Dissertation

Shapes of translators for the mean curvature flow and the inverse mean curvature flow

（平均曲率流と逆平均曲率流のトランスレイターの形状）

March, 2024

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Acknowledgement

First of all, I would like to express my sincere gratitude to my supervisor, Professor Naoyuki Koike for helpful support and valuable comment during my master studies and doctoral studies.

I am greatful to Professor Masahito Ohta, Professor Keiichi Kato, Professor Makiko Tanaka, and Professor Tomomi Yokota for examining this thesis.

Finally, I would like to express my sincere gratitude to my family and my friends for their support and encouragement.

Tomoki Fujii

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Chapter 1

Graphical translators for the mean curvature flow and isoparametric functions

1.1 Introduction

This chapter is based on [7].

Let (N, g) be a n-dimensional Riemannian manifold and $u : M \to \mathbb{R}$ be a smooth function on a domain $M \subset N$. The *graph embedding* f for u is defined as the embedding of M into the product Riemannian manifold $N \times \mathbb{R}$ given by $f(x) := (x, u(x))$ $(x \in M)$. For the simplicity, denote by Γ the graph $f(M)$ of u. If a C^∞-family $\{f_t\}_{t\in I}$ of C^∞-immersions of M into $N \times \mathbb{R}$ (I is an open interval including 0) satisfies

$$\begin{cases} \left(\dfrac{\partial f_t}{\partial t}\right)^{\perp_{f_t}} = H_t \\ f_0 = f, \end{cases} \tag{1.1.1}$$

the family $\{M_t\}_{t\in I}$ of the images $M_t := f_t(M)$ is called the *mean curvature flow starting from* Γ, where H_t is the mean curvature vector field of f_t and $(\bullet)^{\perp_{f_t}}$ is the normal component of $(\bullet)$ with respect to f_t. According to Hungerbühler and Smoczyk [11], we define a soliton for the mean curvature flow as follows. Let $\boldsymbol{X}$ be a Killing vector field on $N \times \mathbb{R}$ and $\{\phi_t\}_{t\in\mathbb{R}}$ be the one-parameter transformation group associated to $\boldsymbol{X}$. If $\{f_t\}_{t\in I}$ satisfies

$$\left(\frac{\partial(\phi_t^{-1} \circ f_t)}{\partial t}\right)^{\perp_{(\phi_t^{-1}\circ f_t)}} = 0, \tag{1.1.2}$$

then Γ is called a *soliton for the mean curvature flow with respect to* $\boldsymbol{X}$. In the sequel, we call such a soliton a $\boldsymbol{X}$*-soliton* simply. In particular, when $X = (0, 1) \in T(N \times \mathbb{R})(= TN \oplus T\mathbb{R})$, we call the $\boldsymbol{X}$-soliton a *translator*.

The translator for $N = \mathbb{R}^n$ has been studied by several authors. When $n = 2$, Shahriyari [23] proved non-existence of complete translating graphs over bounded connected domains of $\mathbb{R}^2$ with smooth boundary. Also, she showed that if a complete translator which is a graph over a domain in $\mathbb{R}^2$, then the domain is a strip, or a halfspace, or $\mathbb{R}^2$. Further, Hoffman, Ilmanen, Martín and White [10] showed that no complete translator is the graph of a function over a halfspace in $\mathbb{R}^2$. Minimal submanifolds is special case of translators. Therefore, translators can be regarded as the generalization of minimal submanifolds. Hence, Bao and Shi [2] showed a Bernstein-type theorem for complete translators in case of codimension one. Also, by Kunikawa [15], a Bernstein-type theorem for complete translators with flat normal bundle in case of higher codimension was shown. For the function $u : \mathbb{R}^n \to \mathbb{R}$ defined by $u(x_1, \ldots, x_n) = -\log \cos x_n$, $(x_1, \ldots, x_n) \in \mathbb{R}^n$, the mean curvature flow starting from the graph of u is the translator. When $n = 1$, the curve of u is called a *grim reaper* (Figure 1.1.1). When $n \geq 2$, the graph of u is called a *grim hyperplane* (Figure 1.1.2). Further, Martin, Savas-Halilaj, and Smoczyk [17] gave the characterization of the grim hyperplane. Clutterbuck, Schnürer and Schulze [4] showed the existence of the complete rotationally symmetric graphical translator which is called bowl soliton (Altschuler and Wu [1] had already showed the existence in the case $n = 2$) and a certain type of stability for the bowl soliton. Further, they showed that bowl solitons have the following asymptotic expansion as r approaches infinity:

$$\frac{r^2}{2(n-1)} - \log r + O(r^{-1}),$$

where r is the distance function in $\mathbb{R}^n$ because u is the composition of r and the solution of a certain ordinary differential equation. Wang [26] showd that when $n = 2$, the bowl soliton is the only convex translator which is an entire graph. Further, Spruck and Xiao [24] showed that the bowl soliton with $n = 2$ is the only complete translator which is an entire graph.

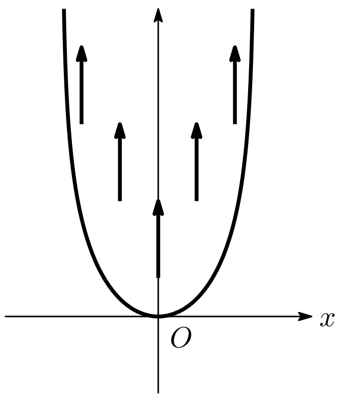

Figure 1.1.1: The grim reaper

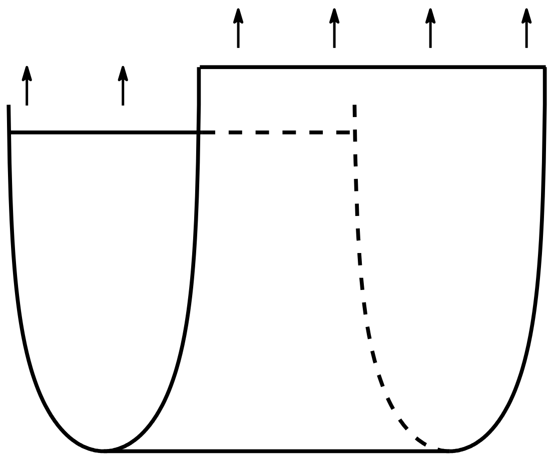
Figure 1.1.2: The grim hyperplane

In this chapter, we consider the case where N is the n-dimensional unit sphere $\mathbb{S}^n$ and u is a composition of an isoparametric function on $\mathbb{S}^n$ and some function. The level sets of the isoparametric functions give compact isoparametric hypersurfaces of $\mathbb{S}^n$. Münzner [19] showed that the number k of distinct principal curvatures of compact isoparametric hypersurfaces of $\mathbb{S}^n$ is 1, 2, 3, 4 or 6 by a topological method. In cases $k = 1, 2, 3$, Cartan [3] classified the isoparametric hypersurfaces. The hypersurfaces are $\mathbb{S}^{n-1} \subset \mathbb{S}^n$ in case $k = 1$, $\mathbb{S}^k \times \mathbb{S}^{n-k-1} \subset \mathbb{S}^n$ in case $k = 2$ and the tubes over the Veronese surfaces $\mathbb{R}P^2 \subset \mathbb{S}^4$, $\mathbb{C}P^2 \subset \mathbb{S}^7$, $\mathbb{Q}P^2 \subset \mathbb{S}^{13}$, $\mathbb{O}P^2 \subset \mathbb{S}^{25}$ (i.e., the principal orbits of the isotropy representations of the rank two symmetric spaces $SU(3)/SO(3)$, $(SU(3) \times SU(3))/SU(3)$, $SU(6)/Sp(3)$, E_6/F_4) in case $k = 3$. These hypersurfaces are homogeneous. In case $k = 6$, the hypersurfaces are homogeneous by the result of Dorfmeister and Neher [5] and Miyaoka [18]. The hypersurfaces are the principal orbits of the isotropy representations of $(G_2 \times G_2)/G_2$, $G_2/SO(4)$. In case $k = 4$, Ozeki and

Takeuchi [21, 22] found that non-homogeneous isoparametric hypersurfaces are constructed as the regular level sets of the restrictions of the Cartan-Münzner polynomial functions to the sphere. In this chapter, we obtain the following result.

Theorem 1.1.1. *Let r be an isoparametric function on $\mathbb{S}^n$ $(n \geq 2)$ and V be a C^∞-function on an interval $J \subset r(\mathbb{S}^n)$. If the mean curvature flow starting from the graph of the function $u = (V \circ r)|_{r^{-1}(J)}$ is a translator, the shape of the graph of V is like one of those defined in* Figures 1.1.3–1.1.9. *The real number $R \in (-1, 1)$ in* Figures 1.1.3–1.1.9 *is given by*

$$R := \begin{cases} 0 & (k = 1, 3, 6) \\ -1 + \dfrac{km}{n-1} & (k = 2, 4), \end{cases}$$

where k is the number of distinct principal curvatures of the compact isoparametric hypersurface defined by the regular level set for r and m is the multiplicity of the smallest principal curvature of the isoparametric hypersurface.

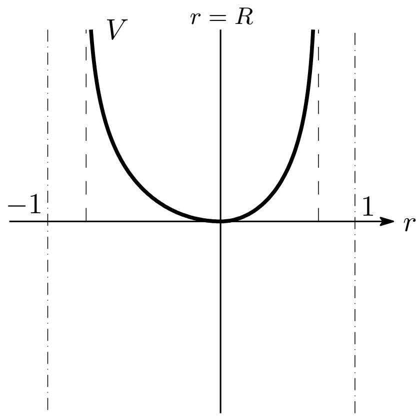

Figure 1.1.3: The graph of V (Type I)

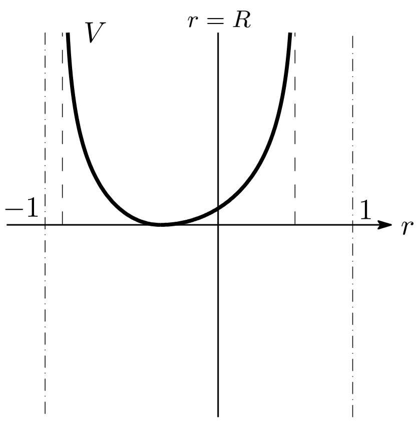

Figure 1.1.4: The graph of V (Type II)

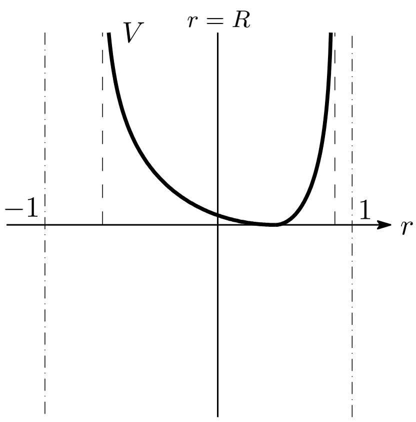

Figure 1.1.5: The graph of V (Type III)

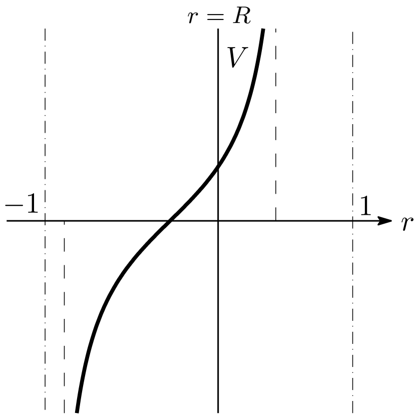

Figure 1.1.6: The graph of V (Type IV)

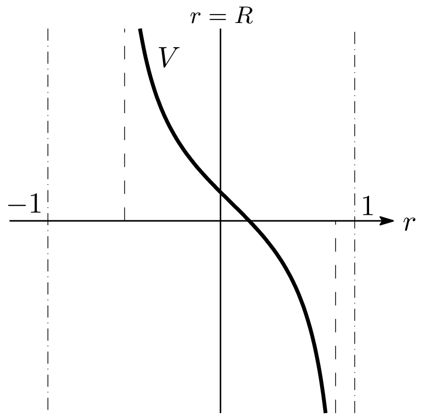

Figure 1.1.7: The graph of V (Type V)

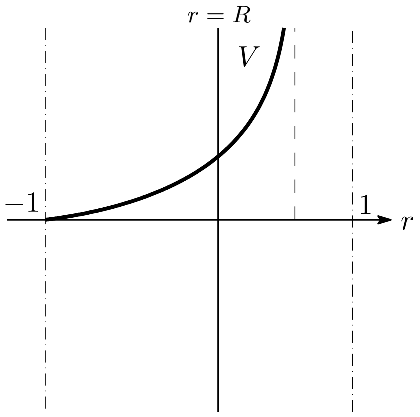

Figure 1.1.8: The graph of V (Type VI)

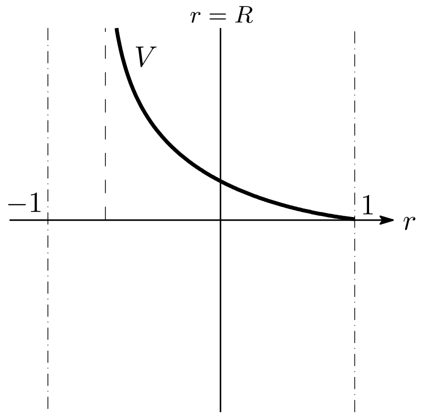

Figure 1.1.9: The graph of V (Type VII)

The function $u = (V \circ r)|_{r^{-1}(J)}$ in Theorem 1.1.1 is constant on the level set of r and its behavior on the normal direction for the level set of r is a little understood from the behavior of V in Figures 1.1.3–1.1.9. In the last section, we investigate the domain of the function u in Theorem 1.1.1.

1.2 Basic facts

Let g be a Riemannian metric of an n-dimensional Riemannian manifold N and $u : M \to \mathbb{R}$ be a function on a domain $M \subset N$. Define the immersion f of M into the product Riemannian manifold $N \times \mathbb{R}$ by $f(x) = (x, u(x))$, $x \in M$. Denote the graph of u by Γ and the mean curvature vector field of f by H. Further, we assume that X is a Killing vector field on $N \times \mathbb{R}$ and $\{\phi_t\}_{t\in\mathbb{R}}$ is the one-parameter transformation associated to X on $N \times \mathbb{R}$. Then, we have the following lemma about the soliton of the mean curvature flow.

Lemma 1.2.1. *If the Γ is X-soliton, f satisfies*

$$(X \circ f)^{\perp_f} = H. \tag{1.2.1}$$

Conversely, if f satisfies (1.2.1)*, the family of the images $\{M_t\}_{t\in\mathbb{R}}$ defined by $f_t = \phi_t \circ f$ and $M_t = f_t(M)$ is the mean curvature flow and Γ is the X-soliton.*

Proof. According to Hungerbühler and Smoczyk [11], we find the first half of the lemma. For the second half of the lemma, since ϕ_t's are isometries and f satisfies (1.2.1), we find that $f_t = \phi_t \circ f$ satisfies

$$\begin{aligned}\left(\frac{\partial f_t}{\partial t}\right)^{\perp_{f_t}} - H_t &= d\phi_t((X \circ f)^{\perp_f} - H)\\ &= 0,\end{aligned}$$

and $\{f_t\}_{t\in\mathbb{R}}$ satisfies (1.1.1). Therefore, $\{M_t\}_{t\in\mathbb{R}}$ is the mean curvature flow. Further, by $\phi_t^{-1} \circ f_t = f$, it turns out that f_t satisfies (1.1.2). So, Γ is the X-soliton. □

Let ∇ and div be the gradient and divergence with respect to g, respectively. For Lemma 1.2.1, considering the case where an X-soliton is a translator, the following lemma is derived.

