = 0$), and the real number λ is chosen by the user. The index $\Phi^{(\lambda)}$ has the following characteristics: (i) $0 \leq \Phi^{(\lambda)} \leq 1$; (ii) $\Phi^{(\lambda)} = 0$ if and only if the symmetry model holds; and (iii) $\Phi^{(\lambda)} = 1$ if and only if $\pi_{ij} = 0$ (then $\pi_{ji} > 0$) or $\pi_{ji} = 0$ (then $\pi_{ij} > 0$) for at least one $i < j$.

2.2.2 Two-Dimensional Index of Symmetry

Assuming that $\pi_{ij} + \pi_{ji} > 0$ for all $i < j$, we propose a two-dimensional index defined by

$$\boldsymbol{\Lambda}^{(\lambda)} = (\Phi^{(\lambda)}, \tau^{(\lambda)})^\top \quad \text{for } \lambda > -1,$$

where $\Phi^{(\lambda)}$ and $\tau^{(\lambda)}$ are described in section 2.2.1 and $\boldsymbol{a}^\top$ is the transpose of $\boldsymbol{a}$. The values at $\lambda = 0$ are taken to be the continuous limit as $\lambda \to 0$. Note that $H_{ij}^{(\lambda)}$ is the diversity index of degree λ in Patil and Taillie (1982) including the Shannon entropy ($\lambda = 0$), where λ is a real number chosen by the user.

By noting that the indexes $\Phi^{(\lambda)}$ and $\tau^{(\lambda)}$ are expressed using the weighted geometric mean of the diversity index, the following theorem concerning the relationship between $\Phi^{(\lambda)}$ and $\tau^{(\lambda)}$ holds.

Theorem 2.1. *The inequality $\tau^{(\lambda)} \leq \Phi^{(\lambda)}$ holds, and that equality holds if, and only if, the conditional difference asymmetry model defined by Tomizawa et al. (2004) as*

$$\pi_{ij} = e^{\Delta_{ij}} \pi_{ji} \quad \textit{for all } i < j,$$

where $|\Delta_{ij}| = \Delta$ and $\{\Delta_{ij}\}$ are unspecified real-valued parameters, holds.

The proof of this theorem is given in Appendix A.1.

Based on the above properties of the elements $\Phi^{(\lambda)}$ and $\tau^{(\lambda)}$ of the proposed two-dimensional index $\mathbf{\Lambda}^{(\lambda)}$, $\mathbf{\Lambda}^{(\lambda)}$ has the following characteristics:

(1) The value of $\mathbf{\Lambda}^{(\lambda)}$ lies on the sides and inside the triangle at vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$;

(2) $\mathbf{\Lambda}^{(\lambda)} = (0, 0)^\top$ if, and only if, the symmetry model holds;

(3) $\mathbf{\Lambda}^{(\lambda)} = (1, 1)^\top$ if, and only if, the degree of asymmetry is maximum in the sense that $\pi_{ij} = 0$ (then $\pi_{ji} > 0$) or $\pi_{ji} = 0$ (then $\pi_{ij} > 0$) for all $i < j$;

(4) $\mathbf{\Lambda}^{(\lambda)} = (t, t)^\top$, where t is a constant for $0 \leq t \leq 1$ if, and only if, the conditional difference asymmetry model holds.

The conditional difference asymmetry model holds if, and only if, the absolute value of the log odds of each symmetric cell with respect to the main diagonal of the table can be expressed by the constant Δ. Namely, the proposed two-dimensional index can represent the degree of departure from the symmetry model in terms of the log odds $\log(\pi_{ij}/\pi_{ji})$.

Remark 2.2. Similar to the index proposed by Tomizawa et al. (1998) (see Appendix A.2), the proposed two-dimensional index represents the degree of departure from the symmetry model, and $\pi_{ij} = 0$ (then $\pi_{ji} > 0$) or $\pi_{ji} = 0$ (then $\pi_{ij} > 0$) for all $i < j$ when the degree of asymmetry is maximum. However, the proposed two-dimensional index represents it in terms of the log odds of each symmetric cell with respect to the main diagonal of the table, which makes the analysis results easier to interpret than the Tomizawa et al. (1998)'s index. Section 2.4 shows the usefulness of the proposed two-dimensional index by using an example. This may be one of the advantages of the proposed two-dimensional index, which cannot be represented only by using the indexes $\Phi^{(\lambda)}$ and $\tau^{(\lambda)}$, which are elements of the proposed two-dimensional index.

2.3 Approximate Confidence Region for the Proposed Index

Let n_{ij} denote the observed frequency in the ith row and jth column of the table $(i, j = 1, \ldots, r)$. Assume that a multinomial distribution applies to the $r \times r$ table. The sample proportions of $\{\pi_{ij}; i, j = 1, \ldots, r\}$ are $\{p_{ij} = n_{ij}/n\}$ with $n = \sum_{i,j} n_{ij}$. The estimator of the proposed index $\mathbf{\Lambda}^{(\lambda)}$, $\widehat{\mathbf{\Lambda}}^{(\lambda)}$ is provided by replacing $\{\pi_{ij}\}$ with $\{p_{ij}\}$.

Let $\boldsymbol{p}$ and $\boldsymbol{\pi}$ be the $r^2 \times 1$ vectors

$$\boldsymbol{p} = (p_{11}, p_{12}, \ldots, p_{1r}, p_{21}, \ldots, p_{rr})^\top \quad \text{and} \quad \boldsymbol{\pi} = (\pi_{11}, \pi_{12}, \ldots, \pi_{1r}, \pi_{21}, \ldots, \pi_{rr})^\top,$$

respectively. Then $\sqrt{n}(\boldsymbol{p} - \boldsymbol{\pi})$ asymptotically (as $n \to \infty$) has a normal distribution with the zero mean vector and the covariance matrix $\mathrm{diag}(\boldsymbol{\pi}) - \boldsymbol{\pi}\boldsymbol{\pi}^\top$, where $\mathrm{diag}\boldsymbol{\pi}$ is a diagonal matrix with the elements of $\boldsymbol{\pi}$ on the main diagonal. Therefore, by the delta method (see, e.g., Agresti, 2013), $\sqrt{n}(\widehat{\mathbf{\Lambda}}^{(\lambda)} - \mathbf{\Lambda}^{(\lambda)})$ asymptotically (as $n \to \infty$) has a bivariate normal distribution with the mean zero vector and the covariance matrix

$$\begin{aligned}
\mathbf{\Sigma}[\mathbf{\Lambda}^{(\lambda)}] &= \left(\frac{\partial \mathbf{\Lambda}^{(\lambda)}}{\partial \boldsymbol{\pi}}\right)^\top \left(\mathrm{diag}(\boldsymbol{\pi}) - \boldsymbol{\pi}\boldsymbol{\pi}^\top\right) \left(\frac{\partial \mathbf{\Lambda}^{(\lambda)}}{\partial \boldsymbol{\pi}}\right) \\
&= \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix},
\end{aligned}$$

where

$$\sigma_{11} = \sum_{i,j} \pi_{ij} \left(\frac{\partial \Phi^{(\lambda)}}{\partial \pi_{ij}} \right)^2 - \left(\sum_{i,j} \pi_{ij} \frac{\partial \Phi^{(\lambda)}}{\partial \pi_{ij}} \right)^2,$$

$$\sigma_{12} = \sum_{i,j} \pi_{ij} \frac{\partial \Phi^{(\lambda)}}{\partial \pi_{ij}} \frac{\partial \tau^{(\lambda)}}{\partial \pi_{ij}} - \left(\sum_{i,j} \pi_{ij} \frac{\partial \Phi^{(\lambda)}}{\partial \pi_{ij}} \right) \left(\sum_{i,j} \pi_{ij} \frac{\partial \tau^{(\lambda)}}{\partial \pi_{ij}} \right)$$

$$= \sigma_{21},$$

$$\sigma_{22} = \sum_{i,j} \pi_{ij} \left(\frac{\partial \tau^{(\lambda)}}{\partial \pi_{ij}} \right)^2 - \left(\sum_{i,j} \pi_{ij} \frac{\partial \tau^{(\lambda)}}{\partial \pi_{ij}} \right)^2$$

