Dissertation

Φ^3 Matrix Model and Φ^3-Φ^4 Hybrid Matrix Model as Grosse-Steinacker-Wulkenhaar Type Noncommutative Scalar Field Theories

(Grosse-Steinacker-Wulkenhaar型非可換スカラー場の理論としてのΦ^3行列模型とΦ^3-Φ^4混合行列模型)

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Contents

Chapter 1

Introduction

1.1. Historical Background

The behavior of elementary particles is governed by four fundamental forces, namely strong interaction, electromagnetic interaction, weak interaction, and gravitational interaction. Quantum field theory provides a theoretical framework to understand the first three interactions. However, the fourth interaction, gravity, poses significant difficulties when it comes to quantization. Incorporating gravity into the quantum framework that has been successfully applied to the other three forces is a challenging task due to notable difficulties. The superstring theory is a way of understanding quantum gravity by imagining elementary particles as one-dimensional strings instead of points. These strings have various vibration modes, which means that a single string can represent many different particles. From this perspective, gravitons, which are hypothetical particles that facilitate gravitational interaction, naturally arise. This approach helps to overcome problems that arise when trying to understand quantum gravity using point particles. However, there is a significant issue in string theory because there are an infinite number of stable vacuums that exist. Each vacuum implies different spacetime dimensions and matter compositions. Understanding which vacuum describes our observable universe requires a non-perturbative approach. In 1974 a string field theory was proposed[38, 39]; by 1985 a free string field theory was proposed[48, 49]. Later, in 1986, two types of string field theories were proposed: HIKKO model[25, 26, 27, 28, 29], and the model devised by Witten[52]. They add an interaction to the action of the free string field theory. In 1996 we saw the development of a non-perturbative formulation of superstring theory using matrices. This matrix model called the IKKT matrix model[34, 3], is an example of a candidate non-perturbative formulation, as are the BFSS model[4, 50] and various string field theories. It is also known that noncommutative spaces appear in various contexts of string theory (matrix models, string field theories, etc.). Noncommutative spaces are spacetimes in which the coordinates of the spacetime are considered to be operators rather than ordinary numbers.

Therefore, it is important to consider quantum field theory on noncommutative spaces. In particular, we focus on scalar ϕ^3 theories on noncommutative spaces. Because scalar ϕ^3 theories are theories with 3-point interaction for a scalar field ϕ^3, and this is the easiest model of quantum field theory with interaction. One problem with quantum field theories on noncommutative spaces is that perturbative renormalization is non-trivial because of the mixing of ultraviolet and infrared divergences (UV/IR mixing). The term 'renormalization' here means that the divergences that appear in the perturbative expansions are pushed into the redefinition of each parameter that appears in the theory. The process of removing the divergences is brought about by adding counter terms to the Lagrangian. However, when the harmonic oscillator potential is added, the UV/IR mixing is resolved and the theory becomes renormalizable. The harmonic oscillator potential is the potential of a system of oscillating objects connected to a spring that causes motion. This scalar ϕ^3 theory on noncommutative spaces with a harmonic oscillator potential is called Grosse-Steinacker-Wulkenhaar Φ^3 model[13, 14, 15, 18, 19]. This thesis will deal with this model.

On the other hand, we have to mention the history of matrix models from the viewpoint of 2-dimensional quantum gravity. We especially see Φ^3 matrix models. Matrix models related to 2-dimensional quantum gravities were well studied in the 1980s and 1990s[8]. 2-dimensional quantum gravities are non-critical string theories, i.e., string theories with $(0+1)$ dimension of target spaces. Each Feynman diagram in perturbative expansions of the matrix models represents a corresponding simplicial decomposition of a two-dimensional surface. Each Feynman diagram is a graphical representation of each term of the perturbative expansion in quantum field theory. In particular, the 2-dimensional surface is represented by Φ^3 matrix model when considered as a discretized graph by triangulation. The sum over two-dimensional surfaces corresponds to path integrals of two-dimensional quantum gravity theories[2, 8]. A monumental achievement of mathematical fact revealed using Φ^3 matrix model is the proof of the Witten conjecture by Kontsevich. Historically, Fukuma, Kawai, and Nakayama proved that the Virasoro constraint is equivalent to the condition that the solution of the KdV hierarchy satisfies the string equation[11, 55]. Dijkgraaf, Verlinde, and Verlinde also derived similar results independently[9]. The KdV equation is a nonlinear partial differential equation that is an example of an integrable system with soliton solutions. The KdV equation includes a family of higher-order KdV equations, which are expressed as differential equations. These equations are treated as a system of simultaneous equations, known as the KdV hierarchy. Witten then showed that the Witten-Kontsevich-τ function satisfies the string equation[53, 55]. Furthermore, Witten predicted that the Witten-Kontsevich-τ function is the τ function of the KdV hierarchy[53, 55]. This is called Witten conjecture. Finally, Kontsevich proved the Witten conjecture using the Kontsevich model[42, 55]. Kontsevich model coincidentally coincides with Grosse-Steinacker-Wulkenhaar Φ^3 model. Therefore, Grosse-Steinacker-Wulkenhaar Φ^3 model is also called Kontsevich model.

Thus, Grosse-Steinacker-Wulkenhaar Φ^3 model has been further developed. In the following, Grosse-Steinacker-Wulkenhaar Φ^3 model is referred to as Φ^3 matrix model. For example, all multipoint correlation functions of Φ^3 matrix model in large N, V limit were computed by Grosse, Sako, and Wulkenhaar by solving exactly Schwinger-Dyson equations[18, 19]. Here N is the size of the matrix and V is the noncommutativity parameter. The multipoint correlation function in Φ^3 matrix model here is a physical quantity in quantum field theory that is the probability amplitude of observing a particle at n points.

A matrix model as a renormalizable Φ^4 theory on Moyal spaces similar to Φ^3 matrix model is

Grosse-Wulkenhaar Φ^4 model[30, 17]. In the following, Grosse-Wulkenhaar Φ^4 model is called Φ^4 matrix model. This model corresponds to the quantum scalar field theories on noncommutative spaces, which is renormalized by adding a harmonic oscillator potential to scalar ϕ^4 theories on Moyal spaces[16]. This is how Φ^4 matrix model came to be considered. The 2-point function of Φ^4 matrix model whose Feynman diagrams in the perturbative expansion in large N, V limit can be drawn in the planar diagrams was solved exactly by Grosse, Wulkenhaar, and Hock[20]. The multipoint correlation functions of Φ^4 matrix model have been solved by Wulkenhaar and Hock[33]. Multipoint correlation function for Φ^4 matrix model with nonplanar Feynman diagrams has been studied by Grosse, Wulkenhaar, Hock, and Branahl using a computational method called "Blobbed Topological Recursion" in [5, 6, 7, 32]. The multitrace matrix model is approximated using analytical methods, and the multitrace matrix model is also studied numerically by Monte Carlo simulations in [46]. In August 2023, Grosse and Sako showed that the partition function of Φ^4 matrix model solves an integrable Schrödinger-type equation for a non-interacting N-body Harmonic oscillator system[21]. In November 2023, Grosse, Sako, Wulkenhaar, and the author of this thesis showed that the partition function of a real symmetric Φ^4 matrix model corresponds to a zero-energy solution of a Schrödinger-type equation with Calogero-Moser Hamiltonian[22]. A family of differential equations satisfied by the partition function was also obtained from the Virasoro algebra (Witt algebra).

1.2. Purpose

The purpose of this thesis is to make progress in Grosse-Steinacker-Wulkenhaar type noncommutative scalar field theories by using the technology of the random matrix theory.

Exact solutions of all multipoint correlation functions of Grosse-Steinacker-Wulkenhaar Φ^3 model were obtained by taking large N, V limit[18, 19]. Taking large N, V limit simplifies the equation for the n-point function, which is given by an integral equation. Because numerous terms in this equation disappear during the limit operation. Exact solutions of all multipoint correlation functions of Grosse-Steinacker-Wulkenhaar Φ^3 model were not obtained for the ones with finite matrix degrees of freedom. Therefore, one of the purposes of this thesis is to obtain exact solutions for all multipoint correlation functions in Grosse-Steinacker-Wulkenhaar Φ^3 model with finite N, V without any limit operations.

Grosse-Wulkenhaar Φ^4 model is more challenging than Grosse-Steinacker-Wulkenhaar Φ^3 model. Therefore, we developed an integrable Φ^3-Φ^4 hybrid matrix model that includes Φ^4 interaction, which has easier aspects than Grosse-Wulkenhaar Φ^4 model. In this thesis, the goal is to develop a perturbation theory for this model since it is a newly created model. The other purpose is also to obtain exact solutions for some of the multipoint correlation functions, which are also required. The following is more detailed descriptions.

In this thesis, we study Φ^3 matrix model in noncommutative scalar fields of Grosse-Steinacker-Wulkenhaar type. As mentioned above, all multipoint correlation functions of Φ^3 matrix model in large N, V limit are computed by Grosse, Sako, and Wulkenhaar by solving exactly Schwinger-Dyson equations [18, 19]. However, its exact solution was not obtained due to the complexity of the matrix with finite degrees of freedom. Therefore, in this thesis, we study the exact solutions of the multipoint correlation function in the finite Grosse-Steinacker-Wulkenhaar Φ^3 model without any limit operation[40]. In calculating the partition function $\mathcal{Z}[J]$, where $J = (J_{ab})$ is an auxiliary matrix, the integration of the off-diagonal elements of the Hermitian matrix is calculated using Harish-Chandra-Itzykson-Zuber integral[36, 51, 24, 56]. On the other hand, the integral of the diagonal elements of the Hermitian matrix is done by using the Airy functions. We use the result to calculate the multipoint correlation function when the external field J is a diagonal matrix. The exact solutions of the multipoint correlation function when the external field J is a diagonal matrix can be obtained by calculating the n-th derivative $\partial^n/\partial J_{a^1a^1}\cdots\partial J_{a^na^n}$ of $\log\mathcal{Z}[J]$. We apply the formula from the previous study [23] to this formula. As a result, we succeed in finding the exact solutions for the multipoint correlation function when the external field J is a diagonal matrix. Since arbitrary multipoint correlation function can be expressed by using the multipoint correlation function when the external field J is a diagonal matrix, we can obtain all the exact solutions of Φ^3 finite matrix model. We also give one-point function, two-point function that has one boundary, two-point function that has two boundaries, and three-point function that has three boundaries as concrete function examples. We also study Φ^3-Φ^4 hybrid matrix model in a noncommutative scalar field of Grosse-Steinacker-Wulkenhaar type[41]. Grosse-Wulkenhaar Φ^4 matrix model has many unanswered questions compared to Φ^3 theory. For example, Grosse-Steinacker-Wulkenhaar Φ^3 model can be transformed using Itzykson-Zuber integral[40]. But Grosse-Wulkenhaar Φ^4 model cannot. So the exact solutions of the multi-point correlation function of Grosse-Wulkenhaar Φ^4 matrix model for the case of finite matrix degrees of freedom have not been obtained. Also, Grosse-Wulkenhaar Φ^4 model has an unknown integrable form by the KdV hierarchy. It seems natural to think that we should study a matrix model that is similar to Grosse-Wulkenhaar Φ^4 model but removed the difficulties of Grosse-Wulkenhaar Φ^4 model. Therefore, we consider a matrix model with mixed potentials, in particular a 1-matrix model with 3-point interaction and 4-point interaction. However, the 3-point interaction of Φ^3 is multiplied by a diagonal matrix M (Φ^3-Φ^4 hybrid matrix model). This model is a generalized Kontsevich-type integrable matrix model corresponding to the higher-order KdV hierarchy. In this study, we construct Feynman rules for this model in terms of quantum field theories. The Φ^3 interaction is $\mathrm{tr} M\Phi^3$ and M is inserted so that it is unconventional Feynman rules. We also compute the connected multipoint correlation function of this model in perturbation theories and give their exact solutions. First, using the Feynman rules of Φ^3-Φ^4 hybrid matrix model, perturbative expansions for one-point function that has one boundary, two-point function that has one boundary, and two-point function that has two boundaries are computed by drawing Feynman diagrams for matrix size $N = 2$ case as pedagogical instructions to understand the way of calculations. Next, we perform perturbative calculations for one-point function that has one boundary, two-point function that has one boundary, and two-point function that has two boundaries in the case that the matrix size is any N. On the other hand, the calculation of the partition function $\mathcal{Z}[J]$ in Φ^3-Φ^4 hybrid matrix model can be carried out rigorously. For the computation of the partition function $\mathcal{Z}[J]$, the integral of the off-diagonal elements of the Hermitian matrix is computed using Itzykson-Zuber integral[36, 51, 10, 56]. In contrast, the integral of the diagonal elements of the Hermite matrix is obtained by using a function

$P(z)$ that is similar to the Pearcey integral. We then use the exact calculated partition function $\mathcal{Z}[J]$ to compute the exact solutions for one-point function that has one boundary, two-point function that has one boundary, two-point function that has two boundaries, and n-point function that has n boundaries for any N matrix size. We verify that the final results of the perturbative expansions for $N = 2$ are in agreement with the saddle point approximation using the results of the exact solutions in one-point function that has one boundary, two-point function that has one boundary, and two-point function that has two boundaries by setting $N = 2$. Finally, we make remarks about contributions from Feynman diagrams of Φ^3-Φ^4 hybrid matrix model corresponding to nonplanar or higher genus surfaces.

1.3. Organization of the thesis

In Chapter 2, we review Grosse-Steinacker-Wulkenhaar Φ^3 model[13, 14, 15, 18, 19, 30]. In the previous study, all multipoint correlation functions of Φ^3 matrix model in large N, V limit were computed by Grosse, Sako, and Wulkenhaar by solving exactly Schwinger-Dyson equations[18, 19]. In Chapter 3, we obtain the exact solutions of the multipoint correlation function of finite Grosse-Steinacker-Wulkenhaar Φ^3 model without any limit operations[40]. It is known that an arbitrary multipoint correlation function can be computed using the multipoint correlation function when the external field J is a diagonal matrix. Therefore, we compute exactly the multipoint correlation function when the external field J is a diagonal matrix. In Chapter 4, we consider a matrix model with mixed potentials, in particular a 1-matrix model with 3-point interaction and 4-point interaction[41]. Note that the 3-point interaction of Φ^3 is multiplied by a diagonal matrix M (Φ^3-Φ^4 hybrid matrix model). This M allows the model to be integrable. We construct Feynman rules for this model in terms of quantum field theories. The Φ^3 interaction is $\mathrm{tr} M\Phi^3$ and M is inserted so that it is unconventional Feynman rules. We also compute the connected multipoint correlation function of this model in perturbation theories and give their exact nonperturbative solutions. In Chapter 5, we summarize this thesis.

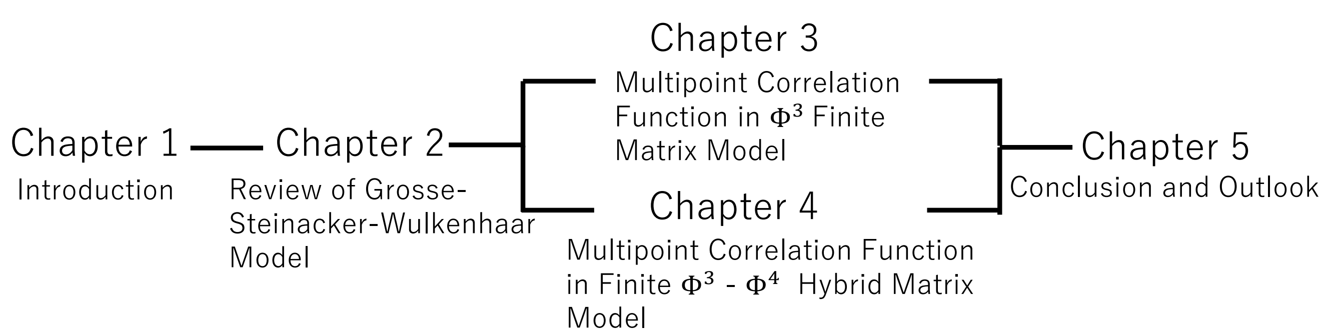


Figure 1.1: connections of the chapters

Chapter 2

Review of Grosse-Steinacker-Wulkenhaar Model

In this chapter, we summarize the definitions and theorems necessary to understand Chapters 3 and 4. We review Grosse–Steinacker-Wulkenhaar Φ^3 model and its research by Grosse, Hock, Sako, Steinacker, and Wulkenhaar[13, 14, 15, 18, 19, 30, 31]. They investigated the model as a renormalizable quantum field theory on Moyal spaces, which historically faced the UV/IR mixing issue when examining quantum field theories on noncommutative spaces. To solve this issue, Grosse and Steinacker modified the behavior of scalar ϕ^3 theories by including a harmonic oscillator potential[13, 14, 15]. This modification helped eliminate the UV/IR mixing issue and allowed for the renormalization of the noncommutative Φ^3 matrix model. Thus, scalar ϕ^3 theories on Moyal spaces are given a harmonic oscillator potential to achieve renormalization.

2.1. QFT on Moyal Spaces and Grosse-Steinacker-Wulkenhaar Model

We review QFT on Moyal Spaces and Grosse-Steinacker-Wulkenhaar Φ^3 model in the two-dimensional case for simplicity[13, 18, 30].

We define Schwartz spaces in the two-dimensional case

$$\mathcal{S}(\mathbb{R}^2) = \left\{ g \in C^\infty(\mathbb{R}^2) \,\middle|\, \sup_{x\in\mathbb{R}^2} |x^\alpha \partial^\beta g(x)| < \infty \quad \forall \alpha, \beta \in \mathbb{N}^2 \right\}, \tag{2.1}$$

where $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$, $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2}$, and $\partial^\beta g := \dfrac{\partial^{\beta_1+\beta_2} g}{\partial x_1^{\beta_1} \partial x_2^{\beta_2}}$. The Moyal product[12, 45] is defined by

$$f(x) \star g(x) = \exp\left\{ \frac{i\theta}{2} \left(\partial_1 \partial_2' - \partial_2 \partial_1'\right) \right\} f(x) g(x') \bigg|_{x'=x}, \tag{2.2}$$

where $\theta \in \mathbb{R}$, $\partial_k = \dfrac{\partial}{\partial x_k}$, $\partial_l' = \dfrac{\partial}{\partial x_l'}$, and $f, g \in \mathcal{S}(\mathbb{R}^2)$. The Moyal product shows that the order of the product of functions is noncommutative. From this, the Moyal plane $(\mathcal{S}(\mathbb{R}^2), \star) := \mathbb{R}^2_\theta$ is a set with a noncommutative algebraic structure[47].

We consider the Moyal plane $\mathbb{R}^2_\theta$. Then, $[f(x), g(x)]_\star$ is calculated as follows:

$$\begin{aligned}[f(x), g(x)]_\star &= f(x) \star g(x) - g(x) \star f(x) \\ =& i\theta \left\{ \partial_1 f(x) \partial_2 g(x) - \partial_2 f(x) \partial_1 g(x) \right\} + \mathcal{O}(\theta^2).\end{aligned} \tag{2.3}$$

The coordinates $x_i \;\; (i = 1, 2)$ on the Moyal plane $\mathbb{R}^2_\theta$ are noncommutative:

$$\begin{aligned}&[x_1, x_2]_\star = x_1 \star x_2 - x_2 \star x_1 \\ =& i\theta \left(\partial_1 x_1 \partial_2 x_2 - \partial_2 x_1 \partial_1 x_2 \right) + 0 \\ =& i\theta.\end{aligned} \tag{2.4}$$

We define $z = x_1 + ix_2$ and $\overline{z} = x_1 - ix_2$. Then

$$\begin{aligned}[z, \overline{z}]_\star =& [x_1 + ix_2, x_1 - ix_2]_\star \\ =& 2\theta,\end{aligned} \tag{2.5}$$

where we use (2.4). Next we define an annihilation operator $a := \dfrac{z}{\sqrt{2\theta}}$ and a creation operator $a^\dagger := \dfrac{\overline{z}}{\sqrt{2\theta}}$. These satisfy the following relationship: $[a, a^\dagger]_\star = \dfrac{1}{2\theta}[z, \overline{z}]_\star = 1$, $[a, a]_\star = \dfrac{1}{2\theta}[z, z]_\star = 0$, and $[a^\dagger, a^\dagger]_\star = \dfrac{1}{2\theta}[\overline{z}, \overline{z}]_\star = 0$. We can use $\dfrac{\partial}{\partial z} = -\dfrac{1}{\sqrt{2\theta}}[a^\dagger, \bullet]_\star$ and $\dfrac{\partial}{\partial \overline{z}} = \dfrac{1}{\sqrt{2\theta}}[a, \bullet]_\star$. For example, $\dfrac{\partial z}{\partial z} = -\dfrac{1}{\sqrt{2\theta}}[a^\dagger, z]_\star = 1$, $\dfrac{\partial \overline{z}}{\partial z} = -\dfrac{1}{\sqrt{2\theta}}[a^\dagger, \overline{z}]_\star = 0$, $\dfrac{\partial z}{\partial \overline{z}} = \dfrac{1}{\sqrt{2\theta}}[a, z]_\star = 0$, and $\dfrac{\partial \overline{z}}{\partial \overline{z}} = \dfrac{1}{\sqrt{2\theta}}[a, \overline{z}]_\star = 1$. Let us introduce Fock basis $|n\rangle := \dfrac{1}{\sqrt{n!}} \underbrace{a^\dagger \star a^\dagger \star \cdots \star a^\dagger}_{n} \star |0\rangle = \dfrac{1}{\sqrt{n!}} (a^\dagger)^n_\star |0\rangle$, where $|0\rangle$ is defined a non-zero vector satisfying $a\,|0\rangle = 0$. A number operator is $\mathcal{N} := a^\dagger a$. The Eigenvalue n of number operator $\mathcal{N}$ satisfy the following relationship: $\mathcal{N}\,|n\rangle = n\,|n\rangle$, $n = 0, 1, 2, \cdots$. $\langle n|$ is a dual vector of $|n\rangle$, where they satisfy the following relationship: $\langle n \,|\, m\rangle = \delta_{nm}$. The scalar field is $\phi = \displaystyle\sum_{n,m=0}^{\infty} \Phi_{nm}\,|m\rangle\,\langle n|$. Here $\Phi_{nm} \in \mathbb{C}$ satisfies $\overline{\Phi}_{nm} = \Phi_{mn}$. The action for scalar ϕ^3 field theories on Moyal spaces with a harmonic oscillator is

$$S[\phi] = \frac{1}{8\pi} \int dx_1 dx_2 \left\{ \kappa\phi - 4\phi \star \left(\frac{\partial}{\partial z} \frac{\partial}{\partial \overline{z}} \right) \phi + \frac{\lambda}{3} \phi \star \phi \star \phi + \frac{\mu_0^2}{2} \phi \star \phi + \frac{4}{\theta} \phi \star a \star \phi \star a^\dagger \right\}, \tag{2.6}$$

where κ is a renormalization constant (real), λ is a coupling constant that is non-zero real, and μ_0^2

is a squared mass. Using $\frac{\partial}{\partial z} = -\frac{1}{\sqrt{2\theta}}[a^\dagger, \bullet]_\star$ and $\frac{\partial}{\partial \bar{z}} = \frac{1}{\sqrt{2\theta}}[a, \bullet]_\star$,

$$
\begin{aligned}
S[\phi] =&(2.6)\\
=&\frac{1}{8\pi}\int dx_1 dx_2 \left\{ \kappa\phi + \frac{2}{\theta}\phi\star[a^\dagger,[a,\phi]_\star]_\star + \frac{\lambda}{3}\phi\star\phi\star\phi + \frac{\mu_0^2}{2}\phi\star\phi + \frac{4}{\theta}\phi\star a\star\phi\star a^\dagger \right\}\\
=&\frac{1}{8\pi}\int dx_1 dx_2 \left\{ \kappa\phi + \frac{4}{\theta}\left(\frac{1}{2}\phi\star a^\dagger\star a\star\phi - \frac{1}{2}\phi\star a\star\phi\star a^\dagger - \frac{1}{2}\phi\star a^\dagger\star\phi\star a + \frac{1}{2}\phi\star\phi\star a\star a^\dagger\right)\right.\\
&\left. + \frac{4}{\theta}\phi\star a\star\phi\star a^\dagger + \frac{\lambda}{3}\phi\star\phi\star\phi + \frac{\mu_0^2}{2}\phi\star\phi \right\}.
\end{aligned} \tag{2.7}
$$

Now, we use $\int dx_1 dx_2 = (2\pi\theta)\mathrm{Tr}$, where Tr satisfy $\mathrm{Tr}\,|n\rangle\langle m| = \delta_{nm}$, and $aa^\dagger = [a, a^\dagger]_\star + a^\dagger a = 1 + \mathcal{N}$. Then (2.7) is rewritten as

$$
\begin{aligned}
S[\Phi] =&\frac{\theta}{4}\sum_{n,m,k=0}^{\infty}\left(\kappa\Phi_{nn} + \frac{\lambda}{3}\Phi_{nm}\Phi_{mk}\Phi_{kn} + \frac{\mu_0^2}{2}\Phi_{nm}\Phi_{mn}\right)\\
&+\frac{\theta}{4}\sum_{n,m,n',m'=0}^{\infty}\mathrm{Tr}\frac{4}{\theta}\left(\frac{1}{2}\Phi_{nm}|n\rangle\langle m|\mathcal{N}\Phi_{n'm'}|n'\rangle\langle m'| + \frac{1}{2}\Phi_{nm}|n\rangle\langle m|\Phi_{n'm'}|n'\rangle\langle m'|\right)\\
&+\frac{\theta}{4}\sum_{n,m,n',m'=0}^{\infty}\mathrm{Tr}\frac{4}{\theta}\left(\frac{1}{2}\Phi_{nm}|n\rangle\langle m|\Phi_{n'm'}|n'\rangle\langle m'|\mathcal{N}\right)\\
=&\frac{\theta}{4}\sum_{n,m,k=0}^{\infty}\left(\kappa\Phi_{nn} + \frac{\lambda}{3}\Phi_{nm}\Phi_{mk}\Phi_{kn} + \frac{\mu_0^2}{2}\Phi_{nm}\Phi_{mn} + \frac{4}{\theta}\frac{1}{2}\Phi_{nm}\Phi_{mn} + \frac{4}{\theta}\Phi_{nm}\Phi_{mn}m\right).
\end{aligned} \tag{2.8}
$$

We define $V = \frac{\theta}{4}$ and $\mu^2 = \frac{1}{V} + \mu_0^2$. Then, we get the action of the matrix model:

$$
S[\Phi] = V\sum_{n,m,k=0}^{N}\left\{\kappa\Phi_{nn} + \frac{\lambda}{3}\Phi_{nm}\Phi_{mk}\Phi_{kn} + \left(\frac{\mu^2}{2} + \frac{m}{V}\right)\Phi_{nm}\Phi_{mn}\right\}. \tag{2.9}
$$

The size of the matrix is set from 0 to ∞, but change it and make this the size N of a finite matrix. We define $E_m = \frac{\mu^2}{2} + \frac{m}{V}$. Let $E = (E_m\delta_{mn})$ be a diagonal matrix for $m, n = 0, \ldots, N$. Then the action for scalar ϕ^3 field theories on Moyal spaces with a harmonic oscillator potential is obtained by the following matrix model:

$$
S[\Phi] = V\mathrm{tr}\left(E\Phi^2 + \kappa\Phi + \frac{\lambda}{3}\Phi^3\right). \tag{2.10}
$$

This is called Grosse-Steinacker-Wulkenhaar Φ^3 Model.

2.2. Setup of Grosse-Steinacker-Wulkenhaar Model

In this section, we review Φ^3 matrix model based on the previous studies[18, 19, 30, 31], and we determine the notation.

Let $\Phi = (\Phi_{ij})$ be a Hermitian matrix for $i, j = 1, 2, \ldots, N$ and E_m be a discretization of a monotonously increasing differentiable function e with $e(0) = 0$,

$$E_m = \mu^2 \left(\frac{1}{2} + e\left(\frac{m}{\mu^2 V} \right) \right), \tag{2.11}$$

where μ^2 is a squared mass and V is a real constant. Let $E = (E_m \delta_{mn})$ be a diagonal matrix for $m, n = 1, \ldots, N$. Let us consider the following action:

$$S[\Phi] = V \mathrm{tr} \left(E\Phi^2 + \kappa\Phi + \frac{\lambda}{3}\Phi^3 \right), \tag{2.12}$$

where κ is a renormalization constant (real), λ is a coupling constant that is non-zero real. By the diagonal matrix E that is not proportional to the unit matrix in general, there is no symmetry for the unitary transformation in $\Phi \to U\Phi U^*$. Here U is a unitary matrix, and U^* is its Hermitian conjugate.

Let $J = (J_{mn})$ be a Hermitian matrix for $m, n = 1, \ldots, N$ as an external field. Let $\mathcal{D}\Phi$ be the integral measure,

$$\mathcal{D}\Phi := \prod_{i=1}^{N} d\Phi_{ii} \prod_{1 \le i < j \le N} d\mathrm{Re}\Phi_{ij} d\mathrm{Im}\Phi_{ij}, \tag{2.13}$$

where each variable is divided into real and imaginary parts $\Phi_{ij} = \mathrm{Re}\Phi_{ij} + i\mathrm{Im}\Phi_{ij}$ with $\mathrm{Re}\Phi_{ij} = \mathrm{Re}\Phi_{ji}$ and $\mathrm{Im}\Phi_{ij} = -\mathrm{Im}\Phi_{ji}$. Let us consider the following partition function:

$$\begin{aligned} \mathcal{Z}[J] &:= \int \mathcal{D}\Phi \exp\left(-S[\Phi] + V\mathrm{tr}(J\Phi)\right) \\ &= \int \mathcal{D}\Phi \exp\left(-V\mathrm{tr}\left(E\Phi^2 + \kappa\Phi + \frac{\lambda}{3}\Phi^3 \right) \right) \exp\left(V\mathrm{tr}\left(J\Phi \right) \right). \end{aligned} \tag{2.14}$$

Using $\log \frac{\mathcal{Z}[J]}{\mathcal{Z}[0]}$, the $\sum_{i=1}^{B} N_i$-point function $G_{|a_1^1 \ldots a_{N_1}^1| \ldots |a_1^B \ldots a_{N_B}^B|}$ is defined as

$$\log \frac{\mathcal{Z}[J]}{\mathcal{Z}[0]} := \sum_{B=1}^{\infty} \sum_{1 \le N_1 \le \cdots \le N_B}^{\infty} \sum_{p_1^1, \ldots, p_{N_B}^B = 1}^{N} V^{2-B} \frac{G_{|p_1^1 \ldots p_{N_1}^1| \ldots |p_1^B \ldots p_{N_B}^B|}}{S_{(N_1, \ldots, N_B)}} \prod_{\beta=1}^{B} \frac{\mathbb{J}_{p_1^\beta \ldots p_{N_\beta}^\beta}}{N_\beta}, \tag{2.15}$$

where N_i is the identical valence number for $i = 1, \ldots, B$, $\mathbb{J}_{p_1^i \ldots p_{N_i}^i} := \prod_{j=1}^{N_i} J_{p_j^i p_{j+1}^i}$ with $N_i + 1 \equiv 1$, $(N_1, \ldots, N_B) = (\underbrace{N_1', \ldots, N_1'}_{\nu_1}, \ldots, \underbrace{N_s', \ldots, N_s'}_{\nu_s})$, and $S_{(N_1, \ldots, N_B)} = \prod_{\beta=1}^{s} \nu_\beta !$. The $\sum_{i=1}^{B} N_i$-point function denoted by $G_{|a_1^1\ldots a_{N_1}^1|\ldots|a_1^B\ldots a_{N_B}^B|}$ is given by the sum over all Feynman diagrams (ribbon graphs) on Riemann surfaces with B-boundaries, and each $|a_1^i\cdots a_{N_i}^i|$ corresponds to the Feynman diagrams having N_i-external ribbons from the i-th boundary. (See Figure 2.1.)

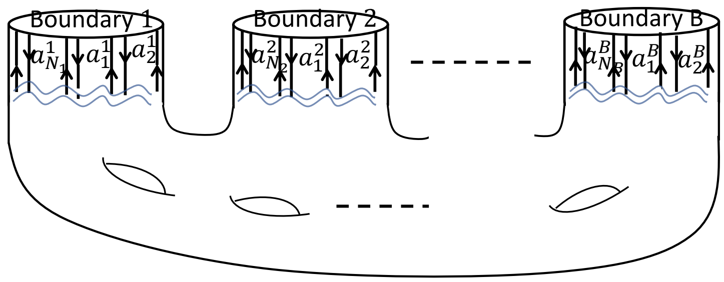


Figure 2.1: The relationship between external ribbons of Feynman diagrams and boundaries as expressed in $G_{|a_1^1\ldots a_{N_1}^1|\ldots|a_1^B\ldots a_{N_B}^B|}$

We give the reason why the Figure 2.1 picture for the Feynman diagram is obtained, in the following. We define a $\displaystyle\sum_{i=1}^{B} N_i$-point cumulant which represent contributions of connected Feynman diagrams as

$$\begin{aligned}&\langle \Phi_{a_1^1a_2^1}\cdots\Phi_{a_{N_1}^1a_1^1}\Phi_{a_1^2a_2^2}\cdots\Phi_{a_{N_2}^2a_1^2}\cdots\Phi_{a_1^Ba_2^B}\cdots\Phi_{a_{N_B}^Ba_1^B}\rangle_c\\ &:=\frac{1}{V^{N_1+\cdots+N_B}}\frac{\partial}{\partial J_{a_2^1a_1^1}}\cdots\frac{\partial}{\partial J_{a_1^Ba_{N_B}^B}}\log\mathcal{Z}[J]\Bigg|_{J=0}.\end{aligned}\tag{2.16}$$

Let us focus on a Feynman diagram with $\mathscr{N}:=\displaystyle\sum_{i=1}^{B}N_i$-external ribbons. Let Σ be the number of loops contained in the Feynman diagram. Let k_3 be the number of $V\mathrm{tr}\Phi^3$ interactions in the Feynman diagram. The contribution from such Feynman diagram has $V^{k_3-\frac{3k_3+\mathscr{N}}{2}}$ since the contribution from vertexes is V^{k_3} and the contribution from propagators is $\left(\dfrac{1}{V}\right)^{\frac{3k_3+\mathscr{N}}{2}}$. Also, the Euler number of a surface with genus "g" and boundaries "B" is $\chi=2-2g-B$. For this Feynman diagram, the corresponding Euler number is given by $\chi=(k_3+\mathscr{N})-\left(\frac{3k_3+\mathscr{N}}{2}+\mathscr{N}\right)+(\mathscr{N}+\sum)$. Here $k_3+\mathscr{N}$ is the number of vertexes, $\left(\frac{3k_3+\mathscr{N}}{2}+\mathscr{N}\right)$ is the number of the edges, and $(\mathscr{N}+\Sigma)$ is the number of the faces in the Feynman diagrams. Note that we count one ribbon as one edge, here. Let us see the reason why the last $+\mathscr{N}$ of $\left(\frac{3k_3+\mathscr{N}}{2}+\mathscr{N}\right)$ appears in the number of edge, and $\mathscr{N}$ also represents the number of faces. For example, we see the i-th boundary. There are N_i faces touching one boundary, since there is N_i external ribbons in the Feynman diagram from the term $\displaystyle\prod_{j=1}^{N_i}J_{p_j^ip_{j+1}^i}$ with $N_i+1\equiv 1$. (See Figure 2.2.)