Lemma 1.2.2. *If the graph Γ of u is a translator, u satisfies*

$$\sqrt{1+\|\nabla u\|^2}\ \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+\|\nabla u\|^2}}\right) = 1. \tag{1.2.2}$$

Conversely, if u satisfies (1.2.2)*, the family of the images $\{M_t\}_{t\in\mathbb{R}}$ defined by $f_t(x) = (x, u(x)+t)$, $x \in M$ and $M_t = f_t(M)$ is the mean curvature flow and Γ is a translator.*

Proof. Let $(x_1, \ldots, x_n, s)$ be local coordinates of $N \times \mathbb{R}$. Define the Killing vector $X = (0, 1) \in T(N \times \mathbb{R}) = TN \oplus T\mathbb{R}$. By $f(x) = (x, u(x))$, $x \in M$ and $X = \frac{\partial}{\partial s}$, we find

$$\begin{aligned}(X \circ f)^{\perp_f} &= \frac{\partial}{\partial s} - \frac{1}{1+\|\nabla u\|^2} df(\nabla u),\\ H &= \sqrt{1+\|\nabla u\|^2}\ \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+\|\nabla u\|^2}}\right)\left(\frac{\partial}{\partial s} - \frac{1}{1+\|\nabla u\|^2} df(\nabla u)\right).\end{aligned}$$

Therefore, we obtain that (1.2.1) and (1.2.2) are equivalent in this case. □

Next, we consider the case where u is a composition of an isoparametric function and some function. Let Δ be the Laplacian with respect to g. A non-constant C^∞-function $r : N \to \mathbb{R}$ is called an isoparametric function if there exist C^∞-functions α, β such that

$$\begin{cases}\|\nabla r\|^2 = \alpha \circ r\\ \Delta r = \beta \circ r.\end{cases}$$

Further, the regular level set of r is called an isoparametric hypersurface.

In case where N is the n-dimensional unit sphere $\mathbb{S}^n$, Münzner [19] showed the following theorem for an isoparametric function on $\mathbb{S}^n$.

Theorem 1.2.3. (Münzner [19]) *(i) An isoparametric function r on $\mathbb{S}^n$ is a restriction to $\mathbb{S}^n$ of a homogeneous polynomial $h : \mathbb{R}^{n+1} \to \mathbb{R}$ which satisfies*

$$\begin{cases}|(\nabla^{\mathbb{R}} h)_x|^2 = k^2|x|^{2k-2}\\ (\Delta_{\mathbb{R}} h)_x = \dfrac{m_2 - m_1}{2} k^2 |x|^{k-2}\end{cases} \qquad (x \in \mathbb{R}^{n+1}), \tag{1.2.3}$$

where $|\bullet|$ is the Euclidean norm and $\nabla^{\mathbb{R}}$ and $\Delta_{\mathbb{R}}$ are the gradient and Laplacian for the Euclidean space $\mathbb{R}^n$. Here, we assume that the isoparametric hypersurface defined by the level set of r has k distinct principal curvatures $\lambda_1 > \cdots > \lambda_k$ with respective multiplicities $m_1, \ldots, m_k$.

(ii) The above natural number k is 1, 2, 3, 4 *or* 6.

Remark 1.2.4. *According to Münzner [19, 20], we find the following two facts.*

(i) If $k = 1,\ 3,\ 6$, then the mulitiplicities are equal. If $k = 2,\ 4$, then there are at most two distinct multiplicities m_1, m_2.

(ii) By (1.2.3)*, we obtain*

$$\begin{cases} \|\nabla r\|^2 = k^2(1-r^2) \\ \Delta r = \dfrac{m_2 - m_1}{2}k^2 - k(n+k-1)r. \end{cases} \tag{1.2.4}$$

From the first equation of (1.2.4)*, we find that $r(\mathbb{S}^n) = [-1, 1]$.*

For Lemma 1.2.2, considering the case where u is the composition of the isoparametric function and some function, the following lemma is derived.

Lemma 1.2.5. *Let $r : N \to \mathbb{R}$ be an isoparametric function on N. If the graph Γ of u is a translator and if there exists a C^∞-function V on $r(M)$ such that $u = (V \circ r)|_M$, the function V satisfies*

$$2\alpha V'' - \alpha(\alpha' - 2\beta)(V')^3 - 2\alpha(V')^2 + 2\beta V' - 2 = 0, \tag{1.2.5}$$

where $'$ denotes derivative on $r(M)$ and α, β are C^∞-functions which satisfy $\|\nabla r\|^2 = \alpha \circ r$, $\Delta r = \beta \circ r$. Conversely, if V satisfies (1.2.5)*, the family of the images $\{M_t\}_{t\in\mathbb{R}}$ defined by $f_t(x) = (x, (V \circ r)(x) + t)$, $x \in M$ and $M_t = f_t(M)$ is the mean curvature flow and Γ is the translator.*

Proof. For the left side of (1.2.2), we have

$$\sqrt{1 + \|\nabla u\|^2}\ \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+\|\nabla u\|^2}}\right) = \Delta u - \frac{1}{2(1+\|\nabla u\|^2)}\nabla u(\|\nabla u\|^2).$$

By $u = V \circ r$, we find

$$\begin{aligned} \|\nabla u\|^2 &= \left(\alpha(V')^2\right) \circ r, \\ \nabla u(\|\nabla u\|^2) &= \left(\alpha(V')^2\left(2\alpha V'' + \alpha' V'\right)\right) \circ r, \\ \Delta u &= \left(\alpha V'' + \beta V'\right) \circ r. \end{aligned}$$

Therefore, (1.2.2) is reduced to the following equation

$$\frac{\alpha V''}{1 + \alpha(V')^2} + \beta V' - \frac{\alpha\alpha'(V')^3}{2\left(1 + \alpha(V')^2\right)} = 1.$$

By this equation, we obtain (1.2.5). ☐

1.3 Proof of Theorem 1.1.1

In this section, we assume that N is the n-dimensional unit sphere $\mathbb{S}^n$ $(n \geq 2)$ and $u = (V \circ r)|_{r^{-1}(J)}$ with an isoparametric function r on $\mathbb{S}^n$ and a C^∞-function V on an interval $J \subset r(\mathbb{S}^n) = [-1, 1]$. By (1.2.4), substituting $\alpha(r) = k^2(1 - r^2)$ and $\beta(r) = \frac{m_2 - m_1}{2}k^2 - k(n + k - 1)r$ for (1.2.5), we obtain

$$\begin{aligned} V''(r) =& k((n-1)r - \frac{m_2 - m_1}{2}k)V'(r)^3 + V'(r)^2 \\ &+ \frac{(n+k-1)r - \frac{m_2-m_1}{2}k}{k(1-r^2)}V'(r) + \frac{1}{k^2(1-r^2)}, \quad r \in (-1, 1). \end{aligned} \tag{1.3.1}$$

The local existence of the solution V of (1.3.1) is clear. By Remark 1.2.4 (i), we find

$$m_2 - m_1 = \begin{cases} 0 & (k = 1, 3, 6) \\ 2(m_2 - \frac{n-1}{k}) & (k = 2, 4). \end{cases}$$

Therefore, (1.3.1) is reduced to

$$\begin{aligned} V''(r) =& k((n-1)(r-R))V'(r)^3 + V'(r)^2 \\ &+ \frac{(n+k-1)r - (n-1)R}{k(1-r^2)} V'(r) + \frac{1}{k^2(1-r^2)}, \quad r \in (-1, 1). \end{aligned} \tag{1.3.2}$$

Here, $R \in (-1, 1)$ is the constant defined by

$$R := \begin{cases} 0 & (k = 1, 3, 6) \\ -1 + \dfrac{km_2}{n-1} & (k = 2, 4), \end{cases}$$

and when $k = 2, 4$, m_2 is equal to the multiplicity of the smallest principal curvature of the isoparametric hypersurface defined by the level set of r. To prove Theorem 1.1.1, we consider the graph of the solution V of (1.3.2). Define $\psi(r) = k\sqrt{1-r^2}V'(r)$. The equation (1.3.2) is reduced to

$$\psi'(r) = \frac{1}{k(1-r^2)} \left(\psi(r)^2 + 1\right) \left((n-1)(r-R)\psi(r) + \sqrt{1-r^2}\right). \tag{1.3.3}$$

Therefore, we consider the behavior of the solution ψ of (1.3.3). Define $\eta(r) = -\frac{\sqrt{1-r^2}}{(n-1)(r-R)}$. Then, the following lemma holds clearly.

Lemma 1.3.1.

(i) When $r \in (R, 1)$:

- *(a) if $\psi(r) > \eta(r)$, then $\psi'(r) > 0$,*
- *(b) if $\psi(r) = \eta(r)$, then $\psi'(r) = 0$,*
- *(c) if $\psi(r) < \eta(r)$, then $\psi'(r) < 0$.*

(ii) When $r \in (-1, R)$:

- *(a) if $\psi(r) < \eta(r)$, then $\psi'(r) > 0$,*
- *(b) if $\psi(r) = \eta(r)$, then $\psi'(r) = 0$,*
- *(c) if $\psi(r) > \eta(r)$, then $\psi'(r) < 0$.*

(iii) When $r = R$ or $\psi(r) = 0$: $\psi'(r) > 0$.

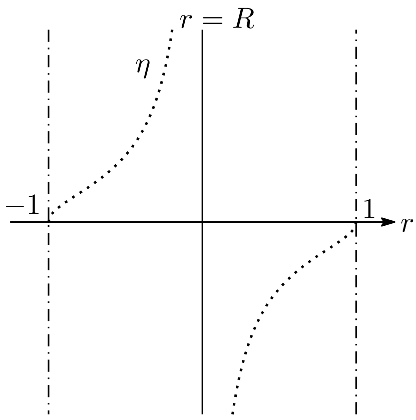

Figure 1.3.1: The graph of η

For the shape of ψ in the case where $\psi > 0$, we obtain the following lemmas.

Lemma 1.3.2. *If there exists $r_0 \in (R, 1)$ with $\psi(r_0) > 0$, there exists $r_1 \in (r_0, 1)$ such that*

$$\lim_{r \uparrow r_1} \psi(r) = +\infty.$$

Proof. For all $r \in (r_0, 1)$, we find $\psi'(r) > 0$ and $\psi(r) > 0$. Also, we have

$$\begin{aligned} \psi'(r) &= \frac{1}{k(1-r^2)} \Big(\psi(r)^2 + 1\Big)\Big((n-1)(r-R)\psi(r) + \sqrt{1-r^2}\Big) \\ &> \frac{(n-1)(r-R)}{k(1-r^2)} \psi(r)^3. \end{aligned}$$

Therefore, we find

$$\frac{\psi'(r)}{\psi(r)^3} > \frac{(n-1)(r-R)}{k(1-r^2)}.$$

By integrating both sides of this inequality from r_0 to r, we have

$$\begin{aligned} \frac{1}{\psi(r)^2} < \frac{(n-1)}{k} \log(1-r^2) + \frac{(n-1)R}{k} \log \frac{1+r}{1-r} \\ - \frac{(n-1)}{k} \log(1-r_0^2) - \frac{(n-1)R}{k} \log \frac{1+r_0}{1-r_0} + \frac{1}{\psi(r_0)^2} =: h_1(r). \end{aligned}$$

Here, h_1 is decreasing on $(r_0, 1)$ and

$$h_1(r_0) = \frac{1}{\psi(r_0)^2} > 0, \quad \lim_{r \uparrow 1} h_1(r) = -\infty.$$

Therefore, there exists $r_1 \in (r_0, 1)$ with $h_1(r_1) = 0$ and

$$\psi(r) > \frac{1}{\sqrt{h_1(r)}} \to +\infty \quad (\ r \ \uparrow \ r_1\ ).$$

The proof is completed. $\square$

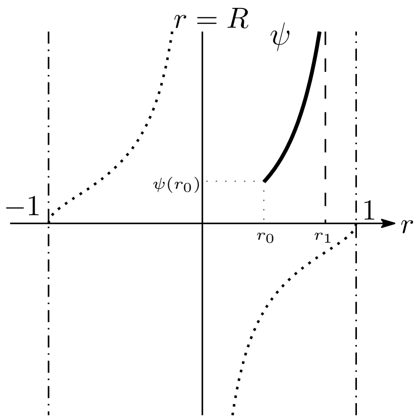

Figure 1.3.2: The behavior of the graph of ψ in Lemma 1.3.2.

Lemma 1.3.3. *If there exists* $r_0 \in (-1, R)$ *with* $0 < \psi(r_0) < \eta(r_0)$*, there exists* $C \in (\psi(r_0), +\infty)$ *such that*

$$\lim_{r \uparrow R} \psi(r) = C.$$

Proof. First, we consider the case $k = 1$. For all $r \in (r_0, 0)$, we find $\psi'(r) > 0$ and $0 < \psi(r) < \eta(r)$. Also, we have

$$\begin{aligned}\psi'(r) &= \frac{1}{1-r^2}\left(\psi(r)^2 + 1\right)\left((n-1)r\psi(r) + \sqrt{1-r^2}\right)\\ &< \frac{1}{\sqrt{1-r^2}}\left(\psi(r)^2 + 1\right).\end{aligned}$$

Therefore, we find

$$\frac{\psi'(r)}{\psi(r)^2 + 1} < \frac{1}{\sqrt{1-r^2}}.$$

By integrating both sides of this inequality from r_0 to r, we have

$$\arctan \psi(r) < \arcsin r - \arcsin r_0 + \arctan \psi(r_0) =: h_2(r).$$

Here, h_2 is increasing on $(r_0, 0)$ and

$$h_2(r_0) = \arctan \psi(r_0), \quad h_2(0) = \arctan \psi(r_0) - \arcsin r_0.$$

Since we find

$$\psi(r_0) < \eta(r_0) = -\frac{\sqrt{1-r_0^2}}{(n-1)r_0} \leq -\frac{\sqrt{1-r_0^2}}{r_0} = \tan\left(\arcsin r_0 + \frac{\pi}{2}\right),$$

we have

$$h_2(0) = \arctan \psi(r_0) - \arcsin r_0 < \frac{\pi}{2}.$$

Therefore, $\tan(h_2(r))$ is defined on $(r_0, 0]$ and

$$\psi(r) < \tan(h_2(r)).$$

The proof is completed for $k = 1$.