with

$$\frac{\partial \Phi^{(\lambda)}}{\partial \pi_{ij}} = \begin{cases} \dfrac{1-\Phi^{(\lambda)}}{\delta} \left[\log \dfrac{1-\Phi^{(\lambda)}}{1-\phi_{ij}^{(\lambda)}} + \dfrac{\lambda+1}{1-2^{-\lambda}} \dfrac{\pi_{ji}^c}{1-\phi_{ij}^{(\lambda)}} \left\{ (\pi_{ij}^c)^\lambda - (\pi_{ji}^c)^\lambda \right\} \right] & \text{for } \lambda \neq 0, \\ \dfrac{1-\Phi^{(0)}}{\delta} \left(\log \dfrac{1-\Phi^{(0)}}{1-\phi_{ij}^{(0)}} + \dfrac{1}{\log 2} \dfrac{\pi_{ji}^c}{1-\phi_{ij}^{(0)}} \log \dfrac{\pi_{ij}^c}{\pi_{ji}^c} \right) & \text{for } \lambda = 0, \end{cases}$$

$$\frac{\partial \tau^{(\lambda)}}{\partial \pi_{ij}} = \begin{cases} \dfrac{\tau^{(\lambda)}}{\delta} \left[\log \dfrac{\phi_{ij}^{(\lambda)}}{\tau^{(\lambda)}} + \dfrac{\lambda+1}{1-2^{-\lambda}} \dfrac{\pi_{ji}^c}{\phi_{ij}^{(\lambda)}} \left\{ (\pi_{ij}^c)^\lambda - (\pi_{ji}^c)^\lambda \right\} \right] & \text{for } \lambda \neq 0, \\ \dfrac{\tau^{(0)}}{\delta} \left(\log \dfrac{\phi_{ij}^{(0)}}{\tau^{(0)}} + \dfrac{1}{\log 2} \dfrac{\pi_{ji}^c}{\phi_{ij}^{(0)}} \log \dfrac{\pi_{ij}^c}{\pi_{ji}^c} \right) & \text{for } \lambda = 0. \end{cases}$$

Let $\widehat{\mathbf{\Sigma}}[\mathbf{\Lambda}^{(\lambda)}]$ denote $\mathbf{\Sigma}[\mathbf{\Lambda}^{(\lambda)}]$ with $\{\pi_{ij}\}$ replaced by $\{p_{ij}\}$. The approximate $100(1-\alpha)\%$ confidence region of $\mathbf{\Lambda}^{(\lambda)}$ is given as

$$n(\widehat{\mathbf{\Lambda}}^{(\lambda)} - \mathbf{\Lambda}^{(\lambda)})^\top \widehat{\mathbf{\Sigma}}[\mathbf{\Lambda}^{(\lambda)}]^{-1} (\widehat{\mathbf{\Lambda}}^{(\lambda)} - \mathbf{\Lambda}^{(\lambda)}) \leq \chi^2_{(1-\alpha;2)},$$

where $\chi^2_{(1-\alpha;2)}$ is the $1-\alpha$ quantile of the chi-square distribution with two degrees of freedom. Note that the confidence region is computable when $0 < \Phi^{(\lambda)} < 1$ and

$0 < \tau^{(\lambda)} < 1$.

Since the delta method for multinomial distributions assumes asymptotic normality for the observed frequencies of each cell, the asymptotic normality of the index obtained by the delta method may be affected when the observed frequencies are small near the corners of the contingency table (see, e.g., Agresti, 2013, p.589).

2.4 Example

This section demonstrates the usefulness of the proposed two-dimensional index $\mathbf{\Lambda}^{(\lambda)}$ compared with the index of Tomizawa et al. (1998) (denoted by $\zeta^{(\lambda)}$; see Appendix A.2), which measures the degree of departure from the symmetry model by using the real data cited from Hashimoto (1999, 2003) (2.1).

These real data are the cross-classification of fathers' and sons' occupational status categories in Japan, which were examined in 1955 and 1995. Their status could be classified as (1) capitalist, (2) new middle, (3) working, (4) self-employed, and (5) farming.

TABLE 2.1: The cross-classification of father's and his son's occupational status categories in Japan which were examined in 1955 and 1995

Father's status	Son's status (1)	(2)	(3)	(4)	(5)	Total
(a) in 1955; source Hashimoto (1999)						
(1)	39	39	39	57	23	197
(2)	12	78	23	23	37	173
(3)	6	16	78	23	20	143
(4)	18	80	79	126	31	334
(5)	28	106	136	122	628	1020
Total	103	319	355	351	739	1867
(b) in 1995; source Hashimoto (2003)						
(1)	68	48	36	23	1	176
(2)	33	191	102	33	3	362
(3)	25	147	229	34	2	437
(4)	48	119	146	129	5	447
(5)	40	126	192	82	88	528
Total	214	631	705	301	99	1950

Table 2.2 represents estimates of indexes $\Phi^{(\lambda)}$ and $\tau^{(\lambda)}$, approximate standard errors for $\widehat{\Phi}^{(\lambda)}$ and $\widehat{\tau}^{(\lambda)}$, and approximate 95% confidence intervals for $\Phi^{(\lambda)}$ and $\tau^{(\lambda)}$ for the real data in Hashimoto (1999, 2003).

Note that from the delta method, the approximate confidence intervals for $\Phi^{(\lambda)}$ and $\tau^{(\lambda)}$ can be obtained by using the $(1,1)$ and $(2,2)$ components of the estimator of the covariance matrix $\widehat{\boldsymbol{\Sigma}}[\boldsymbol{\Lambda}^{(\lambda)}]$ of $\boldsymbol{\Lambda}^{(\lambda)}$, respectively. When $\lambda = 1$, the estimates of the two-dimensional index $\boldsymbol{\Lambda}^{(1)}$ are

$$\widehat{\boldsymbol{\Lambda}}^{(1)}_{1955} = \begin{pmatrix} 0.340 \\ 0.252 \end{pmatrix} \quad \text{and} \quad \widehat{\boldsymbol{\Lambda}}^{(1)}_{1995} = \begin{pmatrix} 0.656 \\ 0.222 \end{pmatrix},$$

respectively, and the estimates of $\boldsymbol{\Sigma}$ are

$$\widehat{\boldsymbol{\Sigma}}_{1955}[\boldsymbol{\Lambda}^{(1)}_{1955}] = \begin{pmatrix} 2.021 & 1.410 \\ 1.410 & 4.700 \end{pmatrix} \quad \text{and} \quad \widehat{\boldsymbol{\Sigma}}_{1995}[\boldsymbol{\Lambda}^{(1)}_{1995}] = \begin{pmatrix} 4.295 & 0.641 \\ 0.641 & 3.291 \end{pmatrix},$$

respectively. Figure 2.1 shows point estimates and approximate 95% confidence regions of the two-dimensional index $\boldsymbol{\Lambda}^{(1)}$ for the real data in Hashimoto (1999, 2003). The vertical and horizontal axes in Figure 2.1 represent the values of indexes $\Phi^{(\lambda)}$ and $\tau^{(\lambda)}$, respectively. Since these confidence regions do not overlap, it is inferred that these data have different probability structures. From Table 2.2 and Figure 2.1, we can see that the degree of departure from the symmetry model for the data in Hashimoto (2003) is lager than for the data in Hashimoto (1999). Additionally, the confidence region of $\boldsymbol{\Lambda}^{(1)}$ for the data in Hashimoto (1999) includes the line passing through the points $(0, 0)$ and $(1, 1)$, but the confidence region of $\boldsymbol{\Lambda}^{(1)}$ for the data in Hashimoto (2003) does not. Therefore, the probability structure of the data in Hashimoto (1999) may have conditional difference asymmetry, and we can see that the father's occupational status in Japan in 1955 has a greater influence on his son's status than in 1995.

On the other hand, even using the existing index $\zeta^{(\lambda)}$ of Tomizawa et al. (1998), which measures the degree of departure from the symmetry model, since confidence intervals of $\zeta^{(1)}$ for the real data in Hashimoto (1999, 2003) are $(0.265, 0.381)$ and $(0.399, 0.482)$, respectively, we can see the degree of departure from symmetry for the data in Hashimoto (2003) is larger than for the data in Hashimoto (1999). However, the existing index $\zeta^{(\lambda)}$ does not allow us to determine which the data in Hashimoto (1999, 2003) show that the father's occupational status has more influence on his son's status.

The proposed two-dimensional index $\mathbf{\Lambda}^{(\lambda)}$ is not only capable of representing the degree of departure from the symmetry model, but also can take into account the log odds of each symmetric cell with respect to the main diagonal of the table (i.e., the degree of departure from the conditional difference asymmetry model). It can, therefore, provide more interpretable analysis results, as above.