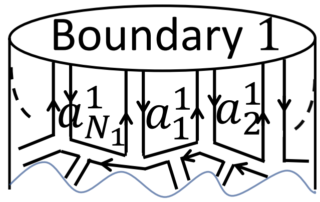


Figure 2.2: The relationship between external ribbons of Feynman diagrams and Boundary 1 as expressed in $G_{|a^1_1\dots a^1_{N_1}|}$

Therefore the number of all surfaces touching the boundary is $\mathscr{N}=\sum_{i=1}^{B}N_i$ in this case, and $\mathscr{N}$ edges appear as not ribbons but segments on boundaries. The contribution from the Feynman diagram has $V^{k_3-\frac{3k_3+\mathscr{N}}{2}}=V^{\chi-\mathscr{N}-\Sigma}=V^{2-2g-B-\mathscr{N}-\Sigma}$. So, we introduce $G_{|a^1_1\dots a^1_{N_1}|\dots|a^B_1\dots a^B_{N_B}|}$ for pairwise different a^i_j $(i=1,\cdots,B,\ \ j=1,\cdots,N_i)$ as in the following equation.

$$\langle\Phi_{a^1_1a^1_2}\cdots\Phi_{a^1_{N_1}a^1_1}\Phi_{a^2_1a^2_2}\cdots\Phi_{a^2_{N_2}a^2_1}\cdots\Phi_{a^B_1a^B_2}\cdots\Phi_{a^B_{N_B}a^B_1}\rangle_c=V^{2-\mathscr{N}-B}G_{|a^1_1\dots a^1_{N_1}|\dots|a^B_1\dots a^B_{N_B}|}. \tag{2.17}$$

Let us check its consistency with (2.15). Note that

$$\begin{aligned}&\frac{1}{V^{\mathscr{N}}}\frac{\partial^{\mathscr{N}}}{\partial J_{a^1_2a^1_1}\cdots\partial J_{a^B_1a^B_{N_B}}}\sum_{B'=1}^{\infty}\sum_{1\le\dots\le N_{B'}}^{\infty}\sum_{p^1_1,\dots,p^{B'}_{N_{B'}}=1}^{N}\prod_{\beta=1}^{B'}\frac{\mathbb{J}_{p^\beta_1\dots p^\beta_{N_\beta}}}{N_\beta}\,G_{|p^1_1\dots p^1_{N_1}|\dots|p^{B'}_1\dots p^{B'}_{N_{B'}}|}\Bigg|_{J=0}\\&=\frac{1}{V^{\mathscr{N}}}G_{|a^1_1\dots a^1_{N_1}|\dots|a^B_1\dots a^B_{N_B}|}\times S_{(N_1,\dots,N_B)}.\end{aligned} \tag{2.18}$$

Then the $\mathscr{N}$-th derivative of the right-hand side of (2.15) with respect to $J_{a^1_2a^1_1},\cdots,J_{a^B_1a^B_{N_B}}$ is given by

$$\frac{1}{V^{\mathscr{N}}}\frac{\partial^{\mathscr{N}}}{\partial J_{a^1_2a^1_1}\cdots\partial J_{a^B_1a^B_{N_B}}}\left(R.H.S\ of\ (2.15)\right)=V^{2-\mathscr{N}-B}G_{|a^1_1\dots a^1_{N_1}|\dots|a^B_1\dots a^B_{N_B}|}, \tag{2.19}$$

and the corresponding one from the left-hand side of (2.15) is given as

$$\frac{1}{V^{\mathscr{N}}}\frac{\partial^{\mathscr{N}}}{\partial J_{a^1_2a^1_1}\cdots\partial J_{a^B_1a^B_{N_B}}}\left(L.H.S\ of\ (2.15)\right)=\langle\Phi_{a^1_1a^1_2}\cdots\Phi_{a^1_{N_1}a^1_1}\Phi_{a^2_1a^2_2}\cdots\Phi_{a^2_{N_2}a^2_1}\cdots\Phi_{a^B_1a^B_2}\cdots\Phi_{a^B_{N_B}a^B_1}\rangle_c. \tag{2.20}$$

Therefore, we found that (2.17) is consistent with (2.15) when all a^i_j are pairwise different.

If there is no condition that any two indexes do not much, then (2.17) is not necessarily correct. $\langle\Phi_{a_1^1 a_2^1}\cdots\Phi_{a_{N_1}^1 a_1^1}\Phi_{a_1^2 a_2^2}\cdots\Phi_{a_{N_2}^2 a_1^2}\cdots\Phi_{a_1^B a_2^B}\cdots\Phi_{a_{N_B}^B a_1^B}\rangle_c$ might include contributions from several types of surfaces classified by their boundaries. For example, let us consider $\langle\Phi_{aa}\Phi_{aa}\rangle_c$. From (2.15), $\langle\Phi_{aa}\Phi_{aa}\rangle_c = \frac{1}{V}G_{|aa|} + \frac{1}{V^2}G_{|a|a|}$. This means that $\langle\Phi_{aa}\Phi_{aa}\rangle_c$ includes contributions from two types of surfaces which are surfaces with one boundary and ones with two boundaries.

From these observations, it is concluded that we should prepare a connected oriented surface with B boundaries for drawing each Feynman diagram to calculate $G_{|a_1^1\ldots a_{N_1}^1|\ldots|a_1^B\ldots a_{N_B}^B|}$. We draw a Feynman diagram with external ribbons with $(a_1^i a_2^i),\cdots,(a_{N_i}^i a_1^i)$ subscripted to each boundary i. For any connected segments in a Feynman diagram, both ends are on the same boundary. $G_{|a_1^1\ldots a_{N_1}^1|\ldots|a_1^B\ldots a_{N_B}^B|}$ is given by the sum over all such Feynman diagrams.

In addition, since it is $V^{k_3-\frac{3k_3+\mathscr{N}}{2}} = V^{\chi-\mathscr{N}-\Sigma} = V^{2-2g-B-\mathscr{N}-\Sigma}$, we can consider "genus expansion" of $G_{|a_1^1\ldots a_{N_1}^1|\ldots|a_1^B\ldots a_{N_B}^B|}$ like [31] as

$$G_{|a_1^1\ldots a_{N_1}^1|\ldots|a_1^B\ldots a_{N_B}^B|} = \sum_{g=0}^{\infty} V^{-2g} G^{(g)}_{|a_1^1\ldots a_{N_1}^1|\ldots|a_1^B\ldots a_{N_B}^B|}. \tag{2.21}$$

2.3. Ward-Takahashi Identity and Schwinger-Dyson Equation

We review the derivation of the Schwinger-Dyson equation and Ward-Takahashi identity for Φ^3 matrix model based on previous studies [18, 30]. First, we calculate the partition function $\mathcal{Z}[J]$ of Φ^3 matrix model.

$$\begin{aligned}
&Z[J] = \int \mathcal{D}\Phi \exp\left(-S[\Phi] + V tr(J\Phi)\right)\\
=&\int \mathcal{D}\Phi \exp\left(-\frac{\lambda}{3V^2}\sum_{m,n,k=1}^{N}\frac{\partial^3}{\partial J_{mn}\partial J_{nk}\partial J_{km}}\right)\\
&\times \exp\sum_{m,n=1}^{N} -\frac{V}{2}\left(\Phi_{mn} + (\kappa\delta_{mn} - J_{mn})\frac{1}{E_m+E_n}\right)(E_m+E_n)\left(\Phi_{nm} + (\kappa\delta_{mn} - J_{nm})\frac{1}{E_m+E_n}\right)\\
&\times \exp\sum_{m,n=1}^{N} \frac{V}{2}(\kappa\delta_{mn} - J_{mn})\frac{1}{E_m+E_n}(\kappa\delta_{mn} - J_{nm})\\
=&C\exp\left(-\frac{\lambda}{3V^2}\sum_{m,n,k=1}^{N}\frac{\partial^3}{\partial J_{mn}\partial J_{nk}\partial J_{km}}\right)Z_{free}[J]\\
=&Ce^{-S_{int}\left(\frac{1}{V}\frac{\partial}{\partial J}\right)}Z_{free}[J],
\end{aligned}\tag{2.22}$$

where C is a constant, $S_{int}\left(\frac{1}{V}\frac{\partial}{\partial J}\right) := \frac{\lambda}{3V^2}\sum_{m,n,k=1}^{N}\frac{\partial^3}{\partial J_{mn}\partial J_{nk}\partial J_{km}}$, and the partition function of the free field $Z_{free}[J]$ is as follows:

$$Z_{free}[J] = \exp\left(\frac{V}{2}\sum_{m,n=1}^{N}\frac{(J_{mn}-\kappa\delta_{mn})(J_{nm}-\kappa\delta_{nm})}{E_n+E_m}\right). \tag{2.23}$$

Here, we introduce the Ward-Takahashi identity. We calculated $\Phi' = U\Phi U^{-1} = (1+iu)\Phi(1-iu)+\mathcal{O}(u^2) = \Phi + iu\Phi - i\Phi u + \mathcal{O}(u^2) = \Phi + i[u,\Phi] + \mathcal{O}(u^2)$, where $U = e^{iu}$ and u is the Hermitian matrix. At order u^1, $S[\Phi'] - S[\Phi] = \delta S$ is calculated as

$$\begin{aligned}\delta S =& Vtr\left(iE[u,\Phi]\Phi + iE\Phi[u,\Phi]\right)\\ =& \sum_{m,k=1}^{N} iu_{mk}V\left\{(E_m-E_k)\sum_{l=1}^{N}\Phi_{kl}\Phi_{lm}\right\}.\end{aligned} \tag{2.24}$$

$\delta(Vtr[J\Phi])$ was calculated as follows:

$$\begin{aligned}\delta(Vtr[J\Phi]) =& Vtr\left(J\Phi' - J\Phi\right)\\ =& \sum_{m,k=1}^{N} iu_{mk}V\left\{\sum_{n=1}^{N}\left(J_{nm}\Phi_{kn} - J_{kn}\Phi_{nm}\right)\right\}.\end{aligned} \tag{2.25}$$

In addition, since $\mathcal{D}\Phi = \mathcal{D}\Phi'$, we obtain the following.

$$\begin{aligned}0 =& \int \mathcal{D}\Phi' e^{-S[\Phi']+Vtr[J\Phi']} - \int \mathcal{D}\Phi e^{-S[\Phi]+Vtr[J\Phi]}\\ =& \int \mathcal{D}\Phi\left(-\delta S + \delta\left(Vtr[J\Phi]\right)\right)e^{-S[\Phi]+Vtr[J\Phi]}.\end{aligned} \tag{2.26}$$

We applied (2.24) and (2.25) to (2.26). Additionally, we obtain the Ward-Takahashi identity of Φ^3 matrix model.

$$\begin{aligned}0 =& \int \mathcal{D}\Phi\left(-\delta S + \delta\left(Vtr[J\Phi]\right)\right)e^{-S[\Phi]+Vtr[J\Phi]}\\ =& \left[-\sum_{m,k=1}^{N} u_{mk}V\left\{(E_m-E_k)\sum_{l=1}^{N}\frac{1}{V}\frac{\partial}{\partial J_{lk}}\frac{1}{V}\frac{\partial}{\partial J_{ml}}\right\}\right.\\ &\left.+\sum_{m,k=1}^{N} u_{mk}V\left\{\sum_{n=1}^{N}\left(J_{nm}\frac{1}{V}\frac{\partial}{\partial J_{nk}} - J_{kn}\Phi_{mn}\right)\right\}\right]Z[J].\end{aligned} \tag{2.27}$$

Thus, the Ward-Takahashi identity of Φ^3 matrix model is given as follows:

$$\sum_{l=1}^{N}\frac{\partial^2}{\partial J_{lk}\partial J_{ml}}Z[J]=\frac{V}{(E_m-E_k)}\left\{\sum_{l=1}^{N}\left(J_{lm}\frac{\partial}{\partial J_{lk}}-J_{kl}\frac{\partial}{\partial J_{ml}}\right)\right\}Z[J]. \tag{2.28}$$

Next, we review the derivation of the Schwinger-Dyson equation of one-point function $G_{|a|}$. One-point function $G_{|a|}$ can be calculated as follows:

$$\begin{aligned}G_{|a|}&=\frac{1}{V}\frac{\partial\log\mathcal{Z}[J]}{\partial J_{aa}}\bigg|_{J=0}\\&=\frac{1}{V}\frac{1}{\mathcal{Z}[0]}\left(Ce^{-S_{int}[\frac{1}{V}\frac{\partial}{\partial J}]}\frac{\partial}{\partial J_{aa}}\mathcal{Z}_{free}[J]\right)\bigg|_{J=0},\end{aligned} \tag{2.29}$$

where $\frac{\partial}{\partial J_{aa}}$ and $e^{-S_{int}[\frac{1}{V}\frac{\partial}{\partial J}]}$ are commutative. Next $\frac{\partial}{\partial J_{aa}}\mathcal{Z}_{free}[J]$ is calculated as follows:

$$\begin{aligned}\frac{\partial}{\partial J_{aa}}\mathcal{Z}_{free}[J]=&\frac{V}{2}\sum_{n,m=1}^{N}\bigg\{\left(-\delta_{ma}\delta_{na}\right)H^{-1}_{mn}\left(\kappa\delta_{mn}-J_{mn}\right)\\&\qquad+\left(\kappa\delta_{mn}-J_{mn}\right)H^{-1}_{mn}\left(-\delta_{ma}\delta_{na}\right)\bigg\}\mathcal{Z}_{free}[J]\\=&-\frac{V}{H_{aa}}\left(\kappa-J_{aa}\right)\mathcal{Z}_{free}[J],\end{aligned} \tag{2.30}$$

where we define $H_{mn}=E_m+E_n$. Applying (2.30) to (2.29) gives

$$\begin{aligned}G_{|a|}&=\frac{1}{V}\frac{1}{\mathcal{Z}[J]}\left\{Ce^{-S_{int}[\frac{1}{V}\frac{\partial}{\partial J}]}\left(-\frac{V}{H_{aa}}\right)(\kappa-J_{aa})\right\}\mathcal{Z}_{free}[J]\bigg|_{J=0}\\&=\frac{1}{\mathcal{Z}[J]}\frac{1}{H_{aa}}\left(-\kappa-\frac{\lambda}{V^2}\sum_{m=1}^{N}\frac{\partial}{\partial J_{ma}}\frac{\partial}{\partial J_{am}}\right)\mathcal{Z}[J]\bigg|_{J=0}.\end{aligned} \tag{2.31}$$

We calculate $\displaystyle\sum_{m=1}^{N}\frac{1}{\mathcal{Z}[J]}\frac{\partial}{\partial J_{ma}}\frac{\partial}{\partial J_{am}}\mathcal{Z}[J]\bigg|_{J=0}$:

$$\begin{aligned}\sum_{m=1}^{N}\frac{1}{\mathcal{Z}[J]}\frac{\partial}{\partial J_{ma}}\frac{\partial}{\partial J_{am}}\mathcal{Z}[J]\bigg|_{J=0}&=\sum_{m=1}^{N}\left(\frac{\partial\log\mathcal{Z}[J]}{\partial J_{ma}}\frac{\partial\log\mathcal{Z}[J]}{\partial J_{am}}\bigg|_{J=0}+\frac{\partial}{\partial J_{ma}}\frac{\partial}{\partial J_{am}}\log\mathcal{Z}[J]\bigg|_{J=0}\right)\\&=V^2G_{|a|}^2+V\sum_{m=1}^{N}G_{|am|}+G_{|a|a|}.\end{aligned} \tag{2.32}$$

We apply (2.32) to (2.31), then the Schwinger-Dyson equation for 1-point function of Φ^3 matrix model are given:

$$G_{|a|} = \frac{1}{H_{aa}} \left\{ -\kappa - \frac{\lambda^{\frac{3}{2}}}{V^2} \left(V^2 G_{|a|}^2 + V \sum_{m=1}^{N} G_{|am|} + G_{|aa|} \right) \right\}. \tag{2.33}$$

Next, we derive the Schwinger-Dyson equation of the two-point function $G_{|ab|}$. When $a \neq b$, two-point function $G_{|ab|}$ can be calculated as follows:

$$\begin{aligned} G_{|ab|} =& \frac{1}{V} \left. \frac{\partial^2 \log Z[J]}{\partial J_{ab} \partial J_{ba}} \right|_{J=0} \\ =& \frac{1}{V} \frac{\partial}{\partial J_{ab}} \left\{ \frac{C}{Z[J]} \frac{\partial}{\partial J_{ba}} \left(e^{-S_{int}\left(\frac{1}{V}\frac{\partial}{\partial J}\right)} Z_{free}[J] \right) \right\} \Bigg|_{J=0}. \end{aligned} \tag{2.34}$$

We derive the following equation for the transformation of the right-hand side of (2.34).

$$\frac{\partial}{\partial J_{ba}} e^{-S_{int}\left(\frac{1}{V}\frac{\partial}{\partial J}\right)} Z_{free}[J] = \frac{V}{H_{ab}} e^{-S_{int}\left(\frac{1}{V}\frac{\partial}{\partial J}\right)} J_{ab} Z_{free}[J], \tag{2.35}$$

where (2.35) is computed as

$$\begin{aligned} \frac{\partial}{\partial J_{ba}} e^{-S_{int}\left(\frac{1}{V}\frac{\partial}{\partial J}\right)} Z_{free}[J] =& e^{-S_{int}\left(\frac{1}{V}\frac{\partial}{\partial J}\right)} \sum_{m,n=1}^{N} \Bigg\{ \left(-\frac{V}{2} \left(\delta_{mb}\delta_{na}\right) H^{-1}_{mn} \left(\kappa \delta_{mn} - J_{nm}\right) \right) \\ &+ \left(-\frac{V}{2} \left(\kappa\delta_{mn} - J_{mn}\right) H^{-1}_{mn} \left(\delta_{nb}\delta_{ma}\right) \right) \Bigg\} Z_{free}[J] \\ =& e^{-S_{int}\left(\frac{1}{V}\frac{\partial}{\partial J}\right)} V H^{-1}_{ab} J_{ab} Z_{free}[J]. \end{aligned} \tag{2.36}$$

We apply (2.36) to (2.34), then

$$\begin{aligned} G_{|ab|} =& \frac{C}{H_{ab}} \frac{\partial}{\partial J_{ab}} \left(\frac{1}{Z[J]} e^{-S_{int}\left(\frac{1}{V}\frac{\partial}{\partial J}\right)} J_{ab} Z_{free}[J] \right) \Bigg|_{J=0} \\ =& \frac{C}{H_{ab}} \left(-\frac{1}{Z[J]^2} \frac{\partial Z[J]}{\partial J_{ab}} e^{-S_{int}\left(\frac{1}{V}\frac{\partial}{\partial J}\right)} J_{ab} Z_{free}[J] \right) \Bigg|_{J=0} \\ &+ \frac{C}{H_{ab}} \frac{1}{Z[0]} \frac{\partial}{\partial J_{ab}} \left(e^{-S_{int}\left(\frac{1}{V}\frac{\partial}{\partial J}\right)} J_{ab} Z_{free}[J] \right) \Bigg|_{J=0}. \end{aligned} \tag{2.37}$$

Next, we calculate

$$[e^{-S_{int}[\frac{1}{V}\frac{\partial}{\partial J}]}, J_{ab}] \Big|_{J=0} = -\frac{\lambda}{V^2} \sum_{m=1}^{N} \frac{\partial}{\partial J_{ma}} \frac{\partial}{\partial J_{bm}} e^{-S_{int}[\frac{1}{V}\frac{\partial}{\partial J}]}, \tag{2.38}$$

where

$$S_{int}\left[\frac{1}{V}\frac{\partial}{\partial J}\right] := \frac{\lambda}{3}V\sum_{n,m,l=1}^{N}\frac{1}{V^3}\frac{\partial}{\partial J_{mn}}\frac{\partial}{\partial J_{lm}}\frac{\partial}{\partial J_{nl}}. \tag{2.39}$$

The calculation of (2.38) can be performed as follows:

$$\begin{aligned}[e^{-S_{int}}, J_{ba}]\big|_{J=0} =& \sum_{k=0}^{\infty}\frac{\left(-\frac{\lambda}{3}V\right)^k}{k!}\left(\sum_{n,m,l=1}^{N}\frac{1}{V^3}\frac{\partial}{\partial J_{mn}}\frac{\partial}{\partial J_{lm}}\frac{\partial}{\partial J_{nl}}\right)^k J_{ab}\Bigg|_{J=0} \\ =& -\frac{\lambda}{V^2}\sum_{m=1}^{N}\frac{\partial}{\partial J_{ma}}\frac{\partial}{\partial J_{bm}}e^{-S_{int}\left[\frac{1}{V}\frac{\partial}{\partial J}\right]}.\end{aligned} \tag{2.40}$$

We apply (2.38) and (2.28) to $C\frac{\partial}{\partial J_{ab}}e^{-S_{int}\left(\frac{1}{V}\frac{\partial}{\partial J}\right)}J_{ab}Z_{free}[J]$ to compute the following:

$$\begin{aligned}C\frac{\partial}{\partial J_{ab}}\left(e^{-S_{int}\left(\frac{1}{V}\frac{\partial}{\partial J}\right)}J_{ab}Z_{free}[J]\right)\Bigg|_{J=0} =& C\frac{\partial}{\partial J_{ab}}\left(\left[e^{-S_{int}\left(\frac{1}{V}\frac{\partial}{\partial J}\right)}, J_{ab}\right]Z_{free}[J]\right) + \frac{\partial}{\partial J_{ab}}\left(J_{ab}Z[J]\right)\Bigg|_{J=0} \\ =& \frac{\partial}{\partial J_{ab}}\left(-\frac{\lambda}{V^2}\sum_{m=1}^{N}\frac{\partial}{\partial J_{ma}}\frac{\partial}{\partial J_{bm}}Z[J]\right)\Bigg|_{J=0} + Z[0] \\ =& \frac{\lambda}{V}\frac{\partial}{\partial J_{ab}}\left\{\frac{1}{E_a - E_b}\sum_{m=1}^{N}\left(J_{mb}\frac{\partial}{\partial J_{ma}} - J_{am}\frac{\partial}{\partial J_{bm}}\right)Z[J]\right\}\Bigg|_{J=0} \\ &+ Z[0].\end{aligned} \tag{2.41}$$

We apply (2.41) to (2.37) to get the following:

$$G_{|ab|} = \frac{1}{H_{ab}} + \frac{1}{H_{ab}Z[0]}\frac{\lambda}{V}\left\{\frac{1}{E_a - E_b}\left(\frac{\partial Z[J]}{\partial J_{aa}}\Bigg|_{J=0} - \frac{\partial Z[J]}{\partial J_{bb}}\Bigg|_{J=0}\right)\right\}. \tag{2.42}$$

We obtain the Schwinger-Dyson equation for the two-point function of Φ^3 matrix model.

$$G_{|ab|} = \frac{1}{H_{ab}}\left[1 + \lambda\left\{\frac{1}{E_a - E_b}\left(G_{|a|} - G_{|b|}\right)\right\}\right]. \tag{2.43}$$

According to (2.15) we can express the connected ($N>2$)-point functions for pairwise different indices as follows:

$$G_{|a_1a_2\ldots a_N|} = \frac{1}{V}\frac{\partial}{\partial J_{a_2a_1}}\frac{\partial}{\partial J_{a_3a_2}}\cdots\frac{\partial}{\partial J_{a_Na_{N-1}}}\frac{\partial}{\partial J_{a_1a_N}}\log\frac{\mathcal{Z}[J]}{\mathcal{Z}[0]}\Bigg|_{J=0}. \tag{2.44}$$

Similar to the previous way,

$$
\begin{aligned}
G_{|a_1a_2\dots a_N|} &= \frac{1}{V}\frac{1}{\mathcal{Z}[0]}\frac{\partial}{\partial J_{a_2a_1}}\frac{\partial}{\partial J_{a_3a_2}}\cdots\frac{\partial}{\partial J_{a_Na_{N-1}}}\frac{\partial}{\partial J_{a_1a_N}}\mathcal{Z}[J]\Big|_{J=0} \\
&= \frac{C}{V}\frac{1}{\mathcal{Z}[0]}\frac{\partial}{\partial J_{a_3a_2}}\cdots\frac{\partial}{\partial J_{a_Na_{N-1}}}\frac{\partial}{\partial J_{a_1a_N}}\left(\frac{\partial}{\partial J_{a_2a_1}}e^{-S_{int}\left(\frac{1}{V}\frac{\partial}{\partial J}\right)}\mathcal{Z}_{free}[J]\right)\Big|_{J=0}.
\end{aligned}
\tag{2.45}
$$

We use (2.36), then

$$
G_{|a_1a_2\dots a_N|} = \frac{C}{\mathcal{Z}[0]}\frac{1}{E_{a_1}+E_{a_2}}\frac{\partial}{\partial J_{a_3a_2}}\cdots\frac{\partial}{\partial J_{a_Na_{N-1}}}\frac{\partial}{\partial J_{a_1a_N}}\left(e^{-S_{int}\left(\frac{1}{V}\frac{\partial}{\partial J}\right)}J_{a_1a_2}\mathcal{Z}_{free}[J]\right)\Big|_{J=0}. \tag{2.46}
$$

From (2.40), $Ce^{-S_{int}\left(\frac{1}{V}\frac{\partial}{\partial J}\right)}J_{ab}\mathcal{Z}_{free}[J] = -\frac{\lambda}{V^2}\sum_{m=1}^{N}\frac{\partial}{\partial J_{ma}}\frac{\partial}{\partial J_{bm}}\mathcal{Z}[J] + J_{ab}\mathcal{Z}[J]$, then

$$
\begin{aligned}
G_{|a_1a_2\dots a_N|} = &\frac{1}{\mathcal{Z}[0]}\frac{1}{E_{a_1}+E_{a_2}}\frac{\partial}{\partial J_{a_3a_2}}\cdots\frac{\partial}{\partial J_{a_Na_{N-1}}}\frac{\partial}{\partial J_{a_1a_N}}\left(-\frac{\lambda}{V^2}\sum_{m=1}^{N}\frac{\partial}{\partial J_{ma_1}}\frac{\partial}{\partial J_{a_2m}}\mathcal{Z}[J]\right)\Bigg|_{J=0} \\
&+\frac{1}{\mathcal{Z}[0]}\frac{1}{E_{a_1}+E_{a_2}}\frac{\partial}{\partial J_{a_3a_2}}\cdots\frac{\partial}{\partial J_{a_Na_{N-1}}}\frac{\partial}{\partial J_{a_1a_N}}\left(J_{a_1a_2}\mathcal{Z}[J]\right)\Big|_{J=0}.
\end{aligned}
\tag{2.47}
$$

Note that the second term is zero because we consider all indices are pairwise different. Using Ward-Takahashi identity (2.28),

$$
\begin{aligned}
G_{|a_1a_2\dots a_N|} &= \lambda\frac{1}{E_{a_1}+E_{a_2}}\left\{\frac{1}{\mathcal{Z}[0]}\frac{1}{V}\frac{\partial}{\partial J_{a_4a_3}}\cdots\frac{\partial}{\partial J_{a_1a_N}}\left(\frac{1}{E_{a_1}-E_{a_2}}\frac{\partial}{\partial J_{a_3a_1}}\mathcal{Z}[J]\right)\right\} \\
&\quad -\lambda\frac{1}{E_{a_1}+E_{a_2}}\left\{\frac{1}{\mathcal{Z}[0]}\frac{1}{V}\frac{\partial}{\partial J_{a_3a_2}}\cdots\frac{\partial}{\partial J_{a_Na_{N-1}}}\left(\frac{1}{E_{a_1}-E_{a_2}}\frac{\partial}{\partial J_{a_2a_N}}\mathcal{Z}[J]\right)\right\} \\
&= \lambda\frac{G_{|a_1a_3\dots a_N|}-G_{|a_2\dots a_N|}}{E_{a_1}^2-E_{a_2}^2}.
\end{aligned}
\tag{2.48}
$$

Proposition 2.1 (Grosse-Sako-Wulkenhaar[18])**.** *Let $G_{|a_1a_2,\dots a_N|}$ be connected $(N{\geq}2)$-point functions for pairwise different indices given by (2.15). Then $G_{|a_1a_2,\dots a_N|}$ is given as*

$$
G_{|a_1a_2,\dots a_N|} = \frac{\lambda^{N-2}}{2}\sum_{k=1}^{N}W_{|a_k|}\prod_{l=1,l\neq k}^{N}P_{a_ka_l}, \tag{2.49}
$$

where $W_{|a|} = 2\lambda G_{|a|} + 2E_a$, and $P_{ab} := \frac{1}{E_a^2-E_b^2}$.

Proof. We confirm Proposition 2.1 using a mathematical induction. When $N = 2$,

$$
\begin{aligned}
G_{|a_1a_2|} =&(2.43) \\
=&\frac{1}{2}\frac{W_{|a_1|}-W_{|a_2|}}{P_{a_1a_2}}.
\end{aligned}
\tag{2.50}
$$

From this, it holds true when $N=2$. Next, we assume that Proposition 2.1 holds true for the fixed N. When $N+1$,

$$\begin{aligned} G_{|a_1\dots a_{N+1}|} &= \lambda P_{a_1a_2}(G_{|a_1a_3\dots a_{N+1}|}-G_{|a_2\dots a_{N+1}|}) \\ &= \frac{\lambda^{N-1}}{2}P_{a_1a_2}\Bigg(\sum_{k=1,k\neq 2}^{N+1}W_{|a_k|}\prod_{l=1,l\notin\{2,k\}}^{N+1}P_{a_ka_l}-\sum_{k=2}^{N+1}W_{|a_k|}\prod_{l=2,l\neq k}^{N+1}P_{a_ka_l},\Bigg). \end{aligned} \tag{2.51}$$

where we use (2.48) and the assumption for N. When we substitute $k=1$ for (2.51), we calculate as follows:

$$\frac{\lambda^{N-1}}{2}W_{|a_1|}\prod_{l=2}^{N+1}P_{a_1a_l}. \tag{2.52}$$

When we substitute $k=2$ for (2.51), we calculate as follows:

$$\frac{\lambda^{N-1}}{2}W_{|a_2|}\prod_{l=1,l\neq 2}^{N+1}P_{a_2a_l}. \tag{2.53}$$

When (2.51) is from $k=3$ to $N+1$, we calculate as follows:

$$\frac{\lambda^{N-1}}{2}\sum_{k=3}^{N+1}W_{|a_k|}P_{a_1a_2}\Bigg(P_{a_ka_1}\prod_{l=3,l\neq k}^{N+1}P_{a_ka_l}-P_{a_ka_2}\prod_{l=3,l\neq k}^{N+1}P_{a_ka_l}\Bigg) \tag{2.54}$$

Using the results above, we rewrite (2.51) as

$$\begin{aligned} G_{|a_1\dots a_{N+1}|} =&(2.51) \\ =&\frac{\lambda^{N-1}}{2}\Bigg\{W_{|a_1|}\prod_{l=2}^{N+1}P_{a_1a_l}+W_{|a_2|}\prod_{l=1,l\neq 2}^{N+1}P_{a_2a_l} \\ &+\sum_{k=3}^{N+1}W_{|a_k|}P_{a_1a_2}\Bigg(P_{a_ka_1}\prod_{l=3,l\neq k}^{N+1}P_{a_ka_l}-P_{a_ka_2}\prod_{l=3,l\neq k}^{N+1}P_{a_ka_l}\Bigg)\Bigg\}, \end{aligned} \tag{2.55}$$

where $P_{a_1a_2}(P_{a_ka_1}-P_{a_ka_2})=P_{a_ka_1}P_{a_ka_2}$. Using this formula for the last line of (2.55), (2.55) is simplified as

$$G_{|a_1\dots a_{N+1}|}=\frac{\lambda^{N-1}}{2}\sum_{k=1}^{N+1}W_{|a_k|}\prod_{l=1,l\neq k}^{N+1}P_{a_ka_l}. \tag{2.56}$$

It holds true when $N\mapsto N+1$. From this, we can derivative (2.49). $\square$

Proposition 2.2 (Grosse-Sako-Wulkenhaar[18])**.** *We define the notation* $\dfrac{\partial^N}{\partial\mathbb{J}_{a_1\dots a_N}}:=\dfrac{\partial^N}{\partial J_{a_1a_2}\dots\partial J_{a_{N-1}a_N}\partial J_{a_Na_1}}$. *For $N_1>1$ and pairwise different indices,*

$$G_{|a_1^1\dots a_{N_1}^1|\dots|a_1^B\dots a_{N_B}^B|}=\lambda\frac{G_{|a_1^1a_3^1\dots a_{N_1}^1|a_1^2\dots a_{N_2}^2|\dots|a_1^B\dots a_{N_B}^B|}-G_{|a_2^1a_3^1\dots a_{N_1}^1|a_1^2\dots a_{N_2}^2|\dots|a_1^B\dots a_{N_B}^B|}}{E_{a_1^1}^2-E_{a_2^1}^2}. \tag{2.57}$$

Proof. For pairwise different indices $G_{|a_1^1 \ldots a_{N_1}^1| \ldots |a_1^B \ldots a_{N_B}^B|}$ can be expressed as

$$G_{|a_1^1 \ldots a_{N_1}^1| \ldots |a_1^B \ldots a_{N_B}^B|} = V^{B-2} \frac{\partial^{N_1}}{\partial \mathbb{J}_{a_1^1 \ldots a_{N_1}^1}} \cdots \frac{\partial^{N_B}}{\partial \mathbb{J}_{a_1^B \ldots a_{N_B}^B}} \log \frac{\mathcal{Z}[J]}{\mathcal{Z}[0]} \bigg|_{J=0}$$

$$= V^{B-2} \frac{C}{\mathcal{Z}[0]} \frac{\partial^{N_1 - 1}}{\partial J_{a_2^1 a_3^1} \cdots \partial J_{a_{N_1}^1 a_1^1}} \frac{\partial^{N_2}}{\partial \mathbb{J}_{a_1^2 \ldots a_{N_2}^2}} \cdots \frac{\partial^{N_B}}{\partial \mathbb{J}_{a_1^B \ldots a_{N_B}^B}} \left(\frac{\partial}{\partial J_{a_1^1 a_2^1}} e^{-S_{int}\left(\frac{1}{V}\frac{\partial}{\partial J}\right)} \mathcal{Z}_{free}[J] \right) \bigg|_{J=0} \tag{2.58}$$

We use (2.36), and then

$$G_{|a_1^1 \ldots a_{N_1}^1| \ldots |a_1^B \ldots a_{N_B}^B|} = (2.58)$$
$$= V^{B-1} \frac{C}{\mathcal{Z}[0]} \frac{1}{E_{a_1^1} + E_{a_2^1}} \frac{\partial^{N_1 - 1}}{\partial J_{a_2^1 a_3^1} \cdots \partial J_{a_{N_1}^1 a_1^1}} \frac{\partial^{N_2}}{\partial \mathbb{J}_{a_1^2 \ldots a_{N_2}^2}} \cdots \frac{\partial^{N_B}}{\partial \mathbb{J}_{a_1^B \ldots a_{N_B}^B}} \left(e^{-S_{int}\left(\frac{1}{V}\frac{\partial}{\partial J}\right)} J_{a_2^1 a_1^1} \mathcal{Z}_{free}[J] \right) \bigg|_{J=0} . \tag{2.59}$$

From (2.40), $Ce^{-S_{int}\left(\frac{1}{V}\frac{\partial}{\partial J}\right)} J_{ab} \mathcal{Z}_{free}[J] = -\frac{\lambda}{V^2} \sum_{m=1}^{N} \frac{\partial}{\partial J_{ma}} \frac{\partial}{\partial J_{bm}} \mathcal{Z}[J] + J_{ab} \mathcal{Z}[J]$, then

$$G_{|a_1^1 \ldots a_{N_1}^1| \ldots |a_1^B \ldots a_{N_B}^B|}$$
$$= V^{B-1} \frac{1}{\mathcal{Z}[0]} \frac{1}{E_{a_1^1} + E_{a_2^1}} \frac{\partial^{N_1 - 1}}{\partial J_{a_2^1 a_3^1} \cdots \partial J_{a_{N_1}^1 a_1^1}} \frac{\partial^{N_2}}{\partial \mathbb{J}_{a_1^2 \ldots a_{N_2}^2}} \cdots \frac{\partial^{N_B}}{\partial \mathbb{J}_{a_1^B \ldots a_{N_B}^B}} \left(-\frac{\lambda}{V^2} \sum_{m=1}^{N} \frac{\partial}{\partial J_{m a_2^1}} \frac{\partial}{\partial J_{a_1^1 m}} \mathcal{Z}[J] \right) \bigg|_{J=0}$$
$$+ V^{B-1} \frac{1}{\mathcal{Z}[0]} \frac{1}{E_{a_1^1} + E_{a_2^1}} \frac{\partial^{N_1 - 1}}{\partial J_{a_2^1 a_3^1} \cdots \partial J_{a_{N_1}^1 a_1^1}} \frac{\partial^{N_2}}{\partial \mathbb{J}_{a_1^2 \ldots a_{N_2}^2}} \cdots \frac{\partial^{N_B}}{\partial \mathbb{J}_{a_1^B \ldots a_{N_B}^B}} \left(J_{a_2^1 a_1^1} \mathcal{Z}[J] \right) \bigg|_{J=0} . \tag{2.60}$$

The second term is zero because we consider all indices are pairwise different. Using Ward-Takahashi identity (2.28),

$$G_{|a_1^1 \ldots a_{N_1}^1| \ldots |a_1^B \ldots a_{N_B}^B|}$$
$$= \frac{V^{B-2}}{\mathcal{Z}[0]} \frac{(-\lambda)}{E_{a_1^1}^2 - E_{a_2^1}^2} \frac{\partial^{N_1 - 1}}{\partial J_{a_2^1 a_3^1} \ldots \partial J_{a_N^1 a_1^1}} \frac{\partial^{N_2}}{\partial \mathbb{J}_{a_1^2 \ldots a_{N_2}^2}} \cdots \frac{\partial^{N_B}}{\partial \mathbb{J}_{a_1^B \ldots a_{N_B}^B}} \sum_{m=1}^{\mathcal{N}} \left(J_{m a_1^1} \frac{\partial \mathcal{Z}}{\partial J_{m a_2^1}} - J_{a_2^1 m} \frac{\partial \mathcal{Z}}{\partial J_{a_1^1 m}} \right) \bigg|_{J=0}$$
$$= V^{B-2} \frac{(-\lambda)}{E_{a_1^1}^2 - E_{a_2^1}^2} \left(\frac{\partial^{N_1 - 1}}{\partial \mathbb{J}_{a_2^1 \ldots a_{N_1}^1}} - \frac{\partial^{N_1 - 1}}{\partial \mathbb{J}_{a_1^1 a_3^1 \ldots a_{N_1}^1}} \right) \frac{\partial^{N_2}}{\partial \mathbb{J}_{a_1^2 \ldots a_{N_2}^2}} \cdots \frac{\partial^{N_B}}{\partial \mathbb{J}_{a_1^B \ldots a_{N_B}^B}} \log \mathcal{Z}[J] \bigg|_{J=0}$$
$$= \lambda \frac{G_{|a_1^1 a_3^1 \ldots a_{N_1}^1| a_1^2 \ldots a_{N_2}^2| \ldots |a_1^B \ldots a_{N_B}^B|} - G_{|a_2^1 a_3^1 \ldots a_{N_1}^1| a_1^2 \ldots a_{N_2}^2| \ldots |a_1^B \ldots a_{N_B}^B|}}{E_{a_1^1}^2 - E_{a_2^1}^2} . \tag{2.61}$$

From this, we can derivative (2.57).