Next, we consider the case $k = 2, 3, 4$ or 6. For all $r \in (r_0, R)$, we find $\psi(r) > 0$ and $0 < \psi(r) < \eta(r)$. Also, we have

$$\begin{aligned}\psi'(r) &= \frac{1}{k(1-r^2)}\Big(\psi(r)^2+1\Big)\Big((n-1)(r-R)\psi(r)+\sqrt{1-r^2}\Big)\\ &< \frac{1}{2\sqrt{1-r^2}}\left(\psi(r)^2+1\right).\end{aligned}$$

Therefore, we find

$$\frac{\psi'(r)}{\psi(r)^2+1} < \frac{1}{2\sqrt{1-r^2}}.$$

By integrating both sides of this inequality from r_0 to r, we have

$$\arctan\psi(r) < \frac{1}{2}\arcsin r - \frac{1}{2}\arcsin r_0 + \arctan\psi(r_0) =: \hat{h}_2(r).$$

Here, $\hat{h}_2$ is increasing on (r_0, R) and

$$\hat{h}_2(r_0) = \arctan\psi(r_0), \quad \hat{h}_2(R) = \arctan\psi(r_0) + \frac{1}{2}\arcsin R - \frac{1}{2}\arcsin r_0.$$

Since we find

$$\begin{aligned}\psi(r_0) < \eta(r_0) &= -\frac{\sqrt{1-r_0^2}}{(n-1)(r_0-R)}\\ &< -\frac{\sqrt{1-r_0^2}+\sqrt{1-R^2}}{r_0-R}\\ &= \tan\left(\frac{1}{2}\arcsin r_0 - \frac{1}{2}\arcsin R + \frac{\pi}{2}\right),\end{aligned}$$

we have

$$\hat{h}_2(R) = \arctan\psi(r_0) - \arcsin r_0 < \frac{\pi}{2}.$$

Therefore, $\tan\left(\hat{h}_2(r)\right)$ is defined on $(r_0, R]$ and

$$\psi(r) < \tan\left(\hat{h}_2(r)\right).$$

The proof is completed. $\square$

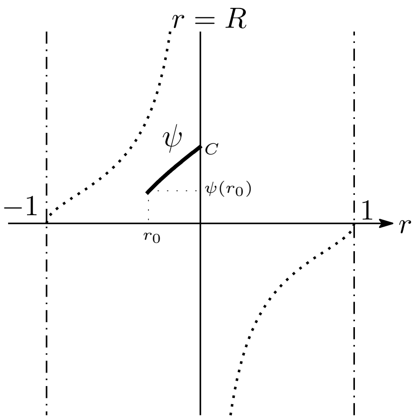

Figure 1.3.3: The behavior of the graph of ψ in Lemma 1.3.3

Lemma 1.3.4. *If there exists $r_0 \in (-1, R)$ with $\psi(r_0) > \eta(r_0)$, there exists $r_1 \in (-1, r_0)$ such that*

$$\lim_{r \downarrow r_1} \psi(r) = +\infty.$$

Proof. For all $r \in (-1, r_0)$, we find $\psi'(r) < 0$ and $\psi(r) > \eta(r)$. Also, we have

$$\begin{aligned}\psi'(r) &= \frac{1}{k(1-r^2)} \left(\psi(r)^2 + 1\right) \left((n-1)(r-R)\psi(r) + \sqrt{1-r^2}\right) \\ &< \frac{1}{k(1-r^2)} \left((n-1)\psi(r_0)(r-R) + \sqrt{1-r^2}\right) \psi(r)^2.\end{aligned}$$

Therefore, we find

$$\frac{\psi'(r)}{\psi(r)^2} < \frac{(n-1)}{k} \psi(r_0) \frac{r-R}{1-r^2} + \frac{1}{k\sqrt{1-r^2}}.$$

By integrating both sides of this inequality from r_0 to r, we have

$$\begin{aligned}\frac{1}{\psi(r)} < \ & \frac{(n-1)\psi(r_0)}{2k} \log\left(1-r^2\right) - \frac{1}{k} \arcsin r \\ & + \frac{(n-1)R\psi(r_0)}{2k} \log \frac{1+r}{1-r} - \frac{(n-1)\psi(r_0)}{2k} \log\left(1-r_0^2\right) \\ & + \frac{1}{k} \arcsin r_0 - \frac{(n-1)R\psi(r_0)}{2k} \log \frac{1+r_0}{1-r_0} + \frac{1}{\psi(r_0)} =: h_3(r).\end{aligned}$$

Here, h_3 is increasing on $(-1, r_0)$ and

$$h_3(r_0) = \frac{1}{\psi(r_0)} > 0, \quad \lim_{r \downarrow -1} h_3(r) = -\infty.$$

Therefore, there exists $r_1 \in (-1, r_0)$ with $h_3(r_1) = 0$ and

$$\psi(r) > \frac{1}{h_3(r)} \to +\infty \quad (\ r \ \downarrow \ r_1\ ).$$

The proof is completed. □

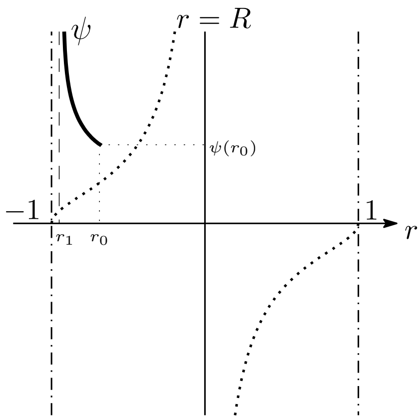

Figure 1.3.4: The behavior of the graph of ψ in Lemma 1.3.4

Since the existence of the solution ψ of (1.3.3) which is defined to $r = -1$ could not be excluded, we consider that case.

Lemma 1.3.5. *If ψ is defined to $r = -1$, $\psi(-1) = 0$ and* $V'(-1) = \frac{1}{k(k+(n-1)(1+R))}$

Proof. It is clear that $\psi(-1) = 0$. By $V'(r) = \frac{1}{k\sqrt{(1-r^2)}}\psi(r)$, we find

$$\begin{aligned}\frac{\psi'(r)}{(k\sqrt{1-r^2}\ )'} &= -\frac{1}{k^2 r}\left(k^2(1-r^2)V'(r)^2+1\right)\left(k(n-1)(r-R)V'(r)+1\right)\\ &\to -\frac{1}{k^2}\left(k(n-1)(1+R)V'(-1)-1\right) \quad (\ r\ \downarrow\ -1\ ).\end{aligned}$$

By l'Hôpital's rule, we find $V'(-1) = \frac{1}{k(k+(n-1)(1+R))}$. □

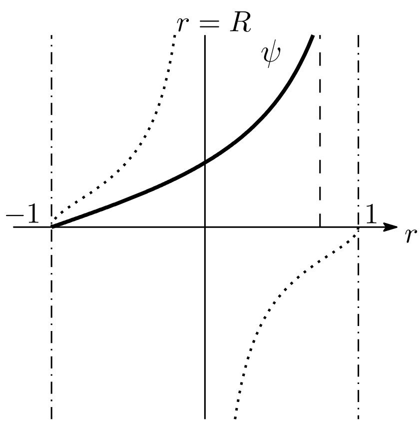

Figure 1.3.5: The behavior of the graph of ψ in Lemma 1.3.5

Also, in the case where $\psi < 0$, by proofs similar to Lemmas 1.3.2–1.3.5, we obtain the following lemmas.

Lemma 1.3.6. *If there exists $r_0 \in (-1, R)$ with $\psi(r_0) < 0$, there exists $r_1 \in (-1, r_0)$ such that*

$$\lim_{r \downarrow r_1} \psi(r) = -\infty.$$

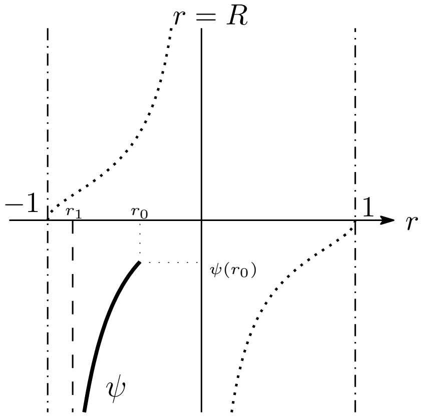

Figure 1.3.6: The behavior of the graph of ψ in Lemma 1.3.6

Lemma 1.3.7. *If there exists $r_0 \in (R, 1)$ with $0 > \psi(r_0) > \eta(r_0)$, there exists $C \in (-\infty, \psi(r_0))$ such that*

$$\lim_{r \downarrow R} \psi(r) = C.$$

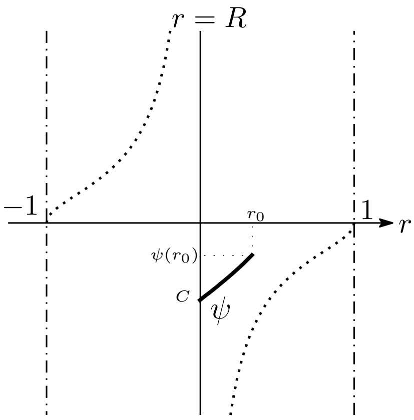

Figure 1.3.7: The behavior of the graph of ψ in Lemma 1.3.7

Lemma 1.3.8. *If there exists $r_0 \in (R, 1)$ with $\psi(r_0) < \eta(r_0)$, there exists $r_1 \in (r_0, 1)$ such that*

$$\lim_{r \uparrow r_1} \psi(r) = -\infty.$$

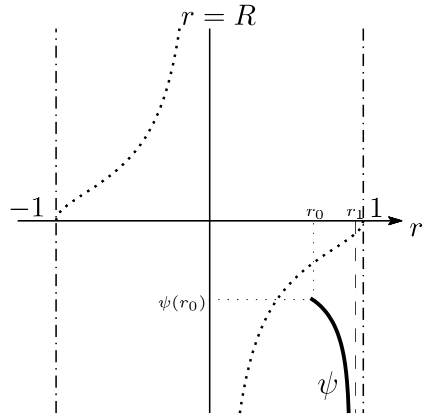

Figure 1.3.8: The behavior of the graph of ψ in Lemma 1.3.8

Lemma 1.3.9. *If ψ is defined to $r = 1$, $\psi(1) = 0$ and $V'(1) = -\frac{1}{k(k+(n-1)(1-R))}$*

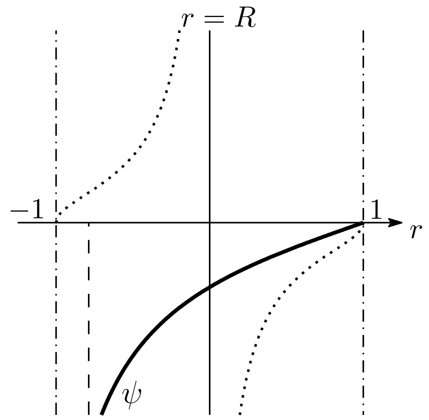

Figure 1.3.9: The behavior of the graph of ψ in Lemma 1.3.9

By Lemmas 1.3.1–1.3.9, we obtain the following proposition for the behavior of the graph of ψ.

Proposition 1.3.10. *For the solution ψ of the equation* (1.3.3)*, the behavior of the graph of ψ is like one of* Figures 1.3.10–1.3.16*.*

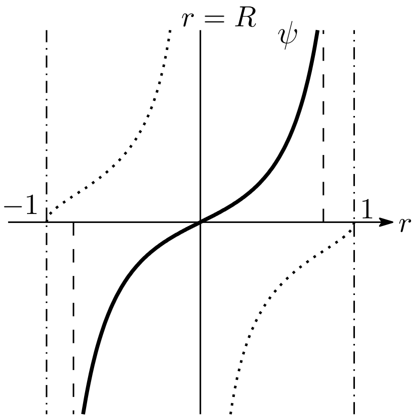

Figure 1.3.10: The graph of ψ (Type I)

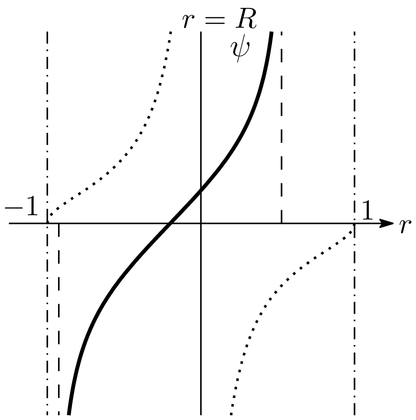

Figure 1.3.11: The graph of ψ (Type II)

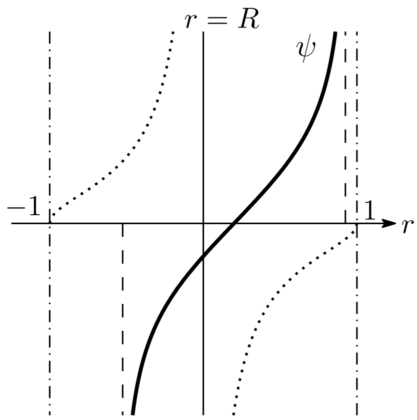

Figure 1.3.12: The graph of ψ (Type III)

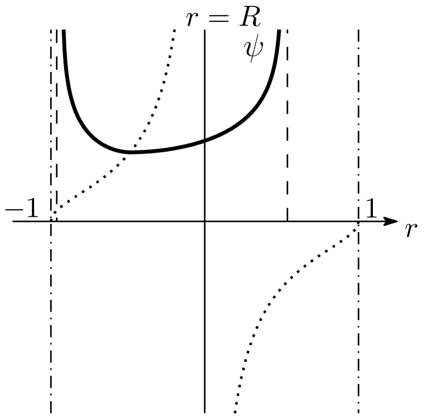

Figure 1.3.13: The graph of ψ (Type IV)

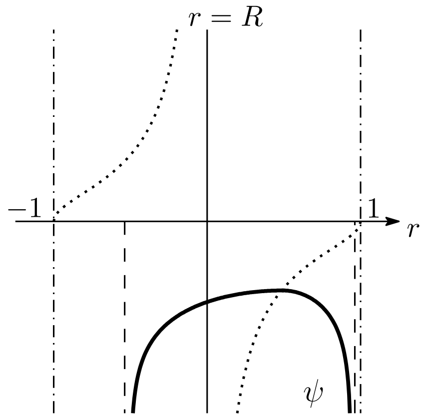

Figure 1.3.14: The graph of ψ (Type V)

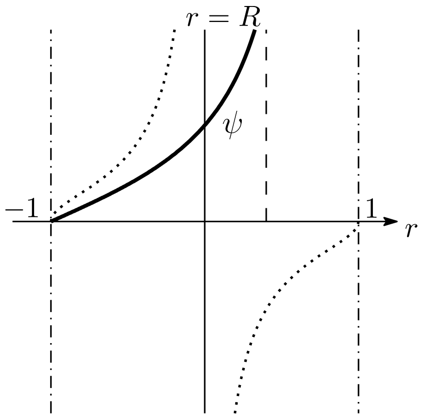

Figure 1.3.15: The graph of ψ (Type VI)

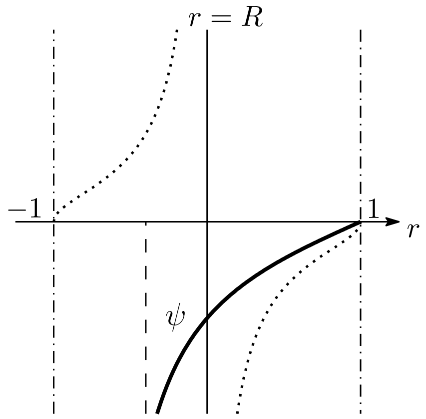

Figure 1.3.16: The graph of ψ (Type VII)

For the graph of ψ in Proposition 1.3.10, we have not yet shown whether ψ in the case of Figures 1.3.15 and 1.3.16 exists or not. By the following lemma, we obtain the existence.

Lemma 1.3.11. *The solutions ψ of the equation* (1.3.3) *in* Figures 1.3.15 *and* 1.3.16 *exist.*

Proof. For the set S of all solutions of the equation (1.3.3), we define sets S_1, S_2, $S_3 \subset S$ by

$$\begin{aligned} S_1 &:= \{\psi \in S | \exists r_0 \in (-1,1) : \psi(r_0) = 0\}, \\ S_2 &:= \{\psi \in S | \exists r_0 \in (-1,1) : \psi(r_0) = \eta(r_0)\}, \\ S_3 &:= \{\psi \in S | \psi(1) = 0 \textit{ or } \psi(-1) = 0\}. \end{aligned}$$

Then, we have

$$(-1,1) \times \mathbb{R} = \bigcup_{\psi \in S_1 \cup S_2 \cup S_3} \mathrm{Graph}(\psi),$$

where $\mathrm{Graph}(\psi)$ is the graph of ψ. Since $\bigcup_{\psi \in S_1} \mathrm{Graph}(\psi)$ and $\bigcup_{\psi \in S_2} \mathrm{Graph}(\psi)$ are open sets and $(-1,1) \times \mathbb{R}$ is connected, we find S_3 is not empty set. The proof is completed. $\square$

Define $\zeta(r) = -\frac{1}{k(n-1)(r-R)}$. By $V'(r) = \frac{1}{k\sqrt{1-r^2}}\psi(r)$ and Proposition 1.3.10, we have the following proposition for the behavior of the graph of V'. Besides, by Proposition 1.3.12, we obtain Theorem 1.1.1.