TABLE 2.2: Estimates of indexes $\Phi^{(\lambda)}$ and $\tau^{(\lambda)}$, approximate standard errors for $\widehat{\Phi}^{(\lambda)}$ and $\widehat{\tau}^{(\lambda)}$, and approximate 95% confidence intervals for $\Phi^{(\lambda)}$ and $\tau^{(\lambda)}$ for the real data in Hashimoto (1999, 2003)

	$\boldsymbol{\lambda}$	**Estimated Index**	**Standard Error**	**Confidence interval**
(a) For Hashimoto (1999)'s data				
$\Phi^{(\lambda)}$	1	0.340	0.033	$(0.276, 0.404)$
	0.5	0.316	0.031	$(0.254, 0.377)$
	0	0.262	0.027	(0.209, 0.316)
$\tau^{(\lambda)}$	1	0.252	0.050	$(0.154, 0.350)$
	0.5	0.233	0.047	$(0.142, 0.325)$
	0	0.193	0.039	$(0.117, 0.270)$
(b) For Hashimoto (2003)'s data				
$\Phi^{(\lambda)}$	1	0.656	0.047	$(0.564, 0.748)$
	0.5	0.631	0.049	$(0.536, 0.726)$
	0	0.558	0.049	(0.462, 0.655)
$\tau^{(\lambda)}$	1	0.222	0.041	$(0.142, 0.303)$
	0.5	0.209	0.039	$(0.133, 0.285)$
	0	0.179	0.034	$(0.113, 0.245)$

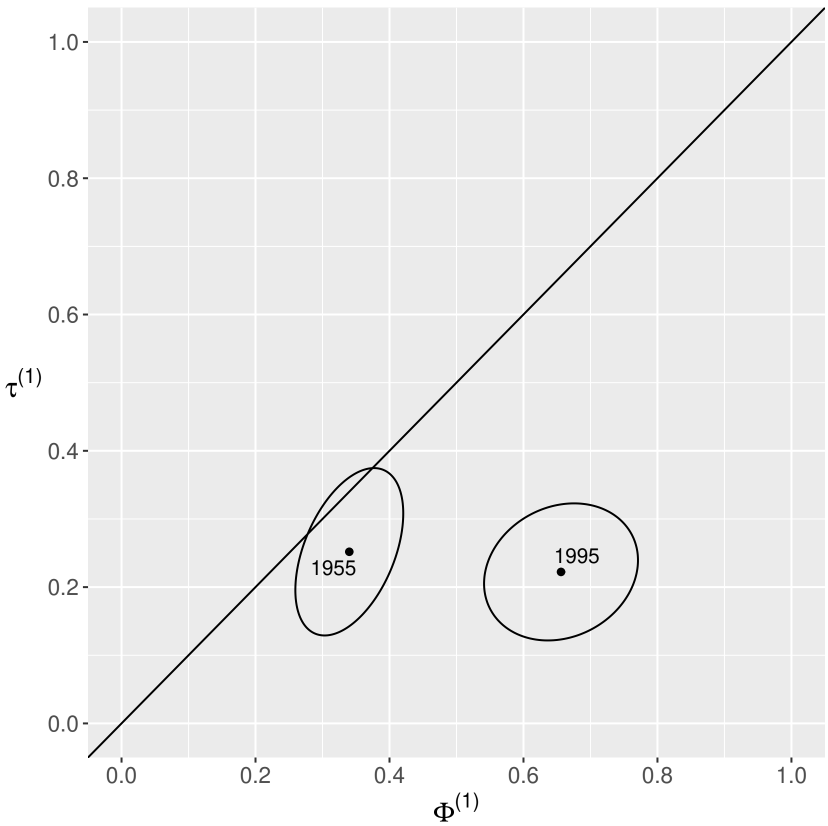

FIGURE 2.1: Point estimates and approximate 95% confidence regions of the two-dimensional index $\mathbf{\Lambda}^{(1)}$ for the real data in Hashimoto (1999, 2003)

2.5 Discussion

This section discusses properties of the proposed index $\mathbf{\Lambda}^{(\lambda)}$ for several asymmetry models. Consider the conditional symmetry model (McCullagh, 1978) and the linear diagonals-parameter symmetry model (Agresti, 1983a) as asymmetry models. The conditional symmetry model is defined by

$$\pi_{ij} = \gamma \pi_{ji} \quad \text{for all } i < j.$$

The linear diagonals-parameter symmetry model is defined by

$$\pi_{ij} = \theta^{j-i}\pi_{ji} \quad \text{for all } i < j.$$

Note that if the conditional symmetry model holds, then the conditional difference asymmetry model holds, but the reverse is not true. In contrast, if the linear diagonals-parameter symmetry model holds, the conditional difference asymmetry model does not always hold.

Figure 2.2 plots the values of the proposed index $\mathbf{\Lambda}^{(1)}$ for $\gamma = 1, 2, 3, 5, 10, 100, 1000$ in the conditional symmetry model. The vertical and horizontal axes in Figure 2.2 represent the values of indexes $\Phi^{(\lambda)}$ and $\tau^{(\lambda)}$, respectively. From Figure 2.2, as the value of γ increases, the value of $\mathbf{\Lambda}^{(1)}$ approaches $(1, 1)$ and lies on the straight line passing through $(0, 0)$ and $(1, 1)$. On the other hand, Figure 2.3 plots the values of the proposed index $\mathbf{\Lambda}^{(1)}$ for $\theta = 1, 2, 3, 5, 10, 100, 1000$ in the linear diagonals-parameter model. The vertical and horizontal axes in Figure 2.3 represent the values of indexes $\Phi^{(\lambda)}$ and $\tau^{(\lambda)}$, respectively. As can be seen in Figure 2.3, the value of $\mathbf{\Lambda}^{(1)}$ approaches $(1, 1)$ as the value of θ increases, but the value of $\mathbf{\Lambda}^{(1)}$ does not lie on the straight line passing through $(0, 0)$ and $(1, 1)$ for all values of θ. Similar results are observed for another value of λ, although the details are omitted. This difference is due to the fact that the proposed index $\mathbf{\Lambda}^{(\lambda)}$ measures the degree of departure from symmetry in terms of the log odds of each symmetric cell with respect to the main diagonal in the table.

Therefore, the proposed index $\mathbf{\Lambda}^{(\lambda)}$ is suitable for measuring the degree of departure from symmetry and can visually distinguish the asymmetry models as described above.