□

Proposition 2.3 (Grosse-Sako-Wulkenhaar[18]). *For pairwise different indices the* $(N_1+\cdots+N_B)$*-point function is*

$$G_{|a_1^1\cdots a_{N_1}^1|\cdots|a_1^B\cdots a_{N_B}^B|}=\lambda^{L-B}\sum_{k_1=1}^{N_1}\cdots\sum_{k_B=1}^{N_B}G_{|a_{k_1}^1|a_{k_2}^2|\cdots|a_{k_B}^B|}\prod_{\beta=1}^{B}\prod_{\substack{l_\beta=1\\ l_\beta\neq k_\beta}}^{N_\beta}\frac{1}{E^2_{a^\beta_{k_\beta}}-E^2_{a^\beta_{l_\beta}}},\tag{2.62}$$

where $L=N_1+\cdots+N_B$.

Proof. Here, we follow the discussion given in Hock[30] for the proof of this proposition. We confirm Proposition 2.3 using a mathematical induction. When $N_1=2$ and $N_\beta=1$ for all $\beta\in\{2,\cdots,B\}$,

$$\begin{aligned}G_{|a_1^1a_2^1|a_1^2|a_1^3|\cdots|a_1^B|}=&\lambda\sum_{k_1=1}^{2}G_{|a_{k_1}^1|a_1^2|a_1^3|\cdots|a_1^B|}\prod_{\beta=1}^{B}\prod_{\substack{l_\beta=1\\ l_\beta\neq k_\beta}}^{N_\beta}\frac{1}{E^2_{a^\beta_{k_\beta}}-E^2_{a^\beta_{l_\beta}}}\\ =&\lambda\frac{G_{|a_1^1|a_1^2|\ldots|a_1^B|}-G_{|a_2^1|a_1^2|\ldots|a_1^B|}}{E^2_{a_1^1}-E^2_{a_2^1}}.\end{aligned}\tag{2.63}$$

From this, it holds true when $N_1=2$ and $N_\beta=1$ for all $\beta\in\{2,\ldots,B\}$. Next, we assume that Proposition 2.3 holds true when N_1. When N_1+1,

$$\begin{aligned}&G_{|a_1^1\cdots a_{N_1+1}^1|\cdots|a_1^B\cdots a_{N_B}^B|}\\ =&-\lambda^{L+1-B}\sum_{k_2=1}^{N_2}\cdots\sum_{k_B=1}^{N_B}\frac{1}{E^2_{a_1^1}-E^2_{a_2^1}}\prod_{\beta=2}^{B}\prod_{\substack{l_\beta=1\\ l_\beta\neq k_\beta}}^{N_\beta}\frac{1}{E^2_{a^\beta_{k_\beta}}-E^2_{a^\beta_{l_\beta}}}\\ &\times\left(\sum_{k_1=2}^{N_1+1}G_{|a_{k_1}^1|a_{k_2}^2|\cdots|a_{k_B}^B|}\prod_{\substack{l_1=2\\ l_1\neq k_1}}^{N_1+1}\frac{1}{E^2_{a^1_{k_1}}-E^2_{a^1_{l_1}}}-\sum_{\substack{k_1=1\\ k_1\neq 2}}^{N_1+1}G_{|a_{k_1}^1|a_{k_2}^2|\cdots|a_{k_B}^B|}\prod_{\substack{l_1=1\\ l_1\neq k_1,l_1\neq 2}}^{N_1+1}\frac{1}{E^2_{a^1_{k_1}}-E^2_{a^1_{l_1}}}\right),\end{aligned}\tag{2.64}$$

where we use (2.61) and the assumption for N_1. When we substitute $k_1=1$ for (2.64), we calculate as follows:

$$-\lambda^{L+1-B}\sum_{k_2=1}^{N_2}\cdots\sum_{k_B=1}^{N_B}\prod_{\beta=2}^{B}\prod_{\substack{l_\beta=1\\ l_\beta\neq k_\beta}}^{N_\beta}\frac{1}{E^2_{a^\beta_{k_\beta}}-E^2_{a^\beta_{l_\beta}}}\times\left(-G_{|a_1^1|a_{k_2}^2|\cdots|a_{k_B}^B|}\prod_{l_1=2}^{N_1+1}\frac{1}{E^2_{a_1^1}-E^2_{a^1_{l_1}}}\right).\tag{2.65}$$

When we substitute $k_1=2$ for (2.64), we calculate as follows:

$$-\lambda^{L+1-B}\sum_{k_2=1}^{N_2}\cdots\sum_{k_B=1}^{N_B}\prod_{\beta=2}^{B}\prod_{\substack{l_\beta=1\\ l_\beta\neq k_\beta}}^{N_\beta}\frac{1}{E^2_{a^\beta_{k_\beta}}-E^2_{a^\beta_{l_\beta}}}\times\left(-G_{|a_2^1|a_{k_2}^2|\cdots|a_{k_B}^B|}\prod_{\substack{l_1=1\\ l_1\neq 2}}^{N_1+1}\frac{1}{E^2_{a_2^1}-E^2_{a^1_{l_1}}}\right).\tag{2.66}$$

When (2.64) is from $k_1 = 3$ to $k_1 = N_1 + 1$, we calculate as follows:

$$
\begin{aligned}
&-\lambda^{L+1-B} \sum_{k_2=1}^{N_2} \cdots \sum_{k_B=1}^{N_B} \prod_{\beta=2}^{B} \prod_{\substack{l_\beta=1 \\ l_\beta \neq k_\beta}}^{N_\beta} \frac{1}{E^2_{a^\beta_{k_\beta}} - E^2_{a^\beta_{l_\beta}}} \\
&\quad \times \left\{ \sum_{k_1=3}^{N_1+1} G_{|a^1_{k_1}|a^2_{k_2}|\cdots|a^B_{k_B}|} \frac{1}{E^2_{a^1_1} - E^2_{a^1_2}} \left(\prod_{\substack{l_1=2 \\ l_1 \neq k_1}}^{N_1+1} \frac{1}{E^2_{a^1_{k_1}} - E^2_{a^1_{l_1}}} - \prod_{\substack{l_1=1 \\ l_1 \neq k_1, l_1 \neq 2}}^{N_1+1} \frac{1}{E^2_{a^1_{k_1}} - E^2_{a^1_{l_1}}} \right) \right\}.
\end{aligned} \tag{2.67}
$$

Using the results above, we rewrite (2.64) as

$$
\begin{aligned}
&G_{|a^1_1 \cdots a^1_{N_1+1}|\cdots|a^B_1 \cdots a^B_{N_B}|} = (2.64) \\
&= -\lambda^{L+1-B} \sum_{k_2=1}^{N_2} \cdots \sum_{k_B=1}^{N_B} \prod_{\beta=2}^{B} \prod_{\substack{l_\beta=1 \\ l_\beta \neq k_\beta}}^{N_\beta} \frac{1}{E^2_{a^\beta_{k_\beta}} - E^2_{a^\beta_{l_\beta}}} \\
&\quad \times \Bigg\{ -G_{|a^1_1|a^2_{k_2}|\cdots|a^B_{k_B}|} \prod_{l_1=2}^{N_1+1} \frac{1}{E^2_{a^1_1} - E^2_{a^1_{l_1}}} - G_{|a^1_2|a^2_{k_2}|\cdots|a^B_{k_B}|} \prod_{\substack{l_1=1 \\ l_1 \neq 2}}^{N_1+1} \frac{1}{E^2_{a^1_2} - E^2_{a^1_{l_1}}} \\
&\quad + \sum_{k_1=3}^{N_1+1} G_{|a^1_{k_1}|a^2_{k_2}|\cdots|a^B_{k_B}|} \frac{1}{E^2_{a^1_1} - E^2_{a^1_2}} \left(\prod_{\substack{l_1=2 \\ l_1 \neq k_1}}^{N_1+1} \frac{1}{E^2_{a^1_{k_1}} - E^2_{a^1_{l_1}}} - \prod_{\substack{l_1=1 \\ l_1 \neq k_1, l_1 \neq 2}}^{N_1+1} \frac{1}{E^2_{a^1_{k_1}} - E^2_{a^1_{l_1}}} \right) \Bigg\},
\end{aligned} \tag{2.68}
$$

where $\frac{1}{E^2_{a^1_1} - E^2_{a^1_2}} \left(\frac{1}{E^2_{a^1_{k_1}} - E^2_{a^1_2}} - \frac{1}{E^2_{a^1_{k_1}} - E^2_{a^1_1}} \right) = -\frac{1}{E^2_{a^1_{k_1}} - E^2_{a^1_2}} \frac{1}{E^2_{a^1_{k_1}} - E^2_{a^1_1}}$ for any $k_1 > 2$. Using this formula for the last line of (2.68), $G_{|a^1_1 \cdots a^1_{N_1+1}|\cdots|a^B_1 \cdots a^B_{N_B}|}$ is simplified as

$$
\begin{aligned}
&G_{|a^1_1 \cdots a^1_{N_1+1}|\cdots|a^B_1 \cdots a^B_{N_B}|} = (2.68) \\
&= \lambda^{L+1-B} \sum_{k_1=1}^{N_1+1} \cdots \sum_{k_B=1}^{N_B} G_{|a^1_{k_1}|a^2_{k_2}|\cdots|a^B_{k_B}|} \prod_{\beta=2}^{B} \prod_{\substack{l_\beta=1 \\ l_\beta \neq k_\beta}}^{N_\beta} \frac{1}{E^2_{a^\beta_{k_\beta}} - E^2_{a^\beta_{l_\beta}}} \prod_{\substack{l_1=1 \\ l_1 \neq k_1}}^{N_1+1} \frac{1}{E^2_{a^1_{k_1}} - E^2_{a^1_{l_1}}}
\end{aligned} \tag{2.69}
$$

It holds true when $N_1 \mapsto N_1 + 1$. And $N_\beta \to N_\beta + 1$ can be calculated in the same way. From this, we can derivative (2.62).

□

Chapter 3

Multipoint Correlation Function in Φ^3 Finite Matrix Model

In this chapter we find the exact solutions of the Φ^3 finite matrix model (Grosse-Steinacker-Wulkenhaar model)[40]. In the Φ^3 finite matrix model, multipoint correlation functions are expressed as $G_{|a_1^1\ldots a_{N_1}^1|\ldots|a_1^B\ldots a_{N_B}^B|}$. The $\sum_{i=1}^{B} N_i$-point function denoted by $G_{|a_1^1\ldots a_{N_1}^1|\ldots|a_1^B\ldots a_{N_B}^B|}$ is given by the sum over all Feynman diagrams (ribbon graphs) on Riemann surfaces with B-boundaries, and each $|a_1^i \cdots a_{N_i}^i|$ corresponds to the Feynman diagrams having N_i-external lines from the i-th boundary. It is known that any $G_{|a_1^1\ldots a_{N_1}^1|\ldots|a_1^B\ldots a_{N_B}^B|}$ can be expressed using $G_{|a^1|\ldots|a^n|}$ type n-point functions. Thus we focus on rigorous calculations of $G_{|a^1|\ldots|a^n|}$. The formula for $G_{|a^1|\ldots|a^n|}$ is obtained, and it is achieved by using the partition function $\mathcal{Z}[J]$ calculated by the Harish-Chandra-Itzykson-Zuber integral. We give $G_{|a|}$, $G_{|ab|}$, $G_{|a|b|}$, and $G_{|a|b|c|}$ as the specific simple examples. All of them are described by using the Airy functions.

3.1. Setup of Φ^3 Matrix Model

In this section, we review the Φ^3 matrix model based on the previous studies[18, 19, 30], and we determine the notation in this chapter.

Let $\Phi = (\Phi_{ij})$ be a Hermitian matrix for $i, j = 1, 2, \ldots, N$ and E_{m-1} be a discretization of a monotonously increasing differentiable function e with $e(0) = 0$,

$$E_{m-1} = \mu^2 \left(\frac{1}{2} + e\left(\frac{m-1}{\mu^2 V} \right) \right), \tag{3.1}$$

where μ^2 is a squared mass and V is a real constant. Let $E = (E_{m-1}\delta_{mn})$ be a diagonal matrix for $m, n = 1, \ldots, N$. Let us consider the following action:

$$S[\Phi] = iV \mathrm{tr} \left(E\Phi^2 + \kappa\Phi + \frac{\lambda}{3}\Phi^3 \right), \tag{3.2}$$

where κ is a renormalization constant (real), λ is a coupling constant that is non-zero real, and $i = \sqrt{-1}$. Compared to the paper[18], the difference is that V is replaced with iV. By the diagonal matrix E that is not proportional to the unit matrix in general, there is no symmetry for the unitary transformation in $\Phi \to U\Phi U^*$. Here U is a unitary matrix, and U^* is its Hermitian conjugate.

Let $J = (J_{mn})$ be a Hermitian matrix for $m, n = 1, \ldots, N$ as an external field. Let $\mathcal{D}\Phi$ be the integral measure,

$$\mathcal{D}\Phi := \prod_{i=1}^{N} d\Phi_{ii} \prod_{1\le i<j\le N} d\mathrm{Re}\Phi_{ij} d\mathrm{Im}\Phi_{ij}, \tag{3.3}$$

where each variable is divided into real and imaginary parts $\Phi_{ij} = \mathrm{Re}\Phi_{ij} + i\mathrm{Im}\Phi_{ij}$ with $\mathrm{Re}\Phi_{ij} = \mathrm{Re}\Phi_{ji}$ and $\mathrm{Im}\Phi_{ij} = -\mathrm{Im}\Phi_{ji}$. Let us consider the following partition function:

$$\begin{aligned}\mathcal{Z}[J] &:= \int \mathcal{D}\Phi \exp\left(-S[\Phi] + iV\mathrm{tr}(J\Phi)\right) \\ &= \int \mathcal{D}\Phi \exp\left(-iV\mathrm{tr}\left(E\Phi^2 + \kappa\Phi + \frac{\lambda}{3}\Phi^3\right)\right)\exp\left(iV\mathrm{tr}\left(J\Phi\right)\right).\end{aligned} \tag{3.4}$$

Using $\log\dfrac{\mathcal{Z}[J]}{\mathcal{Z}[0]}$, the $\displaystyle\sum_{i=1}^{B} N_i$-point function $G_{|a_1^1\ldots a_{N_1}^1|\ldots|a_1^B\ldots a_{N_B}^B|}$ is defined as

$$\log\frac{\mathcal{Z}[J]}{\mathcal{Z}[0]} := \sum_{B=1}^{\infty}\sum_{1\le N_1\le\cdots\le N_B}^{\infty}\sum_{p_1^1,\ldots,p_{N_B}^B=0}^{\mathcal{N}} (iV)^{2-B}\frac{G_{|p_1^1\ldots p_{N_1}^1|\ldots|p_1^B\ldots p_{N_B}^B|}}{S_{(N_1,\ldots,N_B)}}\prod_{\beta=1}^{B}\frac{\mathbb{J}_{p_1^\beta\ldots p_{N_\beta}^\beta}}{N_\beta}, \tag{3.5}$$

where N_i is the identical valence number for $i = 1, \ldots, B$, $\mathbb{J}_{p_1\ldots p_{N_i}} := \displaystyle\prod_{j=1}^{N_i} J_{p_j p_{j+1}}$ with $N_i + 1 \equiv 1$, $(N_1, \ldots, N_B) = (\underbrace{N_1', \ldots, N_1'}_{\nu_1}, \ldots, \underbrace{N_s', \ldots, N_s'}_{\nu_s})$, and $S_{(N_1,\ldots,N_B)} = \displaystyle\prod_{\beta=1}^{s}\nu_\beta!$.

3.2. Calculation of Partition Function $\mathcal{Z}[J]$

In this section, we perform the integration of the partition function $\mathcal{Z}[J]$ by dividing the Hermitian matrix into its diagonal and off-diagonal elements. The off-diagonal elements of the Hermitian matrix in the partition function $\mathcal{Z}[J]$ are integrated using the Harish-Chandra-Izykson-Zuber integral [36, 51, 24, 56], and the integration of the diagonal elements of the Hermitian matrix in the partition function $\mathcal{Z}[J]$ is performed by using Airy functions. The calculations are essentially in line with the calculations of Kontsevich [42]. We write without omitting details because the results are different due to the presence of external fields J and the renormalization term κ.

We introduce a Hermitian matrix $\widetilde{E} = (\widetilde{E_m}\delta_{mn}) = \dfrac{1}{\lambda}E = \left(\dfrac{E_{m-1}}{\lambda}\delta_{mn}\right)$ for $m, n = 1, \ldots, N$ and $\dfrac{\kappa}{\lambda} = \widetilde{\kappa}$. Note that the indices are shifted i.e. $\widetilde{E} = diag\left(\widetilde{E}_1, \cdots, \widetilde{E}_N\right)$ and $E = diag\left(E_0, \cdots, E_{N-1}\right)$.

Then $\mathcal{Z}[J]$ is written as

$$\mathcal{Z}[J] = \int \mathcal{D}\Phi \exp\left(-2i\lambda V \mathrm{tr}\left(\frac{\widetilde{E}\Phi^2}{2} + \frac{\widetilde{\kappa}\Phi}{2} + \frac{1}{3!}\Phi^3\right)\right) \exp\left(iV\mathrm{tr}\left(J\Phi\right)\right). \tag{3.6}$$

We introduce a new variable X by $\Phi = X - \widetilde{E}$. Here $X = (X_{mn})$ is a Hermitian matrix, too. We do a change of variables of the integral measure $\mathcal{D}\Phi$ as

$$d\Phi_{ij} = \sum_{m,n=1}^{N} \frac{\partial \Phi_{ij}}{\partial X_{mn}} dX_{mn} = dX_{ij}. \tag{3.7}$$

By the variable transformation $\mathrm{tr}\left(\frac{\widetilde{E}\Phi^2}{2} + \frac{\widetilde{\kappa}\Phi}{2} + \frac{1}{3!}\Phi^3\right)$ is

$$\mathrm{tr}\left(\frac{\widetilde{E}\Phi^2}{2} + \frac{\widetilde{\kappa}\Phi}{2} + \frac{1}{3!}\Phi^3\right) = \mathrm{tr}\left(\frac{(X)^3 - 3(\widetilde{E})^2X + 2(\widetilde{E})^3 + 3\widetilde{\kappa}X - 3\widetilde{\kappa}\widetilde{E}}{6}\right), \tag{3.8}$$

then $\mathcal{Z}[J]$ is given as

$$\begin{aligned}
\mathcal{Z}[J] &= \int \mathcal{D}X \exp\left(-2i\lambda V \mathrm{tr}\left(\frac{(X)^3 - 3(\widetilde{E})^2X + 2(\widetilde{E})^3 + 3\widetilde{\kappa}X - 3\widetilde{\kappa}\widetilde{E}}{6}\right)\right) \\
&\quad \times \exp\left(iV\mathrm{tr}\left(JX - J\widetilde{E}\right)\right) \\
&= \exp\left(-i\lambda V tr\left(\frac{2}{3}(\widetilde{E})^3 - \widetilde{\kappa}\widetilde{E} + \frac{1}{\lambda}J\widetilde{E}\right)\right) \\
&\quad \times \int \mathcal{D}X \exp\left(+i\lambda V tr\left(-\widetilde{\kappa}X + \frac{1}{\lambda}JX - \frac{1}{3}X^3 + (\widetilde{E})^2 X\right)\right) \\
&= \exp\left(-i\lambda V tr\left(\frac{2}{3}(\widetilde{E})^3 - \widetilde{\kappa}\widetilde{E} + \frac{1}{\lambda}J\widetilde{E}\right)\right) \\
&\quad \times \int \mathcal{D}X \exp\left(-i\frac{\lambda V}{3}\mathrm{tr}(X^3)\right) \exp\left(i\lambda V\widetilde{\kappa}\mathrm{tr}\{(M - I + K)X\}\right).
\end{aligned} \tag{3.9}$$

Here $M = \frac{(\widetilde{E})^2}{\widetilde{\kappa}} = \frac{E^2}{\lambda\kappa}$, $K = \frac{J}{\kappa}$, and I is the unit matrix. Note that

$$\mathcal{D}X = \left(\prod_{i=1}^{N} dx_i\right)\left(\prod_{1\leq k<l\leq N}(x_l - x_k)^2\right) dU, \tag{3.10}$$

where x_i is the eigenvalues of X for $i = 1, \cdots, N$, dU is the Haar probability measure of the unitary group $U(N)$, and U is the unitary matrix which diagonalize X[44, 10]. Then (3.9) can be rewritten as the following:

$$
\begin{aligned}
\mathcal{Z}[J] = & \exp\left(-i\lambda V tr\left(\frac{2}{3}(\widetilde{E})^3 - \widetilde{\kappa}\widetilde{E} + \frac{1}{\lambda}J\widetilde{E}\right)\right) \\
& \int \left(\prod_{i=1}^{\mathrm{N}} dx_i \exp\left(-i\frac{\lambda V}{3}x_i^3\right)\right)\left(\prod_{1\le k<l\le N}(x_l - x_k)^2\right) \\
& \int dU \exp\left(i\lambda V\widetilde{\kappa}\mathrm{tr}\{(M - I + K)U\widetilde{X}U^*\}\right), \qquad (3.11)
\end{aligned}
$$

where $\widetilde{X}$ is the diagonal matrix $\widetilde{X} = U^*XU$. We use the following formula.

Fact. *The Harish-Chandra-Itzykson-Zuber integral [36, 51, 24, 56] for the unitary group $U(n)$ is*

$$
\int_{U(n)} \exp\left(t\mathrm{tr}\left(AUBU^*\right)\right) dU = c_n \frac{\det_{1\le i,j\le n}\left(\exp\left(t\lambda_i(A)\lambda_j(B)\right)\right)}{t^{\frac{(n^2-n)}{2}}\Delta(\lambda(A))\Delta(\lambda(B))}. \qquad (3.12)
$$

Here $A = (A_{ij})$, and $B = (B_{ij})$ are some Hermitian matrices whose eigenvalues denoted by $\lambda_i(A)$ and $\lambda_i(B)$ $(i = 1, \cdots, n)$, respectively. t is the non-zero complex parameter, $\Delta(\lambda(A)) := \prod_{1\le i<j\le n}(\lambda_j(A) - \lambda_i(A))$ is the Vandermonde determinant, and $c_n := \left(\prod_{i=1}^{n-1} i!\right) \times \pi^{\frac{n(n-1)}{2}}$ is the constant. $(\exp\left(t\lambda_i(A)\lambda_j(B)\right))$ is the $n \times n$ matrix with the i-th row and the j-th column being $\exp\left(t\lambda_i(A)\lambda_j(B)\right)$.

Applying Harish-Chandra-Itzykson-Zuber integral (3.12) to $\int dU \exp\left(i\lambda V\widetilde{\kappa}\mathrm{tr}\{(M - I + K)U\widetilde{X}U^*\}\right)$ in (3.11), the result is

$$
\int dU \exp\left(i\lambda V\widetilde{\kappa}\mathrm{tr}\{(\mathrm{M} - \mathrm{I} + \mathrm{K})U\widetilde{X}U^*\}\right) = \frac{C}{N!}\frac{\det_{1\le i,j\le N}\exp\left(i\lambda V\widetilde{\kappa}x_i s_j\right)}{\prod_{i<j}(x_j - x_i)\prod_{i<j}(s_j - s_i)}, \qquad (3.13)
$$

where s_t is the eigenvalues of the matrix $M - I + K$ for $t = 1, \cdots, N$ and $C = \left(\prod_{p=1}^{N} p!\right) \times \left(\frac{\pi}{i\lambda V\widetilde{\kappa}}\right)^{\frac{N(N-1)}{2}}$. $(\exp\left(i\lambda V\widetilde{\kappa}x_i s_j\right))$ denotes the $N\times N$ matrix with the i-th row and the j-th column being $\exp(i\lambda V\widetilde{\kappa}x_i s_j)$. Then the partition function $\mathcal{Z}[J]$ is described as

$$\begin{aligned}
\mathcal{Z}[J] =&\frac{C}{N!}\exp\left(-i\lambda V tr\left(\frac{2}{3}(\widetilde{E})^3-\widetilde{\kappa}\widetilde{E}+\frac{1}{\lambda}J\widetilde{E}\right)\right)\frac{1}{\displaystyle\prod_{1\le t<u\le N}(s_u-s_t)}\\
&\int\left(\prod_{i=1}^N dx_i\exp\left(-i\frac{\lambda V}{3}x_i^3\right)\right)\left(\prod_{1\le k<l\le N}(x_l-x_k)\right)\det_{1\le m,n\le N}\exp(i\lambda V\widetilde{\kappa}x_m s_n)\\
=&C\exp\left(-i\lambda V tr\left(\frac{2}{3}(\widetilde{E})^3-\widetilde{\kappa}\widetilde{E}+\frac{1}{\lambda}J\widetilde{E}\right)\right)\frac{1}{\displaystyle\prod_{1\le t<u\le N}(s_u-s_t)}\\
&\int\left(\prod_{i=1}^N dx_i\exp\left(-i\frac{\lambda V}{3}x_i^3\right)\exp(i\lambda V\widetilde{\kappa}x_i s_i)\right)\prod_{1\le k<l\le N}(x_l-x_k).
\end{aligned}\tag{3.14}$$

Here we use the following result at the second equality in (3.14):

$$\begin{aligned}
&\int\left(\prod_{i=1}^N dx_i\exp\left(-i\frac{\lambda V}{3}x_i^3\right)\right)\left(\prod_{1\le k<l\le N}(x_l-x_k)\right)\det_{1\le m,n\le N}\exp(i\lambda V\widetilde{\kappa}x_m s_n)\\
=&\sum_{\sigma\in S_N}\int\left(\prod_{i=1}^N dx_i\exp\left(-i\frac{\lambda V}{3}x_i^3\right)\right)\left(\prod_{1\le k<l\le N}(x_l-x_k)\right)(-1)^\sigma\left(\prod_{j=1}^N e^{i\lambda V\widetilde{\kappa}x_{\sigma(j)}s_j}\right)\\
=&\sum_{\sigma\in S_N}\int\left(\prod_{i=1}^N dx_i\exp\left(-i\frac{\lambda V}{3}x_i^3\right)\right)\left(\prod_{1\le k<l\le N}(x_l-x_k)\right)(-1)^\sigma(-1)^\sigma\left(\prod_{j=1}^N e^{i\lambda V\widetilde{\kappa}x_j s_j}\right)\\
=&N!\int\left(\prod_{i=1}^N dx_i\exp\left(-i\frac{\lambda V}{3}x_i^3\right)\exp(i\lambda V\widetilde{\kappa}x_i s_i)\right)\prod_{1\le k<l\le N}(x_l-x_k).
\end{aligned}\tag{3.15}$$

In the transformation of the above equation from the second line to the third line, we changed variables as $x_{\sigma(i)}\mapsto x_i$ $(i=1,\cdots,N)$.

Using $\displaystyle\prod_{1\le k<l\le N}(x_l-x_k)=\det_{1\le k,l\le N}\left(x_k^{l-1}\right)$, we calculate the remaining integral in the right-hand side in (3.14) as

$$\begin{aligned}
&\int_{-\infty}^{\infty}\left(\prod_{i=1}^N dx_i\exp\left(-i\frac{\lambda V}{3}x_i^3\right)\exp(i\lambda V\widetilde{\kappa}s_i x_i)\right)\det_{1\le k,l\le \mathrm{N}}\left(x_l^{k-1}\right)\\
=&\sum_{\sigma\in S_\mathrm{N}}\mathrm{sgn}\sigma\prod_{i=1}^N\phi_{\sigma(i)}(s_i)\\
=&\det_{1\le i,j\le N}(\phi_i(s_j)),
\end{aligned}\tag{3.16}$$

where $\phi_k(z)$ is defined by

$$\phi_k(z) = \int_{-\infty}^{\infty} dx \;\; x^{k-1} \exp\left(-i\frac{\lambda V}{3}x^3 + iV\kappa x z\right), \tag{3.17}$$

and $(\phi_i(s_j))$ is the $N \times N$ matrix with the i-th row and the j-th column being $\phi_i(s_j)$. Summarizing the results (3.14) and (3.16), we obtain the following:

Proposition 3.1 (Kontsevich[42], N.K-Sako[40])**.** *Let $\mathcal{Z}[J]$ be the partition function of Φ_2^3 matrix model given by (3.4). Then, $\mathcal{Z}[J]$ is given as*

$$\mathcal{Z}[J] = C \exp\left(-i\lambda V tr\left(\frac{2}{3}(\widetilde{E})^3 - \widetilde{\kappa}\widetilde{E} + \frac{1}{\lambda}J\widetilde{E}\right)\right) \frac{\det\limits_{1\le i,j\le N}(\phi_i(s_j))}{\prod\limits_{1\le t<u\le \mathrm{N}}(s_u - s_t)}. \tag{3.18}$$

Note that $\phi_k(z)$ is expressed as

$$\phi_k(z) = \left(\frac{1}{iV\kappa}\right)^{k-1}\left(\frac{d}{dz}\right)^{k-1}\int_{-\infty}^{\infty} dx \exp\left(-i\frac{\lambda V}{3}x^3 + iV\kappa x z\right). \tag{3.19}$$

We use Airy function:

$$Ai(\gamma L) = \frac{1}{2\pi\gamma}\int_{-\infty}^{\infty} \exp\left[i\left(Lx + \frac{x^3}{3\gamma^3}\right)\right] dx. \tag{3.20}$$

Here $\gamma \in \mathbb{R}\backslash\{0\}$ and $L \in \mathbb{R}$. Substituting (3.20) for (3.19), $\phi_k(z)$ is calculated as follows:

$$\phi_k(z) = \left(\frac{i}{(\lambda V)^{\frac{1}{3}}}\right)^{k-1}\left(\frac{-2\pi}{(\lambda V)^{\frac{1}{3}}}\right)\left(\frac{d}{dy}\right)^{k-1} Ai\,[y]\Bigg|_{y=-\frac{V\kappa z}{(\lambda V)^{\frac{1}{3}}}}. \tag{3.21}$$

N.K-Sako[40] differs from Kontsevich[42] in that $\kappa = J = 0$ is $\kappa = J \neq 0$.

Proposition 3.2 (Kontsevich[42])**.** *Let $\left(Ai^{(j-1)}(y_i)\right)$ be the $N \times N$ matrix with the i-th row and the j-th column being $Ai^{(j-1)}(y_i) = \left(\frac{d}{dy_i}\right)^{j-1} Ai(y_i)$. We then obtain the following:*

$$\det\left(Ai^{(j-1)}(y_i)\right) = \left(\prod_{1\le i<j\le N}\left(\partial_{y_i} - \partial_{y_j}\right)\right) Ai(y_1)\cdots Ai(y_N).$$

The proof is omitted in [42], so it is appended for the reader's convenience.

Proof. Here, we follow the proof given by N.K-Sako in [40]. We calculate $\det\left(Ai^{(j-1)}(y_i)\right)$ according to the definition of the determinant.

$$
\begin{aligned}
\det\left(Ai^{(j-1)}(y_i)\right) &= \sum_{\sigma\in S_N} sgn\sigma \prod_{k=0}^{N-1} Ai^{(k)}(y_{\sigma(k+1)}) \\
&= \sum_{\sigma\in S_N} sgn\sigma \prod_{k=0}^{N-1} \partial^k_{y_{\sigma(k+1)}} Ai(y_{\sigma(k+1)}),
\end{aligned} \tag{3.22}
$$

where S_N is the N-th order symmetry group. For $\prod_{k=0}^{N-1}\partial^k_{y_{\sigma(k+1)}}$ using similar calculation to the Vandermonde determinant, $\det\left(Ai^{(j-1)}(y_i)\right)$ is as follows:

$$
\begin{aligned}
\det\left(Ai^{(j-1)}(y_i)\right) &= \sum_{\sigma\in S_N} sgn\sigma \left(\prod_{k=0}^{N-1}\partial^k_{y_{\sigma(k+1)}}\right) Ai(y_1)\cdots Ai(y_N) \\
&= \left(\prod_{1\le i<j\le N}\left(\partial_{y_i}-\partial_{y_j}\right)\right) Ai(y_1)\cdots Ai(y_N).
\end{aligned} \tag{3.23}
$$

□

We introduce

$$
A_N(y_1,\cdots,y_N) = \left(\prod_{1\le i<j\le N}\left(\partial_{y_i}-\partial_{y_j}\right)\right) Ai(y_1)\cdots Ai(y_N), \tag{3.24}
$$

where $y_j = -\frac{V\kappa s_j}{(\lambda V)^{\frac{1}{3}}}$ for $j=1,\ldots,N$. From this, $\det_{1\le i,j\le N}(\phi_i(s_j))$ is calculated as follows:

$$
\det_{1\le i,j\le N}(\phi_i(s_j)) = \left(\frac{(i)^{\frac{N(N-1)}{2}}(-2\pi)^N}{(\lambda V)^{\frac{N(N+1)}{6}}}\right) A_N(y_1,\cdots,y_N). \tag{3.25}
$$

Summarizing the above results, we obtain the following:

Theorem 3.3 (Kontsevich[42], N.K-Sako[40])**.** *Let $\mathcal{Z}[J]$ be the partition function of the Φ^3_2 matrix model given by (3.4). Then, $\mathcal{Z}[J]$ is given as*

$$
\begin{aligned}
\mathcal{Z}[J] &= \int \mathcal{D}\Phi \exp\left(-iV\mathrm{tr}\left(E\Phi^2+\kappa\Phi+\frac{\lambda}{3}\Phi^3\right)\right)\exp\left(iV\mathrm{tr}\left(J\Phi\right)\right) \\
&= C'\frac{e^{\frac{-iV}{\lambda}\mathrm{tr}(JE)}A_N(y_1,\cdots,y_N)}{\prod_{1\le t<u\le N}(s_u-s_t)}.
\end{aligned} \tag{3.26}
$$

Here $C' = \exp\left(-\frac{iV}{\lambda^2}\mathrm{tr}\left(\frac{2}{3}E^3-\lambda\kappa E\right)\right)\left(\prod_{p=1}^{N}p!\right)\frac{(-2)^N\pi^{\frac{N(N+1)}{2}}}{\lambda^{\frac{N(N+1)}{6}}V^{\frac{N(2N-1)}{3}}\kappa^{\frac{N(N-1)}{2}}}$*, s_t is the eigenvalues of the matrix* $\frac{E^2}{\lambda\kappa}-I+\frac{J}{\kappa}$ *for $t=1,\cdots,N$, and* $y_j=-\frac{V\kappa s_j}{(\lambda V)^{\frac{1}{3}}}$ *for $j=1,\cdots,N$.*

3.3. Calculation of 1-Point Function $G_{|a|}$ and 2-Point Function $G_{|ab|}$

In the calculation of the 1-point function $G_{|a|}$, the external field J can be treated as the diagonal matrix $J = diag\,(J_{11}, \cdots, J_{NN})$. Then the eigenvalues s_t in (3.26) is given $s_t = \dfrac{\lambda(\widetilde{E}_t)^2 + J_{tt}}{\kappa} - 1$. Then, the 1-point function $G_{|a|}$ is calculated as follows:

$$
\begin{aligned}
G_{|a|} &= \frac{1}{iV}\frac{\partial \log \mathcal{Z}[J]}{\partial J_{aa}}\bigg|_{J=0} \\
&= \frac{\left(\dfrac{1}{iV}\right)\dfrac{\partial}{\partial J_{aa}}\left(\dfrac{e^{-iV\mathrm{tr}(\mathrm{J}\widetilde{E})}A_N(y_1,\cdots,y_N)}{\prod\limits_{1\le t<u\le \mathrm{N}}\left(\dfrac{\lambda(\widetilde{E}_u)^2-\lambda(\widetilde{E}_t)^2+(J_{uu}-J_{tt})}{\kappa}\right)}\right)\Bigg|_{J=0}}{\dfrac{A_N(y_1,\cdots,y_N)\Big|_{J=0}}{\prod\limits_{1\le p<q\le \mathrm{N}}\left(\dfrac{\lambda}{\kappa}\{(\widetilde{E}_q)^2-(\widetilde{E}_p)^2\}\right)}},
\end{aligned}
\tag{3.27}
$$

where $y_j = -\dfrac{VE_{j-1}^2}{(\lambda V)^{\frac{1}{3}}\lambda} + \dfrac{V\kappa}{(\lambda V)^{\frac{1}{3}}} - \dfrac{VJ_{jj}}{(\lambda V)^{\frac{1}{3}}}$ for $j = 1, \ldots, N$. Note that

$$
\begin{aligned}
&\frac{\partial}{\partial J_{aa}}\left\{e^{-iV\mathrm{tr}(\mathrm{J}\widetilde{E})}A_N(y_1,\cdots,y_N)\right\} \\
&= -iV\widetilde{E}_{aa}e^{-iV\mathrm{tr}(\mathrm{J}\widetilde{E})}A_N(y_1,\cdots,y_N) + e^{-iV\mathrm{tr}(\mathrm{J}\widetilde{E})}\left(-\frac{V}{(\lambda V)^{\frac{1}{3}}}\right)\left(\partial_a A_N(y_1,\cdots,y_N)\right),
\end{aligned}
\tag{3.28}
$$

where $\partial_a A_N(y_1,\cdots,y_N) = \dfrac{\partial}{\partial y_a}A_N(y_1,\cdots,y_N)$. Next, we use the following formula. Let $v_n = v_n(\vec{x}_n) = \det\limits_{1\le i,j\le N}(x_j)^{i-1}$ be the Vandermonde determinant for $\vec{x}_n = (x_1,\cdots,x_n) \in \mathbb{R}^n$. For any $1 \le k \le n$

$$
\frac{\partial v_n}{\partial x_k} = \sum_{i=1, i\neq k}^{n}\frac{v_n(\vec{x}_n)}{x_k - x_i}. \tag{3.29}
$$

(See for example [43].) Using this formula, we get

$$
\begin{aligned}
&\frac{\partial}{\partial J_{aa}}\left\{\left(\frac{1}{\kappa}\right)^{\frac{N(N-1)}{2}}\det_{1\le i,j\le N}\left(\lambda(\widetilde{E}_j)^2 + J_{jj}\right)^{i-1}\right\} \\
&= \left(\frac{1}{\kappa}\right)^{\frac{N(N-1)}{2}}\sum_{i=1,i\neq a}^{N}\frac{\det\limits_{1\le i,j\le N}\left(\lambda(\widetilde{E}_j)^2+J_{jj}\right)^{i-1}}{(\lambda(\widetilde{E}_a)^2+J_{aa})-(\lambda(\widetilde{E}_i)^2+J_{ii})}.
\end{aligned}
\tag{3.30}
$$

Substituting (3.28) and (3.30) into (3.27), finally $G_{|a|}$ is expressed as

$$G_{|a|} = -\frac{E_{a-1}}{\lambda} - \frac{\lambda}{iV}\sum_{i=1,i\neq a}^{N}\frac{1}{E_{a-1}^2 - E_{i-1}^2} + \left(\frac{1}{i}\right)\left(-\frac{1}{(\lambda V)^{\frac{1}{3}}}\right)\partial_a \log A_N(z_1,\cdots,z_N), \tag{3.31}$$

where $z_j = -\frac{VE_{j-1}^2}{(\lambda V)^{\frac{1}{3}}\lambda} + \frac{V\kappa}{(\lambda V)^{\frac{1}{3}}}$ for $j = 1,\ldots,N$, and $\partial_a = \frac{\partial}{\partial z_a}$.