Proposition 1.3.12. *For the solution V of the equation* (1.3.2)*, the behavior of the graph of V' is like one of* Figures 1.3.17–1.3.23*. Here, the dotted curve in* Figures 1.3.17–1.3.23 *is the graph of ζ.*

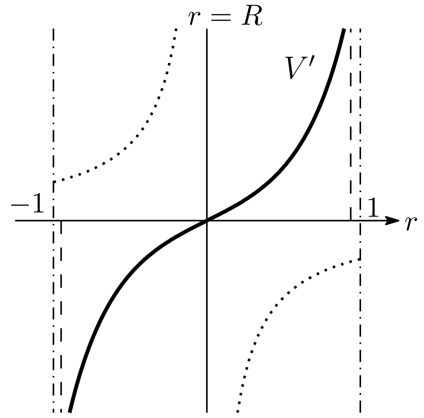

Figure 1.3.17: The graph of V' (Type I)

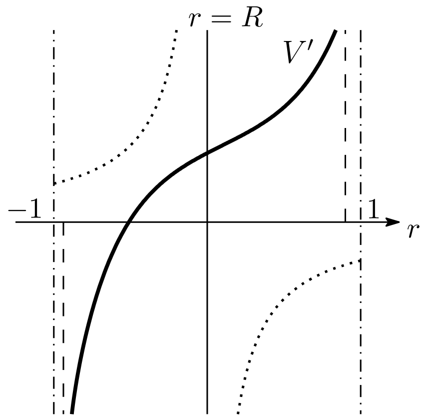

Figure 1.3.18: The graph of V' (Type II)

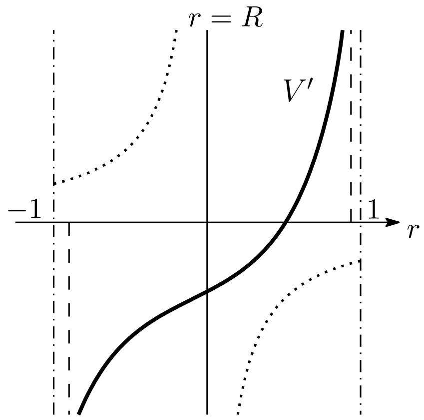

Figure 1.3.19: The graph of V' (Type III)

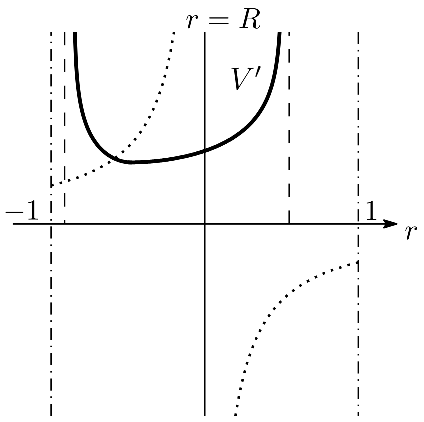

Figure 1.3.20: The graph of V' (Type IV)

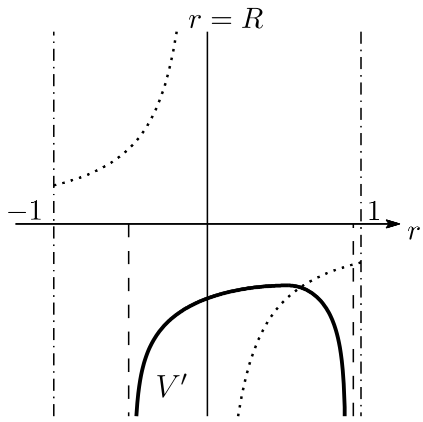

Figure 1.3.21: The graph of V' (Type V)

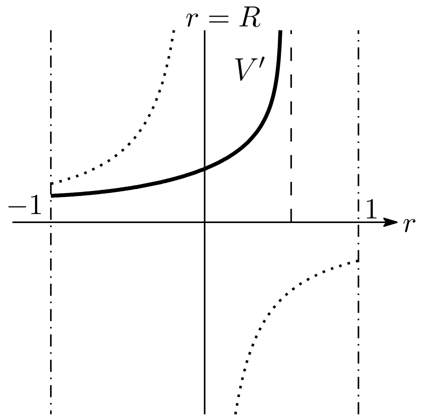

Figure 1.3.22: The graph of V' (Type VI)

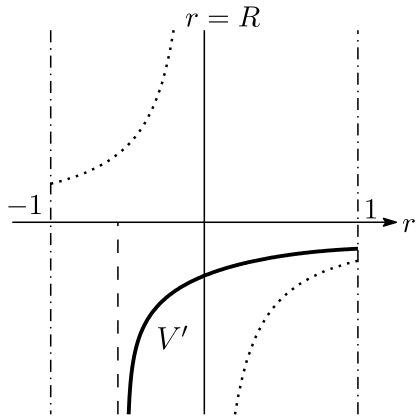

Figure 1.3.23: The graph of V' (Type VII)

1.4 The domain of the function u in Theorem 1.1.1

In this section, we investigate the domain of the function $u = V \circ r$ over $M \subset \mathbb{S}^n$ in Theorem 1.1.1 in the case where the number k of distinct principal curvatures of the isoparametric hypersurface for r is 1, 2 or 3. From the result of Theorem 1.1.1, we find that M does not contain some tubular neighborhoods of the focal submanifolds $r^{-1}(1)$ and $r^{-1}(-1)$ in case that the type of V in Theorem 1.1.1 is I–V. Also, we find that M contains $r^{-1}(-1)$ and does not contain a tubular neighborhood of the focal submanifold $r^{-1}(1)$ in case that the type of V is VI and M contains $r^{-1}(1)$ and does not contain a tubular neighborhood of the focal submanifold $r^{-1}(-1)$ in case that the type of V is VII.

When $k = 1$, the isoparametric function r is defined by

$$r(x_1, \ldots, x_{n+1}) = x_{n+1} \qquad (x_1, \ldots, x_{n+1}) \in \mathbb{S}^n.$$

Therefore, from the result of Theorem 1.1.1, the domain M of u is an open set of $\mathbb{S}^n$ including the set $\{(x_1, \ldots, x_n, 0) \in \mathbb{R}^{n+1} | \, x_1^2 + \cdots + x_n^2 = 1\} \subset \mathbb{S}^n$. Also, as $p = (0, \ldots, 0, 1), q = (0, \ldots, 0, -1)$, we find that $p, q \notin M$ in case that the type of V in Theorem 1.1.1 is I–V, $p \notin M$, $q \in M$ in case that the type of V is VI and $p \in M$, $q \notin M$ in case that the type of V is VII.

When $k=2$, the isoparametric function r is defined by

$$r(x_1,\ldots,x_{n+1})=\sum_{i=1}^{l}x_i^2-\sum_{i=l+1}^{n+1}x_i^2\qquad(x_1,\ldots,x_{n+1})\in\mathbb{S}^n.$$

Here, $l\in\{1,\ldots,n\}$. Since $r^{-1}(t)=\{(x,y)\in\mathbb{R}^l\times\mathbb{R}^{n-l+1}|\,|x|^2=\frac{1+t}{2},|y|^2=\frac{1-t}{2}\}$ for $t\in(-1,1)$, as $S_\theta:=\{((\cos\theta,0,\ldots,0)\mathbf{A},(\sin\theta,0,\ldots,0)\mathbf{B})\in\mathbb{R}^l\times\mathbb{R}^{n-l+1}\mid\mathbf{A}\in SO(l-1),\mathbf{B}\in SO(n-l)\}$, we obtain that $r^{-1}(t)=S_{\theta_t}$ for $\theta_t\in(0,\frac{\pi}{2})$ with $\cos\theta_t=\sqrt{\frac{1+t}{2}}$ and $\sin\theta_t=\sqrt{\frac{1-t}{2}}$. Therefore, from the result of Theorem 1.1.1, we find that the domain M is the open set of $\mathbb{S}^n$ including S_{θ_R}. Also, we find that $M=\bigcup_{\theta\in I}S_\theta$ for an interval $I\subset(0,\frac{\pi}{2})$ in case that the type of V in Theorem 1.1.1 is I–V, $M=\bigcup_{\theta\in(a,\frac{\pi}{2}]}S_\theta$ for some $a\in(0,\frac{\pi}{2})$ in case that the type of V is VI and $M=\bigcup_{\theta\in[0,a)}S_\theta$ for $a\in(0,\frac{\pi}{2})$ in case that the type of V is VII.

When $k=3$, an isoparametric hypersurface is a principal orbit of the isotropy representation of the rank two symmetric space $G/K=SU(3)/SO(3)$, $(SU(3)\times SU(3))/SU(3)$, $SU(6)/Sp(3)$ or E_6/F_4. Since the principal orbit of the isotropy representation intersects with the Weyl domain C at only one point, we find that there exists an open subset $U\subset\overline{C}\cap\mathbb{S}^n$ such that $K\cdot U$ is equal to M. Here, $T_e(G/K)$ for $e\in G/K$ is identified with $\mathbb{R}^{n+1}$. Also, we find that $M\subset K\cdot C$ in case that the type of V in Theorem 1.1.1 is I–V and $M\cap(\overline{C}\setminus C)\neq\emptyset$ in case that the type of V is VI or VII.

In the rest of this section, we shall give explicit descriptions of Weyl domains for the symmetric space $G/K=SU(3)/SO(3)$, $(SU(3)\times SU(3))/SU(3)$ or $SU(6)/Sp(3)$. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively. Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be the canonical decomposition. Denote by $\mathfrak{a}$ the maximal abelian subspace of $\mathfrak{p}$. When $G/K=SU(3)/SO(3)$, we have that $\mathfrak{p}=\{\mathbf{A}\ :\ 3\times 3$ symmetric space purely imaginary matrix such that the trace of $\mathbf{A}=0\}$ and the diagonal matrices in $\mathfrak{p}$ form $\mathfrak{a}$. So, we obtain

$$\mathfrak{a}=\left\{\left.\begin{pmatrix}\sqrt{-1}a&0&0\\0&\sqrt{-1}b&0\\0&0&-\sqrt{-1}(a+b)\end{pmatrix}\right|a,b\in\mathbb{R}\right\}.$$

Define e_i $(1\leq i\leq 3)$ as $e_i(\mathbf{A})$ is the diagonal element of $\mathbf{A}$. Then, we find that for the basis $\{e_1,e_2\}$ the positive restricted root system $\triangle_+=\{\sqrt{-1}(e_1-e_2),\sqrt{-1}(e_1-e_3),\sqrt{-1}(e_2-e_3)\}$. Since the Killing form B is defined by $B(\mathbf{X},\mathbf{Y})=6\mathrm{Tr}(\mathbf{XY})$, as

$$\mathbf{A_1}=\begin{pmatrix}\sqrt{-1}&0&0\\0&-\sqrt{-1}&0\\0&0&0\end{pmatrix},\ \mathbf{A_2}=\begin{pmatrix}\sqrt{-1}&0&0\\0&0&0\\0&0&-\sqrt{-1}\end{pmatrix},\ \mathbf{A_3}=\begin{pmatrix}0&0&0\\0&\sqrt{-1}&0\\0&0&-\sqrt{-1}\end{pmatrix},$$

we obtain the Weyl domain C in Figure 1.4.1. Here, for the angle θ_{ij} with respect to $\mathbf{A_i}$ and $\mathbf{A_j}$, we find $\theta_{12}=\theta_{23}=\frac{\pi}{3}$ and $\theta_{13}=\frac{2\pi}{3}$.

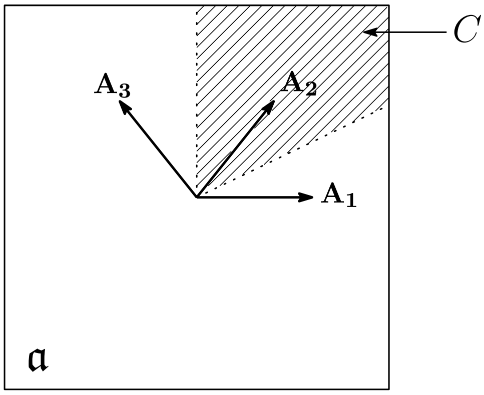

Figure 1.4.1: The Weyl domain C

When $G/K = (SU(3) \times SU(3))/SU(3)$, we have that

$$\mathfrak{p} = \left\{ \begin{pmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & -\mathbf{A} \end{pmatrix} \middle| \, \mathbf{A} : 3 \times 3 \textit{ skew Hermitian matrix}, \textit{ the trace of } \mathbf{A} = 0 \right\}$$

and the diagonal matrices in $\mathfrak{p}$ form $\mathfrak{a}$.

Also, When $G/K = SU(6)/Sp(3)$, we have that

$$\mathfrak{p} = \left\{ \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \overline{\mathbf{B}} & -\overline{\mathbf{A}} \end{pmatrix} \middle| \, \begin{array}{c} \mathbf{A} : 3 \times 3 \textit{ skew Hermitian matrix}, \textit{ the trace of } \mathbf{A} = 0, \\ \mathbf{B} \ : \ 3 \times 3 \textit{ skew symmetric matrix} \end{array} \right\}$$

and the diagonal matrices in $\mathfrak{p}$ form $\mathfrak{a}$.

In a similar way we obtain in the case that $G/K = SU(3)/SO(3)$, that the Weyl domains C for $(SU(3) \times SU(3))/SU(3)$ and $SU(6)/Sp(3)$ are as in Figure 1.4.1.

Chapter 2

Graphical translators for the inverse mean curvature flow and isoparametric functions

2.1 Introduction

In Chapter 1, we classified the shape of the translator for the mean curvature flow given as a graph of a function on a domain in the unit sphere which is a composition of an isoparametric function and some function. In this chapter, we consider the case of the inverse mean curvature flow by the similar way. This chapter is based on [8].

Let N be an n-dimensional Riemannian manifold. Define an immersion f of a domain $M \subset N$ into the product Riemannian manifold $N \times \mathbb{R}$ by $f(x) = (x, u(x))$, $x \in M$ with a smooth function $u : M \to \mathbb{R}$ on M. Also, denote the graph of u (i.e, $f(M)$) by Γ. For a C^∞-family of C^∞-immersions $\{f_t\}_{t\in I}$ of M into $N \times \mathbb{R}$ (I is an open interval including 0) with $f_0 = f$, as $M_t = f_t(M)$, $\{M_t\}_{t\in I}$ is called the inverse mean curvature flow starting from Γ if f_t satisfies

$$\left(\frac{\partial f_t}{\partial t}\right)^{\perp_{f_t}} = -\frac{1}{\|H_t\|^2} H_t, \tag{2.1.1}$$

where H_t is the mean curvature vector field of f_t and $(\bullet)^{\perp_{f_t}}$ is the normal component of $(\bullet)$ with respect to f_t.

Furthermore, according to the definition of a soliton of the mean curvature flow by Hungerbühler and Smoczyk [11], we define a soliton of the inverse mean curvature flow. Let X be a Killing vector field on $N \times \mathbb{R}$ and $\{\phi_t\}_{t\in\mathbb{R}}$ be the one-parameter transformation associated to X on $N \times \mathbb{R}$. Then, Γ is called a *soliton for the inverse mean curvature flow with respect to X* if $\{f_t\}_{t\in I}$ satisfies

$$\left(\frac{\partial(\phi_t^{-1} \circ f_t)}{\partial t}\right)^{\perp_{(\phi_t^{-1}\circ f_t)}} = 0. \tag{2.1.2}$$

In the sequel, we call such soliton an *X-soliton* simply. In particular, when $X = (0, 1) \in T(N \times \mathbb{R}) = TN \oplus T\mathbb{R}$, we call the X-soliton a *translator*.