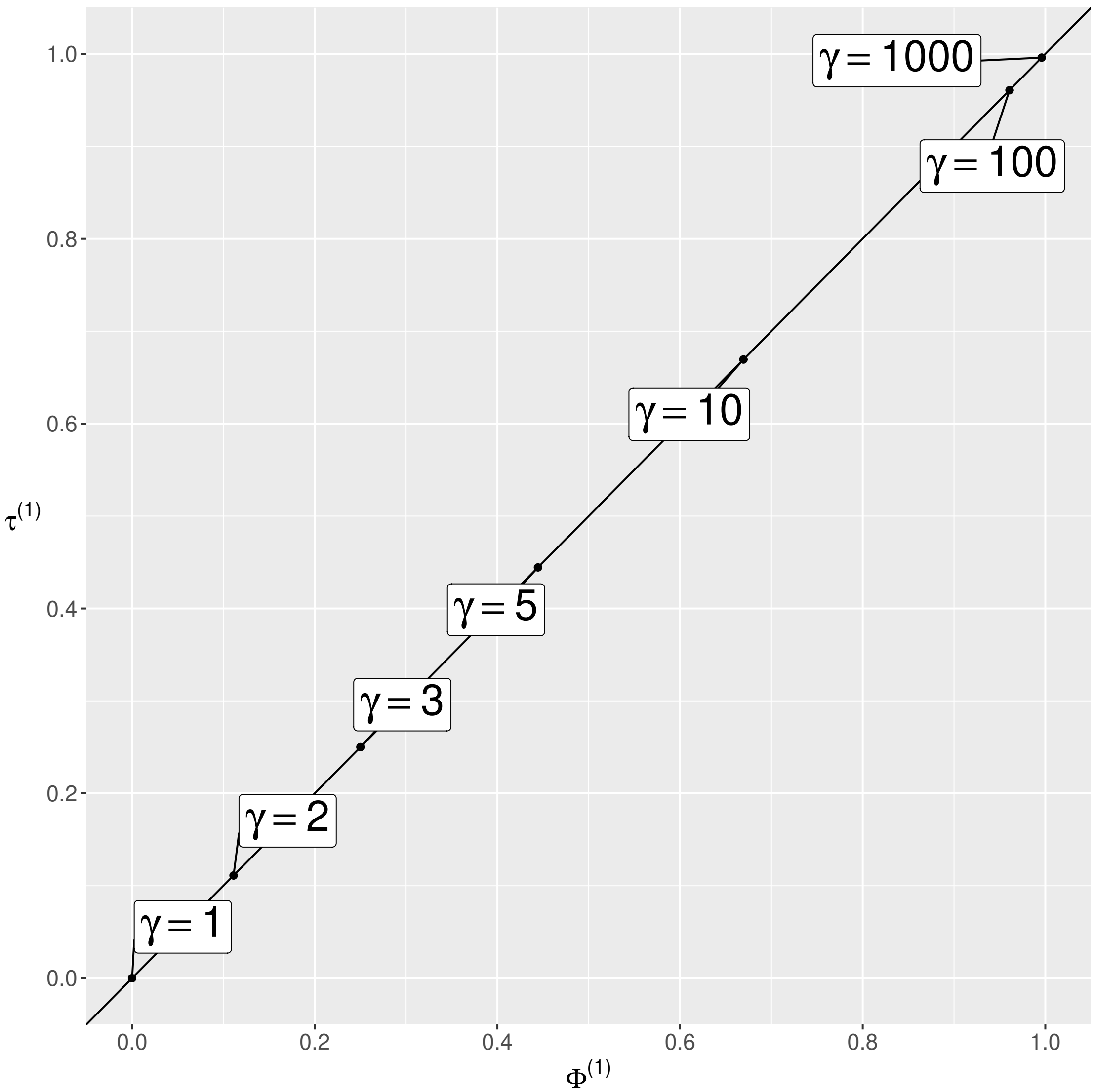

FIGURE 2.2: The values of the proposed index $\mathbf{\Lambda}^{(1)}$ under the conditional symmetry model with parameter γ

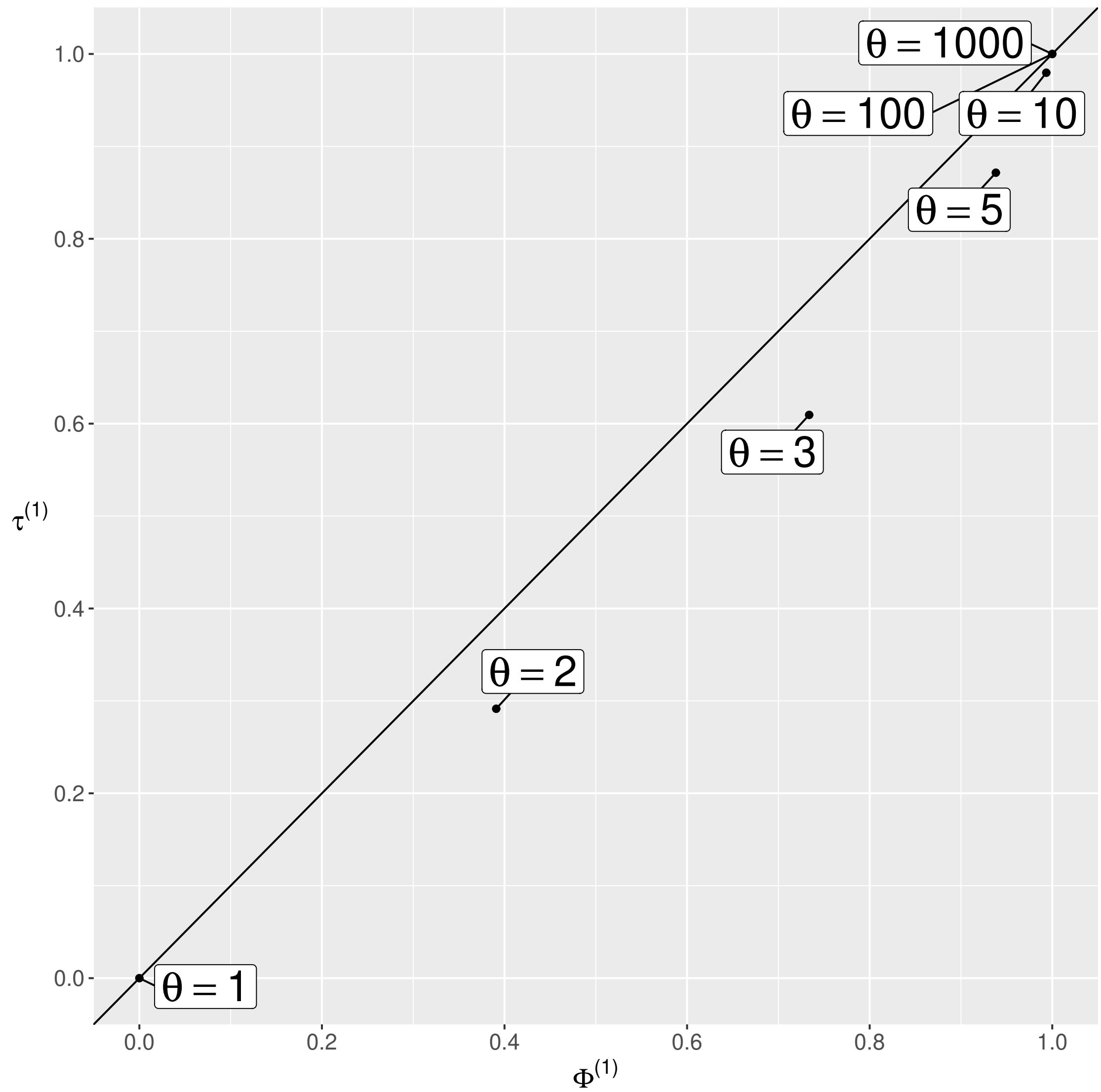

FIGURE 2.3: The values of the proposed index $\mathbf{\Lambda}^{(1)}$ under the linear diagonals-parameter symmetry model with parameter θ

2.6 Conclusions

This paper proposed a two-dimensional index to measure the degree of departure from symmetry in terms of the log odds of each symmetric cell with respect to the main diagonal in the square contingency tables. By measuring the degree of departure from symmetry in terms of the log odds of each symmetric cell, the proposed two-dimensional index provides more interpretable analysis results than the existing index of Tomizawa et al. (1998), which measures the degree of departure from the symmetry model. Additionally, the proposed two-dimensional index allows us to visually compare the degrees of departure from symmetry in multiple data sets using confidence regions and easily interpret the results of data analysis.

The proposed index $\boldsymbol{\Lambda}^{(\lambda)}$ is invariant under arbitrary same permutations of the row and column categories; namely, the value of $\boldsymbol{\Lambda}^{(\lambda)}$ does not depend on the order of the categories. Therefore, it is possible to use the proposed index for data with nominal categories. Moreover, if we may not use the information about the categories' ordering, it is possible to use the proposed index for data on an ordinal scale.

Chapter 3

An Index for the Degree and Directionality of Asymmetry for Square Contingency Tables with Ordered Categories

3.1 Introduction

For two-way contingency tables, an analysis is generally performed to see whether the independence between the row and column classifications holds. On the other hand, for the analysis of square contingency tables with the same row and column classifications, there are many issues related to symmetry rather than independence. This is because, in square contingency tables, there is a strong association between the row and column classifications.

Consider an $r \times r$ square contingency table. Let π_{ij} denote the probability that an observation will fall in the ith row and jth column of the table $(i, j = 1, \dots, r)$. Bowker (1948) proposed the symmetry model defined by

$$\pi_{ij} = \pi_{ji} \quad \text{for all } i < j.$$

The symmetry model can be expressed as

$$U_{ij} = L_{ji} \quad \text{for all } i, j,$$

where

$$U_{ij} = \sum_{s=1}^{i} \sum_{t=j}^{r} \pi_{st}, \quad L_{ji} = \sum_{s=j}^{r} \sum_{t=1}^{i} \pi_{st}.$$

Note that the cumulative probabilities $\{U_{ij}, L_{ji}\}$ that may contain diagonal probabilities $\{\pi_{ii}\}$ can be defined for ordered categories.

The symmetry model, however, often does not hold when applied to real data. When the symmetry model fits real data poorly, other symmetry or asymmetry models (see, e.g., Tahata and Tomizawa, 2014) are applied to real data.