Next, let us consider 2-point functions $G_{|ab|}$ ($a \neq b,\ a,b \in \{1,2,\cdots,N\}$). For the calculation, we put J as the matrix all components without J_{ab}, J_{ba} are zero. Note that $\mathrm{tr}JE = \mathrm{tr}J\widetilde{E} = 0$ for this J.

At first, we estimate eigenvalues s_t for $t = 1,\ldots,N$ of the matrix $M-I+K$. The eigenequation is

$$\begin{aligned}
0 &= \det\left(sI - (M - I + K)\right) \\
&= \left(\prod_{i=1,i\neq a,i\neq b}^{N}\left(s - \frac{E_{i-1}^2}{\lambda\kappa} + 1\right)\right)\left\{s^2 + \left(-\frac{E_{b-1}^2}{\lambda\kappa} - \frac{E_{a-1}^2}{\lambda\kappa} + 2\right)s\right. \\
&\quad \left. + \frac{E_{a-1}^2E_{b-1}^2}{\lambda^2\kappa^2} - \frac{E_{b-1}^2}{\lambda\kappa} - \frac{E_{a-1}^2}{\lambda\kappa} + 1 - \frac{J_{ab}J_{ba}}{\kappa^2}\right\}.
\end{aligned} \tag{3.32}$$

Eigenvalues of the matrix $M - I + K$ are labeled as $s_t = \frac{E_{t-1}^2}{\lambda\kappa} - 1$ for $t \neq a, b$,

$$s_a = \frac{\frac{E_{a-1}^2}{\lambda\kappa} + \frac{E_{b-1}^2}{\lambda\kappa} - 2 + \sqrt{\left(\frac{E_{a-1}^2}{\lambda\kappa} - \frac{E_{b-1}^2}{\lambda\kappa}\right)^2 + 4\times\frac{J_{ab}J_{ba}}{\kappa^2}}}{2}, \tag{3.33}$$

$$s_a|_{J=0} = \frac{E_{a-1}^2}{\lambda\kappa} - 1 \tag{3.34}$$

and

$$s_b = \frac{\frac{E_{a-1}^2}{\lambda\kappa} + \frac{E_{b-1}^2}{\lambda\kappa} - 2 - \sqrt{\left(\frac{E_{a-1}^2}{\lambda\kappa} - \frac{E_{b-1}^2}{\lambda\kappa}\right)^2 + 4\times\frac{J_{ab}J_{ba}}{\kappa^2}}}{2}, \tag{3.35}$$

$$s_b|_{J=0} = \frac{E_{b-1}^2}{\lambda\kappa} - 1. \tag{3.36}$$

Let us calculate $G_{|ab|}$ by using these s_t.

$$\begin{aligned}
G_{|ab|} &= \frac{1}{iV}\frac{\partial^2 \log \mathcal{Z}[J]}{\partial J_{ab}\partial J_{ba}}\bigg|_{J=0} \\
&= \left.\frac{\dfrac{\partial^2}{\partial J_{ab}\partial J_{ba}} A_N(y_1,\cdots,y_N)}{iV \displaystyle\prod_{1\le t<u\le N}(s_u-s_t)}\right|_{J=0} \times \frac{\displaystyle\prod_{1\le t<u\le N}(s_u-s_t)|_{J=0}}{A_N(y_1,\cdots,y_N)|_{J=0}} \\
&\quad - \left.\frac{A_N(y_1,\cdots,y_N)\dfrac{\partial^2}{\partial J_{ab}\partial J_{ba}}\left\{\displaystyle\prod_{1\le t<u\le N}(s_u-s_t)\right\}}{iV\left\{\displaystyle\prod_{1\le t<u\le N}(s_u-s_t)\right\}^2}\right|_{J=0} \times \frac{\displaystyle\prod_{1\le t<u\le N}(s_u-s_t)|_{J=0}}{A_N(y_1,\cdots,y_N)|_{J=0}}.
\end{aligned} \tag{3.37}$$

Here we use $\left.\dfrac{\partial A_N(y_1,\cdots,y_N)}{\partial J_{ab}}\right|_{J=0} = \left.\dfrac{\partial \det\limits_{1\le k,l\le N}(s_l)^{k-1}}{\partial J_{ab}}\right|_{J=0} = 0$, since s_a and s_b are functions of $(J_{ab}J_{ba})$ as we see in (3.33) and (3.35), then $\dfrac{\partial A_N(y_1,\cdots,y_N)}{\partial J_{ab}}$ and $\dfrac{\partial \det\limits_{1\le k,l\le N}(s_l)^{k-1}}{\partial J_{ab}}$ are of the form $J_{ba}\times(\cdots)$.

Recall that $y_k = -\dfrac{V\kappa s_k}{(\lambda V)^{\frac{1}{3}}}$. Using the fact that $\left.\dfrac{\partial y_k}{\partial J_{ab}}\right|_{J=0} = 0$ and $\dfrac{\partial^2 y_a}{\partial J_{ba}\partial J_{ab}} = -\dfrac{V\lambda}{(\lambda V)^{\frac{1}{3}}}\dfrac{1}{E_{a-1}^2-E_{b-1}^2} = -\dfrac{\partial^2 y_b}{\partial J_{ab}\partial J_{ba}}$, we obtain

$$\begin{aligned}
&\left.\frac{\partial^2}{\partial J_{ab}\partial J_{ba}}A_N(y_1,\cdots,y_N)\right|_{J=0} \\
=&\frac{V\lambda}{(\lambda V)^{\frac{1}{3}}}\frac{1}{E_{a-1}^2-E_{b-1}^2}\left(\partial_b A_N(z_1,\ldots,z_N) - \partial_a A_N(z_1,\ldots,z_N)\right),
\end{aligned} \tag{3.38}$$

where $z_j = -\dfrac{VE_{j-1}^2}{(\lambda V)^{\frac{1}{3}}\lambda} + \dfrac{V\kappa}{(\lambda V)^{\frac{1}{3}}}$ for $j=1,\ldots,N$. Similarly, we get

$$\begin{aligned}
&\left.\frac{\partial^2}{\partial J_{ab}\partial J_{ba}}\left\{\prod_{1\le t<u\le N}(s_u-s_t)\right\}\right|_{J=0} \\
&= \frac{\lambda^2}{E_{a-1}^2-E_{b-1}^2}\left(\sum_{i=1,i\ne a}^{N}\frac{1}{E_{a-1}^2-E_{i-1}^2} - \sum_{i=1,i\ne b}^{N}\frac{1}{E_{b-1}^2-E_{i-1}^2}\right)\det_{1\le k,l\le N}(s_k)^{l-1},
\end{aligned} \tag{3.39}$$

where we use the formula (3.29), again. Substituting (3.38) and (3.39) into (3.37), $G_{|ab|}$ ($b < a$, and $E_b < E_a$) is finally obtained as

$$\begin{aligned} G_{|ab|} =& -\frac{\lambda^2}{iV}\sum_{i=1,i\neq a}^{N}\frac{1}{(E_{a-1}^2-E_{i-1}^2)(E_{a-1}^2-E_{b-1}^2)} \\ &+\frac{\lambda^2}{iV}\sum_{i=1,i\neq b}^{N}\frac{1}{(E_{b-1}^2-E_{i-1}^2)(E_{a-1}^2-E_{b-1}^2)} \\ &-\frac{\lambda}{i(\lambda V)^{\frac{1}{3}}}\frac{1}{E_{a-1}^2-E_{b-1}^2}\partial_a \log A_N(z_1,\ldots,z_N) \\ &+\frac{\lambda}{i(\lambda V)^{\frac{1}{3}}}\frac{1}{E_{a-1}^2-E_{b-1}^2}\partial_b \log A_N(z_1,\ldots,z_N). \end{aligned} \tag{3.40}$$

We now refer to Schwinger-Dyson equation

$$G_{|ab|} =\frac{1}{E_{a-1}+E_{b-1}}\left(1+\lambda\frac{\left(G_{|a|}-G_{|b|}\right)}{E_{a-1}-E_{b-1}}\right) \tag{3.41}$$

in reference[18]. Substituting (3.40) for the left side of (3.41) and (3.31) for the right side of (3.41) shows that Schwinger-Dyson equation (3.41) is indeed satisfied.

3.4. Calculation of n-Point Function $G_{|a^1|a^2|\cdots|a^n|}$

The goal of this section is to obtain the explicit formula of n-point function $G_{|a^1|a^2|\cdots|a^n|}$. Here a^β is the pairwise different indices for $\beta = 1,\ldots,n$. From the definition in (3.5), n-point function $G_{|a^1|a^2|\cdots|a^n|}$ is given by

$$G_{|a^1|a^2|\cdots|a^n|} =(iV)^{n-2}\frac{\partial^n}{\partial J_{a^1a^1}\ldots\partial J_{a^na^n}}\log\frac{\mathcal{Z}[J]}{\mathcal{Z}[0]}\bigg|_{J=0}. \tag{3.42}$$

We use the formula in [23]:

$$\frac{\partial^n}{\partial x_1\cdots\partial x_n}f(y) = \sum_{\pi} f^{|\pi|}(y)\prod_{B\in\pi}\frac{\partial^{|B|}y}{\prod_{j\in B}\partial x_j}, \tag{3.43}$$

where $f(y)$ is the differentiable function of the variable $y = y(x_1, x_2, \ldots, x_n)$, $\sum_{\pi}$ means the sum over all partitions π of the set $\{1,\ldots,n\}$, $\prod_{B\in\pi}$ is the product over all of the parts B of the partition π, and $|S|$ denotes the cardinality of any set S. Applying (3.43) to (3.42) n-point functions $G_{|a^1|a^2|\cdots|a^n|}$ is expressed as follows:

$$G_{|a^1|a^2|\cdots|a^n|} =(iV)^{n-2}\sum_{\pi}\left\{\left(\frac{d}{dx}\right)^{|\pi|}(\log x)\bigg|_{x=\mathcal{Z}[0]}\right\}\prod_{B\in\pi}\frac{\partial^{|B|}\mathcal{Z}[J]}{\prod_{j\in B}\partial J_{a^ja^j}}\bigg|_{J=0}. \tag{3.44}$$

After the calculation of $\left.\dfrac{\partial^{|B|}\mathcal{Z}[J]}{\prod\limits_{j\in B}\partial J_{a^j a^j}}\right|_{J=0}$, we get the following result.

Lemma 3.4 (N.K-Sako[40])**.**
We introduce $\mathcal{C}=\exp\left(-\dfrac{iV}{\lambda^2}\mathrm{tr}\left(\dfrac{2}{3}E^3-\lambda\kappa E\right)\right)\left(\prod\limits_{p=1}^{N}p!\right)(-2)^N\dfrac{\pi^{\frac{N(N+1)}{2}}}{V^{\frac{N(2N-1)}{3}}\lambda^{\frac{N(N+1)}{6}}}$. *Then*

$$\left.\frac{\partial^{|B|}\mathcal{Z}[J]}{\prod\limits_{j\in B}\partial J_{a^j a^j}}\right|_{J=0} = \mathcal{C}\sum_{S\subset B}\Bigg(\left(\prod_{i\in S}\left(-iV\frac{E_{a^i-1}}{\lambda}\right)\right)$$

$$\sum_{M\subset\overline{S}}\Bigg(\left(\left\{\prod_{k\in M}\left(-\frac{V}{(\lambda V)^{\frac{1}{3}}}\right)\partial_{a^k}\right\}A_N(z_1,\ldots,z_N)\right)$$

$$\left(\left\{\prod_{q\in\overline{M}}\frac{\partial}{\partial t_{a^q}}\right\}\frac{1}{\det\limits_{1\le l,j\le N}\left(t_l^{j-1}\right)}\right)\Bigg)\Bigg), \tag{3.45}$$

where $z_j=-\dfrac{VE_{j-1}^2}{(\lambda V)^{\frac{1}{3}}\lambda}+\dfrac{V\kappa}{(\lambda V)^{\frac{1}{3}}}$ *for* $j=1,\ldots N$, $\partial_{a^k}=\dfrac{\partial}{\partial z_{a^k}}$ $(k\in M)$, $t_l=\dfrac{(E_{l-1})^2}{\lambda}$ *for* $l=1,\ldots N$, *S runs through the set of all subsets of B, $\overline{S}$ is the complement of S in B, M runs through the set of all subsets of $\overline{S}$, and $\overline{M}=\overline{S}\backslash M$.*

Proof. For the calculation of $G_{|a^1|a^2|\cdots|a^n|}$, we can choose J as a diagonal matrix $diag(J_{11},\cdots,J_{NN})$. Then, $s_t=\dfrac{\lambda(\widetilde{E}_t)^2+J_{tt}}{\kappa}-1$. To calculate

$$\left.\frac{\partial^{|B|}\mathcal{Z}[J]}{\prod\limits_{j\in B}\partial J_{a^j a^j}}\right|_{J=0} = \mathcal{C}\frac{\partial^{|B|}}{\prod\limits_{j\in B}\partial J_{a^j a^j}}\left.\left(\frac{e^{-iVtr(J\widetilde{E})}A_N(y_1,\cdots,y_N)}{\det\limits_{1\le i,j\le N}\left(\lambda(\widetilde{E}_j)^2+J_{jj}\right)^{i-1}}\right)\right|_{J=0} \tag{3.46}$$

we use the formula in [23]:

$$\frac{\partial^n}{\partial x_1\cdots\partial x_n}(uv)=\sum_S\frac{\partial^{|S|}u}{\prod_{j\in S}\partial x_j}\cdot\frac{\partial^{(n-|S|)}v}{\prod_{j\notin S}\partial x_j}, \tag{3.47}$$

where u, and v are differentiable functions of the variable $x=(x_1,x_2,\ldots,x_n)$, and S runs through the set of all subsets of $\{1,\ldots,n\}$.

Using the formula (3.47) twice for (3.46), we obtain the following :

$$\frac{\partial^{|B|}\mathcal{Z}[J]}{\prod_{j\in B}\partial J_{a^j a^j}}\Bigg|_{J=0} = \mathcal{C}\sum_{S\subset B}\frac{\partial^{|S|}e^{-iVtr(J\widetilde{E})}}{\prod_{i\in S}\partial J_{a^i a^i}}\Bigg|_{J=0}\sum_{M\subset\overline{S}}\frac{\partial^{|M|}A_N(y_1,\cdots,y_N)}{\prod_{k\in M}\partial J_{a^k a^k}}\Bigg|_{J=0}$$

$$\frac{\partial^{|\overline{M}|}}{\prod_{q\in\overline{M}}\partial J_{a^q a^q}}\left(\frac{1}{\det_{1\le l,j\le N}\left(\lambda(\widetilde{E}_l)^2+J_{ll}\right)^{j-1}}\right)\Bigg|_{J=0}. \tag{3.48}$$

For the diagonal J, $y_k = \left(-\frac{V\kappa}{(\lambda V)^{\frac{1}{3}}}\right)\left(\frac{E_{k-1}^2}{\lambda\kappa}-1+\frac{J_{kk}}{\kappa}\right)$, then the above is rewritten as

$$\begin{aligned}(3.48) =&\mathcal{C}\sum_{S\subset B}\left(\left(\prod_{i\in S}\left(-iV\frac{E_{a^i-1}}{\lambda}\right)\right)\right.\\ &\sum_{M\subset\overline{S}}\left(\left(\left\{\prod_{k\in M}\left(-\frac{V}{(\lambda V)^{\frac{1}{3}}}\right)\partial_{a^k}\right\}A_N(z_1,\ldots,z_N)\right)\right.\\ &\left.\left.\times\left(\left\{\prod_{q\in\overline{M}}\frac{\partial}{\partial t_{a^q}}\right\}\frac{1}{\det_{1\le l,j\le N}\left(t_l^{j-1}\right)}\right)\right)\right),\end{aligned} \tag{3.49}$$

where $z_j = -\frac{VE_{j-1}^2}{(\lambda V)^{\frac{1}{3}}\lambda}+\frac{V\kappa}{(\lambda V)^{\frac{1}{3}}}$ for $j=1,\ldots N$, $t_l = \frac{(E_{l-1})^2}{\lambda}$ for $l=1,\ldots N$, S runs through the set of all subsets of B, $\overline{S}$ is the complement of S in B, M runs through the set of all subsets of $\overline{S}$, and $\overline{M}=\overline{S}\backslash M$.

□

Note the cases that each set is an empty set,

$$\prod_{i\in\emptyset}\left(-iV\frac{E_{a^i-1}}{\lambda}\right)=1,\qquad \left\{\prod_{k\in\emptyset}\left(-\frac{V}{(\lambda V)^{\frac{1}{3}}}\right)\partial_{a^k}\right\}A_N(z_1,\ldots,z_N)=A_N(z_1,\ldots,z_N),$$

and $\left\{\prod_{q\in\emptyset}\frac{\partial}{\partial t_{a^q}}\right\}\frac{1}{\det_{1\le l,j\le N}\left(t_l^{j-1}\right)}=\frac{1}{\det_{1\le l,j\le N}\left(t_l^{j-1}\right)}$.

Summarizing (3.44) and the result in Lemma 3.4, we obtain the following:

Theorem 3.5 (N.K-Sako[40])**.** *We suppose the partition function $\mathcal{Z}[J]$ of the Φ_2^3 matrix model is* *defined by (3.4) and $G_{|a^1|a^2|\cdots|a^n|}$ is defined by (3.42). In this case,*

$$
\begin{aligned}
&G_{|a^1|a^2|\cdots|a^n|}\\
=&(iV)^{n-2}\mathcal{C}\sum_{\pi}\left\{\left(\frac{d}{dx}\right)^{|\pi|}(\log x)\Big|_{x=\mathcal{Z}[0]}\right\}\prod_{B\in\pi}\sum_{S\subset B}\left(\left(\prod_{i\in S}\left(-iV\frac{E_{a^i-1}}{\lambda}\right)\right)\right.\\
&\times\sum_{M\subset\overline{S}}\left(\left(\left(\left\{\prod_{k\in M}\left(-\frac{V}{(\lambda V)^{\frac{1}{3}}}\right)\partial_{a^k}\right\}A_N(z_1,\ldots,z_N)\right)\right.\right.\\
&\left.\left.\left.\left(\left\{\prod_{q\in\overline{M}}\frac{\partial}{\partial t_{a^q}}\right\}\frac{1}{\prod_{1\le l<j\le N}(t_j-t_l)}\right)\right)\right)\right),
\end{aligned}
\tag{3.50}
$$

where $\sum_{\pi}$ means the sum over all partitions π of the set $\{1,\ldots,n\}$, $\prod_{B\in\pi}$ is over all of the parts B of the partition π, $|S|$ denotes the cardinality of any set S, $\sum_{S\subset B}$ means the sum over all subsets of B, $\sum_{M\subset\overline{S}}$ means the sum over all subsets of $\overline{S}=B\backslash S$, and $t_l=\dfrac{(E_{l-1})^2}{\lambda}$.

Now we refer to the formula in Section 5 in [18].

Theorem 3.6 (Grosse-Sako-Wulkenhaar[18]). *We suppose $G_{|a^1|a^2|\cdots|a^n|}$ is defined by (3.42). In this case,*

$$
\begin{aligned}
&G_{|a^1_1\ldots a^1_{N_1}|\ldots|a^B_1\ldots a^B_{N_B}|}\\
=&\lambda^{N_1+\cdots+N_B-B}\sum_{k_1=1}^{N_1}\cdots\sum_{k_B=1}^{N_B}G_{|a^1_{k_1}|\ldots|a^B_{k_B}|}\left(\prod_{l_1=1,l_1\neq k_1}^{N_1}P_{a^1_{k_1}a^1_{l_1}}\right)\cdots\left(\prod_{l_B=1,l_B\neq k_B}^{N_B}P_{a^B_{k_B}a^B_{l_B}}\right),
\end{aligned}
\tag{3.51}
$$

where $2\le B$, $N_i>1$ for $i=1,\ldots,B$, λ is the coupling constant(real), and $P_{ab}:=\dfrac{1}{E^2_{a-1}-E^2_{b-1}}$.

Substituting (3.50) into (3.51), all the exact solutions of the Φ^3_2 finite matrix model is obtained.

For the later convenience, we introduce a function $F(S,M,\overline{M})$. Let B be a subset of $\{1,\cdots,n\}$. For $S\subset B$, $\overline{S}$ denotes the complement $B\backslash S$.

$$
\begin{aligned}
F(S,M,\overline{M}):=&\left(\prod_{i\in S}\left(-iV\frac{E_{a^i-1}}{\lambda}\right)\right)\left(\left\{\prod_{k\in M}\left(-\frac{V}{(\lambda V)^{\frac{1}{3}}}\right)\partial_{a^k}\right\}A_N(z_1,\ldots,z_N)\right)\\
&\left(\left\{\prod_{q\in\overline{M}}\frac{\partial}{\partial t_{a^q}}\right\}\frac{1}{\det_{1\le l,j\le N}(t_l^{j-1})}\right),
\end{aligned}
\tag{3.52}
$$

where $B = S \sqcup \overline{S}$, $\overline{S} = M \sqcup \overline{M}$, and $\partial_{a^k} = \dfrac{\partial}{\partial z_{a^k}}$. Using this $F(S, M, \overline{M})$, $G_{|a^1|a^2|\cdots|a^n|}$ is expressed as

$$G_{|a^1|a^2|\cdots|a^n|} = (iV)^{n-2}\mathcal{C}\sum_{\pi}\left\{\left(\frac{d}{dx}\right)^{|\pi|}(\log x)\bigg|_{x=\mathcal{Z}[0]}\right\}\prod_{B\in\pi}\sum_{S\subset B}\sum_{M\subset\overline{S}} F(S, M, \overline{M}). \tag{3.53}$$

3.5. Calculation of 2-Point Function $G_{|a|b|}$

The formula (3.53) is used to obtain $G_{|a|b|}$ concretely. We use $a = a^1$ and $b = a^2$ below.
At first, we estimate the case of $\pi = \{\{1, 2\}\}$. In this case $|\pi| = 1$, and it is enough to calculate $F(S, M, \overline{M})$ for $B = \{1, 2\}$. In the context of Theorem 3.5, it corresponds to the part:

$$\left(\frac{d}{dx}\right)^{|\pi|}(\log x)\bigg|_{x=\mathcal{Z}[0]}\prod_{B\in\pi}\frac{\partial^{|B|}\mathcal{Z}[J]}{\prod_{j\in B}\partial J_{a^j a^j}}\bigg|_{J=0} = \frac{1}{\mathcal{Z}[0]}\frac{\partial^2\mathcal{Z}[J]}{\partial J_{aa}\partial J_{bb}}\bigg|_{J=0}. \tag{3.54}$$

Calculating all cases for sets S, M, and $\overline{M}$, we obtain the following results. In the case of $F(\{1, 2\}, \emptyset, \emptyset)$,

$$F(\{1, 2\}, \emptyset, \emptyset) = \left(-iV\frac{E_{a-1}}{\lambda}\right)\left(-iV\frac{E_{b-1}}{\lambda}\right)A_N(z_1, \ldots, z_N)\frac{1}{\det\limits_{1\le l,j\le N}\left(t_l^{j-1}\right)}. \tag{3.55}$$

In the case of $F(\{1\}, \{2\}, \emptyset)$,

$$F(\{1\}, \{2\}, \emptyset) = \left(-iV\frac{E_{a-1}}{\lambda}\right)\left(-\frac{V}{(\lambda V)^{\frac{1}{3}}}\right)\partial_b A_N(z_1, \ldots, z_N)\frac{1}{\det\limits_{1\le l,j\le N}\left(t_l^{j-1}\right)}. \tag{3.56}$$

$F(\{2\}, \{1\}, \emptyset)$ can be calculated in the same way (3.56). The letters a and b in (3.56) are interchanged. In the case of $F(\emptyset, \{1, 2\}, \emptyset)$,

$$F(\emptyset, \{1, 2\}, \emptyset) = \left(-\frac{V}{(\lambda V)^{\frac{1}{3}}}\right)^2\partial_a\partial_b A_N(z_1, \ldots, z_N)\frac{1}{\det\limits_{1\le l,j\le N}\left(t_l^{j-1}\right)}. \tag{3.57}$$

In the case of $F(\{1\}, \emptyset, \{2\})$,

$$F(\{1\}, \emptyset, \{2\}) = \left(-iV\frac{E_{a-1}}{\lambda}\right)A_N(z_1, \ldots, z_N)\frac{-1}{\det\limits_{1\le l,j\le N}\left(t_l^{j-1}\right)}\sum_{i=1, i\ne b}^{N}\frac{1}{t_b - t_i}. \tag{3.58}$$

$F(\{2\}, \emptyset, \{1\})$ can be calculated in the same way (3.58). The letters a and b in (3.58) are interchanged. In the case of $F(\emptyset, \{1\}, \{2\})$,

$$F(\emptyset, \{1\}, \{2\}) = \left(-\frac{V}{(\lambda V)^{\frac{1}{3}}}\right)\partial_a A_N(z_1, \ldots, z_N)\frac{-1}{\det\limits_{1\le l,j\le N}\left(t_l^{j-1}\right)}\sum_{i=1, i\ne b}^{N}\frac{1}{t_b - t_i}. \tag{3.59}$$

$F(\emptyset,\{2\},\{1\})$ can be calculated in the same way (3.59). The letters a and b in (3.59) are interchanged. In the case of $F(\emptyset,\emptyset,\{1,2\})$,

$$\begin{aligned}
&F(\emptyset,\emptyset,\{1,2\}) \\
=&A_N(z_1,\ldots,z_N)\frac{1}{\det\limits_{1\le l,j\le N}\left(t_l^{j-1}\right)}\left(\sum_{i=1,i\neq a}^{N}\frac{1}{t_a-t_i}\sum_{j=1,j\neq b}^{N}\frac{1}{t_b-t_j}-\frac{1}{(t_a-t_b)^2}\right).
\end{aligned} \tag{3.60}$$

From this, (3.54) can be calculated as follows:

$$\begin{aligned}
&\frac{1}{\mathcal{Z}[0]}\frac{\partial^2\mathcal{Z}[J]}{\partial J_{aa}\partial J_{bb}}\bigg|_{J=0} \\
=&\left(\frac{\det\limits_{1\le l,j\le N}\left(t_l^{j-1}\right)}{A_N(z_1,\ldots,z_N)}\right)\Bigg\{F(\{1,2\},\emptyset,\emptyset)+F(\emptyset,\{1,2\},\emptyset)+F(\emptyset,\emptyset,\{1,2\}) \\
&+\sum_{l,n=1,l\neq n}^{2}\Big(F(\{l\},\{n\},\emptyset)+F(\{l\},\emptyset,\{n\})+F(\emptyset,\{l\},\{n\})\Big)\Bigg\}
\end{aligned} \tag{3.61}$$

Next step, let us consider the case $\pi=\{\{1\},\{2\}\}$, $|\pi|=2$, $B=\{1\},or\{2\}$.

The corresponding term $\left(\frac{d}{dx}\right)^{|\pi|}(\log x)\Big|_{x=\mathcal{Z}[0]}\prod_{B\in\pi}\frac{\partial^{|B|}\mathcal{Z}[J]}{\prod\limits_{j\in B}\partial J_{a^ja^j}}\bigg|_{J=0}$ in Theorem 3.5 is as follows:

$$\left(\frac{d}{dx}\right)^{|\pi|}(\log x)\Big|_{x=\mathcal{Z}[0]}\prod_{B\in\pi}\frac{\partial^{|B|}\mathcal{Z}[J]}{\prod\limits_{j\in B}\partial J_{a^ja^j}}\bigg|_{J=0}=-\frac{1}{\mathcal{Z}[0]^2}\frac{\partial\mathcal{Z}[J]}{\partial J_{aa}}\bigg|_{J=0}\frac{\partial\mathcal{Z}[J]}{\partial J_{bb}}\bigg|_{J=0}. \tag{3.62}$$

Calculating all cases for sets S, M, and $\overline{M}$ of $B=\{1\}$, we obtain the following results.

$$F(\{1\},\emptyset,\emptyset)=-iV\frac{E_{a-1}}{\lambda}A_N(z_1,\ldots,z_N)\frac{1}{\det\limits_{1\le l,j\le N}\left(t_l^{j-1}\right)}. \tag{3.63}$$

$$F(\emptyset,\{1\},\emptyset)=\left(-\frac{V}{(\lambda V)^{\frac{1}{3}}}\right)\partial_aA_N(z_1,\ldots,z_N)\frac{1}{\det\limits_{1\le l,j\le N}\left(t_l^{j-1}\right)}. \tag{3.64}$$

$$F(\emptyset,\emptyset,\{1\})=A_N(z_1,\ldots,z_N)\frac{-1}{\det\limits_{1\le l,j\le N}\left(t_l^{j-1}\right)}\sum_{i=1,i\neq a}^{N}\frac{1}{t_a-t_i}. \tag{3.65}$$

These results can be summarized as follows:

$$\frac{1}{\mathcal{Z}[0]}\frac{\partial \mathcal{Z}[J]}{\partial J_{aa}}\bigg|_{J=0} = \left(\frac{\det\limits_{1\le l,j\le N}\left(t_l^{j-1}\right)}{A_N(z_1,\ldots,z_N)}\right)\left\{F(\{1\},\emptyset,\emptyset)+F(\emptyset,\{1\},\emptyset)+F(\emptyset,\emptyset,\{1\})\right\}. \tag{3.66}$$

The same calculation is performed for $B=\{2\}$ as for $B=\{1\}$:

$$\frac{1}{\mathcal{Z}[0]}\frac{\partial \mathcal{Z}[J]}{\partial J_{bb}}\bigg|_{J=0} = \left(\frac{\det\limits_{1\le l,j\le N}\left(t_l^{j-1}\right)}{A_N(z_1,\ldots,z_N)}\right)\left\{F(\{2\},\emptyset,\emptyset)+F(\emptyset,\{2\},\emptyset)+F(\emptyset,\emptyset,\{2\})\right\}. \tag{3.67}$$

Note that (3.66) and (3.67) coincide with iV multiples of the one-point function $G_{|a|}$ and $G_{|b|}$ in Section 3. Substituting (3.66) and (3.67) into (3.62) gives the result :

$$\begin{aligned}&-\frac{1}{\mathcal{Z}[0]^2}\frac{\partial \mathcal{Z}[J]}{\partial J_{aa}}\bigg|_{J=0}\frac{\partial \mathcal{Z}[J]}{\partial J_{bb}}\bigg|_{J=0}\\ &=-\left(\frac{\det\limits_{1\le l,j\le N}\left(t_l^{j-1}\right)}{A_N(z_1,\ldots,z_N)}\right)^2\prod_{l=1}^{2}\Big(F(\{l\},\emptyset,\emptyset)+F(\emptyset,\{l\},\emptyset)+F(\emptyset,\emptyset,\{l\})\Big).\end{aligned} \tag{3.68}$$

Finally, adding (3.61) and (3.68) the result of the two point functions $G_{|a|b|}$ is obtained by

$$\begin{aligned}G_{|a|b|} &= \frac{1}{\mathcal{Z}[0]}\frac{\partial^2 \mathcal{Z}[J]}{\partial J_{aa}\partial J_{bb}}\bigg|_{J=0} - \frac{1}{\mathcal{Z}[0]^2}\frac{\partial \mathcal{Z}[J]}{\partial J_{aa}}\bigg|_{J=0}\frac{\partial \mathcal{Z}[J]}{\partial J_{bb}}\bigg|_{J=0}\\ &= \left(\frac{\det\limits_{1\le l,j\le N}\left(t_l^{j-1}\right)}{A_N(z_1,\ldots,z_N)}\right)\Bigg\{F(\{1,2\},\emptyset,\emptyset)+F(\emptyset,\{1,2\},\emptyset)+F(\emptyset,\emptyset,\{1,2\})\\ &\quad+\sum_{l,n=1,l\neq n}^{2}\Big(F(\{l\},\{n\},\emptyset)+F(\{l\},\emptyset,\{n\})+F(\emptyset,\{l\},\{n\})\Big)\Bigg\}\\ &\quad-\left(\frac{\det\limits_{1\le l,j\le N}\left(t_l^{j-1}\right)}{A_N(z_1,\ldots,z_N)}\right)^2\prod_{l=1}^{2}\Big(F(\{l\},\emptyset,\emptyset)+F(\emptyset,\{l\},\emptyset)+F(\emptyset,\emptyset,\{l\})\Big).\end{aligned} \tag{3.69}$$

For a more complex example, we carry out the calculation for $G_{|a^1|a^2|a^3|}$ in Section 3.6.