Compared with the mean curvature flow, the translator for the inverse mean curvature flow is less studied. For a translator for the inverse mean curvature flow, Drugan, Lee, and Wheeler [6] gave a translator in $\mathbb{R}^2$ which is the cycloid generated by a circle with radius $\frac{1}{4}$ and gave a tilted cycloid product as a translator in $\mathbb{R}^3$. Kim and Pyo [12, 13] showed the existence and classification of rotationally symmetric translators in $\mathbb{R}^{n+1}$ and showed that there is no complete translator for inverse mean curvature flow in $\mathbb{R}^{n+1}$.

In the main theorem of this chapter, we consider the case where N is the n-dimensional unit sphere $\mathbb{S}^n$ and u is a composition of an isoparametric function r on $\mathbb{S}^n$ and some function V. Then, we obtain the following theorem for the shape of the graph of V.

Theorem 2.1.1. *Let r be an isoparametric function on $\mathbb{S}^n$ $(n \geq 2)$ and V be a C^∞-function on an interval $J \subset r(\mathbb{S}^n)$. If the inverse mean curvature flow starting from the graph of the function $u = (V \circ r)|_{r^{-1}(J)}$ is a translator, the shape of the graph of V is like one of those illustrated by* Figures 2.1.1–2.1.5.

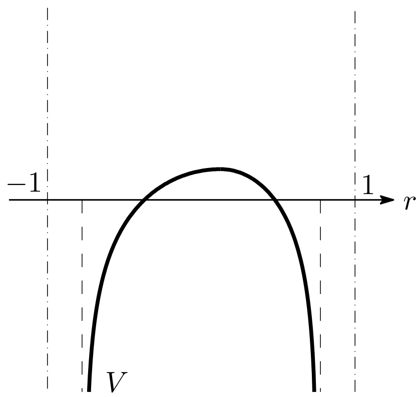

Figure 2.1.1: The graph of V (Type I)

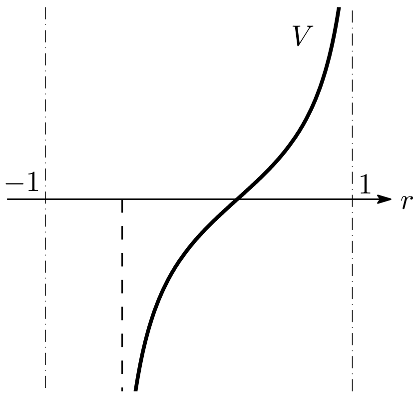

Figure 2.1.2: The graph of V (Type II)

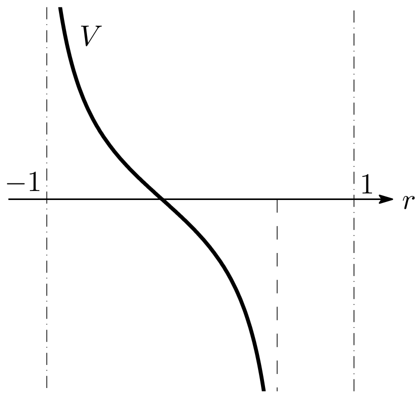

Figure 2.1.3: The graph of V (Type III)

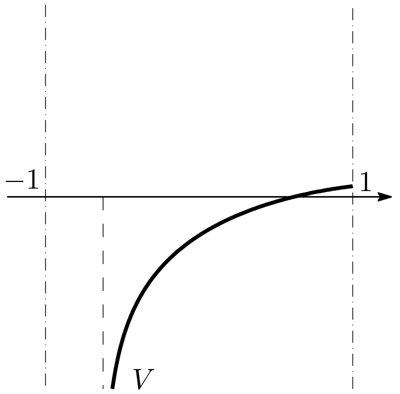

Figure 2.1.4: The graph of V (Type IV)

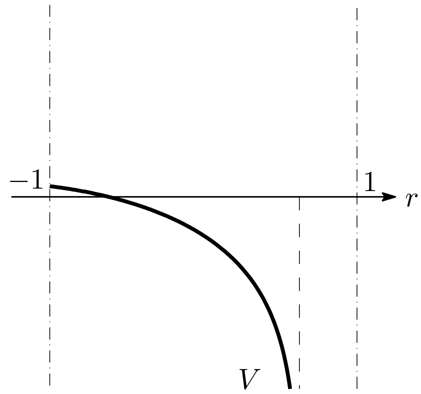

Figure 2.1.5: The graph of V (Type V)

Remark 2.1.2. *For the C^∞-function V in* Theorem 2.1.1*, define a C^∞-function ψ by $\psi(r) = k\sqrt{1-r^2}V'(r)$ and define $\psi_{\min}$, $\psi_{\max}$ by $\psi_{\min} := \min_{\mathrm{r}\in\mathrm{Dom}(\psi)}\psi(\mathrm{r})$, $\psi_{\max} := \max_{\mathrm{r}\in\mathrm{Dom}(\psi)}\psi(\mathrm{r})$, where* $\mathrm{Dom}(\psi)$ *means a domain of ψ. Also, define a constant R and functions η_1, η_2 on $(-1,a]\cup[b,1)$ by*

$$
\begin{aligned}
R &:= \begin{cases} 0 & (k=1,3,6) \\ -1+\dfrac{km}{n-1} & (k=2,4), \end{cases} \\
\eta_1(r) &:= \frac{(n-1)(r-R)-\sqrt{((n-1)^2+4)r^2-2R(n-1)^2r+R^2(n-1)^2-4}}{2\sqrt{1-r^2}}, \\
\eta_2(r) &:= \frac{(n-1)(r-R)+\sqrt{((n-1)^2+4)r^2-2R(n-1)^2r+R^2(n-1)^2-4}}{2\sqrt{1-r^2}}.
\end{aligned}
$$

Here, m is the multiplicity of the smallest principal curvature of the isoparametric hypersurface defined by the level set of the isoparametric function r in Theorem 2.1.1 *and a, b are defined by*

$$
\begin{aligned}
a &:= \frac{(n-1)^2R-2\sqrt{(n-1)^2(1-R^2)+4}}{(n-1)^2+4}, \\
b &:= \frac{(n-1)^2R+2\sqrt{(n-1)^2(1-R^2)+4}}{(n-1)^2+4}.
\end{aligned}
$$

If the graph of V is like one illustrated by Figure 2.1.2*, then we will see that there exists $r_0\in(b,1)$ with $\psi_{\min}=\eta_1(r_0)$ or $\psi_{\min}=\eta_2(r_0)$. If the graph of V is like one illustrated by* Figure 2.1.3*, then we will see that there exists $r_0\in(-1,a)$ with $\psi_{\max}=\eta_1(r_0)$ or $\psi_{\max}=\eta_2(r_0)$. Let the open interval (x,y) be the domain of ψ. Then, it is shown that, for each type of the graph of V, the behavior of the graph of ψ is as in* Table 1*.*

Table 2.1: The behavior of the graph of ψ

the graph of V	$\mathrm{Im}(\psi)$	ψ'	$r \downarrow x$	$r \uparrow y$
Type I	$(-\infty, \infty)$	< 0	∞	$-\infty$
Type II	$[\eta_i(r_0), \infty)$	$-$	∞	∞
Type III	$(-\infty, \eta_i(r_0)]$	$-$	$-\infty$	$-\infty$
Type IV	$[0, \infty)$	< 0	∞	0
Type V	$(-\infty, 0]$	< 0	0	$-\infty$

2.2 Proof of Theorem 2.1.1

Let (N, g) be an n-dimensional Riemannian manifold and $u : M \to \mathbb{R}$ be a function on a domain $M \subset N$. Denote the graph of u by Γ. Also, denote the gradient and Laplacian with respect to g by ∇ and Δ respectively. Then, we have the following lemma about the soliton of the inverse mean curvature flow.

Lemma 2.2.1. *If Γ is a translator, u satisfies*

$$\Delta u + \|\nabla u\|^2 + 1 - \frac{\nabla u(\|\nabla u\|^2)}{2(1 + \|\nabla u\|^2)} = 0. \tag{2.2.1}$$

Conversely, if u satisfies (2.2.1)*, the family of the images $\{M_t\}_{t\in\mathbb{R}}$ definded by $f_t(x) = (x, u(x)+ t)$, $x \in M$ and $M_t = f_t(M)$ is the inverse mean curvature flow and Γ is a translator.*

Proof. Define the immersion f of M into the product Riemannian manifold $N \times \mathbb{R}$ by $f(x) = (x, u(x))$, $x \in M$ and define the Killing vector $X = (0, 1) \in T(N \times \mathbb{R}) = TN \oplus T\mathbb{R}$. Denote the mean curvature vector field of f by H. According to Hungerbühler and Smoczyk [11] in the case of a soliton for the mean curvature flow, if Γ is translator, we find that

$$(X \circ f)^{\perp_f} = -\frac{1}{\|H\|^2} H. \tag{2.2.2}$$

Let $(x^1, \cdots x^n, s)$ be local coordinates of $N \times \mathbb{R}$. By $X = \frac{\partial}{\partial s}$ and $f(x) = (x, u(x))$, $x \in M$, we find

$$(X \circ f)^{\perp_f} = \frac{\partial}{\partial s} - \frac{1}{1 + \|\nabla u\|^2} df(\nabla u),$$
$$\frac{1}{\|H\|^2} H = \frac{1 + \|\nabla u\|^2}{\Delta u - \frac{\nabla u(\|\nabla u\|^2)}{2(1+\|\nabla u\|^2)}} \left(\frac{\partial}{\partial s} - \frac{1}{1 + \|\nabla u\|^2} df(\nabla u) \right).$$

Therefore, we obtain that (2.2.2) is equivalent to (2.2.1).

Conversely, if u satisfies (2.2.1), we find that f satisfies (2.2.2). Then, for the one-parameter transformation $\{\phi_t\}_{t\in\mathbb{R}}$ associated to X on $N \times \mathbb{R}$, since ϕ_t's are isometries and f satisfies (2.2.2), we find that $f_t = \phi_t \circ f$ satisfies

$$\begin{aligned}\left(\frac{\partial f_t}{\partial t}\right)^{\perp_{f_t}} + \frac{1}{\|H_t\|^2} H_t &= d\phi_t \left((X \circ f)^{\perp_f} + \frac{1}{\|H\|^2} H \right) \\ &= 0,\end{aligned}$$

and $\{f_t\}_{t\in\mathbb{R}}$ satisfies (2.1.1). So, $\{M_t\}_{t\in\mathbb{R}}$ is the inverse mean curvature flow. Furthermore, we find that f_t satisfies (2.1.2) from $\phi_t^{-1}\circ f_t = f$. Therefore, $\{M_t\}_{t\in\mathbb{R}}$ is the inverse mean curvature flow and Γ is a translator. Then, we have $f_t(x) = (x, u(x)+t),\ x\in M$. $\square$

We consider the case where u is a composition of an isoparametric function $r : N \to \mathbb{R}$ and some function V. Then, we obtain the following proposition.

Proposition 2.2.2. *Let $r : N \to \mathbb{R}$ be an isoparametric function on N. If Γ is a translator and if there exists a C^∞-function V on an interval $J \subset r(N)$ such that $u = (V\circ r)|_{r^{-1}(J)}$, the function V satisfies*

$$2\alpha V'' + 2\alpha^2 V'^4 + \alpha(2\beta - \alpha')V'^3 + 4\alpha V'^2 + 2\beta V' + 2 = 0, \tag{2.2.3}$$

where $'$ denotes derivative on J and α, β are C^∞-functions which satisfy $\|\nabla r\|^2 = \alpha\circ r,\ \Delta r = \beta\circ r$. Conversely, if V satisfies (2.2.3), the family of the images $\{M_t\}_{t\in\mathbb{R}}$ defined by $f_t(x) = (x, (V\circ r)(x) + t),\ x\in M$ and $M_t = f_t(M)$ is the inverse mean curvature flow and Γ is the translator.

Proof. From (2.2.1), we have

$$2(1+\|\nabla u\|^2)(\Delta u + \|\nabla u\|^2 + 1) - \nabla u(\|\nabla u\|^2) = 0.$$

By $u = V\circ r$, we find

$$\begin{aligned} \|\nabla u\|^2 &= \left(\alpha V'^2\right)\circ r,\\ \nabla u(\|\nabla u\|^2) &= \left(\alpha V'^2\left(2\alpha V'' + \alpha' V'\right)\right)\circ r,\\ \Delta u &= \left(\alpha V'' + \beta V'\right)\circ r. \end{aligned}$$

Therefore, (2.2.1) is reduced to the following equation

$$2(1+\alpha V'^2)(\alpha V'' + \beta V' + \alpha V'^2 + 1) - \alpha V'^2(\alpha' V' + 2\alpha V'') = 0.$$

From this equation, we obtain (2.2.3). $\square$

In the sequel, we assume that N is the n-dimensional unit sphere $\mathbb{S}^n$ $(n\geq 2)$ and $u = (V\circ r)|_{r^{-1}(J)}$ with an isoparametric function $r : \mathbb{S}^n \to \mathbb{R}$ and a C^∞-function V on interval $J \subset r(\mathbb{S}^n) = [-1,1]$. By Remark 1.2.4 (i), we find

$$m_2 - m_1 = \begin{cases} 0 & (k = 1,3,6)\\ 2(m_2 - \frac{n-1}{k}) & (k = 2,4). \end{cases}$$

Therefore, substituting α and β in Remark 1.2.4 (1.2.4) for the equation (2.2.3), we obtain

$$\begin{aligned} V''(r) = &- k^2(1-r^2)V'(r)^4 + k((n-1)(r-R))V'(r)^3 - 2V'(r)^2\\ &+ \frac{(n+k-1)r - (n-1)R}{k(1-r^2)}V'(r) - \frac{1}{k^2(1-r^2)}, \quad r\in(-1,1), \end{aligned} \tag{2.2.4}$$

where $R\in(-1,1)$ is the constant defined by

$$R := \begin{cases} 0 & (k=1,3,6)\\ -1 + \dfrac{km_2}{n-1} & (k=2,4). \end{cases}$$

Here, we note that m_2 is equal to the multiplicity of the smallest principal curvature of the isoparametric hypersurface defined by the level set of r in the case $k = 2, 4$. The local existence of the solution V of (2.2.4) is clear. To prove Theorem 2.1.1, we consider the graph of the solution V of (2.2.4). Define $\psi(r) = k\sqrt{1-r^2}V'(r)$. Then, the equation (2.2.4) is reduced to

$$\psi'(r) = -\frac{1}{k(1-r^2)}\left(\psi(r)^2+1\right)\Big(\sqrt{1-r^2}\psi(r)^2-(n-1)(r-R)\psi(r)+\sqrt{1-r^2}\Big). \quad (2.2.5)$$

Therefore, to obtain the behavior of the graph of V, first we consider the behavior of the solution ψ of (2.2.5). Define the functions η_1 and η_2 on $(-1, a] \cup [b, 1)$by

$$\eta_1(r) := \frac{(n-1)(r-R)-\sqrt{((n-1)^2+4)r^2-2R(n-1)^2r+R^2(n-1)^2-4}}{2\sqrt{1-r^2}},$$
$$\eta_2(r) := \frac{(n-1)(r-R)+\sqrt{((n-1)^2+4)r^2-2R(n-1)^2r+R^2(n-1)^2-4}}{2\sqrt{1-r^2}}.$$

Also, define $a, b \in (-1, 1)$ $(a < b)$ by

$$a := \frac{(n-1)^2R-2\sqrt{(n-1)^2(1-R^2)+4}}{(n-1)^2+4},$$
$$b := \frac{(n-1)^2R+2\sqrt{(n-1)^2(1-R^2)+4}}{(n-1)^2+4}.$$

Then, we find $a < R < b$ and obtain the following lemma.