We are also interested in measuring the degree of asymmetry when the symmetry model does not hold. When the symmetry model does not hold for multiple datasets, we may compare degrees of asymmetry. Over past years, many studies have proposed indexes to represent the degree of asymmetry. Tomizawa et al. (1998) proposed an index based on power divergence (or diversity index) to represent the degree of departure from the symmetry model for square contingency tables with *nominal* categories. Moreover, Iki and Tomizawa (2018) proposed an index based on power divergence (or diversity index) to represent the degree of departure from the symmetry model for square contingency tables with *ordered* categories. For square contingency tables with *ordered* categories, we may be interested in distinguishing two types of maximum asymmetry (i.e., whether (1) all the observations concentrate in the top-right cell in the table, or (2) they concentrate in the bottom-left cell). The index of Iki and Tomizawa (2018), however, cannot distinguish the two directions of maximum asymmetry (see Appendix B.1 for the details of the index).

This paper proposes a directional index that can distinguish the two kinds of maximum asymmetry. Section 3.2 introduces the proposed index and shows the utility of the proposed index. Section 3.3 derives an approximate confidence interval for the proposed index. Section 3.4 shows the usefulness of the proposed index by applying it to real data of the clinical study. Section 3.5 describes the concluding remarks.

3.2 Directional index and its utility

This section proposes a directional index that can distinguish two types of maximum asymmetry and shows the utility of the proposed index.

3.2.1 Directional index via an arc-cosine function

Assuming that $\pi_{1r} + \pi_{r1} > 0$, we propose a directional index defined by

$$\Gamma = \frac{4}{\pi} \sum_{\substack{i,j \\ (i,j)\neq(r,1)}} (U^*_{ij} + L^*_{ji}) \left(\theta_{ij} - \frac{\pi}{4}\right),$$

where

$$U^*_{ij} = \frac{U_{ij}}{\tau}, \quad L^*_{ji} = \frac{L_{ji}}{\tau}, \quad \tau = \sum_{\substack{i,j \\ (i,j)\neq(r,1)}} (U_{ij} + L_{ji}), \quad \theta_{ij} = \arccos\left(\frac{U_{ij}}{\sqrt{U^2_{ij} + L^2_{ji}}}\right).$$

The range of Γ is -1 to 1 since the range of θ_{ij} is $0 \le \theta_{ij} \le \pi/2$. The index Γ has the following characteristics: (i) $\Gamma = -1$ if and only if $\pi_{1r} = 1$, (ii) $\Gamma = 1$ if and only if $\pi_{r1} = 1$; and (iii) if the symmetry model holds then $\Gamma = 0$.

Using the index Γ, we can see whether the degree of asymmetry departs toward the structure such that all the observations concentrate in the top-right cell $(1, r)$ in the table or the structure such that they concentrate in the bottom-left cell $(r, 1)$ in the table. As the index Γ approaches -1, the asymmetry structure is closer to $\pi_{1r} = 1$, and as the index Γ approaches 1, it is closer to $\pi_{r1} = 1$.

Note that for all $\{i, j|(i, j) \neq (r, 1)\}$, $(\theta_{ij} - \pi/4)$ is zero when $U_{ij} = L_{ji}$, negative value when $U_{ij} > L_{ji}$, and positive value when $U_{ij} < L_{ji}$. Since the index Γ is the weighted sum of $(\theta_{ij} - \pi/4)$, the value of Γ is zero if and only if the weighted average of $(\theta_{ij} - \pi/4)$ is zero. If we shall refer to the structure of $\Gamma = 0$ as the average cumulative symmetry, the index Γ represents the degree of departure from the average cumulative symmetry towards the two types of maximum asymmetry.

3.2.2 The utility of the proposed index

This subsection demonstrates the utility of the proposed index. First, we compare the proposed index Γ with the indexes of Iki and Tomizawa (2018) (denoted by $\kappa^{(\lambda)}$; see Appendix B.1) and Tahata et al. (2009) (denoted by φ; see Appendix B.2). Tahata et al. (2009)'s index φ can distinguish whether (I) the complete upper asymmetry or (II) the complete lower asymmetry (i.e., whether (I) all the observations concentrate in the upper right triangle cells in the table, or (II) they concentrate in the lower left triangle cells). We consider the 4×4 structures of probability that have different asymmetric structures in Tables 3.1a to 3.1i. Table 3.2, however, represents that the values of $\kappa^{(1)}$ applied to Tables 3.1a and 3.1i, 3.1b and 3.1h, 3.1c and 3.1g, and 3.1d and 3.1f are equal. Additionally, it is impossible to calculate the values of φ applied to Tables 3.1a, 3.1b, 3.1h, and 3.1i because the existing index φ can be used under the condition $\pi_{ij} + \pi_{ji} > 0$ for all $i < j$. In contrast, all values of Γ are different, and the proposed index Γ can be applied to all of Tables 3.1a through 3.1i. When measuring the degree of asymmetry, it is important to distinguish between the two possible directions of maximum asymmetry, such that $\pi_{1r} = 1$ and $\pi_{r1} = 1$. This is because the interpretation of the result changes depending on whether the degree of asymmetry departs toward $\pi_{1r} = 1$ and $\pi_{r1} = 1$. It is also important to note that the value of the proposed index changes depending on the asymmetric structure. This is useful for comparing the degree of asymmetry among contingency tables. Thus, when measuring the degree of asymmetry, we suggest that analysts use the proposed index that can distinguish between the two directions of maximum asymmetry.

TABLE 3.1: The 4×4 structures of probability

(a)				(b)				(c)			
0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.3333	0.3334	0.1000	0.1000	0.1000	0.1000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.3333	0.0000	0.1000	0.1000	0.1000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.1000	0.1000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.1000
(d)				(e)				(f)			
0.0769	0.0769	0.0770	0.0770	0.0625	0.0625	0.0625	0.0625	0.0769	0.0769	0.0000	0.0000
0.0769	0.0769	0.0769	0.0770	0.0625	0.0625	0.0625	0.0625	0.0769	0.0769	0.0769	0.0000
0.0000	0.0769	0.0769	0.0769	0.0625	0.0625	0.0625	0.0625	0.0770	0.0769	0.0769	0.0769
0.0000	0.0000	0.0769	0.0769	0.0625	0.0625	0.0625	0.0625	0.0770	0.0770	0.0769	0.0769
(g)				(h)				(i)			
0.1000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1000	0.1000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1000	0.1000	0.1000	0.0000	0.3333	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1000	0.1000	0.1000	0.1000	0.3334	0.3333	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000

TABLE 3.2: The values of Γ, $\kappa^{(1)}$, and φ for Table 3.1

Index	Table number								
	3.1a	3.1b	3.1c	3.1d	3.1e	3.1f	3.1g	3.1h	3.1i
Γ	-1.000	-0.916	-0.561	-0.324	0.000	0.324	0.561	0.916	1.000
$\kappa^{(1)}$	1.000	0.846	0.324	0.113	0.000	0.113	0.324	0.846	1.000
φ	NA	NA	-1.000	-0.333	0.000	0.333	1.000	NA	NA

Note that, in addition to the structure of probability where the symmetry model holds as shown in Table 3.1e, there is also a structure of probability where the value of the proposed index Γ is zero as shown in Table 3.3. Average cumulative symmetry includes such a structure of probability.

TABLE 3.3: The 4×4 structure of probability

0.150	0.025	0.050	0.025
0.050	0.150	0.025	0.025
0.025	0.025	0.150	0.050
0.025	0.050	0.025	0.150

3.3 Approximate confidence interval for the proposed index

Let n_{ij} denote the observed frequency in the ith row and jth column of the table $(i, j = 1, \ldots, r)$. Assume that a multinomial distribution applies to the $r \times r$ table. The sample proportions of $\{\pi_{ij}; i, j = 1, \ldots, r\}$ are $\{p_{ij} = n_{ij}/n\}$ with $n = \sum_{i,j} n_{ij}$.

The estimator of the index Γ, $\hat{\Gamma}$ is provided by replacing $\{\pi_{ij}\}$ with $\{p_{ij}\}$. This section derives the asymptotic distribution of $\hat{\Gamma}$ using the delta method (see, e.g., Agresti, 2013), and the approximate confidence interval of Γ.