3.6. Calculation of 3-Point Function $G_{|a^1|a^2|a^3|}$

We calculate the three point functions $G_{|a^1|a^2|a^3|}$ using the formula (3.50) or (3.53). $i,l,k \in \{1,2,3\}$ and $i \neq l \neq k \neq i$ below.

i). We consider the case $\pi = \{\{1,2,3\}\}$,$|\pi| = 1$, and $B = \{1,2,3\}$, then

$$\left(\frac{d}{dx}\right)^{|\pi|} (\log x)\Bigg|_{x=\mathcal{Z}[0]} \prod_{B\in\pi} \frac{\partial^{|B|}\mathcal{Z}[J]}{\prod_{j\in B}\partial J_{a^j a^j}}\Bigg|_{J=0} = \frac{1}{\mathcal{Z}[0]}\frac{\partial^3\mathcal{Z}[J]}{\partial J_{a^1a^1}\partial J_{a^2a^2}\partial J_{a^3a^3}}\Bigg|_{J=0}. \tag{3.70}$$

The calculations required to calculate $\frac{1}{\mathcal{Z}[0]}\frac{\partial^3\mathcal{Z}[J]}{\partial J_{a^1a^1}\partial J_{a^2a^2}\partial J_{a^3a^3}}\Bigg|_{J=0}$ are written below :

$$F(\{1,2,3\},\emptyset,\emptyset) = \left(-iV\frac{E_{a^1-1}}{\lambda}\right)\left(-iV\frac{E_{a^2-1}}{\lambda}\right)\left(-iV\frac{E_{a^3-1}}{\lambda}\right)A_N(z_1,\dots,z_N)\frac{1}{\det\limits_{1\le p,q\le N}\left(t_p^{q-1}\right)}, \tag{3.71}$$

$$F(\{i,l\},\{k\},\emptyset) = \left(-iV\frac{E_{a^i-1}}{\lambda}\right)\left(-iV\frac{E_{a^l-1}}{\lambda}\right)\left(-\frac{V}{(\lambda V)^{\frac{1}{3}}}\right)\partial_{a^k}A_N(z_1,\dots,z_N)\frac{1}{\det\limits_{1\le p,q\le N}\left(t_p^{q-1}\right)}, \tag{3.72}$$

$$F(\{i\},\{l,k\},\emptyset) = \left(-iV\frac{E_{a^i-1}}{\lambda}\right)\left(-\frac{V}{(\lambda V)^{\frac{1}{3}}}\right)^2\partial_{a^l}\partial_{a^k}A_N(z_1,\dots,z_N)\frac{1}{\det\limits_{1\le p,q\le N}\left(t_p^{q-1}\right)}, \tag{3.73}$$

$$F(\emptyset,\{1,2,3\},\emptyset) = \left(-\frac{V}{(\lambda V)^{\frac{1}{3}}}\right)^3\partial_{a^1}\partial_{a^2}\partial_{a^3}A_N(z_1,\dots,z_N)\frac{1}{\det\limits_{1\le p,q\le N}\left(t_p^{q-1}\right)}, \tag{3.74}$$

$$\begin{aligned}&F(\{i\},\{l\},\{k\})\\ &= \left(-iV\frac{E_{a^i-1}}{\lambda}\right)\left(-\frac{V}{(\lambda V)^{\frac{1}{3}}}\right)\partial_{a^l}A_N(z_1,\dots,z_N)\frac{-1}{\det\limits_{1\le p,q\le N}\left(t_p^{q-1}\right)}\sum_{r=1,r\neq a^k}^{N}\frac{1}{t_{a^k}-t_r},\end{aligned} \tag{3.75}$$

$$F(\{i,l\},\emptyset,\{k\}) = \left(-iV\frac{E_{a^i-1}}{\lambda}\right)\left(-iV\frac{E_{a^l-1}}{\lambda}\right)A_N(z_1,\dots,z_N)\frac{-1}{\det\limits_{1\le p,q\le N}\left(t_p^{q-1}\right)}\sum_{r=1,r\neq a^k}^{N}\frac{1}{t_{a^k}-t_r}, \tag{3.76}$$

$$F(\emptyset,\{i,l\},\{k\}) = \left(-\frac{V}{(\lambda V)^{\frac{1}{3}}}\right)^2 \partial_{a^i}\partial_{a^l} A_N(z_1,\ldots,z_N)\frac{-1}{\det\limits_{1\le p,q\le N}\left(t_p^{q-1}\right)}\sum_{r=1,r\neq a^k}^{N}\frac{1}{t_{a^k}-t_r}, \quad (3.77)$$

$$F(\{i\},\emptyset,\{l,k\}) = \left(-iV\frac{E_{a^i-1}}{\lambda}\right) A_N(z_1,\ldots,z_N)$$
$$\frac{1}{\det\limits_{1\le p,q\le N}\left(t_p^{q-1}\right)}\left(\sum_{r=1,r\neq a^l}^{N}\frac{1}{t_{a^l}-t_r}\sum_{w=1,w\neq a^k}^{N}\frac{1}{t_{a^k}-t_w}-\frac{1}{(t_{a^l}-t_{a^k})^2}\right), \quad (3.78)$$

$$F(\emptyset,\{i\},\{l,k\}) = \left(-\frac{V}{(\lambda V)^{\frac{1}{3}}}\right)\partial_{a^i} A_N(z_1,\ldots,z_N)$$
$$\frac{1}{\det\limits_{1\le p,q\le N}\left(t_p^{q-1}\right)}\left(\sum_{r=1,r\neq a^l}^{N}\frac{1}{t_{a^l}-t_r}\sum_{w=1,w\neq a^k}^{N}\frac{1}{t_{a^k}-t_w}-\frac{1}{(t_{a^l}-t_{a^k})^2}\right), \quad (3.79)$$

$$\begin{aligned}F(\emptyset,\emptyset,\{1,2,3\}) = &-\frac{1}{\det\limits_{1\le p,q\le N}\left(t_p^{q-1}\right)}\\ &\times\sum_{r,w,f=1,r\neq a^3,w\neq a^2,f\neq a^1}^{N}\frac{1}{t_{a^3}-t_r}\frac{1}{t_{a^2}-t_w}\frac{1}{t_{a^1}-t_f}A_N(z_1,\ldots,z_N)\\ &+\frac{1}{\det\limits_{1\le p,q\le N}\left(t_p^{q-1}\right)}\sum_{r=1,r\neq a^1}^{N}\frac{1}{(t_{a^2}-t_{a^3})^2}\frac{1}{t_{a^1}-t_r}A_N(z_1,\ldots,z_N)\\ &+\frac{1}{\det\limits_{1\le p,q\le N}\left(t_p^{q-1}\right)}\sum_{w=1,w\neq a^2}^{N}\frac{1}{(t_{a^1}-t_{a^3})^2}\frac{1}{t_{a^2}-t_w}A_N(z_1,\ldots,z_N)\\ &+\frac{1}{\det\limits_{1\le p,q\le N}\left(t_p^{q-1}\right)}\sum_{f=1,f\neq a^3}^{N}\frac{1}{(t_{a^1}-t_{a^2})^2}\frac{1}{t_{a^3}-t_f}A_N(z_1,\ldots,z_N).\end{aligned} \quad (3.80)$$

If we sum up all the cases for sets S, M, and $\overline{M}$ that we have calculated so far and multiply by $\frac{1}{\mathcal{Z}[0]}$, we get the result of (3.70):

$$\frac{1}{\mathcal{Z}[0]}\frac{\partial^3\mathcal{Z}[J]}{\partial J_{a^1a^1}\partial J_{a^2a^2}\partial J_{a^3a^3}}\bigg|_{J=0}$$
$$=\left(\frac{\det\limits_{1\le p,q\le N}\left(t_p^{q-1}\right)}{A_N(z_1,\ldots,z_N)}\right)\Bigg\{F(\{1,2,3\},\emptyset,\emptyset)+F(\emptyset,\{1,2,3\},\emptyset)+F(\emptyset,\emptyset,\{1,2,3\})$$

$$+\sum_{i,l,k=1,i\neq l\neq k\neq i}^{3}\Bigg(F(\{i\},\{l\},\{k\})+\frac{F(\{i,l\},\{k\},\emptyset)}{2}+\frac{F(\{i\},\{l,k\},\emptyset)}{2}+\frac{F(\{i,l\},\emptyset,\{k\})}{2}$$

$$+\frac{F(\emptyset,\{i,l\},\{k\})}{2}+\frac{F(\{i\},\emptyset,\{l,k\})}{2}+\frac{F(\emptyset,\{i\},\{l,k\})}{2}\Bigg)\Bigg\}. \tag{3.81}$$

ii). We consider the case $\pi=\{\{i\},\{l,k\}\}$, $|\pi|=2$, and $B=\{i\},\{l,k\}$, then

$$\left(\frac{d}{dx}\right)^{|\pi|}(\log x)\bigg|_{x=\mathcal{Z}[0]}\prod_{B\in\pi}\frac{\partial^{|B|}\mathcal{Z}[J]}{\prod_{j\in B}\partial J_{a^j a^j}}\Bigg|_{J=0}$$
$$=\left(-\frac{1}{\mathcal{Z}[0]^2}\right)\left(\frac{\partial\mathcal{Z}[J]}{\partial J_{a^i a^i}}\bigg|_{J=0}\right)\left(\frac{\partial^2\mathcal{Z}[J]}{\partial J_{a^l a^l}\partial J_{a^k a^k}}\bigg|_{J=0}\right). \tag{3.82}$$

The calculations required to calculate $\left(-\frac{1}{\mathcal{Z}[0]^2}\right)\left(\frac{\partial\mathcal{Z}[J]}{\partial J_{a^i a^i}}\Big|_{J=0}\right)\left(\frac{\partial^2\mathcal{Z}[J]}{\partial J_{a^l a^l}\partial J_{a^k a^k}}\Big|_{J=0}\right)$ are written below :

$$F(\{i\},\emptyset,\emptyset)=-\,iV\frac{E_{a^i-1}}{\lambda}A_N(z_1,\ldots,z_N)\frac{1}{\det\limits_{1\leq p,q\leq N}\left(t_p^{q-1}\right)}. \tag{3.83}$$

$$F(\emptyset,\{i\},\emptyset)=\left(-\frac{V}{(\lambda V)^{\frac{1}{3}}}\right)\partial_{a^i}A_N(z_1,\ldots,z_N)\frac{1}{\det\limits_{1\leq p,q\leq N}\left(t_p^{q-1}\right)}. \tag{3.84}$$

$$F(\emptyset,\emptyset,\{i\})=A_N(z_1,\ldots,z_N)\frac{-1}{\det\limits_{1\leq p,q\leq N}\left(t_p^{q-1}\right)}\sum_{r=1,r\neq a^i}^{N}\frac{1}{t_{a^i}-t_r}. \tag{3.85}$$

$$F(\{l,k\},\emptyset,\emptyset)=\left(-iV\frac{E_{a^l-1}}{\lambda}\right)\left(-iV\frac{E_{a^k-1}}{\lambda}\right)A_N(z_1,\ldots,z_N)\frac{1}{\det\limits_{1\leq p,q\leq N}\left(t_p^{q-1}\right)}, \tag{3.86}$$

$$F(\{l\},\{k\},\emptyset)=\left(-iV\frac{E_{a^l-1}}{\lambda}\right)\left(-\frac{V}{(\lambda V)^{\frac{1}{3}}}\right)\partial_{a^k}A_N(z_1,\ldots,z_N)\frac{1}{\det\limits_{1\leq p,q\leq N}\left(t_p^{q-1}\right)}, \tag{3.87}$$

$$F(\emptyset,\{l,k\},\emptyset)=\left(-\frac{V}{(\lambda V)^{\frac{1}{3}}}\right)^2\partial_{a^l}\partial_{a^k}A_N(z_1,\ldots,z_N)\frac{1}{\det\limits_{1\leq p,q\leq N}\left(t_p^{q-1}\right)}, \tag{3.88}$$

$$F(\{l\},\emptyset,\{k\}) = \left(-iV\frac{E_{a^l-1}}{\lambda}\right) A_N(z_1,\dots,z_N)\frac{-1}{\det\limits_{1\le p,q\le N}\left(t_p^{q-1}\right)} \sum_{r=1,r\neq a^k}^{N} \frac{1}{t_{a^k}-t_r}, \tag{3.89}$$

$$F(\emptyset,\{l\},\{k\}) = \left(-\frac{V}{(\lambda V)^{\frac{1}{3}}}\right) \partial_{a^l} A_N(z_1,\dots,z_N)\frac{-1}{\det\limits_{1\le p,q\le N}\left(t_p^{q-1}\right)} \sum_{r=1,r\neq a^k}^{N} \frac{1}{t_{a^k}-t_r}, \tag{3.90}$$

$$\begin{aligned} &F(\emptyset,\emptyset,\{l,k\}) \\ =&A_N(z_1,\dots,z_N)\frac{1}{\det\limits_{1\le p,q\le N}\left(t_p^{q-1}\right)}\left(\sum_{r=1,r\neq a^l}^{N}\frac{1}{t_{a^l}-t_r}\sum_{j=1,j\neq a^k}^{N}\frac{1}{t_{a^k}-t_j}-\frac{1}{(t_{a^l}-t_{a^k})^2}\right). \end{aligned} \tag{3.91}$$

If we sum up all the cases for sets S, M, and $\overline{M}$ that we have calculated so far and multiply by $-\dfrac{1}{\mathcal{Z}[0]^2}$, we get the result of (3.82):

$$\begin{aligned} &\left(-\frac{1}{\mathcal{Z}[0]^2}\right)\left(\frac{\partial\mathcal{Z}[J]}{\partial J_{a^ia^i}}\bigg|_{J=0}\right)\left(\frac{\partial^2\mathcal{Z}[J]}{\partial J_{a^la^l}\partial J_{a^ka^k}}\bigg|_{J=0}\right) \\ =&-\left(\frac{\det\limits_{1\le p,q\le N}\left(t_p^{q-1}\right)}{A_N(z_1,\dots,z_N)}\right)^2\sum_{i,l,k=1,i\neq l\neq k\neq i}^{3}\Bigg[\Big\{F(\{i\},\emptyset,\emptyset)+F(\emptyset,\{i\},\emptyset)+F(\emptyset,\emptyset,\{i\})\Big\} \\ &\Big\{F(\{l\},\{k\},\emptyset)+F(\{l\},\emptyset,\{k\})+F(\emptyset,\{l\},\{k\}) \\ &+\frac{F(\emptyset,\emptyset,\{l,k\})}{2}+\frac{F(\{l,k\},\emptyset,\emptyset)}{2}+\frac{F(\emptyset,\{l,k\},\emptyset)}{2}\Big\}\Bigg]. \end{aligned} \tag{3.92}$$

iii). We consider the case $\pi = \{\{1\},\{2\},\{3\}\}$, $|\pi| = 3$, and $B = \{1\},\{2\},\{3\}$, then

$$\begin{aligned} &\left(\frac{d}{dx}\right)^{|\pi|}(\log x)\bigg|_{x=\mathcal{Z}[0]}\prod_{B\in\pi}\frac{\partial^{|B|}\mathcal{Z}[J]}{\prod\limits_{j\in B}\partial J_{a^ja^j}}\Bigg|_{J=0} \\ =&\frac{2}{\mathcal{Z}[0]^3}\left(\frac{\partial\mathcal{Z}[J]}{\partial J_{a^1a^1}}\bigg|_{J=0}\right)\left(\frac{\partial\mathcal{Z}[J]}{\partial J_{a^2a^2}}\bigg|_{J=0}\right)\left(\frac{\partial\mathcal{Z}[J]}{\partial J_{a^3a^3}}\bigg|_{J=0}\right). \end{aligned} \tag{3.93}$$

The calculations required to calculate $\dfrac{2}{\mathcal{Z}[0]^3}\left(\dfrac{\partial\mathcal{Z}[J]}{\partial J_{a^1a^1}}\bigg|_{J=0}\right)\left(\dfrac{\partial\mathcal{Z}[J]}{\partial J_{a^2a^2}}\bigg|_{J=0}\right)\left(\dfrac{\partial\mathcal{Z}[J]}{\partial J_{a^3a^3}}\bigg|_{J=0}\right)$ are written below :

$$F(\{i\},\emptyset,\emptyset) = -iV\frac{E_{a^i-1}}{\lambda}A_N(z_1,\dots,z_N)\frac{1}{\det\limits_{1\le p,q\le N}\left(t_p^{q-1}\right)}. \tag{3.94}$$

$$F(\emptyset,\{i\},\emptyset)=\left(-\frac{V}{(\lambda V)^{\frac{1}{3}}}\right)\partial_{a^i}A_N(z_1,\ldots,z_N)\frac{1}{\det\limits_{1\le p,q\le N}\left(t_p^{q-1}\right)}. \tag{3.95}$$

$$F(\emptyset,\emptyset,\{i\})=A_N(z_1,\ldots,z_N)\frac{-1}{\det\limits_{1\le p,q\le N}\left(t_p^{q-1}\right)}\sum_{r=1,r\neq a^i}^{N}\frac{1}{t_{a^i}-t_r}. \tag{3.96}$$

If we sum up all the cases for sets S, M, and $\overline{M}$ that we have calculated so far and multiply by $\frac{2}{\mathcal{Z}[0]^3}$, we get the result of (3.93):

$$\begin{aligned}&\frac{2}{\mathcal{Z}[0]^3}\left(\left.\frac{\partial\mathcal{Z}[J]}{\partial J_{a^1a^1}}\right|_{J=0}\right)\left(\left.\frac{\partial\mathcal{Z}[J]}{\partial J_{a^2a^2}}\right|_{J=0}\right)\left(\left.\frac{\partial\mathcal{Z}[J]}{\partial J_{a^3a^3}}\right|_{J=0}\right)\\&=2\left(\frac{\det\limits_{1\le l,j\le N}\left(t_l^{j-1}\right)}{A_N(z_1,\ldots,z_N)}\right)^3\prod_{i=1}^{3}\Bigg\{F(\{i\},\emptyset,\emptyset)+F(\emptyset,\{i\},\emptyset)+F(\emptyset,\emptyset,\{i\})\Bigg\}.\end{aligned} \tag{3.97}$$

From i), ii), and iii), all results up to now are combined to obtain the calculation result of the three-point functions $G_{|a^1|a^2|a^3|}$.

$$\begin{aligned}&G_{|a^1|a^2|a^3|}\\&=\frac{iV}{\mathcal{Z}[0]}\left.\frac{\partial^3\mathcal{Z}[J]}{\partial J_{a^1a^1}\partial J_{a^2a^2}\partial J_{a^3a^3}}\right|_{J=0}-\frac{iV}{\mathcal{Z}[0]^2}\left(\left.\frac{\partial\mathcal{Z}[J]}{\partial J_{a^1a^1}}\right|_{J=0}\right)\left(\left.\frac{\partial^2\mathcal{Z}[J]}{\partial J_{a^2a^2}\partial J_{a^3a^3}}\right|_{J=0}\right)\\&\quad-\frac{iV}{\mathcal{Z}[0]^2}\left(\left.\frac{\partial\mathcal{Z}[J]}{\partial J_{a^2a^2}}\right|_{J=0}\right)\left(\left.\frac{\partial^2\mathcal{Z}[J]}{\partial J_{a^1a^1}\partial J_{a^3a^3}}\right|_{J=0}\right)-\frac{iV}{\mathcal{Z}[0]^2}\left(\left.\frac{\partial\mathcal{Z}[J]}{\partial J_{a^3a^3}}\right|_{J=0}\right)\left(\left.\frac{\partial^2\mathcal{Z}[J]}{\partial J_{a^1a^1}\partial J_{a^2a^2}}\right|_{J=0}\right)\\&\quad+\frac{2iV}{\mathcal{Z}[0]^3}\left(\left.\frac{\partial\mathcal{Z}[J]}{\partial J_{a^1a^1}}\right|_{J=0}\right)\left(\left.\frac{\partial\mathcal{Z}[J]}{\partial J_{a^2a^2}}\right|_{J=0}\right)\left(\left.\frac{\partial\mathcal{Z}[J]}{\partial J_{a^3a^3}}\right|_{J=0}\right)\\&=(iV)\left(\frac{\det\limits_{1\le p,q\le N}\left(t_p^{q-1}\right)}{A_N(z_1,\ldots,z_N)}\right)\\&\quad\Bigg\{F(\{1,2,3\},\emptyset,\emptyset)+F(\emptyset,\{1,2,3\},\emptyset)+F(\emptyset,\emptyset,\{1,2,3\})\\&\quad+\sum_{i,l,k=1,i\neq l\neq k\neq i}^{3}\Bigg(F(\{i\},\{l\},\{k\})+\frac{F(\{i,l\},\{k\},\emptyset)}{2}+\frac{F(\{i\},\{l,k\},\emptyset)}{2}+\frac{F(\{i,l\},\emptyset,\{k\})}{2}\\&\quad+\frac{F(\emptyset,\{i,l\},\{k\})}{2}+\frac{F(\{i\},\emptyset,\{l,k\})}{2}+\frac{F(\emptyset,\{i\},\{l,k\})}{2}\Bigg)\Bigg\}\\&\quad-(iV)\left(\frac{\det\limits_{1\le p,q\le N}\left(t_p^{q-1}\right)}{A_N(z_1,\ldots,z_N)}\right)^2\sum_{i,l,k=1,i\neq l\neq k\neq i}^{3}\Bigg[\Bigg\{F(\{i\},\emptyset,\emptyset)+F(\emptyset,\{i\},\emptyset)+F(\emptyset,\emptyset,\{i\})\Bigg\}\end{aligned}$$

$$
\Bigg\{F(\{l\},\{k\},\emptyset)+F(\{l\},\emptyset,\{k\})+F(\emptyset,\{l\},\{k\})
$$

$$
+\frac{F(\emptyset,\emptyset,\{l,k\})}{2}+\frac{F(\{l,k\},\emptyset,\emptyset)}{2}+\frac{F(\emptyset,\{l,k\},\emptyset)}{2}\Bigg\}\Bigg]
$$

$$
+(2iV)\left(\frac{\det\limits_{1\leq p,q\leq N}\left(t_p^{q-1}\right)}{A_N(z_1,\ldots,z_N)}\right)^3\left(\prod_{i=1}^{3}\Bigg\{F(\{i\},\emptyset,\emptyset)+F(\emptyset,\{i\},\emptyset)+F(\emptyset,\emptyset,\{i\})\Bigg\}\right). \tag{3.98}
$$

Chapter 4

Multipoint Correlation Function in Finite Φ^3-Φ^4 Hybrid Matrix Model

In this chapter, there is a matrix model corresponding to a scalar field theory on noncommutative spaces called Grosse-Wulkenhaar model (Φ^4 matrix model), which is renormalizable by adding a harmonic oscillator potential to scalar ϕ^4 theory on Moyal spaces. There are more unknowns in Φ^4 matrix model than in Φ^3 matrix model. We then construct a one-matrix model (Φ^3-Φ^4 hybrid matrix model) with multiple potentials, which is a combination of a 3-point interaction and a 4-point interaction, where the 3-point interaction of Φ^3 is multiplied by some positive definite diagonal matrix M[41]. This model is solvable due to the effect of this M. In particular, the connected $\sum_{i=1}^{B} N_i$-point function $G_{|a^1_{N_1}\cdots a^1_{N_1}|\cdots|a^B_1\cdots a^B_{N_B}|}$ of Φ^3-Φ^4 hybrid matrix model is studied in detail. This $\sum_{i=1}^{B} N_i$-point function can be interpreted geometrically and corresponds to the sum over all Feynman diagrams (ribbon graphs) drawn on Riemann surfaces with B boundaries (punctures). Each $|a^i_1 \cdots a^i_{N_i}|$ represents N_i external lines coming from the i-th boundary (puncture) in each Feynman diagram. First, we construct Feynman rules for Φ^3-Φ^4 hybrid matrix model and calculate perturbative expansions of some multipoint functions in ordinary methods. Second, we calculate the path integral of the partition function $\mathcal{Z}[J]$ and use the result to compute exact solutions for 1-point function $G_{|a|}$ with 1-boundary, 2-point function $G_{|ab|}$ with 1-boundary, 2-point function $G_{|a|b|}$ with 2-boundaries, and n-point function $G_{|a^1|a^2|\cdots|a^n|}$ with n-boundaries. They include contributions from Feynman diagrams corresponding to nonplanar Feynman diagrams or higher genus surfaces.

4.1. Setup of Φ^3-Φ^4 Hybrid Matrix Model

In this section, we define Φ^3-Φ^4 hybrid matrix model, and we determine our notations in this chapter.

Let $\Phi = (\Phi_{ij})$ be a Hermitian matrix for $i, j = 1, 2, \ldots, N$ and E_{m-1} be a discretization of a monotonously increasing differentiable function e with $e(0)=0$,

$$E_{m-1} = \mu^2\left(\frac{1}{2}+e\left(\frac{m-1}{\mu^2 V}\right)\right), \tag{4.1}$$

where μ^2 is a squared mass, and V is a real constant. Let $E=(E_{m-1}\delta_{mn})$ be a diagonal matrix for $m,n=1,\ldots,N$ and $M=(\sqrt{E_{k-1}}\delta_{kl})$ be a diagonal matrix for $k,l=1,\ldots,N$, i.e. $E=M^2$. Let us consider the following action:

$$S[\Phi]=V\mathrm{tr}\left(E\Phi^2+\kappa\Phi+\frac{1}{2}M\Phi M\Phi+\sqrt{\lambda}M\Phi^3+\frac{\lambda}{4}\Phi^4\right), \tag{4.2}$$

where κ is a constant (real), and λ is a coupling constant that is non-zero real. To avoid confusion later, λ and V are assumed to be positive. When we consider the perturbation theory in Section 4.2, we put $\kappa=0$. Let $J=(J_{mn})$ be a Hermitian matrix for $m,n=1,\ldots,N$ as an external field. Let $\mathcal{D}\Phi$ be the integral measure,

$$\mathcal{D}\Phi := \prod_{i=1}^{N} d\Phi_{ii}\prod_{1\le i<j\le N} d\mathrm{Re}\Phi_{ij}d\mathrm{Im}\Phi_{ij}, \tag{4.3}$$

where each variable is divided into real and imaginary parts $\Phi_{ij}=\mathrm{Re}\Phi_{ij}+i\mathrm{Im}\Phi_{ij}$ with $\mathrm{Re}\Phi_{ij}=\mathrm{Re}\Phi_{ji}$ and $\mathrm{Im}\Phi_{ij}=-\mathrm{Im}\Phi_{ji}$. Let us consider the following partition function:

$$\begin{aligned}\mathcal{Z}[J] &:= \int \mathcal{D}\Phi \exp\left(-S[\Phi]+V\mathrm{tr}(J\Phi)\right)\\ &= \int \mathcal{D}\Phi \exp\left(-V\mathrm{tr}\left(E\Phi^2+\kappa\Phi+\frac{1}{2}M\Phi M\Phi+\sqrt{\lambda}M\Phi^3+\frac{\lambda}{4}\Phi^4\right)\right)\exp\left(V\mathrm{tr}\left(J\Phi\right)\right). \end{aligned} \tag{4.4}$$

4.2. Perturbation Theory of Φ^3-Φ^4 Hybrid Matrix Model

The aim of this section is to understand Φ^3-Φ^4 hybrid matrix model by usual perturbative methods in field theories. For this purpose, we make its Feynman rules and calculate one-point functions and two types of two-point functions perturbatively. It is made in the same way as Feynman rules for well-known matrix models[10, 30]. However, it differs slightly from the usual one due to the presence of M found in (4.4). We then construct the perturbative theories a little more carefully for readers who are not familiar with perturbative theories of these matrix models.

4.2.1. Feynman Rules of Φ^3-Φ^4 Hybrid Matrix Model ($\kappa=0$)

We consider the theory of $S_{free}=V\mathrm{tr}\left(E\Phi^2+\frac{1}{2}M\Phi M\Phi\right)$, that is no interaction theory, to consider the perturbation theory of Φ^3-Φ^4 Hybrid-Matrix-Model at first. We calculate $\mathcal{Z}_{free}[J]$;

$$\mathcal{Z}_{free}[J]=\int \mathcal{D}\Phi\exp\left(-V\mathrm{tr}\left(E\Phi^2+\frac{1}{2}M\Phi M\Phi\right)\right)\exp\left(V\mathrm{tr}(J\Phi)\right)$$

$$=\mathcal{C}'\exp\left(\frac{V}{2}\sum_{n,m=1}^{N}J_{mn}\frac{1}{E_{n-1}+E_{m-1}+\sqrt{E_{n-1}}\sqrt{E_{m-1}}}J_{nm}\right). \tag{4.5}$$

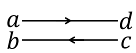
Here $\mathcal{C}' = \mathcal{Z}_{free}[0] = \left(\prod_{n=1}^{N}\sqrt{\frac{2\pi}{3VE_{n-1}}}\right)\left(\prod_{1\le n<m\le N}\frac{\pi}{V(E_{n-1}+E_{m-1}+\sqrt{E_{n-1}E_{m-1}})}\right)$. We introduce the free n-point functions:

$$\left\langle\prod_{k=1}^{n}\Phi_{i_kj_k}\right\rangle_{free} := \frac{1}{\mathcal{Z}_{free}[0]}\int\mathcal{D}\Phi\ \Phi_{i_1j_1}\Phi_{i_2j_2}\cdots\Phi_{i_nj_n}\exp\left(-V\mathrm{tr}\left(E\Phi^2+\frac{1}{2}M\Phi M\Phi\right)\right). \tag{4.6}$$

In particular, the propagator is given as

$$\langle\Phi_{ba}\Phi_{dc}\rangle_{free} = \frac{1}{V}\frac{\delta_{ad}\delta_{bc}}{E_{d-1}+E_{c-1}+\sqrt{E_{c-1}}\sqrt{E_{d-1}}}. \tag{4.7}$$

The Feynman graph of the propagator (ribbon) is then defined as follows:



$$= \langle\Phi_{ba}\Phi_{dc}\rangle_{free} = \frac{1}{V}\frac{\delta_{ad}\delta_{bc}}{E_{c-1}+E_{d-1}+\sqrt{E_{c-1}}\sqrt{E_{d-1}}}. \tag{4.8}$$

In this paper, we do not distinguish Feynman graphs from the functions (or operations) corresponding to the Feynman graphs, for simplicity. Next, we consider the case of (4.4) with the condition $\kappa = 0$. (4.4) can be written as follows:

$$\mathcal{Z}[J] = \int\mathcal{D}\Phi\exp\left(-S_{int}[\Phi]\right)\exp\left(-S_{free}[\Phi]\right)\exp\left(+V\mathrm{tr}J\Phi\right). \tag{4.9}$$

Here $S_{int} = V\mathrm{tr}\left(\frac{\lambda}{4}\Phi^4+\sqrt{\lambda}M\Phi^3\right)$. Using (4.9), as in ordinary field theory, if we consider the n-point function,

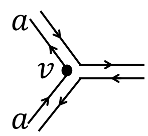
$$\left\langle\prod_{k=1}^{n}\Phi_{i_kj_k}\right\rangle := \frac{1}{\mathcal{Z}[0]}\int\mathcal{D}\Phi\ \Phi_{i_1j_1}\Phi_{i_2j_2}\cdots\Phi_{i_nj_n}\exp\left(-S_{int}\left[\Phi\right]\right)\exp\left(-S_{free}[\Phi]\right), \tag{4.10}$$

then

$$\begin{aligned}\left\langle\prod_{k=1}^{n}\Phi_{i_kj_k}\right\rangle :=& \frac{1}{\mathcal{Z}[0]}\int\mathcal{D}\Phi\ \frac{\partial}{\partial J_{j_1i_1}}\frac{\partial}{\partial J_{j_2i_2}}\cdots\frac{\partial}{\partial J_{j_ni_n}}\exp\left(-S_{int}\left[\frac{1}{V}\frac{\partial}{\partial J}\right]\right)\\ &\times\exp\left(-S_{free}[\Phi]\right)\exp\left(+V\mathrm{tr}J\Phi\right)\\ =&\frac{1}{\mathcal{Z}[0]}\frac{\partial}{\partial J_{j_1i_1}}\frac{\partial}{\partial J_{j_2i_2}}\cdots\frac{\partial}{\partial J_{j_ni_n}}\exp\left(-S_{int}\left[\frac{1}{V}\frac{\partial}{\partial J}\right]\right)\mathcal{Z}_{free}[J].\end{aligned} \tag{4.11}$$

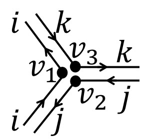
From this, we also obtain the Feynman rule for interactions, which is as follows. First, we consider the three-point interactions.

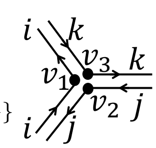
From $-V\mathrm{tr}\sqrt{\lambda}M\Phi^3 = -V\sqrt{\lambda}\sum_{k,l,m=1}^{N}\sqrt{E_{k-1}}\Phi_{kl}\Phi_{lm}\Phi_{mk}$, the vertex weight of the three-point interaction is determined:

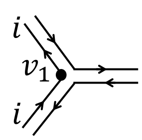


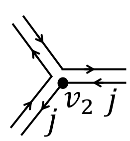
$$= -V\sqrt{\lambda E_{a-1}}. \tag{4.12}$$

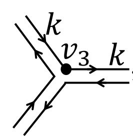
The black dot v corresponds to $\sqrt{E_{a-1}}$. Note that this Feynman rule corresponding interaction does not consider statistical factors. In other words, for all Wick contractions with $\mathrm{tr}\sqrt{\lambda}M\Phi^3$, we shall add up all graphs with this weight. In this paper, we use the following notation:

$$\sum_{v\in\{\{v_1,v_2,v_3\}\}}$$


$$:= \sum_{v\in\{\{i,j,k\}\}}$$


$$:=$$


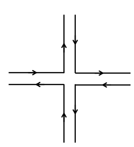
$$+$$


$$+$$


$$, \tag{4.13}$$

where $\{\{v_1, v_2, v_3\}\}$ means multi set. For example, $\sum_{v\in\{\{a,a,n\}\}} v = a + a + n$. So even if the cases $i = j$ and so on, the definition (4.13) is not changed.

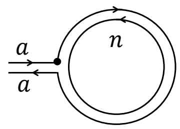
Next, we consider the four-point interactions. From $-V\mathrm{tr}\frac{\lambda}{4}\Phi^4$, the vertex weight of the four-point interaction is obtained:



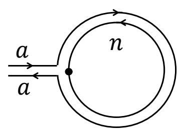
$$= -\frac{V\lambda}{4}. \tag{4.14}$$

Note that this Feynman rule corresponding to this interaction does not consider statistical factors, too. For all Wick contractions with $\mathrm{tr}\frac{\lambda}{4}\Phi^4$, we have to sum all terms with this weight.

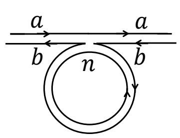
For each loop, we add $\sum_{a=1}^{N}$ to sum over all elements. Note that summation $\sum_{a=1}^{N}$ should be carried out after multiplying $\sqrt{E_{a-1}}$ in (4.12) for each black dot in any loop. See the following examples:



$$= -\frac{\sqrt{\lambda}\sqrt{E_{a-1}}}{V3E_{a-1}} \sum_{n=1}^{N} \frac{1}{E_{a-1}+E_{n-1}+\sqrt{E_{a-1}E_{n-1}}}, \tag{4.15}$$



$$= -\frac{\sqrt{\lambda}}{V3E_{a-1}} \sum_{n=1}^{N} \frac{\sqrt{E_{n-1}}}{E_{a-1}+E_{n-1}+\sqrt{E_{a-1}E_{n-1}}}, \tag{4.16}$$



$$= -\frac{\lambda}{4V^2(E_{a-1}+E_{b-1}+\sqrt{E_{a-1}E_{b-1}})^2} \sum_{n=1}^{N} \frac{1}{E_{b-1}+E_{n-1}+\sqrt{E_{b-1}E_{n-1}}}. \tag{4.17}$$

4.2.2. Cumulant of Φ^3-Φ^4 Hybrid Matrix Model ($\kappa = 0$)

Using $\log \dfrac{\mathcal{Z}[J]}{\mathcal{Z}[0]}$, the $\displaystyle\sum_{i=1}^{B} N_i$-point function $G_{|a_1^1\ldots a_{N_1}^1|\ldots|a_1^B\ldots a_{N_B}^B|}$ is defined as

$$\log \frac{\mathcal{Z}[J]}{\mathcal{Z}[0]} := \sum_{B=1}^{\infty} \sum_{1\le N_1\le\cdots\le N_B}^{\infty} \sum_{p_1^1,\ldots,p_{N_B}^B=1}^{N} V^{2-B} \frac{G_{|p_1^1\ldots p_{N_1}^1|\ldots|p_1^B\ldots p_{N_B}^B|}}{S_{(N_1,\ldots,N_B)}} \prod_{\beta=1}^{B} \frac{\mathbb{J}_{p_1^\beta\ldots p_{N_\beta}^\beta}}{N_\beta}, \tag{4.18}$$

where N_i is the identical valence number for $i = 1,\ldots,B$, $\mathbb{J}_{p_1^i\ldots p_{N_i}^i} := \displaystyle\prod_{j=1}^{N_i} J_{p_j^i p_{j+1}^i}$ with $N_i + 1 \equiv 1$, $(N_1,\ldots,N_B) = (\underbrace{N_1',\ldots,N_1'}_{\nu_1},\ldots,\underbrace{N_s',\ldots,N_s'}_{\nu_s})$, and $S_{(N_1,\ldots,N_B)} = \displaystyle\prod_{\beta=1}^{s} \nu_\beta!$. The $\displaystyle\sum_{i=1}^{B} N_i$-point function denoted by $G_{|a_1^1\ldots a_{N_1}^1|\ldots|a_1^B\ldots a_{N_B}^B|}$ is given by the sum over all Feynman diagrams (ribbon graphs) on Riemann surfaces with B-boundaries, and each $|a_1^i\cdots a_{N_i}^i|$ corresponds to the Feynman diagrams having N_i-external ribbons from the i-th boundary. (See Figure 4.1.)