Lemma 2.2.3.

(i) When $r \in (-1, a] \cup [b, 1)$,

- *(a) if $\eta_1(r) < \psi(r) < \eta_2(r)$, then $\psi'(r) > 0$,*
- *(b) if $\psi(r) = \eta_1(r)$ or $\psi(r) = \eta_2(r)$, then $\psi'(r) = 0$,*
- *(c) if $\psi(r) < \eta_1(r)$ or $\psi(r) > \eta_2(r)$, then $\psi'(r) < 0$.*

(ii) When $r \in (a, b)$, $\psi'(r) < 0$.

Proof. Define $A(x, r)$ and $B(r)$ by

$$A(x,r) := \sqrt{1-r^2}x^2-(n-1)(r-R)x+\sqrt{1-r^2}, \quad (x,r) \in \mathbb{R}\times(-1,1),$$
$$B(r) := ((n-1)^2+4)r^2-2(n-1)^2Rr+(n-1)^2R^2-4, \quad r \in (-1,1).$$

Then, we have

$$A(x,r) = \sqrt{1-r^2}\left(x-\frac{(n-1)(r-R)}{2\sqrt{1-r^2}}\right)^2-\frac{1}{4\sqrt{1-r^2}}B(r),$$
$$B(r) = \left((n-1)^2+4\right)\left(r-\frac{(n-1)^2R}{(n-1)^2+4}\right)^2-\frac{1}{(n-1)^2+4}\left(4(n-1)^2(1-R^2)+16\right).$$

Therefore, we find that if $r \in (-1, a] \cup [b, 1)$, then $B(r) > 0$, if $r \in (a, b)$, then $B(r) < 0$, and if $r \in \{a, b\}$, then $B(r) = 0$. Furthermore, we find that when $r \in (-1, a] \cup [b, 1)$, if $x \in (\eta_1(r), \eta_2(r))$, then $A(x, r) < 0$, if $x \in (-\infty, \eta_1(r)) \cup (\eta_2(r), \infty)$, then $A(x, r) > 0$, and if

$x \in \{\eta_1(r), \eta_2(r)\}$, then $A(x, r) = 0$. Also, when $r \in (a, b)$, we find that $A(x, r) > 0$. Since the equation (2.2.5) is reduced to

$$\psi'(r) = -\frac{1}{k(1-r^2)} \left(\psi(r)^2 + 1\right) A(\psi(r), r),$$

we obtain the statement of this lemma. $\square$

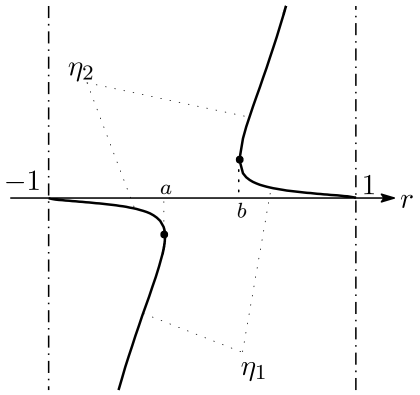

Figure 2.2.1: The graph of η_1 and η_2

For the behavior of the graph of the solution ψ of (2.2.5), we obtain following lemmas.

Lemma 2.2.4. *If there exists $r_0 \in (-1, a]$ with $\psi(r_0) < \eta_1(r_0)$, or if there exists $r_0 \in (a, 1)$ with $\psi(r_0) < 0$, then there exists $r_1 \in (r_0, 1)$ such that*

$$\lim_{r \uparrow r_1} \psi(r) = -\infty.$$

Proof. When $r > r_0$, we find $\psi'(r) < 0$ and $\psi(r) < \psi(r_0)$. Define $\eta_3(r)$ by

$$\eta_3(r) := \frac{(n-1)(r-R)}{2\sqrt{1-r^2}}.$$

Then, we find $\eta_3(r) = \frac{1}{2}(\eta_1(r) + \eta_2(r))$ on $(-1, a] \cup [b, 1)$. In the case where $\psi(r_0) \leq \eta_3(r_0)$, we have

$$\begin{aligned}\psi'(r) &= -\frac{1}{k(1-r^2)} \left(\psi(r)^2 + 1\right) \left(\sqrt{1-r^2}\psi(r)^2 - (n-1)(r-R)\psi(r) + \sqrt{1-r^2}\right) \\ &< -\frac{1}{k(1-r^2)} \left(\psi(r)^2 + 1\right) \left((1+\psi(r_0)^2)\sqrt{1-r^2} - \psi(r_0)(n-1)(r-R)\right).\end{aligned}$$

Therefore, we find

$$\frac{\psi'(r)}{1+\psi(r)^2} < -\frac{1+\psi(r_0)^2}{k\sqrt{1-r^2}} + \frac{\psi(r_0)(n-1)r}{k(1-r^2)} - \frac{\psi(r_0)(n-1)R}{k(1-r^2)}.$$

Integrating from r_0 to r, we have

$$\begin{aligned}\arctan\psi(r) < &-\frac{1+\psi(r_0)^2}{k}\arcsin r - \frac{\psi(r_0)(n-1)}{2k}\log(1-r^2) - \frac{\psi(r_0)(n-1)R}{2k}\log\frac{1+r}{1-r} \\ &+ \frac{1+\psi(r_0)^2}{k}\arcsin r_0 + \frac{\psi(r_0)(n-1)}{2k}\log(1-r_0^2) + \frac{\psi(r_0)(n-1)R}{2k}\log\frac{1+r_0}{1-r_0} \\ &+ \arctan\psi(r_0) =: h_4(r).\end{aligned}$$

Then, h_4 is decreasing on $(r_0, 1)$ and $h_4(r_0) = \arctan\psi(r_0)$, $\lim_{r\uparrow 1} h_4(r) = -\infty$. Therefore, there exists $\bar{r}_1 \in (r_0, 1)$ with $h_4(\bar{r}_1) = -\frac{\pi}{2}$ and

$$\psi(r) < \tan h_4(r) \to -\infty \quad (\ r \ \uparrow \ \bar{r}_1\ ).$$

Also, in the case where $\psi(r_0) > \eta_3(r_0)$, there exists $\bar{r}_0 \in (r_0, R)$ with $\psi(\bar{r}_0) < \eta_3(\bar{r}_0)$. By replacing r_0 by $\bar{r}_0$, the proof is reduced in the case where $\psi(r_0) \leq \eta_3(r_0)$. □

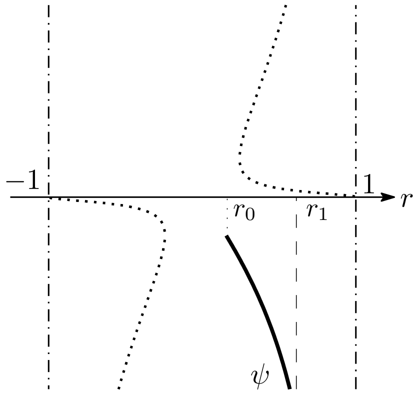

Figure 2.2.2: The behavior of the graph of ψ in Lemma 2.2.4

Lemma 2.2.5. *If there exists $r_0 \in (b, 1)$ with $\eta_1(r_0) < \psi(r_0) < \eta_2(r_0)$, then*

$$\lim_{r\uparrow 1} \psi(r) = \infty.$$

Proof. Assume that there exists a constant $C > 0$ such that $\psi(r) < C$ for all $r \in (r_0, 1)$. Then, there exists $\bar{r}_0 \in (r_0, 1)$ such that $\psi(\bar{r}_0) < \psi(r) < \eta_3(r)$ for all $r \in (\bar{r}_0, 1)$. Therefore, we have

$$\begin{aligned}
\psi'(r) &= -\frac{1}{k(1-r^2)}\left(\psi(r)^2 + 1\right)\left(\sqrt{1-r^2}\psi(r)^2 - (n-1)(r-R)\psi(r) + \sqrt{1-r^2}\right) \\
&> -\frac{1}{k(1-r^2)}\left(\psi(r)^2 + 1\right)\left((1+\psi(\bar{r}_0)^2)\sqrt{1-r^2} - \psi(\bar{r}_0)(n-1)(r-R)\right).
\end{aligned}$$

Then, we find

$$\frac{\psi'(r)}{1+\psi(r)^2} > -\frac{1+\psi(\bar{r}_0)^2}{k\sqrt{1-r^2}} + \frac{\psi(\bar{r}_0)(n-1)r}{k(1-r^2)} - \frac{\psi(\bar{r}_0)(n-1)R}{k(1-r^2)}.$$

Integrating from $\bar{r}_0$ to r, we have

$$\begin{aligned}
\arctan\psi(r) > &-\frac{1+\psi(\bar{r}_0)^2}{k}\arcsin r - \frac{\psi(\bar{r}_0)(n-1)}{2k}\log(1-r^2) - \frac{\psi(\bar{r}_0)(n-1)R}{2k}\log\frac{1+r}{1-r} \\
&+ \frac{1+\psi(\bar{r}_0)^2}{k}\arcsin\bar{r}_0 + \frac{\psi(\bar{r}_0)(n-1)}{2k}\log(1-\bar{r}_0^2) + \frac{\psi(\bar{r}_0)(n-1)R}{2k}\log\frac{1+\bar{r}_0}{1-\bar{r}_0} \\
&+ \arctan\psi(\bar{r}_0) =: h_5(r).
\end{aligned}$$

Then, h_5 is increasing on $(\bar{r}_0, 1)$ and $h_5(\bar{r}_0) = \arctan\psi(\bar{r}_0)$, $\lim_{r\uparrow 1} h_5(r) = \infty$. Therefore, there exists $\bar{r}_1 \in (r_0, 1)$ with $h_5(\bar{r}_1) = \frac{\pi}{2}$ and

$$\psi(r) > \tan h_5(r) \to \infty \quad (\ r \ \uparrow \ \bar{r}_1\ ).$$

This contradicts the assumption that $\psi(r) < C$ for all $r \in (r_0, 1)$. □

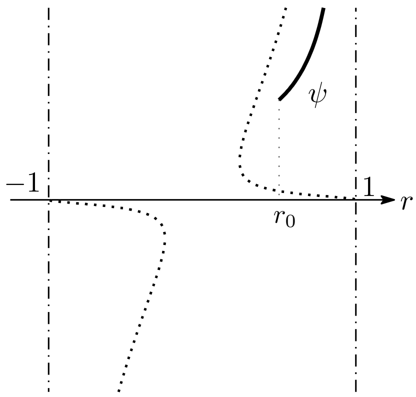

Figure 2.2.3: The behavior of the graph of ψ in Lemma 2.2.5

By proofs similar to Lemma 2.2.4 and Lemma 2.2.5, we obtain the following lemmas.

Lemma 2.2.6. *If there exists* $r_0 \in (-1, b]$ *with* $\psi(r_0) > 0$ *or if there exists* $r_0 \in (b, 1)$ *with* $\psi(r) > \eta_2(r)$, *then there exists* $r_1 \in (-1, r_0)$ *such that*

$$\lim_{r \downarrow r_1} \psi(r) = \infty.$$

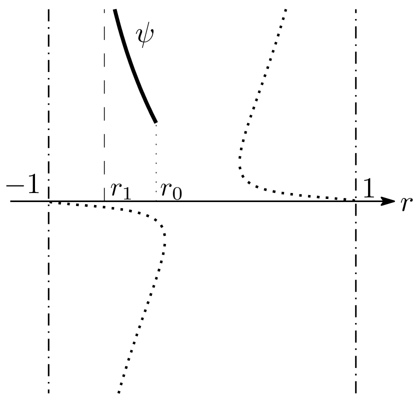

Figure 2.2.4: The behavior of the graph of ψ in Lemma 2.2.6

Lemma 2.2.7. *If there exists* $r_0 \in (-1, a)$ *with* $\eta_1(r_0) < \psi(r_0) < \eta_2(r_0)$, *then*

$$\lim_{r \downarrow -1} \psi(r) = -\infty.$$

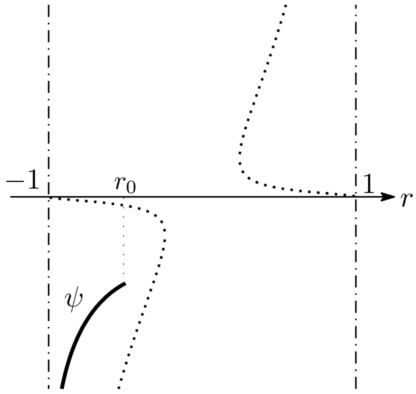

Figure 2.2.5: The behavior of the graph of ψ in Lemma 2.2.7

By lemmas 2.2.3-2.2.7, we obtain the following proposition for the behavior of the graph of the solution ψ of (2.2.5).

Proposition 2.2.8. *For the solution ψ of the equation* (2.2.5)*, the behavior of the graph of ψ is like one of those illustrated by* Figures 2.2.6-2.2.10*.*

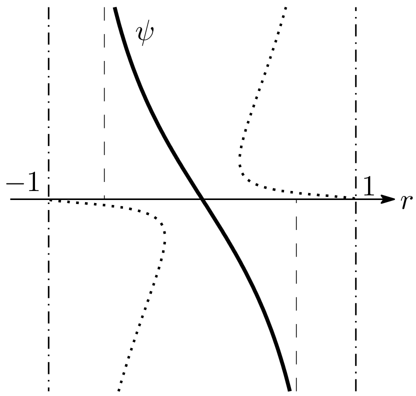

Figure 2.2.6: The graph of ψ (Type I)

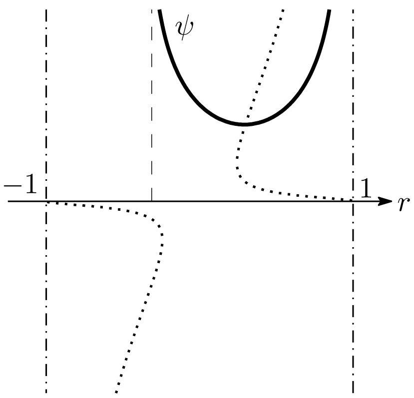

Figure 2.2.7: The graph of ψ (Type II)

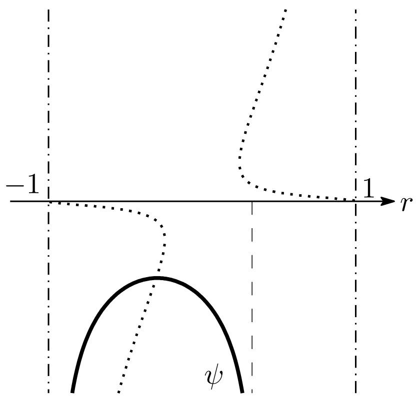

Figure 2.2.8: The graph of ψ (Type III)

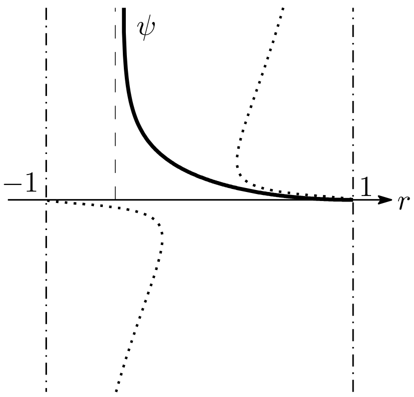

Figure 2.2.9: The graph of ψ (Type IV)

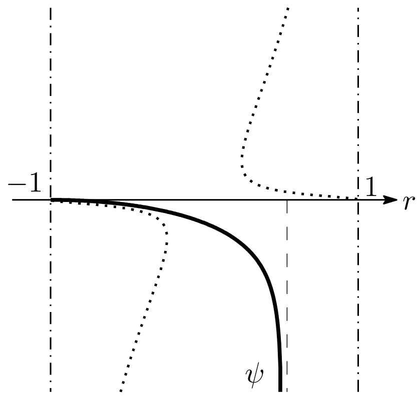

Figure 2.2.10: The graph of ψ (Type V)

For the graph of ψ in Proposition 2.2.8, we have not yet shown whether ψ in the case of Figures 2.2.9 and 2.2.10 exists or not. From the following lemma, we obtain the existence.