Let $\boldsymbol{p}$ and $\boldsymbol{\pi}$ be the $r^2 \times 1$ vectors

$$\boldsymbol{p} = (p_{11}, p_{12}, \ldots, p_{1r}, p_{21}, \ldots, p_{rr})^\top \quad \text{and} \quad \boldsymbol{\pi} = (\pi_{11}, \pi_{12}, \ldots, \pi_{1r}, \pi_{21}, \ldots, \pi_{rr})^\top$$

respectively, where $\boldsymbol{a}^\top$ is the transpose of $\boldsymbol{a}$. Then $\sqrt{n}(\boldsymbol{p} - \boldsymbol{\pi})$ has an asymptotically normal distribution with the mean zero vector and the covariance matrix $\mathrm{diag}(\boldsymbol{\pi}) - \boldsymbol{\pi}\boldsymbol{\pi}^\top$, where $\mathrm{diag}(\boldsymbol{\pi})$ is a diagonal matrix with the elements of $\boldsymbol{\pi}$ on the main diagonal. Since

$$\hat{\Gamma} = \Gamma + d(\boldsymbol{\pi})(\boldsymbol{p} - \boldsymbol{\pi}) + o_p(1)$$

with $d(\boldsymbol{\pi}) = \partial\Gamma/\partial\boldsymbol{\pi}^\top$, $\sqrt{n}(\hat{\Gamma} - \Gamma)$ asymptotically (as $n \to \infty$) has a normal distribution with the mean zero and the variance

$$\begin{aligned}\sigma^2[\Gamma] &= d(\boldsymbol{\pi})(\mathrm{diag}(\boldsymbol{\pi}) - \boldsymbol{\pi}\boldsymbol{\pi}^\top)d(\boldsymbol{\pi})^\top \\ &= \sum_{k,l} \pi_{kl} \left(\frac{\partial\Gamma}{\partial\pi_{kl}}\right)^2 - \left(\sum_{k,l} \pi_{kl} \frac{\partial\Gamma}{\partial\pi_{kl}}\right)^2,\end{aligned}$$

where

$$\begin{aligned}\frac{\partial\Gamma}{\partial\pi_{kl}} =& \frac{4}{\pi\tau} \sum_{\substack{i,j \\ (i,j)\neq(r,1)}} (I[k \leq i, j \leq l] + I[j \leq k, l \leq i]) \left(\theta_{ij} - \frac{\pi}{4}(\Gamma+1)\right) \\ &+ \frac{4}{\pi\tau} \sum_{\substack{i,j \\ (i,j)\neq(r,1)}} (U_{ij} + L_{ji}) \frac{U_{ij} I[j \leq k, l \leq i] - L_{ji} I[k \leq i, j \leq l]}{U_{ij}^2 + L_{ji}^2}\end{aligned}$$

and $I[\cdot]$ is the function, $I[\cdot] = 1$ if true, 0 if not.

Let $\hat{\sigma}^2[\Gamma]$ denote $\sigma^2[\Gamma]$ with $\{\pi_{ij}\}$ replaced by $\{p_{ij}\}$. The square root of $\hat{\sigma}^2[\Gamma]/n$ is an estimated standard error of $\hat{\Gamma}$, and

$$\hat{\Gamma} \pm z_{\alpha/2}\sqrt{\hat{\sigma}^2[\Gamma]/n}$$

is an approximate $100(1-\alpha)\%$ confidence interval for Γ, where $z_{\alpha/2}$ is the percentage point of the standard normal distribution corresponding to a two-tail probability of α.

3.4 Examples

Example 1: First, consider the data in Tables 3.4a and 3.4b, cited from Sugano et al. (2012). In this study, 343 Japanese adult patients (aged $\geq$ 20 years) with a history of peptic ulcers were randomly assigned to treatment (esomeprazole, 175 patients; placebo, 168 patients). The modified LANZA score indicates that "0" is the best score and "+4" is the worst score. Thus, for the data in Tables 3.4a and 3.4b, the more all the observations concentrate in the top-right cell in the tables, the more improvement is shown. On the contrary, the more all the observations concentrate in the bottom-left cell in the tables, the more ingravescence is shown. As a matter of clinical interest, we would like to determine whether patients in the esomeprazole group improved more than patients in the placebo group. Therefore, for these data in Tables 3.4a and 3.4b, we are interested in comparing the degree of asymmetry as well as distinguishing the directionality for the two types of asymmetry. Table 3.5 represents that (i) the degree of asymmetry for the data in Table 3.4a departs toward the asymmetric structure, where all the observations concentrating in the top-right cell in the table since the confidence interval for Γ is negative, and (ii) the degree of asymmetry for the data in Table 3.4b departs toward the asymmetric structure, where all the observations concentrating in the bottom-left cell in the table since the confidence interval for Γ is positive. Thus, using the proposed index, we can see that patients in the esomeprazole group have experienced an improvement in terms of the change from baseline to endpoint in the modified LANZA score, and patients in the placebo group have experienced an ingravescence.

Note that the Tahata et al. (2009)'s index φ can be used only under the condition $\pi_{ij} + \pi_{ji} > 0$ for all $i < j$. Therefore, it is impossible to calculate the values of φ applied to the data in Table 3.4 because $p_{25}+p_{52} = 0$ for the data in the esomeprazole group and $p_{14} + p_{41} = 0$ for the data in the placebo group. In contrast, the proposed index Γ can apply to the data if the data satisfy the only condition $p_{1r} + p_{r1} > 0$. Comparing these conditions, $\pi_{1r} + \pi_{r1} > 0$ and $\pi_{ij} + \pi_{ji} > 0$ for all $i < j$, the proposed index Γ has high applicability and can be easily applied to the sparse data like the data in Table 3.4.

Table 3.4: Change in the modified LANZA score from baseline to end of study in (a) the esomeprazole group and (b) the placebo group; source Sugano et al. (2012)

End of study	Baseline					Total
	0	+1	+2	+3	+4	
(a) Esomeprazole group						
0	78	9	26	3	1	117
+1	1	5	6	4	0	16
+2	9	1	10	3	1	24
+3	1	0	1	0	0	2
+4	3	0	1	1	2	7
Total	92	15	44	11	4	166
(b) Placebo group						
0	41	2	19	0	0	62
+1	8	0	4	0	0	12
+2	12	4	14	3	0	33
+3	0	1	1	3	0	5
+4	29	7	11	6	0	53
Total	90	14	49	12	0	165

TABLE 3.5: Estimates of indexes Γ and $\kappa^{(1)}$, approximate standard errors for $\hat{\Gamma}$ and $\hat{\kappa}^{(1)}$, and approximate 95% confidence interval for Γ and $\kappa^{(1)}$ for the data in Tables 3.4a and 3.4b

Estimated index		Standard error	Confidence interval
(a) For Table 3.4a			
Γ	-0.198	0.056	$(-0.308, -0.088)$
$\kappa^{(1)}$	0.053	0.024	$(0.005, 0.101)$
(b) For Table 3.4b			
Γ	0.430	0.057	$(0.317, 0.542)$
$\kappa^{(1)}$	0.217	0.037	$(0.144, 0.291)$

Example 2: Next, consider the clinical trial data in Tables 3.6a and 3.6b, cited from Lundorff et al. (2005). This study was a randomized (surgery plus Oxiplexw/Ap Gel group and surgery only group), the third party blinded, parallel-group design conducted at four centers in Europe. Patients were 18-46 years old requiring peritoneal cavity surgery by way of laparoscopy and expected to undergo a second-look laparoscopy as part of their treatment plan 6-10 weeks after the initial surgery. The American Fertility Society (AFS) adnexal adhesion score is determined by assessing the extent (area of adnexal organ covered by adhesions) and severity (severe: if the adhesion requires cutting to remove or tears peritoneal surfaces when removed bluntly or requires hemostasis; filmy if not severe) of the adhesions involving the Fallopian tube and ovary. Summing the scores for the Fallopian tube and the ovary provided a clinical category for the adhesion score: Minimum (0-5), Mild (6-10), Moderate (11-20), and Severe (21-32). Thus, for the data in Tables 3.6a and 3.6b, the more all the observations concentrate in the top-right cell in the tables, the more

ingravescence is shown. On the contrary, the more all the observations concentrate in the bottom-left cell in the tables, the more improvement is shown. As a matter of clinical interest, we would like to determine whether patients in the surgery plus Oxiplexw/AP Gel group improved more than patients in the surgery-only group. Therefore, for these data in Tables 3.6a and 3.6b, we are interested in comparing the degree of asymmetry as well as distinguishing the directionality of the two types of asymmetry. Table 3.7 represents that (i) the degree of asymmetry for the data in Table 3.6a departs toward the asymmetric structure, where all the observations concentrating in the bottom-left cell in the table since the confidence interval for Γ is positive, and (ii) the degree of asymmetry for the data in Table 3.6b departs toward the asymmetric structure, where all the observations concentrating in the top-right cell in the table since the confidence interval for Γ is negative. Thus, using the proposed index, we can see that patients in the surgery plus Oxiplexw/AP Gel group have experienced an improvement in terms of the change from baseline to second-look in the AFS score, and patients in the surgery-only group have experienced an ingravescence.