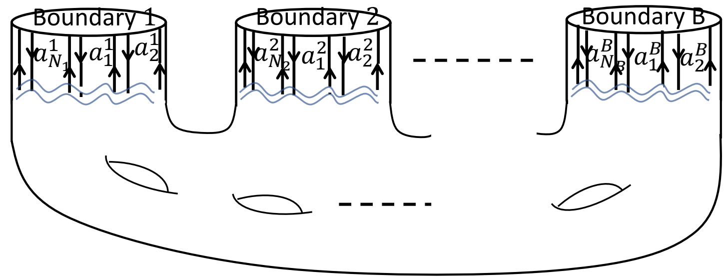


Figure 4.1: The relationship between external ribbons of Feynman diagrams and boundaries as expressed in $G_{|a^1_1 \ldots a^1_{N_1}|\ldots|a^B_1 \ldots a^B_{N_B}|}$

We give the reason why the Figure 4.1 picture for the Feynman diagram is obtained, in the following. We define a $\sum_{i=1}^{B} N_i$-point cumulant which represent contributions of connected Feynman diagrams as

$$\begin{aligned}&\langle \Phi_{a^1_1 a^1_2} \cdots \Phi_{a^1_{N_1} a^1_1} \Phi_{a^2_1 a^2_2} \cdots \Phi_{a^2_{N_2} a^2_1} \cdots \Phi_{a^B_1 a^B_2} \cdots \Phi_{a^B_{N_B} a^B_1} \rangle_c \\ &:= \frac{1}{V^{N_1+\cdots+N_B}} \frac{\partial}{\partial J_{a^1_2 a^1_1}} \cdots \frac{\partial}{\partial J_{a^B_1 a^B_{N_B}}} \log \mathcal{Z}[J] \bigg|_{J=0} . \end{aligned} \tag{4.19}$$

Let us focus on a Feynman diagram with $\mathscr{N} := \sum_{i=1}^{B} N_i$-external ribbons. Let Σ be the number of loops contained in the Feynman diagram. Let k_3 and k_4 be the number of $V \mathrm{tr}\left(\sqrt{\lambda} M \Phi^3\right)$ interactions and the number of $V \mathrm{tr}\left(\frac{\lambda}{4}\Phi^4\right)$ interactions in the Feynman diagram, respectively. The contribution from such Feynman diagram has $V^{k_3+k_4-\frac{3k_3+4k_4+\mathscr{N}}{2}}$ since the contribution from vertexes is $V^{k_3+k_4}$ and the contribution from propagators is $\left(\frac{1}{V}\right)^{\frac{3k_3+4k_4+\mathscr{N}}{2}}$. Also, the Euler number of a surface with genus "g" and boundaries "B" is $\chi = 2-2g-B$. For this Feynman diagram, the corresponding Euler number is given by $\chi = (k_3+k_4+\mathscr{N}) - \left(\frac{3k_3+4k_4+\mathscr{N}}{2} + \mathscr{N}\right) + (\mathscr{N} + \sum)$. Here $k_3+k_4+\mathscr{N}$ is the number of vertexes, $\left(\frac{3k_3+4k_4+\mathscr{N}}{2} + \mathscr{N}\right)$ is the number of the edges, and $(\mathscr{N}+\Sigma)$ is the number of the faces in the Feynman diagrams. Note that we count one ribbon as one edge, here. Let us see the reason why the last $+\mathscr{N}$ of $\left(\frac{3k_3+4k_4+\mathscr{N}}{2} + \mathscr{N}\right)$ appears in the number of edge, and $\mathscr{N}$ also represents the number of faces. For example, we see the i-th boundary. There are N_i faces touching one boundary, since there is N_i external ribbons in the Feynman diagram from the term $\prod_{j=1}^{N_i} J_{p^i_j p^i_{j+1}}$ with $N_i+1 \equiv 1$. (See Figure 4.2.)

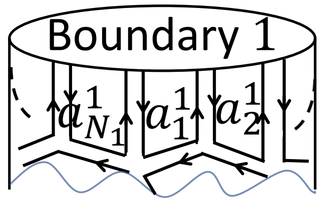


Figure 4.2: The relationship between external ribbons of Feynman diagrams and Boundary 1 as expressed in $G_{|a^1_1\dots a^1_{N_1}|}$

Therefore the number of all surfaces touching the boundary is $\mathscr{N}=\sum_{i=1}^{B}N_i$ in this case, and $\mathscr{N}$ edges appear as not ribbons but segments on boundaries. The contribution from the Feynman diagram has $V^{k_3+k_4-\frac{3k_3+4k_4+\mathscr{N}}{2}}=V^{\chi-\mathscr{N}-\Sigma}=V^{2-2g-B-\mathscr{N}-\Sigma}$. So, we introduce $G_{|a^1_1\dots a^1_{N_1}|\dots|a^B_1\dots a^B_{N_B}|}$ for pairwise different a^i_j $(i=1,\cdots,B,\ \ j=1,\cdots,N_i)$ as in the following equation.

$$\langle\Phi_{a^1_1a^1_2}\cdots\Phi_{a^1_{N_1}a^1_1}\Phi_{a^2_1a^2_2}\cdots\Phi_{a^2_{N_2}a^2_1}\cdots\Phi_{a^B_1a^B_2}\cdots\Phi_{a^B_{N_B}a^B_1}\rangle_c=V^{2-\mathscr{N}-B}G_{|a^1_1\dots a^1_{N_1}|\dots|a^B_1\dots a^B_{N_B}|}. \tag{4.20}$$

Let us check its consistency with (4.18). Note that

$$\begin{aligned}&\frac{1}{V^{\mathscr{N}}}\frac{\partial^{\mathscr{N}}}{\partial J_{a^1_2a^1_1}\cdots\partial J_{a^B_1a^B_{N_B}}}\sum_{B'=1}^{\infty}\sum_{1\le\dots\le N_{B'}}^{\infty}\sum_{p^1_1,\dots,p^{B'}_{N_{B'}}=1}^{N}\prod_{\beta=1}^{B'}\frac{\mathbb{J}_{p^\beta_1\dots p^\beta_{N_\beta}}}{N_\beta}\,G_{|p^1_1\dots p^1_{N_1}|\dots|p^{B'}_1\dots p^{B'}_{N_{B'}}|}\Bigg|_{J=0}\\ =&\frac{1}{V^{\mathscr{N}}}G_{|a^1_1\dots a^1_{N_1}|\dots|a^B_1\dots a^B_{N_B}|}\times S_{(N_1,\dots,N_B)}.\end{aligned} \tag{4.21}$$

Then the $\mathscr{N}$-th derivative of the right-hand side of (4.18) with respect to $J_{a^1_2a^1_1},\cdots,J_{a^B_1a^B_{N_B}}$ is given by

$$\frac{1}{V^{\mathscr{N}}}\frac{\partial^{\mathscr{N}}}{\partial J_{a^1_2a^1_1}\cdots\partial J_{a^B_1a^B_{N_B}}}\left(R.H.S\ of\ (4.18)\right)=V^{2-\mathscr{N}-B}G_{|a^1_1\dots a^1_{N_1}|\dots|a^B_1\dots a^B_{N_B}|}, \tag{4.22}$$

and the corresponding one from the left-hand side of (4.18) is given as

$$\frac{1}{V^{\mathscr{N}}}\frac{\partial^{\mathscr{N}}}{\partial J_{a^1_2a^1_1}\cdots\partial J_{a^B_1a^B_{N_B}}}\left(L.H.S\ of\ (4.18)\right)=\langle\Phi_{a^1_1a^1_2}\cdots\Phi_{a^1_{N_1}a^1_1}\Phi_{a^2_1a^2_2}\cdots\Phi_{a^2_{N_2}a^2_1}\cdots\Phi_{a^B_1a^B_2}\cdots\Phi_{a^B_{N_B}a^B_1}\rangle_c. \tag{4.23}$$

Therefore, we found that (4.20) is consistent with (4.18) when all a^i_j are pairwise different.

If there is no condition that any two indexes do not much, then (4.20) is not necessarily correct. $\langle \Phi_{a_1^1 a_2^1} \cdots \Phi_{a_{N_1}^1 a_1^1} \Phi_{a_1^2 a_2^2} \cdots \Phi_{a_{N_2}^2 a_1^2} \cdots \Phi_{a_1^B a_2^B} \cdots \Phi_{a_{N_B}^B a_1^B} \rangle_c$ might include contributions from several types of surfaces classified by their boundaries. For example, let us consider $\langle \Phi_{aa}\Phi_{aa}\rangle_c$. From (4.18), $\langle \Phi_{aa}\Phi_{aa}\rangle_c = \frac{1}{V} G_{|aa|} + \frac{1}{V^2} G_{|a|a|}$. This means that $\langle \Phi_{aa}\Phi_{aa}\rangle_c$ includes contributions from two types of surfaces which are surfaces with one boundary and ones with two boundaries.

From these observations, it is concluded that we should prepare a connected oriented surface with B boundaries for drawing each Feynman diagram to calculate $G_{|a_1^1 \ldots a_{N_1}^1|\ldots|a_1^B \ldots a_{N_B}^B|}$. We draw a Feynman diagram with external ribbons with $(a_1^i a_2^i), \cdots, (a_{N_i}^i a_1^i)$ subscripted to each boundary i. For any connected segments in a Feynman diagram, both ends are on the same boundary. $G_{|a_1^1 \ldots a_{N_1}^1|\ldots|a_1^B \ldots a_{N_B}^B|}$ is given by the sum over all such Feynman diagrams.

In addition, since it is $V^{k_3+k_4-\frac{3k_3+4k_4+\mathcal{N}}{2}} = V^{\chi-\mathcal{N}-\Sigma} = V^{2-2g-B-\mathcal{N}-\Sigma}$, we can consider "genus expansion" of $G_{|a_1^1 \ldots a_{N_1}^1|\ldots|a_1^B \ldots a_{N_B}^B|}$ like [31] as

$$G_{|a_1^1 \ldots a_{N_1}^1|\ldots|a_1^B \ldots a_{N_B}^B|} = \sum_{g=0}^{\infty} V^{-2g} G^{(g)}_{|a_1^1 \ldots a_{N_1}^1|\ldots|a_1^B \ldots a_{N_B}^B|}. \tag{4.24}$$

We will discuss contributions from nontrivial topology surfaces in Section 4.6.

4.2.3. Perturbative Expansion of 1-Point Function $G_{|1|}$ $(N = 1)$

For $N = 1$, we calculate the 1-point function $G_{|1|}$ using perturbative expansion.

$$\begin{aligned}
G_{|1|} =& \frac{1}{V}\frac{\partial \log \mathcal{Z}[J]}{\partial J}\bigg|_{J=0} \\
=& \frac{1}{\mathcal{Z}[0]}\int_{-\infty}^{\infty} dx\, x \left(\sum_{k=0}^{\infty}\frac{(-V\lambda)^k}{k!4^k}x^{4k}\right)\left(\sum_{l=0}^{\infty}\frac{(-V)^l}{l!}(\sqrt{\lambda})^l(\sqrt{E_0})^l x^{3l}\right)\exp\left(-V\frac{3}{2}E_0x^2\right) \\
=& \frac{1}{\mathcal{Z}[0]}\int_{-\infty}^{\infty}\left(\frac{-V\sqrt{\lambda}\sqrt{E_0}}{1!}\right)x^4\exp\left(-V\frac{3}{2}E_0x^2\right)dx \\
&+ \frac{V^2\lambda\sqrt{\lambda}\sqrt{E_0}}{4\mathcal{Z}[0]}\int_{-\infty}^{\infty}x^8\exp\left(-V\frac{3}{2}E_0x^2\right)dx \\
&+ \frac{1}{\mathcal{Z}[0]}\int_{-\infty}^{\infty}\frac{(-V\sqrt{\lambda}\sqrt{E_0})^3}{3!}x^{10}\exp\left(-V\frac{3}{2}E_0x^2\right)dx + \mathcal{O}(\lambda^2\sqrt{\lambda}).
\end{aligned} \tag{4.25}$$

(4.25) is calculated directly as follows:

$$
\begin{aligned}
G_{|1|} =& (4.25) \\
=& \left\{ -\frac{\sqrt{E_0}\sqrt{\lambda}}{3VE_0^2} - \frac{\sqrt{E_0}\lambda\sqrt{\lambda}}{36V^2E_0^4} + \frac{5\sqrt{E_0}\lambda\sqrt{\lambda}}{54V^2E_0^4} \right\} + \frac{35\sqrt{E_0}\lambda\sqrt{\lambda}}{108V^2E_0^4} - \frac{35\sqrt{E_0}\lambda\sqrt{\lambda}}{54V^2E_0^4} + \mathcal{O}(\lambda^2\sqrt{\lambda}) \\
=& -\frac{\sqrt{\lambda}}{3VE_0\sqrt{E_0}} - \frac{7\lambda\sqrt{\lambda}}{27V^2E_0^{\frac{7}{2}}} + \mathcal{O}(\lambda^2\sqrt{\lambda}). \qquad (4.26)
\end{aligned}
$$

This result is verified from the exact solution of $G_{|1|}$ for $N = 1$ that is given in Subsection 4.5.1.

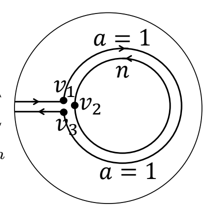
4.2.4. Perturbative Expansion of 1-Point Function $G_{|1|}$, 2-Point Function $G_{|21|}$, 2-Point Function $G_{|2|1|}$ ($N = 2$)

In order to familiarize readers with the perturbation calculations for $G_{|a_1^1 \dots a_{N_1}^1|\dots|a_1^B \dots a_{N_B}^B|}$, several specific example calculations are performed in this Subsection. For simple exercises, $N = 2$ case is calculated in Subsection 4.2.4.

The results obtained in Subsection 4.2.4 will be used in Section 4.5 as a check that the exact solutions are obtained correctly.

We calculate the 1-point function $G_{|1|}$ using perturbative expansion, at first. We compute each term of this expansion by drawing Feynman diagrams on surfaces with one boundary.

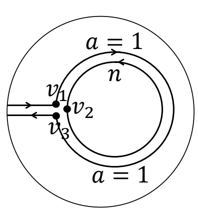
$$G_{|1|} = \sum_{n=1}^{2}\sum_{v\in\mathscr{V}_n}$$



$$+ \mathcal{O}(\lambda\sqrt{\lambda}), \qquad (4.27)$$

where $\mathscr{V}_n = \{\{1, 1, n\}\}$. The circle around the Feynman diagram is the boundary. Feynman diagrams and each term of perturbative expansions have a one-to-one correspondence as follows:

$$\sum_{n=1}^{2}\sum_{v\in\mathscr{V}_n}$$



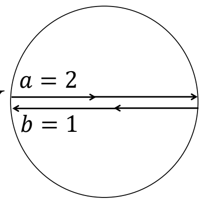
$$= -\frac{\sqrt{\lambda}}{3E_0V}\sum_{n=1}^{2}\sum_{v\in\{\{1,1,n\}\}}\frac{\sqrt{E_{v-1}}}{E_{n-1} + E_0 + \sqrt{E_0}\sqrt{E_{n-1}}}. \qquad (4.28)$$

From this, $G_{|1|}$ becomes as follows:

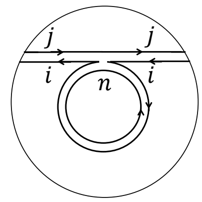
$$
\begin{aligned}
G_{|1|} =& -\frac{\sqrt{\lambda}}{V}\frac{\sqrt{E_0}}{3E_0^2} - \frac{\sqrt{\lambda}}{V}\frac{\sqrt{E_0}}{3E_0}\frac{2}{E_0 + E_1 + \sqrt{E_0}\sqrt{E_1}} - \frac{\sqrt{\lambda}}{V}\frac{\sqrt{E_1}}{3E_0}\frac{1}{E_0 + E_1 + \sqrt{E_0}\sqrt{E_1}} \\
&+ \mathcal{O}(\lambda\sqrt{\lambda}). \qquad (4.29)
\end{aligned}
$$

Next, we calculate the 2-point function $G_{|21|}$ on surfaces with one boundary. We compute each term from the expansion of the 2-point function $G_{|21|}$ by drawing Feynman diagrams.

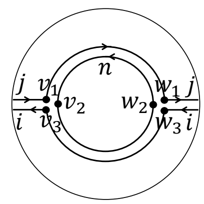
$$G_{|21|} = V$$



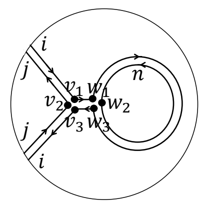
$$+ 4V \sum_{n=1}^{2} \sum_{(i,j)\in\{(1,2),(2,1)\}}$$



$$+ \frac{V}{2} \sum_{(i,j)\in\{(1,2),(2,1)\}} \sum_{n=1}^{2} \sum_{\omega\in\{\{i,j,n\}\}} \sum_{v\in\{\{i,j,n\}\}}$$



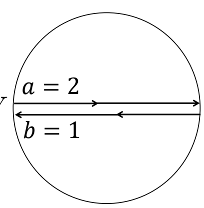
$$+ \frac{V}{2} \times 2 \sum_{(i,j)\in\{(1,2),(2,1)\}} \sum_{n=1}^{2} \sum_{\omega\in\{\{i,i,n\}\}} \sum_{v\in\{\{j,i,i\}\}}$$



$$+ \mathcal{O}(\lambda^2). \tag{4.30}$$

The first diagram in (4.30) is given as follows:

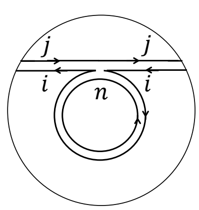
$$V$$



$$= \frac{1}{E_0 + E_1 + \sqrt{E_0}\sqrt{E_1}}. \tag{4.31}$$

The second term in (4.30) is written as follows:

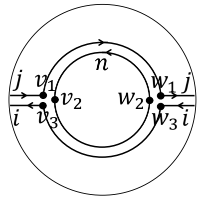
$$4V \sum_{n=1}^{2} \sum_{(i,j)\in\{(1,2),(2,1)\}}$$



$$= -\frac{4\lambda}{4V} \sum_{i,j=1,2} \sum_{i\neq j} \sum_{n=1}^{2} \left(\frac{1}{E_{i-1} + E_{j-1} + \sqrt{E_{i-1}}\sqrt{E_{j-1}}} \right)^2 \left(\frac{1}{E_{i-1} + E_{n-1} + \sqrt{E_{i-1}}\sqrt{E_{n-1}}} \right). \tag{4.32}$$

The third term in (4.30) is expressed as follows:

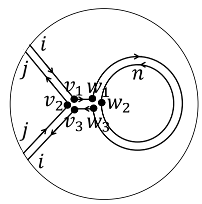
$$\frac{V}{2}\sum_{(i,j)\in\{(1,2),(2,1)\}}\sum_{n=1}^{2}\sum_{\omega\in\{\{i,j,n\}\}}\sum_{v\in\{\{i,j,n\}\}}$$



$$=+\frac{\lambda}{2V}\sum_{(i,j)\in\{(1,2),(2,1)\}}\sum_{n=1}^{2}\sum_{\omega\in\{\{i,j,n\}\}}\sum_{v\in\{\{i,j,n\}\}}\left(\frac{1}{E_{i-1}+E_{j-1}+\sqrt{E_{i-1}}\sqrt{E_{j-1}}}\right)^2$$
$$\times\sqrt{E_{v-1}}\sqrt{E_{w-1}}\left(\frac{1}{E_{i-1}+E_{n-1}+\sqrt{E_{i-1}}\sqrt{E_{n-1}}}\right)\left(\frac{1}{E_{j-1}+E_{n-1}+\sqrt{E_{j-1}}\sqrt{E_{n-1}}}\right). \qquad (4.33)$$

The fourth term in (4.30) is given as follows:

$$2\times\frac{V}{2}\sum_{(i,j)\in\{(1,2),(2,1)\}}\sum_{n=1}^{2}\sum_{\omega\in\{\{i,i,n\}\}}\sum_{v\in\{\{j,i,i\}\}}$$



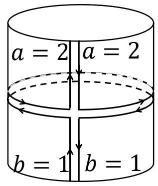
$$=\frac{\lambda}{V}\sum_{(i,j)\in\{(1,2),(2,1)\}}\sum_{n=1}^{2}\sum_{\omega\in\{\{i,i,n\}\}}\sum_{v\in\{\{j,i,i\}\}}\left(\frac{1}{3E_{i-1}}\right)\left(\frac{1}{E_{i-1}+E_{j-1}+\sqrt{E_{i-1}}\sqrt{E_{j-1}}}\right)^2$$
$$\sqrt{E_{v-1}}\sqrt{E_{w-1}}\left(\frac{1}{E_{i-1}+E_{n-1}+\sqrt{E_{i-1}}\sqrt{E_{n-1}}}\right). \qquad (4.34)$$

From (4.31)-(4.34), $G_{|21|}$ is given as follows:

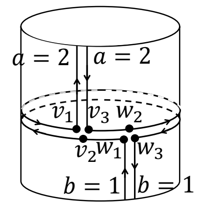
$$\begin{aligned}G_{|21|}=&\frac{1}{E_0+E_1+\sqrt{E_0}\sqrt{E_1}}+\frac{\lambda}{3VE_0(E_0+E_1+\sqrt{E_0}\sqrt{E_1})^2}\\&+\frac{\lambda}{3VE_1(E_0+E_1+\sqrt{E_0}\sqrt{E_1})^2}+\frac{8\sqrt{E_0}\sqrt{E_1}\lambda}{3VE_1(E_0+E_1+\sqrt{E_0}\sqrt{E_1})^3}\\&+\frac{8\sqrt{E_0}\sqrt{E_1}\lambda}{3VE_0(E_0+E_1+\sqrt{E_0}\sqrt{E_1})^3}+\frac{\sqrt{E_0}\sqrt{E_1}\lambda}{3VE_1^2(E_0+E_1+\sqrt{E_0}\sqrt{E_1})^2}\\&+\frac{\sqrt{E_0}\sqrt{E_1}\lambda}{3VE_0^2(E_0+E_1+\sqrt{E_0}\sqrt{E_1})^2}+\frac{2\lambda E_0}{3VE_1(E_0+E_1+\sqrt{E_0}\sqrt{E_1})^3}\\&+\frac{2\lambda E_1}{3VE_0(E_0+E_1+\sqrt{E_0}\sqrt{E_1})^3}+\frac{10\lambda}{3V(E_0+E_1+\sqrt{E_0}\sqrt{E_1})^3}+\mathcal{O}(\lambda^2).\end{aligned} \qquad (4.35)$$

Next, let us calculate the 2-point function $G_{|2|1|}$ that has two boundaries using perturbative expansion. We compute each term of this expansion by drawing Feynman diagrams on surfaces with two boundaries.

$$G_{|2|1|} = 4V^2$$



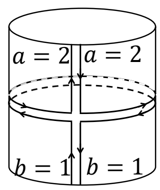
$$+V^2 \sum_{\omega \in \{\{1,1,2\}\}} \sum_{v \in \{\{2,2,1\}\}}$$



$$+\mathcal{O}(\lambda^2) \tag{4.36}$$

There is a one-to-one correspondence between the Feynman diagram and each term in the perturbation expansion. The first diagram in (4.36) is given as follows:

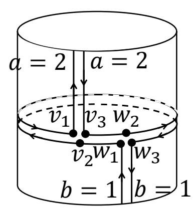
$$4V^2$$



$$= -\frac{\lambda}{9E_0E_1(E_0+E_1+\sqrt{E_0}\sqrt{E_1})} \tag{4.37}$$

The second term in (4.36) is obtained as follows:

$$V^2 \sum_{\omega \in \{\{1,1,2\}\}} \sum_{v \in \{\{2,2,1\}\}}$$



$$\begin{aligned} =&\frac{5\sqrt{E_0}\sqrt{E_1}\lambda}{9E_0E_1(E_0+E_1+\sqrt{E_0}\sqrt{E_1})^2}+\frac{2\lambda}{9E_0(E_0+E_1+\sqrt{E_0}\sqrt{E_1})^2} \\ &+\frac{2\lambda}{9E_1(E_0+E_1+\sqrt{E_0}\sqrt{E_1})^2} \end{aligned} \tag{4.38}$$

From (4.37)-(4.38), $G_{|2|1|}$ becomes as follows:

$$\begin{aligned} G_{|2|1|} =&\frac{4\sqrt{E_0}\sqrt{E_1}\lambda}{9E_0E_1(E_0+E_1+\sqrt{E_0}\sqrt{E_1})^2}+\frac{\lambda}{9E_0(E_0+E_1+\sqrt{E_0}\sqrt{E_1})^2} \\ &+\frac{\lambda}{9E_1(E_0+E_1+\sqrt{E_0}\sqrt{E_1})^2}+\mathcal{O}(\lambda^2). \end{aligned} \tag{4.39}$$

4.2.5. Perturbative Expansion of 1-Point Function $G_{|a|}$, 2-Point Function $G_{|ab|}$, 2-Point Function $G_{|a|b|}$

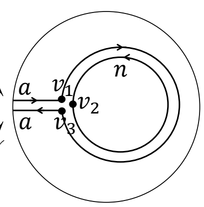
In Subsection 4.2.4, we carried out the calculation for $G_{|1|}$, $G_{|21|}$, $G_{|2|1|}$ perturbatively in the $N=2$ case. In this section, we summarize the similar results for arbitrary N case.

At first, we calculate the connected 1-point function $G_{|a|}$ using perturbative expansion.

$$\begin{aligned}
G_{|a|} =& \frac{1}{V}\frac{\partial \log \mathcal{Z}[J]}{\partial J_{aa}}\bigg|_{J=0} \\
=& \frac{1}{\mathcal{Z}[0]}\int \mathcal{D}\Phi\ \Phi_{aa}\left(\sum_{k=0}^{\infty}\frac{(-V\lambda)^k}{k!4^k}\left(\sum_{n_1,n_2,n_3,n_4=1}^{N}\Phi_{n_1n_2}\Phi_{n_2n_3}\Phi_{n_3n_4}\Phi_{n_4n_1}\right)^k\right) \\
&\times\left(\sum_{l=0}^{\infty}\frac{(-V\sqrt{\lambda})^l}{l!}\left(\sum_{m_1,m_3,m_4=1}^{N}\sqrt{E_{m_1-1}}\Phi_{m_1m_3}\Phi_{m_3m_4}\Phi_{m_4m_1}\right)^l\right) \\
&\times \exp\left(-V\mathrm{tr}\left(E\Phi^2+\frac{\lambda}{2}M\Phi M\Phi\right)\right) \\
=& -V\sqrt{\lambda}\frac{\mathcal{Z}_{free}[0]}{Z[0]}\sum_{m_1,m_3,m_4=1}^{N}\sqrt{E_{m_1-1}}\langle\Phi_{aa}\Phi_{m_1m_3}\Phi_{m_3m_4}\Phi_{m_4m_1}\rangle_{free}+\mathcal{O}(\lambda\sqrt{\lambda}). \qquad (4.40)
\end{aligned}$$

We compute each term of this expansion by drawing perturbative expansions of the 1-point function $G_{|a|}$ in Feynman diagrams. The corresponding Feynman diagrams are drawn on oriented surfaces with one boundary with one connected ribbon graph inserted with an index "a".

$$G_{|a|} = \sum_{n=1}^{N}\sum_{v\in\mathscr{V}}$$



$$+\,\mathcal{O}(\lambda\sqrt{\lambda})$$

$$= -\frac{\sqrt{\lambda}}{3E_{a-1}V}\sum_{n=1}^{N}\sum_{v\in\{\{a,a,n\}\}}\frac{\sqrt{E_{v-1}}}{E_{n-1}+E_{a-1}+\sqrt{E_{a-1}E_{n-1}}}+\mathcal{O}(\lambda\sqrt{\lambda}). \qquad (4.41)$$

Here $\mathscr{V}=\{\{v_1,v_2,v_3\}\}=\{\{a,a,n\}\}$. The circle around the Feynman diagram in (4.41) represents the boundary. So the external lines (Ribbon) grow out of the circle.

Next, we calculate the 2-point function $G_{|ab|}(a\neq b)$ using perturbative expansion.

$$\begin{aligned}
G_{|ab|} =& \frac{1}{V}\frac{\partial^2 \log \mathcal{Z}[J]}{\partial J_{ab}\partial J_{ba}}\bigg|_{J=0} \\
=& \frac{V}{\mathcal{Z}[0]}\int \mathcal{D}\Phi\ \Phi_{ab}\Phi_{ba}\left(\sum_{k=0}^{\infty}\frac{(-V\lambda)^k}{k!4^k}\left(\sum_{n_1,n_2,n_3,n_4=1}^{N}\Phi_{n_1n_2}\Phi_{n_2n_3}\Phi_{n_3n_4}\Phi_{n_4n_1}\right)^k\right) \\
&\times\left(\sum_{l=0}^{\infty}\frac{(-V\sqrt{\lambda})^l}{l!}\left(\sum_{m_1,m_3,m_4=1}^{N}\sqrt{E_{m_1-1}}\Phi_{m_1m_3}\Phi_{m_3m_4}\Phi_{m_4m_1}\right)^l\right)
\end{aligned}$$

$$
\times \exp\left(-V \mathrm{tr}\left(E\Phi^2 + \frac{\lambda}{2} M\Phi M\Phi\right)\right)
$$

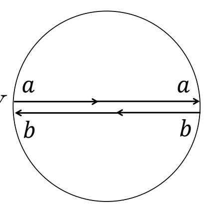
$$
= \frac{V\mathcal{Z}_{free}[0]}{Z[0]} \Bigg\{ \langle \Phi_{ab}\Phi_{ba}\rangle_{free} + \frac{(-V\lambda)^1}{1!4^1} \sum_{n_1,n_2,n_3,n_4=1}^{N} \langle \Phi_{ab}\Phi_{ba}\Phi_{n_1n_2}\Phi_{n_2n_3}\Phi_{n_3n_4}\Phi_{n_4n_1}\rangle_{free}
$$

$$
+ \frac{(-V\sqrt{\lambda})^2}{2!} \sum_{m_1,m_3,m_4,n_1,n_3,n_4=1}^{N} \sqrt{E_{m_1-1}}\sqrt{E_{n_1-1}}
$$

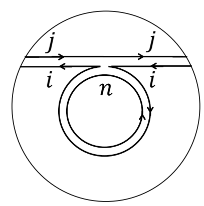
$$
\times \langle \Phi_{ab}\Phi_{ba}\Phi_{m_1m_3}\Phi_{m_3m_4}\Phi_{m_4m_1}\Phi_{n_1n_3}\Phi_{n_3n_4}\Phi_{n_4n_1}\rangle_{free} \Bigg\} + \mathcal{O}(\lambda^2). \tag{4.42}
$$

We compute each term of (4.42) by drawing Feynman diagrams. The corresponding Feynman diagram should be a connected graph with two ribbon graphs with indices a and b inserted at two points from a single boundary:

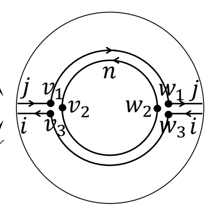
$$
G_{|ab|} = V
$$



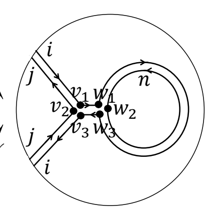
$$
+ 4V \sum_{(i,j)\in\{(a,b),(b,a)\}} \sum_{n=1}^{N}
$$



$$
+ \frac{V}{2!} \sum_{(i,j)\in\{(a,b),(b,a)\}} \sum_{n=1}^{N} \sum_{\omega\in\mathscr{W}} \sum_{v\in\mathscr{V}}
$$



$$
+ 2 \times \frac{V}{2!} \sum_{(i,j)\in\{(a,b),(b,a)\}} \sum_{n=1}^{N} \sum_{\omega\in\mathscr{W}} \sum_{v\in\mathscr{V}}
$$



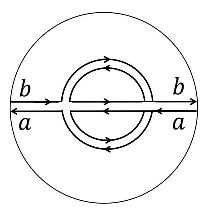
$$
+ \mathcal{O}(\lambda^2) \tag{4.43}
$$

Note that "4" in the 2nd term and "2" in the 4th term are the statistical factors.

$$
\begin{aligned}
G_{|ab|} =& \frac{1}{E_{a-1} + E_{b-1} + \sqrt{E_{a-1}}\sqrt{E_{b-1}}} \\
& - \frac{\lambda}{V} \sum_{(i,j)\in\{(a,b),(b,a)\}} \sum_{n=1}^{N} \left(\frac{1}{E_{i-1} + E_{j-1} + \sqrt{E_{i-1}}\sqrt{E_{j-1}}}\right)^2 \\
& \times \left(\frac{1}{E_{i-1} + E_{n-1} + \sqrt{E_{i-1}}\sqrt{E_{n-1}}}\right) \\
& + \frac{\lambda}{2V} \sum_{(i,j)\in\{(a,b),(b,a)\}} \sum_{n=1}^{N} \sum_{\omega\in\{\{i,j,n\}\}} \sum_{v\in\{\{i,j,n\}\}} \left(\frac{1}{E_{i-1} + E_{j-1} + \sqrt{E_{i-1}}\sqrt{E_{j-1}}}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&\times \sqrt{E_{v-1}}\sqrt{E_{w-1}} \left(\frac{1}{E_{i-1} + E_{n-1} + \sqrt{E_{i-1}}\sqrt{E_{n-1}}} \right) \\
&\times \left(\frac{1}{E_{j-1} + E_{n-1} + \sqrt{E_{j-1}}\sqrt{E_{n-1}}} \right) \\
&+ 2 \times \frac{\lambda}{2V} \sum_{(i,j)\in\{(a,b),(b,a)\}} \sum_{n=1}^{N} \sum_{\omega\in\{\{i,i,n\}\}} \sum_{v\in\{\{i,i,j\}\}} \left(\frac{1}{3E_{i-1}} \right) \\
&\times \left(\frac{1}{E_{i-1} + E_{j-1} + \sqrt{E_{i-1}}\sqrt{E_{j-1}}} \right)^2 \sqrt{E_{v-1}} \\
&\times \sqrt{E_{w-1}} \left(\frac{1}{E_{i-1} + E_{n-1} + \sqrt{E_{i-1}}\sqrt{E_{n-1}}} \right) + \mathcal{O}(\lambda^2). \qquad (4.44)
\end{aligned}
$$

Remark. *We give reasons why a nonplanar Feynman diagram does not appear in (4.43). You might think that the Wick expansion of* $\sum_{n_1,n_2,n_3,n_4=1}^{N} \langle \Phi_{ab}\Phi_{ba}\Phi_{n_1n_2}\Phi_{n_2n_3}\Phi_{n_3n_4}\Phi_{n_4n_1} \rangle$ *would yield a term like* $\sum_{n_1,n_2,n_3,n_4=1}^{N} \langle \Phi_{ab}\Phi_{n_1n_2} \rangle \langle \Phi_{ba}\Phi_{n_3n_4} \rangle \langle \Phi_{n_2n_3}\Phi_{n_4n_1} \rangle$. *In the corresponding Feynman diagram, it is like (4.45), but since b and a are connected by a line, δ_{ab} is generated and 0 is obtained from $a \neq b$.*



$$= 0 \qquad (4.45)$$

As the third example, we calculate the 2-point function $G_{|a|b|}(a \neq b)$ using perturbative expansion. In this case, corresponding Feynman diagrams are drawn on surfaces with two boundaries like a cylinder. The external lines $\begin{array}{c} a \longrightarrow a \\ a \longleftarrow a \end{array}$ and $\begin{array}{c} b \longrightarrow b \\ b \longleftarrow b \end{array}$ are grown out from different boundaries, respectively, and the two boundaries are not connected by any line in any non-zero Feynman diagram.

$$
\begin{aligned}
G_{|a|b|} &= \left. \frac{\partial^2 \log \mathcal{Z}[J]}{\partial J_{aa} \partial J_{bb}} \right|_{J=0} \\
&= -\frac{1}{\mathcal{Z}[0]^2} \left. \frac{\partial \mathcal{Z}[J]}{\partial J_{aa}} \right|_{J=0} \left. \frac{\partial \mathcal{Z}[J]}{\partial J_{bb}} \right|_{J=0} + \frac{1}{\mathcal{Z}[0]} \left. \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{aa} \partial J_{bb}} \right|_{J=0} \\
&= \frac{-V^2}{\mathcal{Z}[0]^2} \int \mathcal{D}\Phi \, \Phi_{aa} \left(\sum_{k=0}^{\infty} \frac{(-V\lambda)^k}{k! 4^k} \left(\sum_{n_1,n_2,n_3,n_4=1}^{N} \Phi_{n_1n_2}\Phi_{n_2n_3}\Phi_{n_3n_4}\Phi_{n_4n_1} \right)^k \right)
\end{aligned}
$$

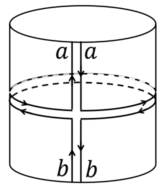
$$\times\left(\sum_{l=0}^{\infty}\frac{(-V\sqrt{\lambda})^l}{l!}\left(\sum_{m_1,m_3,m_4=1}^{N}\sqrt{E_{m_1-1}}\Phi_{m_1m_3}\Phi_{m_3m_4}\Phi_{m_4m_1}\right)^l\right)$$
$$\times\exp\left(-V\operatorname{tr}\left(E\Phi^2+\frac{\lambda}{2}M\Phi M\Phi\right)\right)$$
$$\times\int\mathcal{D}\Phi\ \Phi_{bb}\left(\sum_{k=0}^{\infty}\frac{(-V\lambda)^k}{k!4^k}\left(\sum_{n_1,n_2,n_3,n_4=1}^{N}\Phi_{n_1n_2}\Phi_{n_2n_3}\Phi_{n_3n_4}\Phi_{n_4n_1}\right)^k\right)$$
$$\times\left(\sum_{l=0}^{\infty}\frac{(-V\sqrt{\lambda})^l}{l!}\left(\sum_{m_1,m_3,m_4=1}^{N}\sqrt{E_{m_1-1}}\Phi_{m_1m_3}\Phi_{m_3m_4}\Phi_{m_4m_1}\right)^l\right)$$
$$\times\exp\left(-V\operatorname{tr}\left(E\Phi^2+\frac{\lambda}{2}M\Phi M\Phi\right)\right)$$
$$+\frac{V^2}{\mathcal{Z}[0]}\int\mathcal{D}\Phi\ \Phi_{aa}\Phi_{bb}\left(\sum_{k=0}^{\infty}\frac{(-V\lambda)^k}{k!4^k}\left(\sum_{n_1,n_2,n_3,n_4=1}^{N}\Phi_{n_1n_2}\Phi_{n_2n_3}\Phi_{n_3n_4}\Phi_{n_4n_1}\right)^k\right)$$
$$\times\left(\sum_{l=0}^{\infty}\frac{(-V\sqrt{\lambda})^l}{l!}\left(\sum_{m_1,m_3,m_4=1}^{N}\sqrt{E_{m_1-1}}\Phi_{m_1m_3}\Phi_{m_3m_4}\Phi_{m_4m_1}\right)^l\right)$$
$$\times\exp\left(-V\operatorname{tr}\left(E\Phi^2+\frac{\lambda}{2}M\Phi M\Phi\right)\right). \tag{4.46}$$

So, we estimate the following.