Lemma 2.2.9. *The solution ψ of the equation* (2.2.5) *in* Figures 2.2.9 *and* Figure 2.2.10 *exists.*

Proof. For the set S of all solutions of the equation (2.2.5), we define sets S_1, S_2, $S_3 \subset S$ by

$$\begin{aligned} S_1 &:= \{\psi \in S | \exists r_0 \in (-1,1) : \psi(r_0) = 0\}, \\ S_2 &:= \{\psi \in S | \exists r_0 \in (-1,1) : \psi(r_0) = \eta_1(r_0) \textit{ or } \psi(r_0) = \eta_2(r_0)\}, \\ S_3 &:= \{\psi \in S | \psi(1) = 0 \textit{ or } \psi(-1) = 0\}. \end{aligned}$$

Then, we have

$$(-1,1) \times \mathbb{R} = \bigcup_{\psi \in S_1 \cup S_2 \cup S_3} \mathrm{Graph}(\psi).$$

Since $\bigcup_{\psi \in S_1} \mathrm{Graph}(\psi)$ and $\bigcup_{\psi \in S_2} \mathrm{Graph}(\psi)$ are open sets and $(-1,1) \times \mathbb{R}$ is connected, we find S_3 is not an empty set and we obtain the statement of this lemma. □

Define ζ_1 and ζ_2 by $\zeta_i(r) = \eta_i(r)/(k\sqrt{1-r^2})$, $(i = 1, 2)$. By Proposition 2.2.8, we obtain the following proposition.

Proposition 2.2.10. *For the solution V of the equation* (2.2.4)*, the behavior of the graph of V' is like one of those illustrated by* Figures 2.2.11-2.2.19*. Here, the dotted curves in* Figures 2.2.11-2.2.19 *are the graphs of ζ_1 and ζ_2.*

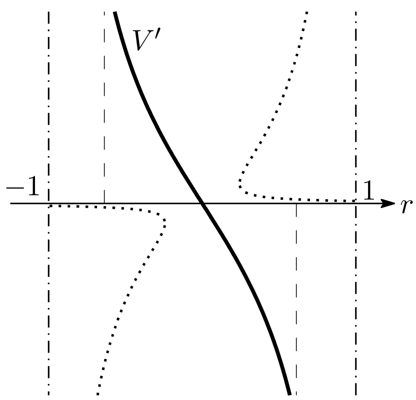

Figure 2.2.11: The graph of V' (Type I)

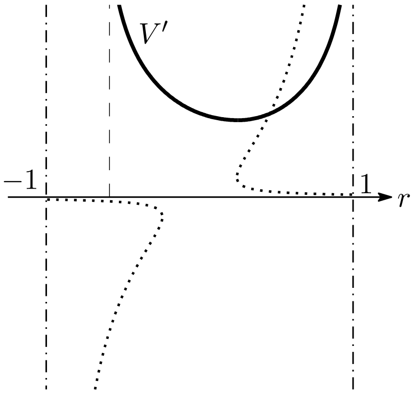

Figure 2.2.12: The graph of V' (Type II)

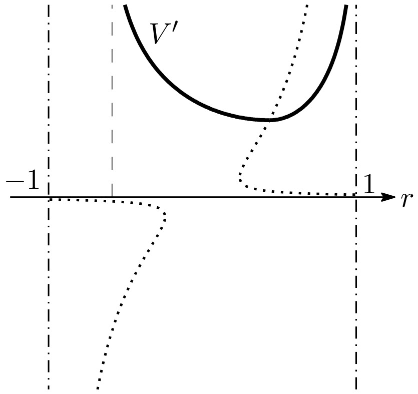

Figure 2.2.13: The graph of V' (Type II$'$)

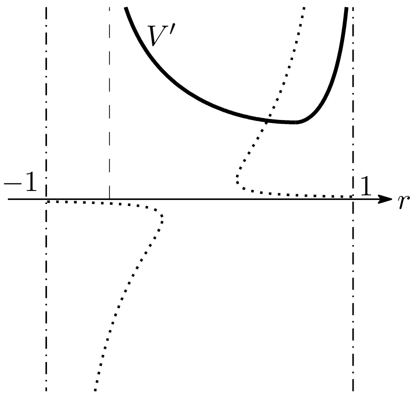

Figure 2.2.14: The graph of V' (Type II$''$)

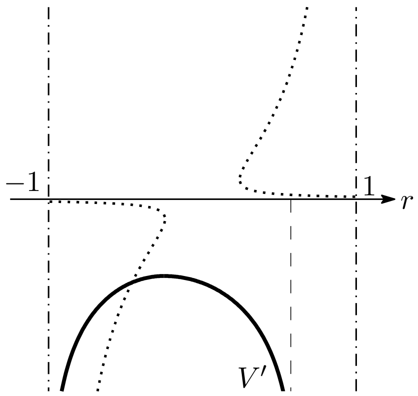

Figure 2.2.15: The graph of V' (Type III)

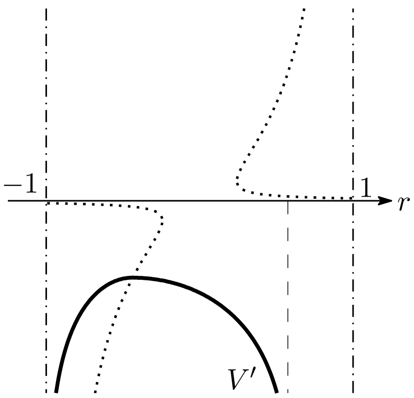

Figure 2.2.16: The graph of V' (Type III$'$)

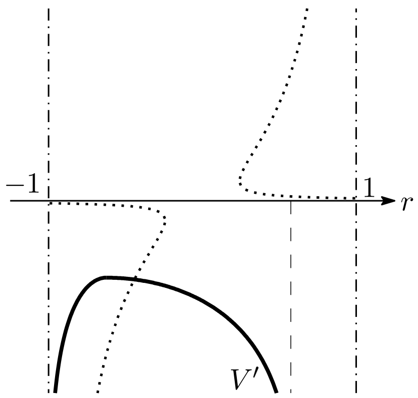

Figure 2.2.17: The graph of V' (Type III$''$)

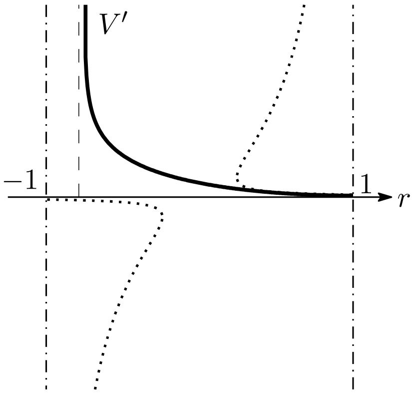

Figure 2.2.18: The graph of V' (Type IV)

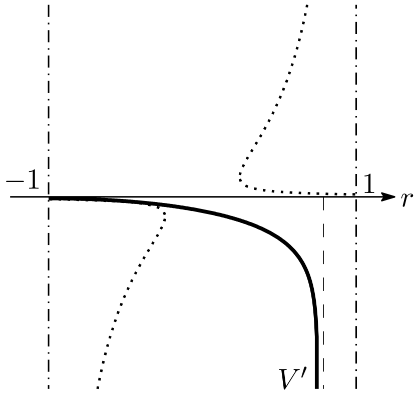

Figure 2.2.19: The graph of V' (Type V)

Proof. For the solution V of the equation (2.2.4), we have $V'(r) = \psi(r)/(k\sqrt{1-r^2})$ and ψ is the solution of the equation (2.2.5). Therefore, when the graph of ψ is like one of those illustrated by Figure 2.2.6, 2.2.9 and 2.2.10, it is clear that the graph of V' is like one illustrated by Figure 2.2.11, 2.2.18 and 2.2.19 respectively. In the case where the graph of ψ is like one of those illustrated by Figure 2.2.7 and 2.2.8, there exists $r_0 \in (-1, a] \cup [b, 1)$ with $\psi(r_0) = \eta_1(r_0)$ or $\psi(r_0) = \eta_2(r_0)$ and we find $\psi'(r_0) = 0$ by Lemma 2.2.3. Then, we obtain $V''(r_0) = r_0\psi(r_0)/(k(1-r_0^2)^{\frac{3}{2}}) + \psi'(r_0)/(k\sqrt{1-r_0^2}) = r_0\psi(r_0)/(k(1-r_0^2)^{\frac{3}{2}})$. Therefore, when the graph of ψ is like one illustrated by Figure 2.2.7, if $r_0 > 0$, then $V''(r_0) > 0$ and the graph of V' is like one illustrated by Figure 2.2.12, if $r_0 = 0$, then $V''(r_0) = 0$ and the graph of V' is like one illustrated by Figure 2.2.13, and if $r_0 < 0$, then $V''(r_0) < 0$ and the graph of V' is like one illustrated by Figure 2.2.14. Also, when the graph of ψ is like one illustrated by Figure 2.2.8, if $r_0 < 0$, then $V''(r_0) > 0$ and the graph of V' is like one illustrated by Figure 2.2.15, if $r_0 = 0$, then $V''(r_0) = 0$ and the graph of V' is like one illustrated by Figure 2.2.16, and if $r_0 > 0$, then $V''(r_0) < 0$ and the graph of V' is like one illustrated by Figure 2.2.17. In the case $k = 1, 3, 6$, we find $a < R = 0 < b$. Therefore, when the graph of ψ is like one of those illustrated by Figure 2.2.7 and Figure 2.2.8, the graph of V' is like one illustrated by Figure 2.2.12 and Figure 2.2.15 respectively if $k = 1, 3, 6$. □

By Proposition 2.2.10, we obtain Theorem 2.1.1. For the solution V of the equation (2.2.4), when the graph of V' is like one of those illustrated by Figure 2.2.11, 2.2.18 and 2.2.19, it is clear that the graph of V is like one illustrated by Figure 2.1.1, 2.1.4 and 2.1.5 respectively. When the graph of V' is like one of those illustrated by Figure 2.2.12, 2.2.13 and 2.2.14, the graph of V is like one illustrated by Figure 2.1.2. Also, when the graph of V' is like one of those illustrated by Figure 2.2.15, 2.2.16 and 2.2.17, the graph of V is like one illustrated by Figure 2.1.3.

Chapter 3

Translators invariant under hyperpolar actions

3.1 Introduction

This chapter is based on [9].

In this chapter, we consider the case where N is a symmetric space G/K of compact type, where we give G/K the G-invariant metric induced from the (-1)-multiple of the Killing form of the Lie algebra $\mathfrak{g}$. When the rank of G/K is equal to one (i.e., $G/K = SO(n+1)/SO(n)$, $SU(n+1)/S(U(1)\times U(n))$, $Sp(n+1)/(Sp(1)\times Sp(n))$ or $F_4/\mathrm{Spin}(9)$), we can take the function $r : G/K \to \mathbb{R}$ with $\|\nabla r\| = 1$ whose level sets are the orbits of the isotropy group action $K \curvearrowright G/K$. Then, a function u on a K-invariant domain M of G/K which is constant along each orbit of the isotropy group action is given by $u := V \circ r$ for some function V on $r(M)$. It is clear that the shape of the graph of u is dominated by that of V. Hence we suffice to classify the shape of the graph of V to classify that of u. Lawn and Ortega [16] studied the translator given by a function invariant under a cohomegeneity one action on a pseudo-Riemannian manifold. They showed that the graph of the function gives a translator if and only if the function is a solution of some ODE. From the ODE, we obtain the following classification theorem for the shape of the graph of V.

Theorem 3.1.1. *The graph of V is given by one of the curves obtained by parallel translating curves as in* Figures 3.1.1-3.1.5 *in the vertical direction. The value α in* Figures 3.1.1-3.1.5 *is the constant given by*

$$\alpha = \begin{cases} \sqrt{\dfrac{n-1}{2}}\pi & (\text{when } G/K = SO(n+1)/SO(n)) \\ \sqrt{n+1}\pi & (\text{when } G/K = SU(n+1)/S(U(1)\times U(n))) \\ \sqrt{2(n+2)}\pi & (\text{when } G/K = Sp(n+1)/(Sp(1)\times Sp(n))) \\ \dfrac{a\pi}{4} & (\text{when } G/K = F_4/\mathrm{Spin}(9)), \end{cases}$$

where a is the positive constant such that a^2 is equal to the $\frac{1}{4}$-multiple of the maximal sectional curvature of $F_4/\mathrm{Spin}(9)$.

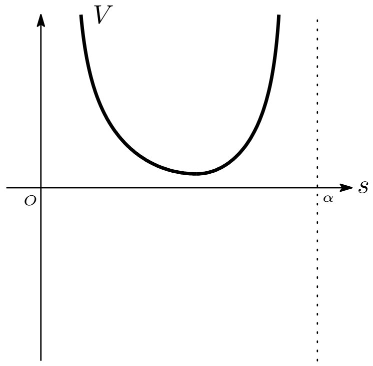

Figure 3.1.1: The graph of V (Type I)

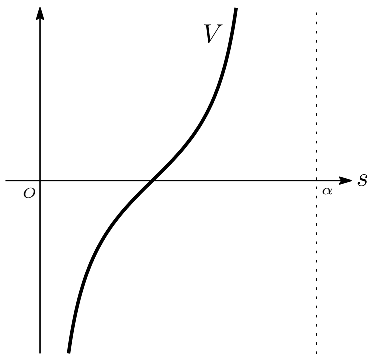

Figure 3.1.2: The graph of V (Type II)

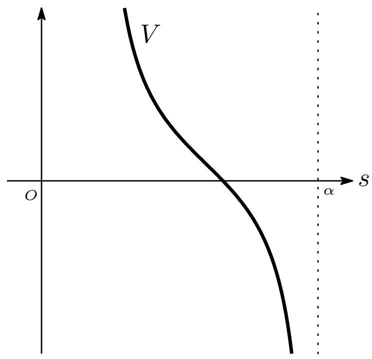

Figure 3.1.3: The graph of V (Type III)

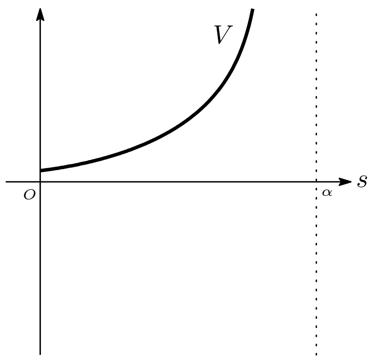

Figure 3.1.4: The graph of V (Type IV)

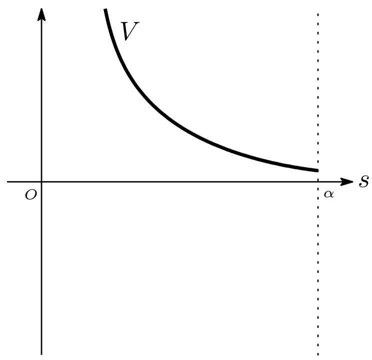

Figure 3.1.5: The graph of V (Type V)

Next we consider the case where G/K is a higher rank irreducible symmetric space of compact type and a translator (for the mean curvature flow) given by a graph of a function on G/K which is invariant under a Hermann action $H \curvearrowright G/K$ of cohomegeneity two, where *Hermann action* means that H is a symmetric subgroup of G. Assume that $H \curvearrowright G/K$ is commutative, that is, $\theta_K \circ \theta_H = \theta_H \circ \theta_K$ holds for the involutions θ_K and θ_H of G satisfying $(\mathrm{Fix}\,\theta_K)_0 \subset K \subset \mathrm{Fix}\,\theta_K$ and $(\mathrm{Fix}\,\theta_H)_0 \subset H \subset \mathrm{Fix}\,\theta_H$, where $\mathrm{Fix}(\cdot)$ is the fixed point group of $(\cdot)$ and $(\cdot)_0$ is the identity component of $(\cdot)$. Here we note that Hermann actions are hyperpolar actions, where a *hyperpolar action* means an isometric action of a compact Lie group on G/K which admits a complete flat totally geodesic submanifold in G/K meeting all orbits of the action orthogonally. The complete flat totally geodesic submanifold is called a *flat section* of this action.