Table 3.6: Change in the American Fertility Society (AFS) score category from baseline to second-look in (a) the surgery plus Oxiplexw/AP Gel group and (b) the surgery-only group; source Lundorff et al. (2005)

Baseline	Second-look				Total
	Minimal	Mild	Moderate	Severe	
(a) Surgery plus Oxiplexw/AP Gel group					
Minimal	22	1	0	0	23
Mild	2	2	1	0	5
Moderate	2	1	1	1	5
Severe	1	1	2	8	12
Total	27	5	4	9	45
(b) Surgery-only group					
Minimal	13	7	1	2	23
Mild	0	0	3	1	4
Moderate	0	0	1	4	5
Severe	0	0	1	8	9
Total	13	7	6	15	41

TABLE 3.7: Estimates of indexes Γ, $\kappa^{(1)}$, and φ, approximate standard errors for $\hat{\Gamma}$, $\hat{\kappa}^{(1)}$, and $\hat{\varphi}$, and approximate 95% confidence interval for Γ, $\kappa^{(1)}$, and φ for the data in Tables 3.6a and 3.6b

Estimated index		Standard error	Confidence interval
(a) For Table 3.6a			
Γ	0.195	0.081	$(0.036, 0.355)$
$\kappa^{(1)}$	0.063	0.043	$(-0.021, 0.147)$
φ	0.538	0.285	$(-0.022, 1.099)$
(b) For Table 3.6b			
Γ	-0.394	0.070	$(-0.532, -0.256)$
$\kappa^{(1)}$	0.206	0.054	$(0.100, 0.313)$
φ	-0.918	0.094	$(-1.101, -0.734)$

3.5 Concluding remarks

This paper proposed a directional index based on an arc-cosine function that can distinguish between the two directions of maximum asymmetry, namely, the structure such that all the observations concentrate in the top-right cell $(1, r)$ or the structure such that they concentrate in the bottom-left cell $(r, 1)$ in the table. Numerical examples demonstrated the utility of the proposed index by showing that the values of the proposed index for some asymmetric probability structures were all different, while the values of the existing index were partially the same. We recommend the proposed index for measuring the degree of asymmetry because the proposed index provides an easier interpretation for the analysis results than the existing index. Since the proposed index lies between -1 and 1 not depending on the number of categories r and the sample size n, the proposed index is useful to compare the degree of asymmetry for several square contingency tables. Note that the value of the proposed index Γ depends on the main diagonal cell probabilities $\{\pi_{ii}\}$, but does not the Tahata et al. (2009)'s index φ. Therefore, Γ rather than φ would be useful when one wants to utilize the information of the observations in the main diagonal cells of the table.

The proposed index should be applied to the tables with the ordered categories of the same row and column because the proposed index is not invariant under arbitrary similar permutations of the row and column categories.

Chapter 4

Discussion and Concluding Remarks

4.1 Discussion

Our proposed index $\boldsymbol{\Lambda}^{(\lambda)}$ in Chapter 2 depends on the real value $\lambda(> -1)$ chosen by the user. Analysts may be interested in which λ value to use. Momozaki et al. (2023) and Urasaki et al. (2023) discuss the choice, and I agree with them.

Momozaki et al. (2023) and Urasaki et al. (2023) recommend the use of various values of λ, not just one value. Suppose we have two square contingency table data (A and B), and we compare their degree of departure from the symmetry model using $\boldsymbol{\Lambda}^{(\lambda)}$. The analysts who compare them with the value λ_1 obtain the result that the degree of departure from the symmetry model is larger for data A than for data B. On the other hand, for those who compare with the value λ_2, the degree is larger for data B than for data A. Thus, using only one value of λ may miss the contradictory results. I argue that we should use multiple values of λ to check whether the results are consistent.

4.2 Concluding Remarks

This thesis proposed two indexes for models of symmetry: (i) a two-dimensional vector type index that measures the degree of departure from the symmetry model for square contingency tables with nominal categories (Chapter 2), and (ii) an index of the degree and direction of the two asymmetries for square contingency tables with ordered categories (Chapter 3). By combining the two indexes, the vector type index can represent the degree of departure from the symmetry model to the maximum asymmetry structure defined by Tomizawa et al. (1998) in terms of the log odds of each symmetric cell probability, and therefore provides more interpretable analysis results than the existing index. The index proposed in Chapter 3 can distinguish between two kinds of maximum asymmetry structures defined in Iki and Tomizawa (2018) while the index in Iki and Tomizawa (2018) cannot. Hence the proposed index is useful when one is interested in which maximum asymmetric structure the probability structure is closer to. As described in Chapter 1, there has been an enormous amount of research on indexes in contingency tables. I would like to discuss some cautions in the use of these indexes.

It is important to note whether the indexes are suitable for contingency tables composed of nominal or ordinal. As mentioned in Chapter 1, nominal categorical variables have no order among categories, while ordinal categorical variables have order. Some indexes may or may not be invariant under arbitrary permutations of the row and column categories. As discussed in Sections 2.6 and 3.5, those indexes whose values are invariant under arbitrary permutation of the row and column categories are applicable to nominal categorical contingency tables, and those whose values are not invariant are applicable to ordinal categorical contingency tables. Note that the indexes for nominal category contingency tables are also applicable to ordinal category contingency tables if the information about the order among categories is not used.

We should also choose the indexes to be used according to the purpose of the data analysis, such as which model to measure the degree of departure and in which direction. For example, even if we measure the degree of departure from the symmetry model, the maximum asymmetry structures defined in Tomizawa et al. (2001) and Iki and Tomizawa (2018) are different, and the interpretation of the analysis results will clearly differ from each other. If we want to distinguish the structures of the maximum asymmetry, we may consider using the index of Tahata et al. (2009) or the index proposed in Chapter 3. In this way, one should understand the properties of the index (e.g., which model is the degree of departure, what is the defined structure of the maximum asymmetry, etc.) and choose the index according to the purpose of the data analysis.

Lastly, we discuss the development of some of indexes in contingency tables for future research. The small number of observations per cell in contingency tables affects poorly not only the accuracy of estimation of cell probabilities, but also the accuracy of estimation of indexes. Such a possibility is more likely to occur as the number of categories increases, and is also a very important topic in multiway contingency tables, although it was not covered in this thesis. As one of the solutions for estimation of indexes, Momozaki et al. (2023) proposed an estimation procedure using Bayesian estimators of cell probabilities that minimize the mean squared error of the indexes. Another possible solution is that instead of using the large sample theory ($n \to \infty$) in deriving the asymptotic distribution of index, an asymptotic theory that also addresses the increasing number of categories (r and/or $c \to \infty$) would be necessary. See Chapter 10 of Kateri (2014) and Chapters 16 and 17 of

Agresti (2013) for other discussions on inference for the small sample and high-dimensional categorical data.

Some may consider merging categories in contingency tables to avoid small cell observation frequencies. In addition, merging categories is easier to interpret, and in clinical research they are often merged to form 3×3 table, and such a contingency table is called a "collapsed" table. However, it may be unreasonable to do so without due care and attention. Tomizawa (1994)'s index to measure the degree of departure from the symmetry model is defined using the Kullback-Leibler divergence, and it is possible that the degree of departure increases as the number of categories increases due to the properties of the Kullback-Leibler divergence. Namely, if the categories are inadvertently merged, the index may possibly underestimate the degree of departure from the symmetry model. See Tahata et al. (2008), Yamamoto et al. (2012), Yamamoto et al. (2013), and Yamamoto et al. (2016) for works on models of symmetry for collapsed tables, and Miyamoto and Tomizawa (2005), Yamamoto et al. (2010), Yamamoto et al. (2015), Yamamoto et al. (2020), Aizawa et al. (2021), Iki et al. (2021), and Shinoda et al. (2023) for works on indexes for models of symmetry.

The development of indexes in contingency tables is expected to continue for the future. According to the applied fields, the probability structures (models) of interest are different, and the desired properties of the indexes can also change with them. There is a need to come up with new models as necessary. We will discuss closely with the applied researchers to develop new indexes and their usefulness, and possibly develop packages in popular programming languages such as `R` and `Python` so that anyone can use the indexes.