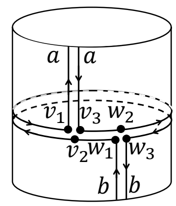
$$G_{|a|b|}=-\frac{V^2\mathcal{Z}_{free}[0]^2}{\mathcal{Z}[0]^2}\times\left((-V\sqrt{\lambda})^1\sum_{m_1,m_3,m_4=1}^{N}\sqrt{E_{m_1-1}}\langle\Phi_{aa}\Phi_{m_1m_3}\Phi_{m_3m_4}\Phi_{m_4m_1}\rangle_{free}\right)$$
$$\times\left((-V\sqrt{\lambda})^1\sum_{m_1,m_3,m_4=1}^{N}\sqrt{E_{m_1-1}}\langle\Phi_{bb}\Phi_{m_1m_3}\Phi_{m_3m_4}\Phi_{m_4m_1}\rangle_{free}\right)$$
$$+\frac{V^2\mathcal{Z}_{free}[0]}{Z[0]}\Bigg\{\langle\Phi_{aa}\Phi_{bb}\rangle_{free}$$
$$+\frac{(-V\lambda)^1}{1!4^1}\sum_{n_1,n_2,n_3,n_4=1}^{N}\langle\Phi_{aa}\Phi_{bb}\Phi_{n_1n_2}\Phi_{n_2n_3}\Phi_{n_3n_4}\Phi_{n_4n_1}\rangle_{free}$$
$$+\frac{(-V\sqrt{\lambda})^2}{2!}\sum_{m_1,m_3,m_4,n_1,n_3,n_4=1}^{N}\sqrt{E_{m_1-1}}\sqrt{E_{n_1-1}}$$
$$\times\langle\Phi_{aa}\Phi_{bb}\Phi_{m_1m_3}\Phi_{m_3m_4}\Phi_{m_4m_1}\Phi_{n_1n_3}\Phi_{n_3n_4}\Phi_{n_4n_1}\rangle_{free}\Bigg\}+\mathcal{O}(\lambda^2). \tag{4.47}$$

We compute each term of this expansion by drawing Feynman diagrams.

$$G_{|a|b|} = 4V^2$$



$$+ V^2 \sum_{\omega\in\mathscr{W}} \sum_{v\in\mathscr{V}}$$



$$+ \mathcal{O}(\lambda^2)$$

$$\begin{aligned} &= -\frac{\lambda}{9E_{a-1}E_{b-1}\left(E_{a-1}+E_{b-1}+\sqrt{E_{a-1}}\sqrt{E_{b-1}}\right)} \\ &\quad + \sum_{\omega\in\mathscr{W}} \sum_{v\in\mathscr{V}} \frac{\sqrt{E_{v-1}}\sqrt{E_{w-1}}\lambda}{9E_{a-1}E_{b-1}\left(E_{a-1}+E_{b-1}+\sqrt{E_{a-1}}\sqrt{E_{b-1}}\right)^2} \\ &\quad + \mathcal{O}(\lambda^2), \end{aligned} \tag{4.48}$$

where $\mathscr{V} = \{\{a, a, b\}\}$ and $\mathscr{W} = \{\{b, b, a\}\}$. More explicitly, this is rewritten as

$$\begin{aligned} G_{|a|b|} = &-\frac{\lambda}{9E_{a-1}E_{b-1}(E_{a-1}+E_{b-1}+\sqrt{E_{a-1}}\sqrt{E_{b-1}})} \\ &+ \frac{5\sqrt{E_{a-1}}\sqrt{E_{b-1}}\lambda}{9E_{a-1}E_{b-1}(E_{a-1}+E_{b-1}+\sqrt{E_{a-1}}\sqrt{E_{b-1}})^2} \\ &+ \frac{2\lambda}{9E_{a-1}(E_{a-1}+E_{b-1}+\sqrt{E_{a-1}}\sqrt{E_{b-1}})^2} \\ &+ \frac{2\lambda}{9E_{b-1}(E_{a-1}+E_{b-1}+\sqrt{E_{a-1}}\sqrt{E_{b-1}})^2} + \mathcal{O}(\lambda^2). \end{aligned} \tag{4.49}$$

4.3. Exact Calculation of Partition Function $\mathcal{Z}[J]$

In this section, the calculation of the partition function[1] is carried out rigorously for any N. The flow of computations is similar to that of [40].

We introduce a new variable X by $\Phi = X - \frac{1}{\sqrt{\lambda}}M$. Here $X = (X_{mn})$ is a Hermitian matrix, too. We do a change of variables of the integral measure $\mathcal{D}\Phi$ as $d\Phi_{ij} = \sum_{m,n=1}^{N} \frac{\partial \Phi_{ij}}{\partial X_{mn}} dX_{mn} = dX_{ij}$. Then $\mathcal{Z}[J]$ is given as

$$\begin{aligned} \mathcal{Z}[J] = &\int \mathcal{D}\Phi \exp\left(-V\mathrm{tr}\left(E\Phi^2 + \kappa\Phi + \frac{\lambda}{4}\Phi^4 + \sqrt{\lambda}M\Phi^3 + \frac{1}{2}M\Phi M\Phi\right)\right) \exp\left(V\mathrm{tr}\left(J\Phi\right)\right) \\ = &\exp\left(-V\mathrm{tr}\left(\frac{3}{4\lambda}M^3 - \frac{\kappa}{\sqrt{\lambda}}I + \frac{1}{\sqrt{\lambda}}J\right)M\right) \end{aligned}$$

[1] For the case with $J = 0$ and $\kappa = 0$, the partition function of this model derives a higher KdV hierarchy. See for example[1, 35, 42, 54].

$$\int \mathcal{D}X \exp\left(-\frac{\lambda V}{4}\mathrm{tr}(X^4)\right)\exp\left(V\mathrm{tr}\left\{\left(\frac{1}{\sqrt{\lambda}}M^3-\kappa I+J\right)X\right\}\right). \tag{4.50}$$

Here I is the unit matrix. Note that

$$\mathcal{D}X=\left(\prod_{i=1}^{N}dx_i\right)\left(\prod_{1\le k<l\le N}(x_l-x_k)^2\right)dU,$$

where x_i is the eigenvalues of X for $i=1,\cdots,N$, dU is the Haar probability measure of the unitary group $U(N)$, and U is the unitary matrix which diagonalize X[10]. Then (4.50) can be rewritten as the following:

$$\begin{aligned}\mathcal{Z}[J]=&\exp\left(-V\mathrm{tr}\left(\frac{3}{4\lambda}M^3-\frac{\kappa}{\sqrt{\lambda}}I+\frac{1}{\sqrt{\lambda}}J\right)M\right)\\ &\int\left(\prod_{i=1}^{N}dx_i\exp\left(-\frac{\lambda V}{4}x_i^4\right)\right)\left(\prod_{1\le k<l\le N}(x_l-x_k)^2\right)\\ &\int_{U(N)}dU\exp\left(V\mathrm{tr}\left\{\left(\frac{1}{\sqrt{\lambda}}M^3-\kappa I+J\right)U\widetilde{X}U^*\right\}\right),\end{aligned} \tag{4.51}$$

where $\widetilde{X}$ is the diagonal matrix $\widetilde{X}=U^*XU$. We use the following formula.

The Harish-Chandra-Itzykson-Zuber integral [36, 51, 56] for the unitary group $U(n)$ is

$$\int_{U(n)}\exp\left(t\mathrm{tr}\left(AUBU^*\right)\right)dU=c_n\frac{\det\limits_{1\le i,j\le n}\left(\exp\left(t\lambda_i(A)\lambda_j(B)\right)\right)}{t^{\frac{(n^2-n)}{2}}\Delta(\lambda(A))\Delta(\lambda(B))}. \tag{4.52}$$

Here $A=(A_{ij})$, and $B=(B_{ij})$ are some Hermitian matrices whose eigenvalues denoted by $\lambda_i(A)$ and $\lambda_i(B)$ $(i=1,\cdots,n)$, respectively. t is the non-zero complex parameter, $\Delta(\lambda(A)):=\prod\limits_{1\le i<j\le n}(\lambda_j(A)-\lambda_i(A))$ is the Vandermonde determinant, and $c_n:=\left(\prod\limits_{i=1}^{n-1}i!\right)\times\pi^{\frac{n(n-1)}{2}}$ is the constant. $(\exp\left(t\lambda_i(A)\lambda_j(B)\right))$ is the $n\times n$ matrix with the i-th row and the j-th column being $\exp\left(t\lambda_i(A)\lambda_j(B)\right)$.

Applying the Harish-Chandra-Itzykson-Zuber integral (4.52) to $\int dU\exp\left(V\mathrm{tr}\left\{\left(\frac{1}{\sqrt{\lambda}}M^3-\kappa I+J\right)U\widetilde{X}U^*\right\}\right)$ in (4.51), the result is

$$\int_{U(N)}dU\exp\left(V\mathrm{tr}\left\{\left(\frac{1}{\sqrt{\lambda}}M^3-\kappa I+J\right)U\widetilde{X}U^*\right\}\right)=\frac{C}{N!}\frac{\det\limits_{1\le i,j\le N}\exp\left(Vx_is_j\right)}{\prod\limits_{i<j}(x_j-x_i)\prod\limits_{i<j}(s_j-s_i)}, \tag{4.53}$$

where s_t is the eigenvalues of the matrix $\frac{1}{\sqrt{\lambda}}M^3 - \kappa I + J$ for $t = 1, \cdots, N$ and $C = \left(\prod_{p=1}^{N} p!\right) \times \left(\frac{\pi}{V}\right)^{\frac{N(N-1)}{2}}$. $(\exp(Vx_i s_j))$ denotes the $N \times N$ matrix with the i-th row and the j-th column being $\exp(Vx_i s_j)$. Then the partition function $\mathcal{Z}[J]$ is described as

$$\begin{aligned}\mathcal{Z}[J] =& \frac{C}{N!} \exp\left(-V \operatorname{tr}\left(\frac{3}{4\lambda}M^3 - \frac{\kappa}{\sqrt{\lambda}}I + \frac{1}{\sqrt{\lambda}}J\right) M\right) \frac{1}{\prod_{1\le t<u\le N}(s_u - s_t)} \\ & \int \left(\prod_{i=1}^{N} dx_i \exp\left(-\frac{\lambda V}{4}x_i^4\right)\right) \left(\prod_{1\le k<l\le N}(x_l - x_k)\right) \det_{1\le m,n\le N} \exp(Vx_m s_n).\end{aligned} \tag{4.54}$$

Let us transform the part of the x_i integrations in (4.54). By definition of the determinant,

$$\begin{aligned}& \int \left(\prod_{i=1}^{N} dx_i \exp\left(-\frac{\lambda V}{4}x_i^4\right)\right) \left(\prod_{1\le k<l\le N}(x_l - x_k)\right) \det_{1\le i,j\le N} \exp(Vx_i s_j) \\ =& \sum_{\sigma\in S_N} \int \left(\prod_{i=1}^{N} dx_i \exp\left(-\frac{\lambda V}{4}x_i^4\right)\right) \left(\prod_{1\le k<l\le N}(x_l - x_k)\right) (-1)^\sigma \left(\prod_{j=1}^{N} e^{Vx_{\sigma(j)}s_j}\right).\end{aligned} \tag{4.55}$$

Here S_N is a symmetric group. Next we changed variables as $x_{\sigma(i)} \mapsto x_i$ $(i = 1, \cdots, N)$. Note that the Vandermonde determinant $\det\left(x_{\sigma(i)}^{j-1}\right) = (-1)^\sigma \det x_i^{j-1}$. The above is written as

$$\begin{aligned}& \sum_{\sigma\in S_N} \int \left(\prod_{i=1}^{N} dx_i \exp\left(-\frac{\lambda V}{4}x_i^4\right)\right) \left(\prod_{1\le k<l\le N}(x_l - x_k)\right) (-1)^\sigma(-1)^\sigma \left(\prod_{j=1}^{N} e^{Vx_j s_j}\right) \\ =& N! \int \left(\prod_{i=1}^{N} dx_i \exp\left(-\frac{\lambda V}{4}x_i^4\right) \exp(Vx_i s_i)\right) \prod_{1\le k<l\le N}(x_l - x_k).\end{aligned} \tag{4.56}$$

From this, the partition function $\mathcal{Z}[J]$ becomes as follows:

$$\begin{aligned}\mathcal{Z}[J] =& C \exp\left(-V \operatorname{tr}\left(\frac{3}{4\lambda}M^3 - \frac{\kappa}{\sqrt{\lambda}}I + \frac{1}{\sqrt{\lambda}}J\right) M\right) \frac{1}{\prod_{1\le t<u\le N}(s_u - s_t)} \\ & \int \left(\prod_{i=1}^{N} dx_i \exp\left(-\frac{\lambda V}{4}x_i^4\right) \exp(Vx_i s_i)\right) \prod_{1\le k<l\le N}(x_l - x_k).\end{aligned} \tag{4.57}$$

Using $\prod_{1\le k<l\le N}(x_l-x_k)=\det_{1\le k,l\le N}\left(x_k^{l-1}\right)$, we calculate the remaining integral in the right-hand side in (4.54) as

$$\begin{aligned}
&\int_{-\infty}^{\infty}\left(\prod_{i=1}^{N}dx_i\exp\left(-\frac{\lambda V}{4}x_i^4\right)\exp\left(Vx_is_i\right)\right)\det_{1\le k,l\le \mathrm{N}}\left(x_l^{k-1}\right)\\
&=\sum_{\sigma\in S_{\mathrm{N}}}\mathrm{sgn}\sigma\prod_{i=1}^{N}\phi_{\sigma(i)}(s_i)\\
&=\det_{1\le i,j\le N}\left(\phi_i(s_j)\right),
\end{aligned}\tag{4.58}$$

where $\phi_k(z)$ is defined by

$$\phi_k(z)=\int_{-\infty}^{\infty}dx\ \ x^{k-1}\exp\left(-\frac{\lambda V}{4}x^4+Vxz\right),\tag{4.59}$$

and $(\phi_i(s_j))$ is the $N\times N$ matrix with the i-th row and the j-th column being $\phi_i(s_j)$. Summarizing the results (4.54) and (4.58), we obtain the following:

Proposition 4.1 (N.K-Sako[41])**.** *Let $\mathcal{Z}[J]$ be the partition function of Φ^3-Φ^4 hybrid matrix model given by (4.4). Then, $\mathcal{Z}[J]$ is given as*

$$\mathcal{Z}[J]=C\exp\left(-V\mathrm{tr}\left(\frac{3}{4\lambda}M^3-\frac{\kappa}{\sqrt{\lambda}}I+\frac{1}{\sqrt{\lambda}}J\right)M\right)\frac{\det_{1\le i,j\le N}\left(\phi_i(s_j)\right)}{\prod_{1\le t<u\le \mathrm{N}}(s_u-s_t)}.$$

Note that $\phi_k(z)$ is expressed as

$$\phi_k(z)=\left(\frac{1}{V}\right)^{k-1}\left(\frac{d}{dz}\right)^{k-1}\int_{-\infty}^{\infty}dx\exp\left(-\frac{\lambda V}{4}x^4+Vxz\right).\tag{4.60}$$

We use

$$P(z)=\int_{-\infty}^{\infty}dx\exp\left(-\frac{\lambda V}{4}x^4+Vxz\right).\tag{4.61}$$

If V is a pure imaginary number, this $P(z)$ is a special case of the following:

$$P(x,y):=\int_{-\infty}^{\infty}dt\exp\left(i\left(t^4+xt^2+yt\right)\right).\tag{4.62}$$

This is called Pearcey integral[37], where $0\le\arg x\le\pi$ and $y\in\mathbb{R}$. Substituting (4.61) for (4.60), $\phi_k(z)$ is calculated as follows:

$$\phi_k(z)=\left(\frac{1}{V}\right)^{k-1}\left(\frac{d}{dz}\right)^{k-1}P(z).\tag{4.63}$$

Proposition 4.2 (N.K-Sako[41]). *Let* $\left(P^{(j-1)}(s_i)\right)$ *be the* $N\times N$ *matrix with the* i*-th row and the* j*-th column being* $P^{(j-1)}(s_i)=\left(\frac{d}{ds_i}\right)^{j-1}P(s_i)$. *We then obtain the following:*

$$\det\left(P^{(j-1)}(s_i)\right)=\left(\prod_{1\le i<j\le N}\left(\partial_{s_i}-\partial_{s_j}\right)\right)P(s_1)\cdots P(s_N).$$

This proposition is identical to Proposition 4.2 in [40] and its proof is also given in [40]. We introduce

$$\begin{aligned}P_N(s_1,\cdots,s_N)=&\left(\prod_{1\le i<j\le N}(\partial_{s_i}-\partial_{s_j})\right)P(s_1)\cdots P(s_N)\\=&\det\begin{pmatrix}P(s_1)&\cdots&P(s_N)\\P^{(1)}(s_1)&\cdots&P^{(1)}(s_N)\\\vdots&&\vdots\\P^{(N-1)}(s_1)&\cdots&P^{(N-1)}(s_N)\end{pmatrix}.\end{aligned}\tag{4.64}$$

From this, $\det\limits_{1\le i,j\le N}(\phi_i(s_j))$ is calculated as follows:

$$\det_{1\le i,j\le N}(\phi_i(s_j))=\frac{1}{V^{\frac{N(N-1)}{2}}}P_N(s_1,\cdots,s_N).\tag{4.65}$$

Summarizing the above results, we obtain the following:

Theorem 4.3 (N.K-Sako[41]). *Let* $\mathcal{Z}[J]$ *be the partition function of* Φ^3*-*Φ^4 *hybrid matrix model given by (4.4). Then,* $\mathcal{Z}[J]$ *is given as*

$$\begin{aligned}\mathcal{Z}[J]=&\int\mathcal{D}\Phi\exp\left(-V\mathrm{tr}\left(E\Phi^2+\kappa\Phi+\frac{\lambda}{4}\Phi^4+\sqrt{\lambda}M\Phi^3+\frac{1}{2}M\Phi M\Phi\right)\right)\exp\left(V\mathrm{tr}\left(J\Phi\right)\right)\\=&C'\frac{e^{\frac{-V}{\sqrt{\lambda}}\mathrm{tr}(JM)}P_N(s_1,\cdots,s_N)}{\prod\limits_{1\le t<u\le N}(s_u-s_t)}.\end{aligned}\tag{4.66}$$

Here $C'=\exp\left(-V\mathrm{tr}\left(\frac{3}{4\lambda}M^3-\frac{\kappa}{\sqrt{\lambda}}I\right)M\right)\left(\prod_{p=1}^{N}p!\right)\frac{\pi^{\frac{N(N-1)}{2}}}{V^{N(N-1)}}$, *and* s_t *is the eigenvalues of the matrix* $\frac{1}{\sqrt{\lambda}}M^3-\kappa I+J$ *for* $t=1,\cdots,N$.

4.4. Exact Calculations of 1-Point Function $G_{|a|}$, 2-Point Function $G_{|ab|}$, 2-Point Function $G_{|a|b|}$, and $G_{|a^1|a^2|\cdots|a^n|}$ $(3\le n)$

In this section, $G_{|a|}$, $G_{|ab|}$, $G_{|a|b|}$, and $G_{|a_1|a_2|\cdots|a_n|}$ are calculated exactly, by using Theorem 4.3.

4.4.1. 1-Point Function $G_{|a|}$

In the calculation of the 1-point function $G_{|a|}$, the external field J can be treated as the diagonal matrix $J = diag\left(J_{11}, \cdots, J_{NN}\right)$. Then the eigenvalues s_t in (4.66) are given $s_t = \frac{E_{t-1}\sqrt{E_{t-1}}}{\sqrt{\lambda}} + J_{tt} - \kappa$. From Theorem 4.3, the 1-point function $G_{|a|}$ is calculated as follows:

$$G_{|a|} = \frac{1}{V}\frac{\partial \log \mathcal{Z}[J]}{\partial J_{aa}}\Bigg|_{J=0} = \frac{\frac{1}{V}\frac{\partial}{\partial J_{aa}}\left(\frac{e^{\frac{-V}{\sqrt{\lambda}}\mathrm{tr}(JM)}P_N(s_1,\cdots,s_N)}{\prod\limits_{1\le t<u\le N}(s_u - s_t)}\right)\Bigg|_{J=0}}{\frac{P_N(s_1,\cdots,s_N)\Big|_{J=0}}{\prod\limits_{1\le p<q\le N}(s_q - s_p)\Big|_{J=0}}}. \tag{4.67}$$

Note that

$$\begin{aligned}&\frac{\partial}{\partial J_{aa}}\left\{e^{\frac{-V}{\sqrt{\lambda}}\mathrm{tr(JM)}}P_N(s_1,\cdots,s_N)\right\}\\ &= -\frac{V}{\sqrt{\lambda}}\sqrt{E_{a-1}}e^{\frac{-V}{\sqrt{\lambda}}\mathrm{tr}(JM)}P_N(s_1,\cdots,s_N) + e^{\frac{-V}{\sqrt{\lambda}}\mathrm{tr}(JM)}\left(\partial_a P_N(s_1,\cdots,s_N)\right),\end{aligned} \tag{4.68}$$

where $\partial_a P_N(s_1,\cdots,s_N) = \frac{\partial}{\partial s_a}P_N(s_1,\cdots,s_N)$. Next, we use the following formula. Let $\Delta = \Delta(\vec{x}_n) = \det\limits_{1\le i,j\le n}\left((x_j)^{i-1}\right)$ be the Vandermonde determinant for $\vec{x}_n = (x_1,\cdots,x_n) \in \mathbb{R}^n$. For any $1 \le k \le n$

$$\frac{\partial M_n}{\partial x_k} = \sum_{i=1, i\ne k}^{n} \frac{M_n(\vec{x}_n)}{x_k - x_i}. \tag{4.69}$$

Using this formula, we get

$$\begin{aligned}\frac{\partial}{\partial J_{aa}}\prod_{1\le i,j\le N}(s_j - s_i)\Bigg|_{J=0} &= \frac{\partial}{\partial J_{aa}}\left\{\det_{1\le i,j\le N}\left(\left(\frac{E_{j-1}\sqrt{E_{j-1}}}{\sqrt{\lambda}} + J_{jj} - \kappa\right)^{i-1}\right)\right\}\Bigg|_{J=0}\\ &= \sum_{i=1,i\ne a}^{N}\frac{\sqrt{\lambda}\det\limits_{1\le i,j\le N}\left(\left(\frac{E_{j-1}\sqrt{E_{j-1}}}{\sqrt{\lambda}} - \kappa\right)^{i-1}\right)}{E_{a-1}\sqrt{E_{a-1}} - E_{i-1}\sqrt{E_{i-1}}}.\end{aligned} \tag{4.70}$$

Substituting (4.70) into (4.67), finally $G_{|a|}$ is expressed as

$$G_{|a|} = -\frac{\sqrt{E_{a-1}}}{\sqrt{\lambda}} - \frac{1}{V}\sum_{i=1,i\ne a}^{N}\frac{\sqrt{\lambda}}{E_{a-1}\sqrt{E_{a-1}} - E_{i-1}\sqrt{E_{i-1}}} + \frac{1}{V}\partial_a \log P_N(z_1,\cdots,z_N), \tag{4.71}$$

where $z_j = \dfrac{E_{j-1}\sqrt{E_{j-1}}}{\sqrt{\lambda}} - \kappa$ for $j = 1, \ldots, N$, and $\partial_a = \dfrac{\partial}{\partial z_a}$.

4.4.2. 2-Point Function $G_{|ab|}$

Let us consider 2-point function $G_{|ab|}$ $(a \neq b,\ a, b \in \{1, 2, \cdots, N\})$. For the calculation, we put J as all components without J_{ab}, J_{ba} are zero. Note that $\mathrm{tr} JM = 0$ for this J.

At first, we estimate eigenvalues s_t for $t = 1, \ldots, N$ of the matrix $\dfrac{1}{\sqrt{\lambda}}M^3 - \kappa I + J$. The eigenequation is

$$
\begin{aligned}
0 &= \det\left(sI - \left(\frac{1}{\sqrt{\lambda}}M^3 - \kappa I + J\right)\right) \\
&= \left(\prod_{i=1, i\neq a, i\neq b}^{N}\left(s - \frac{1}{\sqrt{\lambda}}E_{i-1}\sqrt{E_{i-1}} + \kappa\right)\right) \\
&\quad \times \Bigg\{ s^2 + \left(-\frac{1}{\sqrt{\lambda}}E_{b-1}\sqrt{E_{b-1}} - \frac{1}{\sqrt{\lambda}}E_{a-1}\sqrt{E_{a-1}} + 2\kappa\right)s \\
&\quad - \frac{1}{\sqrt{\lambda}}E_{a-1}\sqrt{E_{a-1}}\kappa - \frac{1}{\sqrt{\lambda}}E_{b-1}\sqrt{E_{b-1}}\kappa + \frac{1}{\lambda}E_{a-1}E_{b-1}\sqrt{E_{a-1}}\sqrt{E_{b-1}} \\
&\quad + \kappa^2 - J_{ab}J_{ba}\Bigg\}.
\end{aligned}
\tag{4.72}
$$

We label the eigenvalues as $s_t = \dfrac{1}{\sqrt{\lambda}}E_{t-1}\sqrt{E_{t-1}} - \kappa$ for $t \neq a, b$,

$$
\begin{aligned}
s_a =& \frac{\frac{1}{\sqrt{\lambda}}E_{a-1}\sqrt{E_{a-1}} + \frac{1}{\sqrt{\lambda}}E_{b-1}\sqrt{E_{b-1}} - 2\kappa}{2} \\
&+ \frac{\sqrt{\left(\frac{1}{\sqrt{\lambda}}E_{a-1}\sqrt{E_{a-1}} - \frac{1}{\sqrt{\lambda}}E_{b-1}\sqrt{E_{b-1}}\right)^2 + 4J_{ab}J_{ba}}}{2},
\end{aligned}
\tag{4.73}
$$

and

$$
\begin{aligned}
s_b =& \frac{\frac{1}{\sqrt{\lambda}}E_{a-1}\sqrt{E_{a-1}} + \frac{1}{\sqrt{\lambda}}E_{b-1}\sqrt{E_{b-1}} - 2\kappa}{2} \\
&- \frac{\sqrt{\left(\frac{1}{\sqrt{\lambda}}E_{a-1}\sqrt{E_{a-1}} - \frac{1}{\sqrt{\lambda}}E_{b-1}\sqrt{E_{b-1}}\right)^2 + 4J_{ab}J_{ba}}}{2}.
\end{aligned}
\tag{4.74}
$$

Let us calculate $G_{|ab|}$ by using these s_t. From Theorem 3.3,

$$\begin{aligned}
G_{|ab|} &= \frac{1}{V}\frac{\partial^2 \log \mathcal{Z}[J]}{\partial J_{ab}\partial J_{ba}}\bigg|_{J=0} \\
&= \frac{1}{V}\frac{\partial^2}{\partial J_{ab}\partial J_{ba}}\left\{\log P_N\left(s_1,\cdots,s_N\right) - \log \prod_{1\le t<u\le N}(s_u - s_t)\right\}\bigg|_{J=0} \\
&= \frac{1}{V}\left\{ \frac{\frac{\partial^2}{\partial J_{ab}\partial J_{ba}}P_N(s_1,\cdots,s_N)}{P_N(s_1,\cdots,s_N)}\Bigg|_{J=0} - \frac{\frac{\partial^2}{\partial J_{ab}\partial J_{ba}}\left\{\prod_{1\le t<u\le N}(s_u-s_t)\right\}}{\prod_{1\le t<u\le N}(s_u-s_t)}\Bigg|_{J=0}\right\}.
\end{aligned} \tag{4.75}$$

Here we use $\frac{\partial P_N(s_1,\cdots,s_N)}{\partial J_{ab}}\bigg|_{J=0} = \frac{\partial \det\limits_{1\le k,l\le N}\left((s_l)^{k-1}\right)}{\partial J_{ab}}\Bigg|_{J=0} = 0$, since s_a and s_b are functions of $(J_{ab}J_{ba})$ as we see in (4.73) and (4.74), then $\frac{\partial P_N(s_1,\cdots,s_N)}{\partial J_{ab}}$ and $\frac{\partial \det\limits_{1\le k,l\le N}\left((s_l)^{k-1}\right)}{\partial J_{ab}}$ are of the form $J_{ba}\times(\cdots)$.

Using the fact that $\frac{\partial s_k}{\partial J_{ab}}\bigg|_{J=0} = 0$ and $\frac{\partial^2 s_a}{\partial J_{ba}\partial J_{ab}}\bigg|_{J=0} = \frac{\sqrt{\lambda}}{\left|E_{a-1}\sqrt{E_{a-1}} - E_{b-1}\sqrt{E_{b-1}}\right|} = -\frac{\partial^2 s_b}{\partial J_{ab}\partial J_{ba}}\bigg|_{J=0}$, we obtain

$$\begin{aligned}
\frac{\partial^2}{\partial J_{ab}\partial J_{ba}}P_N(s_1,\cdots,s_N)\bigg|_{J=0} =& \frac{\sqrt{\lambda}}{\left|E_{a-1}\sqrt{E_{a-1}} - E_{b-1}\sqrt{E_{b-1}}\right|} \\
&\times\left(\partial_a P_N\left(z_1,\ldots,z_N\right) - \partial_b P_N\left(z_1,\ldots,z_N\right)\right),
\end{aligned} \tag{4.76}$$

where $z_j = \frac{E_{j-1}\sqrt{E_{j-1}}}{\sqrt{\lambda}} - \kappa$ for $j = 1,\ldots,N$. Similarly, we get

$$\begin{aligned}
\frac{\partial^2}{\partial J_{ab}\partial J_{ba}}\left\{\prod_{1\le t<u\le N}(s_u-s_t)\right\}\bigg|_{J=0} =& \frac{(\sqrt{\lambda})^2}{\left|E_{a-1}\sqrt{E_{a-1}} - E_{b-1}\sqrt{E_{b-1}}\right|}\det_{1\le k,l\le N}\left((s_k)^{l-1}\right) \\
&\times\left(\sum_{i=1,i\ne a}^{N}\frac{1}{E_{a-1}\sqrt{E_{a-1}} - E_{i-1}\sqrt{E_{i-1}}} - \sum_{i=1,i\ne b}^{N}\frac{1}{E_{b-1}\sqrt{E_{b-1}} - E_{i-1}\sqrt{E_{i-1}}}\right),
\end{aligned} \tag{4.77}$$

where we use the formula (4.69), again. Substituting (4.76) and (4.77) into (4.75), $G_{|ab|}$ ($b < a$, i.e.$E_b < E_a$) is finally obtained as

$$\begin{aligned}G_{|ab|} =&\frac{\sqrt{\lambda}}{V(E_{a-1}\sqrt{E_{a-1}}-E_{b-1}\sqrt{E_{b-1}})}\Bigg\{\left(\frac{\partial_a P_N(z_1,\cdots,z_N)}{P_N(z_1,\cdots,z_N)}-\frac{\partial_b P_N(z_1,\cdots,z_N)}{P_N(z_1,\cdots,z_N)}\right)\\ &-\sqrt{\lambda}\left(\sum_{i=1,i\neq a}^{N}\frac{1}{E_{a-1}\sqrt{E_{a-1}}-E_{i-1}\sqrt{E_{i-1}}}-\sum_{i=1,i\neq b}^{N}\frac{1}{E_{b-1}\sqrt{E_{b-1}}-E_{i-1}\sqrt{E_{i-1}}}\right)\Bigg\}. \qquad (4.78)\end{aligned}$$

4.4.3. 2-Point Function $G_{|a|b|}$

In the calculation of the 2-point functions $G_{|a|b|}$, the external field J can be treated as the diagonal matrix $J = diag\,(J_{11},\cdots,J_{NN})$. Then the eigenvalues s_t in (4.66) are given $s_t = \dfrac{E_{t-1}\sqrt{E_{t-1}}}{\sqrt{\lambda}} + J_{tt} - \kappa$ for $t = 1,\cdots,N$. Then, the 2-point function $G_{|a|b|}$ is calculated as follows:

$$\begin{aligned}G_{|a|b|} &= \left.\frac{\partial^2 \log \mathcal{Z}[J]}{\partial J_{aa}\partial J_{bb}}\right|_{J=0}\\ &= \frac{\partial^2}{\partial J_{aa}\partial J_{bb}}\left\{\log P_N(s_1,\cdots,s_N) - \log\prod_{1\le t<u\le N}(s_u-s_t)\right\}\Bigg|_{J=0}\\ &= -\left.\frac{\partial_a P_N(s_1,\cdots,s_N)}{P_N(s_1,\cdots,s_N)}\right|_{J=0}\left.\frac{\partial_b P_N(s_1,\cdots,s_N)}{P_N(s_1,\cdots,s_N)}\right|_{J=0}+\left.\frac{\partial_a\partial_b P_N(s_1,\cdots,s_N)}{P_N(s_1,\cdots,s_N)}\right|_{J=0}\\ &\quad-\frac{\lambda}{(E_{a-1}\sqrt{E_{a-1}}-E_{b-1}\sqrt{E_{b-1}})^2}. \qquad (4.79)\end{aligned}$$

Finally $G_{|a|b|}$ is expressed as

$$\begin{aligned}G_{|a|b|} =& -\frac{\partial_a P_N(z_1,\cdots,z_N)}{P_N(z_1,\cdots,z_N)}\frac{\partial_b P_N(z_1,\cdots,z_N)}{P_N(z_1,\cdots,z_N)}+\frac{\partial_a\partial_b P_N(z_1,\cdots,z_N)}{P_N(z_1,\cdots,z_N)}\\ &-\frac{\lambda}{(E_{a-1}\sqrt{E_{a-1}}-E_{b-1}\sqrt{E_{b-1}})^2}, \qquad (4.80)\end{aligned}$$

where $z_j = \dfrac{E_{j-1}\sqrt{E_{j-1}}}{\sqrt{\lambda}} - \kappa$ for $j = 1,\ldots,N$,$\partial_a = \dfrac{\partial}{\partial z_a}$, and $\partial_b = \dfrac{\partial}{\partial z_b}$.

4.4.4. n-point function $G_{|a^1|a^2|\cdots|a^n|}$ $(3 \le n)$

Let us calculate $G_{|a^1|a^2|\cdots|a^n|}$ for $3 \le n$. Here a^β is the pairwise different indices for $\beta = 1,\ldots,n$. To calculate $G_{|a^1|a^2|\cdots|a^n|}$, it is enough to take J as a diagonal matrix $J = diag(J_{11},\cdots,J_{NN})$. Then the eigenvalues s_t in (4.66) are given $s_t = \dfrac{E_{t-1}\sqrt{E_{t-1}}}{\sqrt{\lambda}} + J_{tt} - \kappa$ for $t = 1,2,\cdots,N$. From the definition in (2.15), the n-point function $G_{|a^1|a^2|\cdots|a^n|}$ is given by

$$
\begin{aligned}
G_{|a^1|a^2|\cdots|a^n|} =& V^{n-2}\frac{\partial^n}{\partial J_{a^1a^1}\cdots\partial J_{a^na^n}}\log\frac{\mathcal{Z}[J]}{\mathcal{Z}[0]}\bigg|_{J=0} \\
=& V^{n-2}\frac{\partial^n}{\partial J_{a^1a^1}\cdots\partial J_{a^na^n}}\log\left(\frac{e^{\frac{-V}{\sqrt{\lambda}}\mathrm{tr}(JM)}P_N(s_1,\cdots,s_N)}{\prod_{1\le t<u\le N}(s_u-s_t)}\right)\Bigg|_{J=0}.
\end{aligned} \tag{4.81}
$$

Since $s_t=\frac{E_{t-1}\sqrt{E_{t-1}}}{\sqrt{\lambda}}+J_{tt}-\kappa$, $\frac{\partial^3}{\partial J_{a^1a^1}\partial J_{a^2a^2}\partial J_{a^3a^3}}\left(\sum_{1\le t<u\le N}\log(s_u-s_t)\right)\bigg|_{J=0}=0$. In addition

$$
\frac{\partial^2}{\partial J_{a^1a^1}\partial J_{a^2a^2}}\log\exp\left(-\frac{V}{\sqrt{\lambda}}\mathrm{tr}JM\right)=-\frac{\partial^2}{\partial J_{a^1a^1}\partial J_{a^2a^2}}\frac{V}{\sqrt{\lambda}}\sum_{i=1}^{N}J_{ii}\sqrt{E_{i-1}}=0. \tag{4.82}
$$

Then, the n-point function $G_{|a^1|a^2|\cdots|a^n|}$ $(3\le n)$ is obtained as follows:

$$
G_{|a^1|a^2|\cdots|a^n|}=V^{n-2}\frac{\partial^n}{\partial z_{a^1}\cdots\partial z_{a^n}}\log P_N(z_1,\cdots,z_N), \tag{4.83}
$$

where $z_i=\frac{E_{i-1}\sqrt{E_{i-1}}}{\sqrt{\lambda}}-\kappa$.