Also, vector type indexes are composed of a combination of various indexes, and there may be countless possible combinations. Conventional vector type indexes have been developed with the motivation of measuring the degree of departure from certain models and representing the degree of asymmetry and its direction simultaneously, or motivated by the decomposition of models. Meanwhile, the vector type index proposed in Chapter 2 is motivated to provide a new perspective on the degree of departure from the symmetry model by defining a new index that is best suited to be combined with the existing index. Chapter 3 proposed a directional index that can distinguish whether all the observations concentrate in the top-right cell $(1, r)$ or the bottom-left cell $(r, 1)$ in the table. This index does not exactly measure the degree of departure from the symmetry model, since the necessary and sufficient condition for the value of 0 is not that the symmetry model holds. Similar to the index proposed in Ando et al. (2017), we can propose a vector-type index that can measure the degree of departure from the symmetry model, distinguish the directionality for two maximum asymmetry structures, and represent the degree by using Iki and Tomizawa (2018)'s index and our proposed index in Chapter 3.

Appendix A

Appendix of Chapter 2

A.1 The Proof of Theorem 2.1

Proof of Theorem 2.1. Since

$$\Phi^{(\lambda)} \geq 1 - \sum_{i<j}(\pi^*_{ij} + \pi^*_{ji})\left[1 - \phi^{(\lambda)}_{ij}\right], \quad \tau^{(\lambda)} \leq \sum_{i<j}(\pi^*_{ij} + \pi^*_{ji})\phi^{(\lambda)}_{ij}$$

holds from Jensen's inequality, $\tau^{(\lambda)} \leq \Phi^{(\lambda)}$ holds. This equality holds if, and only if, $\{\phi^{(\lambda)}_{ij}\}$ are constant. Moreover, $\{\phi^{(\lambda)}_{ij}\}$ are constant if, and only if, $\{H^{(\lambda)}_{ij}\}$ are constant, which is equivalent to the fact that the conditional difference asymmetry model holds. □

A.2 The Index of Symmetry

For an $r \times r$ square contingency table with nominal categories, the index to represent the degree of departure from the symmetry model is proposed by Tomizawa et al.

(1998), as follows. Assuming that $\pi_{ij} + \pi_{ji} > 0$ for all $i < j$, for $\lambda > -1$,

$$\zeta^{(\lambda)} = \sum_{i<j} (\pi^*_{ij} + \pi^*_{ji}) \phi^{(\lambda)}_{ij},$$

where

$$\phi^{(\lambda)}_{ij} = 1 - \frac{\lambda 2^{\lambda}}{2^{\lambda} - 1} H^{(\lambda)}_{ij}, \quad H^{(\lambda)}_{ij} = \frac{1}{\lambda} \left[1 - (\pi^c_{ij})^{\lambda+1} - (\pi^c_{ji})^{\lambda+1} \right],$$

$$\pi^c_{ij} = \frac{\pi_{ij}}{\pi_{ij} + \pi_{ji}}, \quad \pi^*_{ij} = \frac{\pi_{ij}}{\delta}, \quad \delta = \sum_{i \neq j} \pi_{ij}.$$

The $\zeta^{(0)}$ is defined as

$$\lim_{\lambda \to 0} \zeta^{(\lambda)} = \sum_{i<j} (\pi^*_{ij} + \pi^*_{ji}) \phi^{(0)}_{ij},$$

where

$$\phi^{(0)}_{ij} = 1 - \frac{1}{\log 2} H^{(0)}_{ij}, \quad H^{(0)}_{ij} = -\pi^c_{ij} \log \pi^c_{ij} - \pi^c_{ji} \log \pi^c_{ji}.$$

Note that $H^{(\lambda)}_{ij}$ is Patil and Taillie (1982)'s diversity index of degree λ including the Shannon entropy ($\lambda = 0$), and the real number λ is chosen by the user. The index $\zeta^{(\lambda)}$ has the following characteristics: (i) $0 \leq \zeta^{(\lambda)} \leq 1$; (ii) $\zeta^{(\lambda)} = 0$ if, and only if, the symmetry model holds; and (iii) $\zeta^{(\lambda)} = 1$ if, and only if, the degree of asymmetry is maximum in the sense that $\pi_{ij} = 0$ (then $\pi_{ji} > 0$) or $\pi_{ji} = 0$ (then $\pi_{ij} > 0$) for all $i < j$.

Appendix B

Appendix of Chapter 3

B.1 Index based on the power divergence

Iki and Tomizawa (2018) proposed the index to represent the degree of departure from the symmetry model as follows. Assuming that $\pi_{1r} + \pi_{r1} > 0$, for $\lambda > -1$,

$$\kappa^{(\lambda)} = \frac{\lambda(\lambda+1)}{2^{\lambda}-1} I^{(\lambda)},$$

where

$$I^{(\lambda)} = \frac{1}{\lambda(\lambda+1)} \sum_{\substack{i,j \\ (i,j)\neq(r,1)}} \left\{ U^*_{ij}\left[\left(\frac{U^*_{ij}}{W_{ij}}\right)^{\lambda} - 1\right] + L^*_{ji}\left[\left(\frac{L^*_{ji}}{W_{ij}}\right)^{\lambda} - 1\right]\right\},$$

with

$$U^*_{ij} = \frac{U_{ij}}{\tau}, \quad L^*_{ji} = \frac{L_{ji}}{\tau}, \quad U_{ij} = \sum_{s=1}^{i}\sum_{t=j}^{r}\pi_{st}, \quad L_{ji} = \sum_{s=j}^{r}\sum_{t=1}^{i}\pi_{st},$$

$$\tau = \sum_{\substack{i,j \\ (i,j)\neq(r,1)}} (U_{ij} + L_{ji}), \quad W_{ij} = \frac{U^*_{ij} + L^*_{ji}}{2}.$$

The $\kappa^{(0)}$ is defined as

$$\lim_{\lambda \to 0} \kappa^{(\lambda)} = \frac{1}{\log 2} I^{(0)},$$

where

$$I^{(0)} = \sum_{\substack{i,j \\ (i,j) \neq (r,1)}} \left(U_{ij}^{*} \log \frac{U_{ij}^{*}}{W_{ij}} + L_{ji}^{*} \log \frac{L_{ji}^{*}}{W_{ij}} \right).$$

The $I^{(\lambda)}$ is the power divergence between $\{U_{ij}^{*}, L_{ji}^{*}\}$ and $\{W_{ij}, W_{ji}\}$, and especially, $I^{(0)}$ is the Kullback Leibler information between them. The index $\kappa^{(\lambda)}$ has the following characteristics: (i) $0 \leq \kappa^{(\lambda)} \leq 1$; (ii) $\kappa^{(\lambda)} = 0$ if and only if the symmetry model holds; and (iii) $\kappa^{(\lambda)} = 1$ if and only if the degree of asymmetry is maximum in the sense that $\pi_{1r} = 1$ or $\pi_{r1} = 1$. However, index $\kappa^{(\lambda)}$ cannot distinguish between the two directions of maximum asymmetry. Namely, we cannot see whether the degree of asymmetry increases toward the structure such that all the observations concentrate in the top-right cell $(1, r)$ in the table or toward the structure such that all the observations concentrate in the bottom-left cell $(r, 1)$ in the table.

B.2 Index of complete asymmetry

Tahata et al. (2009) proposed the directional index to distinguish whether (I) the complete upper asymmetry or (II) the complete lower asymmetry (i.e., whether (I) all the observations concentrate in the upper right triangle cells in the table, or (II) they concentrate in the lower left triangle cells) as follows. Assuming that $\pi_{ij} + \pi_{ji} > 0$ for all $i < j$,

$$\varphi = \frac{4}{\pi} \sum_{i<j} (\pi_{ij}^{*} + \pi_{ji}^{*}) \left(\omega_{ij} - \frac{\pi}{4} \right)$$

where

$$\pi_{ij}^{*} = \frac{\pi_{ij}}{\delta}, \quad \delta = \sum_{i \neq j} \pi_{ij}, \quad \omega_{ij} = \arccos\left(\frac{\pi_{ij}}{\sqrt{\pi_{ij}^{2} + \pi_{ji}^{2}}}\right).$$

The directional index φ has the following characteristics: (i) $-1 \leq \varphi \leq 1$; (ii) $\varphi = -1$ if and only if $\pi_{ij} > 0$ for all $i < j$ (then $\pi_{ji} = 0$ for all $i < j$), say, complete-upper-asymmetry; (iii) $\varphi = 1$ if and only if $\pi_{ji} > 0$ for all $i < j$ (then $\pi_{ij} = 0$ for all $i < j$), say, complete-lower-asymmetry; and (iv) if the symmetry model holds then $\varphi = 0$.

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