4.5. Approximations from Exact Solutions by Saddle Point Method

To ensure that perturbative calculations in Section 2 and exact results in Section 4 are consistent, we shall reproduce the contents of Section 2 by approximating the results of Section 4. Thereafter, the calculations are performed with $\kappa=0$.

By the change of variable $x=\frac{1}{\left(V\frac{\lambda}{4}\right)^{\frac{1}{4}}}k$, $P(z)=\int_{-\infty}^{\infty}dx\exp\left(-V\frac{\lambda}{4}x^4+Vxz\right)$ is transformed into

$$
P\left(\frac{z\lambda^{\frac{1}{4}}}{V^{\frac{3}{4}}\sqrt{2}}\right)=\frac{1}{\left(V\frac{\lambda}{4}\right)^{\frac{1}{4}}}\int_{-\infty}^{\infty}dk\exp\left(-k^4+zk\right). \tag{4.84}
$$

The function under integration has three saddle points. We choose an integral path through one of them, $k=\frac{z^{\frac{1}{3}}}{2^{\frac{2}{3}}}$. To estimate the integral around the neighborhood of $k=\frac{z^{\frac{1}{3}}}{2^{\frac{2}{3}}}$ we put $k=\frac{z^{\frac{1}{3}}}{2^{\frac{2}{3}}}+\xi$. Then (4.84) can be evaluated as

$$
\begin{aligned}
P\left(\frac{z\lambda^{\frac{1}{4}}}{V^{\frac{3}{4}}\sqrt{2}}\right) =& \frac{1}{\left(V\frac{\lambda}{4}\right)^{\frac{1}{4}}}\int_{-\infty}^{\infty} d\xi \exp\left(-\left(\frac{z^{\frac{1}{3}}}{2^{\frac{2}{3}}}+\xi\right)^4 + z\left(\frac{z^{\frac{1}{3}}}{2^{\frac{2}{3}}}+\xi\right)\right) \\
=&\mathcal{C}(z)\left(\frac{\sqrt{2}}{\lambda^{\frac{1}{4}}V^{\frac{1}{4}}} + \frac{7}{9\times 2^{\frac{5}{6}}V^{\frac{1}{4}}\lambda^{\frac{1}{4}}z^{\frac{4}{3}}} + \frac{385}{648\times 2^{\frac{1}{6}}\lambda^{\frac{1}{4}}V^{\frac{1}{4}}z^{\frac{8}{3}}}\right) + \mathcal{O}\left(\frac{1}{z^4}\right). \qquad (4.85)
\end{aligned}
$$

Here $\mathcal{C}(z) = \dfrac{2^{\frac{1}{6}}\sqrt{\pi}}{\sqrt{3}z^{\frac{1}{3}}}\exp\left(-\dfrac{z^{\frac{4}{3}}}{2^{\frac{8}{3}}}+\dfrac{z^{\frac{4}{3}}}{2^{\frac{2}{3}}}\right)$. To evaluate n-point functions, $z = \dfrac{\sqrt{2}V^{\frac{3}{4}}E_{i-1}^{\frac{3}{2}}}{\lambda^{\frac{3}{4}}}$ cases are used. For the case $z_i = \dfrac{\sqrt{2}V^{\frac{3}{4}}E_{i-1}^{\frac{3}{2}}}{\lambda^{\frac{3}{4}}}$,

$$
P\left(\frac{E_{i-1}\sqrt{E_{i-1}}}{\sqrt{\lambda}}\right) = \mathcal{C}'(E_i)\left(\frac{\sqrt{2}}{\lambda^{\frac{1}{4}}V^{\frac{1}{4}}} + \frac{7\lambda^{\frac{3}{4}}}{18\sqrt{2}E_{i-1}^2V^{\frac{5}{4}}} + \frac{385\lambda^{\frac{7}{4}}}{1296\sqrt{2}E_{i-1}^4V^{\frac{9}{4}}}\right) + \mathcal{O}(\lambda^{\frac{11}{4}}), \qquad (4.86)
$$

where $\mathcal{C}'(E_i) = \sqrt{\dfrac{\pi\lambda^{\frac{1}{2}}}{3V^{\frac{1}{2}}E_{i-1}}}\exp\left(\dfrac{3VE_{i-1}^2}{4\lambda}\right)$.

Next, we estimate $\partial P(z)$, similarly. After changing variable as $x = \dfrac{1}{\left(V\frac{\lambda}{4}\right)^{\frac{1}{4}}}k$,

$\partial P(z) = V\displaystyle\int_{-\infty}^{\infty} dx\, x \exp\left(-V\frac{\lambda}{4}x^4 + Vxz\right)$ is transformed into

$$
(\partial P)\left(\frac{z\lambda^{\frac{1}{4}}}{V^{\frac{3}{4}}\sqrt{2}}\right) = \frac{V}{\left(V\frac{\lambda}{4}\right)^{\frac{1}{2}}}\int_{\infty}^{\infty} dk\, k \exp\left(-k^4 + zk\right). \qquad (4.87)
$$

The saddle point on the integral path is $k = \dfrac{z^{\frac{1}{3}}}{2^{\frac{2}{3}}}$. In the neighborhood of the saddle point $k = \dfrac{z^{\frac{1}{3}}}{2^{\frac{2}{3}}}$, we put $k = \dfrac{z^{\frac{1}{3}}}{2^{\frac{2}{3}}} + \xi$, then (4.87) can be evaluated as

$$
\begin{aligned}
(\partial P)\left(\frac{z\lambda^{\frac{1}{4}}}{V^{\frac{3}{4}}\sqrt{2}}\right) =& \frac{V}{\left(V\frac{\lambda}{4}\right)^{\frac{1}{2}}}\int_{-\infty}^{\infty} d\xi \left(\frac{z^{\frac{1}{3}}}{2^{\frac{2}{3}}}+\xi\right)\exp\left(-\left(\frac{z^{\frac{1}{3}}}{2^{\frac{2}{3}}}+\xi\right)^4 + z\left(\frac{z^{\frac{1}{3}}}{2^{\frac{2}{3}}}+\xi\right)\right) \\
=&V\mathcal{C}(z)\left(\frac{2^{\frac{1}{3}}z^{\frac{1}{3}}}{\sqrt{\lambda}\sqrt{V}} - \frac{5}{18\sqrt{\lambda}\sqrt{V}z} - \frac{455}{648\times 2^{\frac{1}{3}}\sqrt{\lambda}\sqrt{V}z^{\frac{7}{3}}}\right) + \mathcal{O}\left(\frac{1}{z^{\frac{11}{3}}}\right). \qquad (4.88)
\end{aligned}
$$

For the case $z_i = \frac{\sqrt{2}V^{\frac{3}{4}}E_{i-1}^{\frac{3}{2}}}{\lambda^{\frac{3}{4}}}$,

$$\begin{aligned}(\partial P)\left(\frac{E_{i-1}\sqrt{E_{i-1}}}{\sqrt{\lambda}}\right) =&V\mathcal{C}'(E_i)\left(\frac{\sqrt{2}\sqrt{E_{i-1}}}{\lambda^{\frac{3}{4}}V^{\frac{1}{4}}} - \frac{5\lambda^{\frac{1}{4}}}{18\sqrt{2}E_{i-1}^{\frac{3}{2}}V^{\frac{5}{4}}} - \frac{455\lambda^{\frac{5}{4}}}{1296\sqrt{2}E_{i-1}^{\frac{7}{2}}V^{\frac{9}{4}}}\right)\\ &+\mathcal{O}(\lambda^{\frac{9}{4}}).\end{aligned} \tag{4.89}$$

Next, we consider $\partial^2 P(z)$. $\partial^2 P(z) = V^2 \int_{-\infty}^{\infty} dx\, x^2 \exp\left(-V\frac{\lambda}{4}x^4 + Vxz\right)$ is transformed similarly into

$$\left(\partial^2 P\right)\left(\frac{z\lambda^{\frac{1}{4}}}{V^{\frac{3}{4}}\sqrt{2}}\right) =\frac{V^2}{\left(V\frac{\lambda}{4}\right)^{\frac{3}{4}}}\int_{\infty}^{\infty} dk\, k^2 \exp\left(-k^4 + zk\right). \tag{4.90}$$

In the neighborhood of $k = \frac{z^{\frac{1}{3}}}{2^{\frac{2}{3}}}$, (4.90) can be evaluated as

$$\begin{aligned}\left(\partial^2 P\right)\left(\frac{z\lambda^{\frac{1}{4}}}{V^{\frac{3}{4}}\sqrt{2}}\right) =&\frac{V^2}{\left(V\frac{\lambda}{4}\right)^{\frac{3}{4}}}\int_{-\infty}^{\infty} d\xi\left(\frac{z^{\frac{1}{3}}}{2^{\frac{2}{3}}}+\xi\right)^2 \exp\left(-\left(\frac{z^{\frac{1}{3}}}{2^{\frac{2}{3}}}+\xi\right)^4 + z\left(\frac{z^{\frac{1}{3}}}{2^{\frac{2}{3}}}+\xi\right)\right)\\ =&V^2\mathcal{C}(z)\left(\frac{2^{\frac{1}{6}}z^{\frac{2}{3}}}{\lambda^{\frac{3}{4}}V^{\frac{3}{4}}} - \frac{2^{\frac{5}{6}}}{3\lambda^{\frac{3}{4}}V^{\frac{3}{4}}z^{\frac{2}{3}}} + \frac{7}{18\times 2^{\frac{1}{6}}\lambda^{\frac{3}{4}}V^{\frac{3}{4}}z^{\frac{2}{3}}}\right.\\ &\left.-\frac{35\sqrt{2}}{27\lambda^{\frac{3}{4}}V^{\frac{3}{4}}z^2} + \frac{1705}{648\sqrt{2}\lambda^{\frac{3}{4}}V^{\frac{3}{4}}z^2}\right) + \mathcal{O}\left(\frac{1}{z^{\frac{10}{3}}}\right).\end{aligned} \tag{4.91}$$

For the case $z_i = \frac{\sqrt{2}V^{\frac{3}{4}}E_{i-1}^{\frac{3}{2}}}{\lambda^{\frac{3}{4}}}$,

$$\begin{aligned}\left(\partial^2 P\right)\left(\frac{E_{i-1}\sqrt{E_{i-1}}}{\sqrt{\lambda}}\right) =&V^2\mathcal{C}'(E_i)\left(\frac{\sqrt{2}E_{i-1}}{\lambda^{\frac{5}{4}}V^{\frac{1}{4}}} - \frac{5}{18\sqrt{2}\lambda^{\frac{1}{4}}E_{i-1}V^{\frac{5}{4}}} + \frac{25\lambda^{\frac{3}{4}}}{1296\sqrt{2}E_{i-1}^{3}V^{\frac{9}{4}}}\right)\\ &+\mathcal{O}(\lambda^{\frac{7}{4}}).\end{aligned} \tag{4.92}$$

We use these approximate quantities in the following subsections.

4.5.1. Approximation of 1-Point Function $G_{|1|}$ by Saddle Point Method ($N = 1$)

We consider the 1-point function $G_{|1|}$ in the case of (4.71) in $N = 1$.

$$G_{|1|} = -\frac{\sqrt{E_0}}{\sqrt{\lambda}} + \frac{1}{V}\frac{\partial_1 P_1(z_1)}{P_1(z_1)}, \tag{4.93}$$

where $z_1 = \frac{\sqrt{2}V^{\frac{3}{4}}E_0^{\frac{3}{2}}}{\lambda^{\frac{3}{4}}}$, and $z_2 = \frac{\sqrt{2}V^{\frac{3}{4}}E_1^{\frac{3}{2}}}{\lambda^{\frac{3}{4}}}$. Using (4.86) and (4.89), we approximate 1-Point Function $G_{|1|}$ ($N = 1$) as

$$\begin{aligned}
G_{|1|} =& -\frac{\sqrt{E_0}}{\sqrt{\lambda}} + \frac{\sqrt{E_0}}{\sqrt{\lambda}} - \frac{\sqrt{\lambda}}{12VE_0\sqrt{E_0}} - \frac{\sqrt{\lambda}}{3VE_0\sqrt{E_0}} + \frac{5\sqrt{\lambda}}{18VE_0\sqrt{E_0}} + \frac{35\lambda\sqrt{\lambda}}{54 \times 2^4 \times E_0^{\frac{7}{2}}V^2} \\
&+ \frac{35\lambda\sqrt{\lambda}}{27 \times 2^2 \times E_0^{\frac{7}{2}}V^2} - \frac{35\lambda\sqrt{\lambda}}{V^2 \times 2^2 \times 18 \times E_0^{\frac{7}{2}}} - \frac{35\lambda\sqrt{\lambda}}{V^2 \times 54 \times E_0^{\frac{7}{2}}} + \frac{11 \times 35\lambda\sqrt{\lambda}}{V^2 \times 2^2 \times E_0^{\frac{7}{2}} \times 2 \times 3^4} \\
&+ \frac{\sqrt{E_0}}{\sqrt{\lambda}}\left\{\frac{\lambda}{12E_0^2V} - \frac{5\lambda}{18E_0^2V} - \frac{35\lambda^2}{V^2 \times 216 \times 4E_0^4} + \frac{35\lambda^2}{72V^2E_0^4} - \frac{11 \times 35\lambda^2}{V^2 \times 8 \times 3^4E_0^4}\right\} \\
&- \frac{\sqrt{\lambda}}{12VE_0\sqrt{E_0}}\left\{\frac{\lambda}{12E_0^2V} - \frac{5\lambda}{18E_0^2V}\right\} - \frac{\sqrt{\lambda}}{3VE_0\sqrt{E_0}}\left\{\frac{\lambda}{12E_0^2V} - \frac{5\lambda}{18E_0^2V}\right\} \\
&+ \frac{5\sqrt{\lambda}}{18VE_0\sqrt{E_0}}\left\{\frac{\lambda}{12E_0^2V} - \frac{5\lambda}{18VE_0^2}\right\} + \frac{\sqrt{E_0}}{\sqrt{\lambda}}\left\{\frac{\lambda^2}{144E_0^4V^2} + \frac{25\lambda^2}{18^2E_0^4V^2} - \frac{10\lambda^2}{12 \times 18E_0^4V^2}\right\} \\
&+ \mathcal{O}(\lambda^2\sqrt{\lambda}) \\
=& -\frac{\sqrt{\lambda}}{3VE_0\sqrt{E_0}} - \frac{7\lambda\sqrt{\lambda}}{27V^2E_0^{\frac{7}{2}}} + \mathcal{O}(\lambda^2\sqrt{\lambda}).
\end{aligned} \tag{4.94}$$

This is consistent with the calculation of 1-point function $G_{|1|}$ ($N = 1$) using perturbative expansion i.e. (4.26) = (4.94).

4.5.2. Approximation of 1-Point Function $G_{|1|}$ by Saddle Point Method ($N = 2$)

We consider the 1-point function $G_{|1|}$ in the case of (4.71) in $N = 2$.

$$G_{|1|} = -\frac{\sqrt{E_0}}{\sqrt{\lambda}} + \frac{1}{V}\partial_1 \log P_2(z_1, z_2) - \frac{1}{V}\frac{\sqrt{\lambda}}{E_0\sqrt{E_0} - E_1\sqrt{E_1}}, \tag{4.95}$$

where $z_1 = \frac{\sqrt{2}V^{\frac{3}{4}}E_0^{\frac{3}{2}}}{\lambda^{\frac{3}{4}}}$, and $z_2 = \frac{\sqrt{2}V^{\frac{3}{4}}E_1^{\frac{3}{2}}}{\lambda^{\frac{3}{4}}}$. Using (4.86), (4.89) and (4.92), we approximate 1-Point Function $G_{|1|}$ $(N = 2)$ as

$$
\begin{aligned}
G_{|1|} &= -\frac{\sqrt{E_0}}{\sqrt{\lambda}} + \frac{1}{V}\frac{\partial_1 P_2(z_1, z_2)}{P_2(z_1, z_2)} - \frac{1}{V}\frac{\sqrt{\lambda}}{E_0\sqrt{E_0} - E_1\sqrt{E_1}} \\
&= -\frac{\sqrt{\lambda}}{V}\frac{\sqrt{E_0}}{3E_0^2} - \frac{\sqrt{\lambda}}{V}\frac{\sqrt{E_0}}{3E_0}\frac{2}{E_0 + E_1 + \sqrt{E_0}\sqrt{E_1}} - \frac{\sqrt{\lambda}}{V}\frac{\sqrt{E_1}}{3E_0}\frac{1}{E_0 + E_1 + \sqrt{E_0}\sqrt{E_1}} \\
&\quad + \mathcal{O}(\lambda\sqrt{\lambda}).
\end{aligned}
\tag{4.96}
$$

This is consistent with the calculation of the 1-point function $G_{|1|}$ $(N = 2)$ using perturbative expansion as (4.29) = (4.96).

4.5.3. Approximation of 2-Point Function $G_{|21|}$ by Saddle Point Method $(N = 2)$

We consider the 2-point function $G_{|21|}$ in the case of (4.78) in $N = 2$.

$$
\begin{aligned}
G_{|21|} = &-\frac{1}{V}\frac{\sqrt{\lambda}}{E_1\sqrt{E_1} - E_0\sqrt{E_0}}\frac{\partial_1 P_2(z_1, z_2)}{P_2(z_1, z_2)} + \frac{1}{V}\frac{\sqrt{\lambda}}{E_1\sqrt{E_1} - E_0\sqrt{E_0}}\frac{\partial_2 P_2(z_1, z_2)}{P_2(z_1, z_2)} \\
&- \frac{2}{V}\frac{\lambda}{(E_1\sqrt{E_1} - E_0\sqrt{E_0})^2},
\end{aligned}
\tag{4.97}
$$

where $z_1 = \frac{\sqrt{2}V^{\frac{3}{4}}E_0^{\frac{3}{2}}}{\lambda^{\frac{3}{4}}}$, and $z_2 = \frac{\sqrt{2}V^{\frac{3}{4}}E_1^{\frac{3}{2}}}{\lambda^{\frac{3}{4}}}$. Using (4.86), (4.89) and (4.92), we approximate 2-Point Function $G_{|21|}$ $(N = 2)$ as

$$
\begin{aligned}
G_{|21|} =& \frac{1}{E_0 + E_1 + \sqrt{E_0}\sqrt{E_1}} + \frac{\sqrt{E_1}\lambda}{3V(\sqrt{E_1} - \sqrt{E_0})E_0\sqrt{E_0}(E_1\sqrt{E_1} - E_0\sqrt{E_0})} \\
&+ \frac{\sqrt{E_0}\lambda}{3V(\sqrt{E_1} - \sqrt{E_0})E_1\sqrt{E_1}(E_1\sqrt{E_1} - E_0\sqrt{E_0})} - \frac{2}{V}\frac{\lambda}{(E_1\sqrt{E_1} - E_0\sqrt{E_0})^2} \\
&+ \mathcal{O}(\lambda^2) \\
=& \frac{1}{E_0 + E_1 + \sqrt{E_0}\sqrt{E_1}} + \frac{\lambda}{3VE_0(E_0 + E_1 + \sqrt{E_0}\sqrt{E_1})^2} \\
&+ \frac{\lambda}{3VE_1(E_0 + E_1 + \sqrt{E_0}\sqrt{E_1})^2} + \frac{8\sqrt{E_0}\sqrt{E_1}\lambda}{3VE_1(E_0 + E_1 + \sqrt{E_0}\sqrt{E_1})^3} \\
&+ \frac{8\sqrt{E_0}\sqrt{E_1}\lambda}{3VE_0(E_0 + E_1 + \sqrt{E_0}\sqrt{E_1})^3} + \frac{\sqrt{E_0}\sqrt{E_1}\lambda}{3VE_1^2(E_0 + E_1 + \sqrt{E_0}\sqrt{E_1})^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\sqrt{E_0}\sqrt{E_1}\lambda}{3VE_0^2(E_0+E_1+\sqrt{E_0}\sqrt{E_1})^2}+\frac{2\lambda E_0}{3VE_1(E_0+E_1+\sqrt{E_0}\sqrt{E_1})^3}\\
&+\frac{2\lambda E_1}{3VE_0(E_0+E_1+\sqrt{E_0}\sqrt{E_1})^3}+\frac{10\lambda}{3V(E_0+E_1+\sqrt{E_0}\sqrt{E_1})^3}+\mathcal{O}(\lambda^2).
\end{aligned}
\tag{4.98}
$$

This is consistent with the result of the 2-point function $G_{|21|}$ $(N=2)$ using perturbative expansion i.e. (4.35) = (4.98).

4.5.4. Approximation of 2-Point Function $G_{|2|1|}$ by Saddle Point Method $(N=2)$

We consider the 2-point function $G_{|2|1|}$ in the case of (4.39) in $N=2$.

$$
\begin{aligned}
G_{|2|1|}=&-\frac{\partial_1 P_2(z_1,z_2)}{P_2(z_1,z_2)}\frac{\partial_2 P_2(z_1,z_2)}{P_2(z_1,z_2)}+\frac{\partial_1\partial_2 P_2(z_1,z_2)}{P_2(z_1,z_2)}\\
&-\frac{\lambda}{E_0^3+E_1^3-2E_0\sqrt{E_0}E_1\sqrt{E_1}},
\end{aligned}
\tag{4.99}
$$

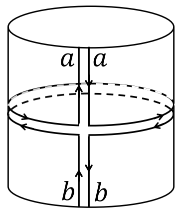
where $z_1=\dfrac{\sqrt{2}V^{\frac{3}{4}}E_0^{\frac{3}{2}}}{\lambda^{\frac{3}{4}}}$, and $z_2=\dfrac{\sqrt{2}V^{\frac{3}{4}}E_1^{\frac{3}{2}}}{\lambda^{\frac{3}{4}}}$. Using (4.86), (4.89) and (4.92), we approximate 2-Point Function $G_{|21|}$ $(N=2)$ as

$$
\begin{aligned}
G_{|2|1|}=&\frac{4\sqrt{E_0}\sqrt{E_1}\lambda}{9E_0E_1(E_0+E_1+\sqrt{E_0}\sqrt{E_1})^2}+\frac{\lambda}{9E_0(E_0+E_1+\sqrt{E_0}\sqrt{E_1})^2}\\
&+\frac{\lambda}{9E_1(E_0+E_1+\sqrt{E_0}\sqrt{E_1})^2}+\mathcal{O}(\lambda^2).
\end{aligned}
\tag{4.100}
$$

Then, we verified the consistency between the exact solution and the perturbative calculation for $G_{|2|1|}$ by (4.39) = (4.100).

4.6. Contributions from Non-trivial Topology Surfaces

In this section, we make remarks about contributions from Feynman diagrams corresponding to nonplanar or higher genus surfaces. As we saw in Section 2, perturbative expansions of Φ^3-Φ^4 hybrid matrix model are given by the sum over not only planar but also non-planar Feynman diagrams. Nonplanar graphs are diagrams that cannot be drawn on a plane. For example, a nonplanar graph in Figure 4.3 appeared when the 2-point function $G_{|a|b|}$ was calculated.

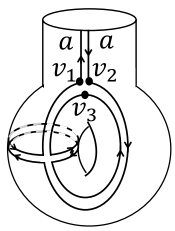


$$= -\frac{\lambda}{36V^2E_{a-1}E_{b-1}(E_{a-1}+E_{b-1}+\sqrt{E_{a-1}}\sqrt{E_{b-1}})}$$

Figure 4.3: The surface of 2-point function $G_{|a|b|}$ with two boundaries

Calculations in this paper are carried out for finite N, so nonplanar Feynman diagrams are taken into account.

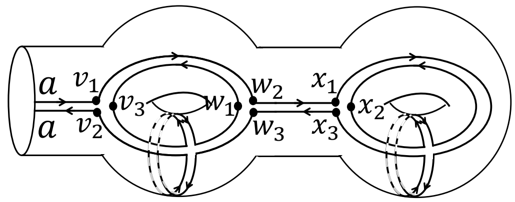
Next, let us consider contributions from higher genus surfaces. Specifically, we consider "genus expansion" in one-point function $G_{|a|}$. First, we consider the contribution to the one-point function $G_{|a|}$ from a surface of one genus. (See Figure 4.4.)



$$= +\frac{\lambda\sqrt{\lambda}}{324V^2E_{a-1}^4}\sum_{v\in\{\{a,a,a\}\}}\sqrt{E_{v-1}}$$

Figure 4.4: A surface for 1-point function $G_{|a|}$ with one boundary and one genus

This diagram is constructed by using $\Sigma = 0$, $k_3 = 1$, $k_4 = 1$, and $\mathscr{N} = 1$, where we use the notation in Subsection 2.2. The contribution from the Feynman diagram has $V^{k_3+k_4-\frac{3k_3+4k_4+\mathscr{N}}{2}}$ as we saw in Subsection 2.2. From this formula, $V^{k_3+k_4-\frac{3k_3+4k_4+\mathscr{N}}{2}} = V^{1+1-\frac{3+4+1}{2}} = V^{-2}$. Also $V^{k_3+k_4-\frac{3k_3+4k_4+\mathscr{N}}{2}} = V^{\chi-\mathscr{N}-\Sigma} = V^{2-2g-B-\mathscr{N}-\Sigma}$. From this, $V^{2-2g-1-1-0} = V^{-2}$. It is consistent with Figure 4.4 that the genus of the surface is $g = 1$. Second, we consider the contribution for $G_{|a|}$ from a two genus surface. (See Figure 4.5.)



$$= -\frac{\lambda^3\sqrt{\lambda}}{16\times 3^9 E_{a-1}^9 V^4}\sum_{v,w,x\in\{\{a,a,a\}\}}\sqrt{E_{v-1}}\sqrt{E_{w-1}}\sqrt{E_{x-1}}$$

Figure 4.5: The surface for 1-point function $G_{|a|}$ with one boundary and two genus

From this diagram, we find that $\Sigma = 0$, $k_3 = 3$, $k_4 = 2$, and $\mathscr{N} = 1$. From them, $V^{k_3+k_4-\frac{3k_3+4k_4+\mathscr{N}}{2}} = V^{3+2-\frac{9+8+1}{2}} = V^{-4}$. Also $V^{k_3+k_4-\frac{3k_3+4k_4+\mathscr{N}}{2}} = V^{\chi-\mathscr{N}-\Sigma} = V^{2-2g-B-\mathscr{N}-\Sigma}$. On the other hand, Figure 4.5 implies $V^{2-2g-1-1-0} = V^{-4}$. It is consistent with Figure 4.5 that the genus of the surface is $g = 2$. These observations show that our calculations in Section 4.4 took into account any contributions from higher genus surfaces.

In other words, if we expand (4.71),(4.78),(4.80),(4.83), and so on as (4.24), then each $G^{(g)}_{|a_1^1\dots a_{N_1}^1|\dots|a_1^B\dots a_{N_B}^B|}$ is obtained as the contribution from the fixed genus g.

Chapter 5

Conclusion and Outlook

We provide a summary of this thesis in this chapter. In this thesis, the matrix models corresponding to Grosse-Steinacker-Wulkenhaar type noncommutative scalar field theories (Grosse-Steinacker-Wulkenhaar Φ^3 matrix model and Φ^3-Φ^4 hybrid matrix model) are studied in the case where both the matrix size N and the noncommutative parameter V are finite, then various progresses are made. If the partition function $\mathcal{Z}[J]$ as the generating function of multipoint correlation functions with an external field J can be obtained in a concrete form, all the desired information as quantum field theories can be obtained. In this thesis, we sought a method to obtain the partition function $\mathcal{Z}[J]$ directly using Itzykson-Zuber integral. Even if Itzykson-Zuber integral can be applied, there is a problem that the partition function $\mathcal{Z}[J]$ can not be determined specifically for a general external field J unless all eigenvalues of a certain matrix can be obtained. There are cases where this problem can be avoided, such as when the external field J is a diagonal matrix, or when the external field J is not a diagonal matrix but is limited to a simple one, in which case the eigenvalue problem can be solved. Using this method, we have succeeded in obtaining exact multipoint correlation functions for these cases. These results, especially in the case of Grosse-Steinacker-Wulkenhaar Φ^3 model, together with the results of previous studies, lead to the arbitrary multi-point correlation functions. Φ^3-Φ^4 hybrid matrix model, inspired by the experience of obtaining the direct partition function $\mathcal{Z}[J]$ using Itzykson-Zuber integral in Φ^3 model, opens up new areas including the new model construction, perturbation theories, and other broad topics. This model is expected to develop further. In the following, we summarize the contents in more detail according to the structure of this thesis.

In Chapter 1, we explained the background of this thesis and the motives for the study. Next, we explained the organization of this thesis.

In Chapter 2, we summarized the definitions and theorems necessary to understand Chapters 3 and 4. We reviewed Grosse–Steinacker-Wulkenhaar Φ^3 model and its research by Grosse, Hock, Sako, Steinacker, and Wulkenhaar[13, 14, 15, 18, 19, 30, 31].

In Chapter 3, we found the exact solutions of correlation functions of Φ^3 finite matrix model (Grosse-Steinacker-Wulkenhaar model)[40]. In Φ^3 finite matrix model, multipoint correlation functions were expressed as $G_{|a_1^1 \ldots a_{N_1}^1|\ldots|a_1^B \ldots a_{N_B}^B|}$ defined in Chapter 2. In terms of Feynman diagram, it corresponds with sum of connected Feynman diagrams on Riemann surfaces with B-boundaries.

The indices of $G_{|a^1_1\ldots a^1_{N_1}|\ldots|a^B_1\ldots a^B_{N_B}|}$ mean that corresponding connected Feynman diagrams have B-boundaries, and i-th boundary has N_i-external lines. It is known that any $G_{|a^1_1\ldots a^1_{N_1}|\ldots|a^B_1\ldots a^B_{N_B}|}$ can be expressed using $G_{|a^1|\ldots|a^n|}$ type n-point functions as (3.51). Thus we focused on rigorous calculations of $G_{|a^1|\ldots|a^n|}$.

In Section 3.2, the integration of the off-diagonal elements of the Hermitian matrix was calculated using Harish-Chandra-Itzykson-Zuber integral[36, 51] in calculating the partition function $\mathcal{Z}[J]$. Next, the integral of the diagonal elements of the Hermitian matrix was calculated using the Airy functions as similar to [42]. In Section 3.3, 3.4, and 3.5, we used the obtained partition function $\mathcal{Z}[J]$ to calculate $G_{|a^1|\ldots|a^n|}$ type n-point functions and $G_{|ab|}$. The exact solutions of $G_{|a^1|\ldots|a^n|}$ type n-point functions were obtained by calculating the n-th derivative $\partial^n/\partial J_{a^1a^1}\cdots\partial J_{a^na^n}$ of $\log\mathcal{Z}[J]$ with the external field J as a diagonal matrix. The result of the calculations for $G_{|a^1|\ldots|a^n|}$ was described in Theorem 3.5. In the formula for $G_{|a^1|\ldots|a^n|}$ in Theorem 3.5, no integral remains. More concletely, the n-point function was determined by a function $F(S,M,\overline{M})$ whose variables are $S\subset B$, M, and $\overline{M}$ $\left(B\backslash S=M\sqcup\overline{M}\right)$ as formula (3.53), where B is an element of a partition of $\{1,\cdots,n\}$. Since the algorithm for finding the exact solutions of $G_{|a^1|\ldots|a^n|}$ type n-point functions is explicitly determined in the formula of Theorem 3.5, the exact solutions can be obtained automatically. Indeed, the calculations for $G_{|a|b|}$ and $G_{|a|b|c|}$ were carried out in Section 3.5 and Section 3.6, respectively. Since a general $(N_1+\cdots+N_B)$-point function $G_{|a^1_1\ldots a^1_{N_1}|\ldots|a^B_1\ldots a^B_{N_B}|}$ is expressed by using $G_{|a^1|\ldots|a^B|}$ type B-point functions, we can obtain all the exact solutions of the Φ^3 finite matrix model.

In Chapter 4, we made Φ^3-Φ^4 hybrid matrix model and studied its basic properties. We constructed Feynman rules, and calculated several multipoint correlation functions perturbatively. We also solved the partition function $\mathcal{Z}[J]$ rigorously to obtain the exact solutions of several multipoint correlation functions. In Subsection 4.2.1, we constructed Feynman rules of Φ^3-Φ^4 hybrid matrix model in a well-known way in terms of quantum field theories[41]. The Φ^3 interaction caused unconventional Feynman rules because of the insertion of M as $\mathrm{tr}M\Phi^3$. Therefore, we discussed it carefully without omission. Particular attention was paid to the connected $\displaystyle\sum_{i=1}^{B}N_i$-point function $G_{|a^1_{N_1}\cdots a^1_{N_1}|\cdots|a^B_1\cdots a^B_{N_B}|}$. Its details were defined and discussed in Subsection 4.2.2, but this function can be interpreted geometrically and corresponds to the sum over all Feynman diagrams (ribbon graphs) drawn in certain rules on Riemann surfaces with B-boundaries (punctures). Each $|a^i_1\cdots a^i_{N_i}|$ corresponds to N_i external lines coming from the i-th boundary (puncture) in the Feynman diagrams (ribbon graphs). First, using the Feynman rules of Φ^3-Φ^4 hybrid matrix model, perturbative expansions for one-point function $G_{|1|}$, two-point function $G_{|21|}$, and two-point function $G_{|2|1|}$ were computed by drawing Feynman diagrams for matrix size $N=2$ case as pedagogical instructions to understand the way of calculations. Next, we performed perturbative calculations for $G_{|a|}$,$G_{|ab|}$, and $G_{|a|b|}$ in the case that the matrix size is any N.

On the other hand, the calculation of the partition function $\mathcal{Z}[J]$ in Φ^3-Φ^4 hybrid matrix model was carried out rigorously. For the computation of the partition function $\mathcal{Z}[J]$, the integral of the off-diagonal elements of the Hermitian matrix was computed using Itzykson-Zuber integral[36, 51].

In contrast, the integral of the diagonal elements of the Hermite matrix was obtained by using a function $P(z)$ that is similar to the Pearcey integral. We then used the exact calculated partition function $\mathcal{Z}[J]$ to compute the exact solutions for $G_{|a|}$, $G_{|ab|}$, $G_{|a|b|}$, and n-point function $G_{|a^1|a^2|\cdots|a^n|}$ for any N matrix size. We verified that the final results of the perturbative expansions for $N = 2$ are in agreement with the saddle point approximation using the results of the exact solutions in $G_{|1|}$, $G_{|21|}$, and $G_{|2|1|}$ by setting $N = 2$. Finally, we made remarks about contributions from Feynman diagrams of Φ^3-Φ^4 hybrid matrix model corresponding to nonplanar or higher genus surfaces.

Finally, we discuss the prospects for future research. As discussed in Chapter 2, there is a kind of noncommutative space called Moyal spaces, in which the fields are discretized and the matrix appears. Therefore, the scalar ϕ^3 theories on Moyal spaces are replaced by the matrix model. The matrix model takes the matrix size N to infinity and the noncommutative parameter V is usually finite. One of a goal is to find exact solutions of multipoint correlation functions in the scalar ϕ^3 theories on Moyal spaces. In this thesis, the exact solutions of the multipoint correlation functions in Grosse-Steinacker-Wulkenhaar Φ^3 model were obtained when the matrix size N is finite and the noncommutative parameter V is also finite. The exact solutions of the multi-point correlation functions in Grosse-Steinacker-Wulkenhaar Φ^3 model by taking the limit of matrix size N to infinity but keeping finite V corresponds to the exact solutions of the multi-point correlation functions in the scalar ϕ^3 theories on Moyal spaces. We need to compute the exact solutions in this case, but so far it seems to be difficult. It is not naively expected from the results of this thesis. In the future, we plan to develop new methods to obtain such large N limit.

It is also necessary to study the renormalizability in large N limit of Φ^3-Φ^4 hybrid matrix model. The term 'renormalization' here means that the divergences that appear in the perturbative expansions are pushed into the redefinition of each parameter that appears in the theory. The process of removing the divergences is brought about by adding counter terms to the Lagrangian. First, we check if the Feynman diagrams of the one-loop that appears in the perturbative expansion of Φ^3-Φ^4 hybrid matrix model can be renormalized. We investigate whether one-loop calculations can produce counter terms without divergence. Next, we compute in the case of the Feynman diagrams of the higher loop and check if it can be renormalized. The possibility of eliminating infinities arising from perturbing multipoint correlation functions of Φ^3-Φ^4 hybrid matrix model should be investigated. By introducing finite numbers of counter terms, we aim to ascertain the potential for renormalizability in Φ^3-Φ^4 hybrid matrix model.

As described above, many issues remain in the analysis of scalar fields on noncommutative spaces, and further progress is desirable.

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