Let $r = (r_1, r_2) : G/K \to \mathbb{R}^2$ be a map on G/K with $g(\nabla r_i, \nabla r_j) = \delta_{ij}$ $(i, j \in \{1, 2\})$ whose level sets are the orbits of the action $H \curvearrowright G/K$. Then, a function u on a H-invariant domain M of G/K is invariant under the action $H \curvearrowright G/K$ if and only if u is described as $u = V \circ r$ for some function V on $r(M)$. Let Σ be the flat section of the H-action through $o := eK$, were we note that Σ is diffeomorphic to a torus $T^2 (= S^1 \times S^1)$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the canonical decomposition associated to the symmetric pair (G, K). The space $\mathfrak{p}$ is identified with the tangent space $T_o(G/K)$ through the restriction $\pi_{*e}|_{\mathfrak{p}}$ of the differential π_{*e} of the natural projection $\pi : G \to G/K$. There exists the maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ satisfying $\exp_o(\mathfrak{a}) = \Sigma$, where $\exp_o$ is the exponential map of G/K at o. Let $\mathcal{C}(\subset \mathfrak{a})$ be a Weyl domain and W be the Weyl group. Denote by $\boldsymbol{X}$ the tangent vector field on $\exp_o(\mathcal{C})$ defined by assigning the mean curvature vector of the orbit $H \cdot w$ at w to each $w \in \exp_o(\mathcal{C})$. By identifying $\exp_o(\mathcal{C})$ with $\mathcal{C}$, we regard $\boldsymbol{X}$ as a tangent vector field on $\mathcal{C}$.

Theorem 3.1.2. *The graph Γ of $u = V \circ r$ is a translator if and only if V satisfies*

$$\sum_{i,j=1}^{2} \frac{\partial^2 V}{\partial x_i \partial x_j} \frac{\partial V}{\partial x_i} \frac{\partial V}{\partial x_j} - \left(1 + |\nabla V|^2\right) \left(\sum_{i=1}^{2} X_i \frac{\partial V}{\partial x_i} + \Delta V - 1 \right) = 0, \tag{3.1.1}$$

where (x_1, x_2) is the Euclidean coordinate of $\mathfrak{a}$, V is regarded as a function on $\mathfrak{a}$ through $(x_1, x_2) : \mathfrak{a} \to \mathbb{R}^2$ and X_i $(i = 1, 2)$ are the components of the tangent vector field $\boldsymbol{X}$ on $\mathcal{C}$ with respect to the Euclidean coordinate (x_1, x_2) of $\mathfrak{a}$ (i.e., $\boldsymbol{X} = \sum_{i=1}^{2} X_i \frac{\partial}{\partial x_i}$).

For all commutative Hermann actions of cohomogeneity two on an irreducible symmetric space of compact type, the explicit descriptions of the component (X_1, X_2) of the above tangent vector field $\boldsymbol{X}$ on $\mathcal{C}$ are given in [14]. By using the explicit description, we can describe the PDE (3.1.1) of order two explicitly. Clearly we can choose the above function r as $r|_{\exp_o(\mathcal{C})} = (x_1, x_2) \circ (\exp_o|_{\mathcal{C}})^{-1}$ holds. According to [14], $\boldsymbol{X}$ is described as $\boldsymbol{X} = \nabla \rho$ for some convex function ρ on $\mathcal{C}$. Next we consider the case where V is constant along each level set of ρ. In this case, ∇V is described as $\nabla V = F\boldsymbol{X}$ for some function F on $\mathcal{C}$. It is clear that the shape of the graph of u is dominated by F. Hence we suffice to investigate F to classify the shape of the graph of u. In this case, we obtain the following fact.

Theorem 3.1.3. *Assume that V is constant along each level set of ρ and let F be the function on $\mathcal{C}$ satisfying $\nabla V = F\boldsymbol{X}$. Then the graph Γ of $u = V \circ r$ is a translator if and only if F satisfies*

$$\boldsymbol{X}(F) = \frac{1}{2} \boldsymbol{X}(|\boldsymbol{X}|^2) F^3 - (1 + |\boldsymbol{X}|^2 F^2)((|\boldsymbol{X}|^2 + \operatorname{div} \boldsymbol{X}) F - 1). \tag{3.1.2}$$

By using the explicit descriptions of $\boldsymbol{X}$ in [14], we can describe the PDE (3.1.2) of order one explicitly. As one cexample we investigate the shape of the graph of V in the case where the Hermann action $H \curvearrowright G/K$ is the dual action of the Hermann type action $SO_0(1, 2) \curvearrowright SL(3, \mathbb{R})/SO(3)$.

In Section 2, we investigate translators which are invariant under the isotropy group action of rank one symmetric spaces of compact type and prove Theorem 3.1.1. In Section 3, we investigate translators which are invariant under Hermann action of cohomogeneity two on higher rank symmetric spaces of compact type and prove Theorems 3.1.2 and 3.1.3.

3.2 The case of cohomogenity one

Let (N, g) be an n-dimensional Riemannian manifold and $u : M \to \mathbb{R}$ be a (C^∞)-function on a domain M of N. Let f be the graph embedding of u, that is, the embedding $f : M \hookrightarrow (N, g)$ defined by $f(x) = (x, u(x))$ $(x \in M)$. Denote by Γ the graph of u and H the mean curvature vector field of f. Also, denote by $\nabla(\cdot)$ and $\mathrm{div}(\cdot)$ be the gradient vector field and the divergence of $(\cdot)$ with respect to g, respectively. For the translatority of Γ, the following fact holds (see [4], [7] and [16]).

Lemma 3.2.1. *If Γ is a translator, u satisfies*

$$\sqrt{1 + \|\nabla u\|^2}\ \mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + \|\nabla u\|^2}}\right) = 1. \tag{3.2.1}$$

Conversely, if u satisfies (3.2.1), *Γ is a translator.*

We consider the case where (N, g) is a rank one symmetric space G/K of compact type and u is a function on a K-invariant domain M of G/K which is invariant under the isotropy group action $K \curvearrowright G/K$, where we give G/K the G-invariant metric induced from the (-1)-multiple of the Killing form of $\mathfrak{g}$. Let $r : G/K \to \mathbb{R}$ be the function on G/K with $\|\nabla r\| = 1$ whose level sets are the orbits of $K \curvearrowright G/K$. Then, since u is invariant under the action $K \curvearrowright G/K$, u is described as $u = V \circ r$ for some function V on $r(M)$. According to the result by Lawn and Ortega (Theorem 3.5 of [16]) for the graph of a function on a pseudo-Riemaniann manifold which is invariant under a cohomogeneity one proper isometric action of a Lie group, we obtain the following fact.

Proposition 3.2.2. *The graph Γ of $V \circ r$ is a translator if and only if V satisfies*

$$V''(s) = \left(1 + V'(s)^2\right)\left(1 - h(s)V'(s)\right) \tag{3.2.2}$$

where $(\cdot)'$ denotes the derivative of $(\cdot)$ and $h(s)$ is the constant mean curvature of the orbit $r^{-1}(s)$ of $K \curvearrowright G/K$.

Let $\mathfrak{p}$ be as in Introduction. $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and $\triangle_+$ be the positive root system with respect to $\mathfrak{a}$. Then we have

$$\triangle_+ = \begin{cases} \left\{\dfrac{1}{\sqrt{2(n-1)}}\langle \boldsymbol{e}, \cdot\rangle\right\} & (\text{when } G/K = SO(n+1)/SO(n)) \\ \left\{\dfrac{1}{4\sqrt{n+1}}\langle \boldsymbol{e}, \cdot\rangle,\ \dfrac{1}{2\sqrt{n+1}}\langle \boldsymbol{e}, \cdot\rangle\right\} & (\text{when } G/K = SU(n+1)/S(U(1)\times U(n))) \\ \left\{\dfrac{1}{4\sqrt{2(n+2)}}\langle \boldsymbol{e}, \cdot\rangle,\ \dfrac{1}{2\sqrt{2(n+2)}}\langle \boldsymbol{e}, \cdot\rangle\right\} & (\text{when } G/K = Sp(n+1)/(Sp(1)\times Sp(n))) \\ \{a\langle \boldsymbol{e}, \cdot\rangle,\ 2a\langle \boldsymbol{e}, \cdot\rangle\} & (\text{when } G/K = F_4/\mathrm{Spin}(9)), \end{cases} \tag{3.2.3}$$

where $\boldsymbol{e}$ is a unit normal vector of $\mathfrak{a}$, $\langle\ ,\ \rangle$ is the restriction of the (-1)-multiple of of the Killing form to $\mathfrak{a}$ and a is the positive constant stated in Theorem 3.1.1. Thus, in the case of $G/K = SU(n+1)/S(U(1) \times U(n))$, $Sp(n+1)/(Sp(1) \times Sp(n))$ or $F_4/\mathrm{Spin}(9)$, we write $\triangle_+$ as $\triangle_+ = \{\lambda, 2\lambda\}$. The multiplicity $m_{2\lambda}$ of 2λ is given by

$$m_{2\lambda} = \begin{cases} 1 & (\text{when } G/K = SU(n+1)/S(U(1)\times U(n))) \\ 3 & (\text{when } G/K = Sp(n+1)/(Sp(1)\times Sp(n))) \\ 7 & (\text{when } G/K = F_4/\mathrm{Spin}(9)) \end{cases} \tag{3.2.4}$$

Verhóczki [25] described explicitly the eigenvalues (i.e., the principal curvatures) of the shape operators of the orbits of the isotropy group actions $K \curvearrowright G/K$ by using the positive restricted roots (i.e., the elements of $\triangle_+$ (see Theorem 1 of [25]). By using the explicit descriptions of principal curvatures, (3.2.3) and (3.2.4), we can explicitly described the above constant mean curvature $h(s)$ in the case of $G/K = SO(n+1)/SO(n)$, $SU(n+1)/S(U(1) \times U(n))$, $Sp(n+1)/(Sp(1) \times Sp(n))$ or $F_4/\mathrm{Spin}(9)$ as follows.

Lemma 3.2.3. *The mean curvatyre $h(s)$ of the principle orbit $r^{-1}(s)$ is given by*

$$h(s) = \begin{cases} \sqrt{\dfrac{n-1}{2}} \cdot \dfrac{1}{\tan \frac{s}{\sqrt{2(n-1)}}} & (G/K = SO(n+1)/SO(n)) \\ \dfrac{2n-1-\tan^2 \frac{s}{2\sqrt{n+1}}}{2\sqrt{n+1} \cdot \tan \frac{s}{2\sqrt{n+1}}} & (G/K = SU(n+1)/S(U(1) \times U(n))) \\ \dfrac{4n-1-3\tan^2 \frac{s}{2\sqrt{2(n+2)}}}{2\sqrt{2(n+2)} \cdot \tan \frac{s}{2\sqrt{2(n+2)}}} & (G/K = Sp(n+1)/(Sp(1) \times Sp(n))) \\ \dfrac{(16 - 7\tan^2 as)a}{\tan as} & (G/K = F_4/\mathrm{Spin}(9)) \end{cases} \tag{3.2.5}$$

From Proposition 3.2.2 and Lemma 3.2.3, we prove Theorem 3.1.1.

Proof of Theorem 3.1.1. We consider the case of $G/K = SU(n+1)/S(U(1) \times U(n))$. Define a function ψ by

$$\psi(x) := V'(2\sqrt{n+1}\arctan x) \quad (x \in r(M)).$$

From (3.2.2) and (3.2.5), it is shown that ψ satisfies the following ODE:

$$\psi'(x) = \frac{2\sqrt{n+1}}{1+x^2}\left(1+\psi(x)^2\right)\left(1 - \frac{2n-1-x^2}{2\sqrt{n+1}\,x}\psi(x)\right) \quad (x > 0) \tag{3.2.6}$$

We shall analyze the shape of the solution ψ of (3.2.6) to recognize the shape of V. Define a function $\eta : [0, \infty) \setminus \{\sqrt{2n-1}\} \to \mathbb{R}$ by $\eta(x) := \frac{2\sqrt{n+1}x}{2n-1-x^2}$. From (3.2.6), we can show the following facts directly:

- (i) When $x \in (0, \sqrt{2n-1})$,
  - (i-a) if $\psi(x) > \eta(x)$, then $\psi'(x) < 0$
  - (i-b) if $\psi(x) < \eta(x)$, then $\psi'(x) > 0$
  - (i-c) if $\psi(x) = \eta(x)$, then $\psi'(x) = 0$;
- (ii) When $x \in (\sqrt{2n-1}, \infty)$,
  - (ii-a) if $\psi(x) > \eta(x)$, then $\psi'(x) > 0$
  - (ii-b) if $\psi(x) < \eta(x)$, then $\psi'(x) < 0$
  - (ii-c) if $\psi(x) = \eta(x)$, then $\psi'(x) = 0$
- (iii) When $x = \sqrt{2n-1}$, $\psi'(x) > 0$.

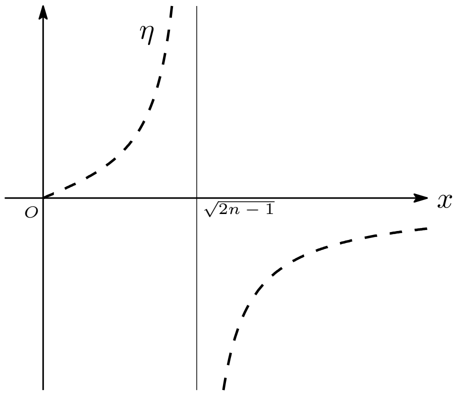

Figure 3.2.1: The graph of η

Next we shall show that the following fact for ψ holds.

($*_1$) *If there exists* $x_0 \in (0, \sqrt{2n-1})$ *with* $\psi(x_0) > \eta(x_0)$, $\lim_{x \downarrow x_1} \psi(x) = \infty$ *holds for some* $x_1 \in (0, x_0)$ *(see Figure 3.2.2).*

Take any $x \in (0, x_0)$. Then, by using (i-a), we can show

$$\begin{aligned}\psi'(x) &= \frac{2\sqrt{n+1}}{1+x^2}\left(1+\psi(x)^2\right)\left(1-\frac{\psi(x)}{\eta(x)}\right) \\ &< \frac{2\sqrt{n+1}}{1+x^2}\left(1+\psi(x)^2\right)\left(1-\frac{\psi(x_0)}{\eta(x)}\right).\end{aligned}$$

Therefore, we have

$$\frac{\psi'(x)}{1+\psi(x)^2} < \frac{2\sqrt{n+1}}{1+x^2} - \frac{(2n-1)\psi(x_0)}{x(1+x^2)} + \frac{\psi(x_0)x}{1+x^2}.$$

By integrating both-hand sides of this inequality from x to x_0, we obtain

$$\begin{aligned}\arctan\psi(x_0) - \arctan\psi(x) < &-2\sqrt{n+1}\arctan x + (2n-1)\psi(x_0)\log\frac{x}{\sqrt{1+x^2}} \\ &- \frac{\psi(x_0)}{2}\log(1+x^2) + 2\sqrt{n+1}\arctan x_0 \\ &- (2n-1)\psi(x_0)\log\frac{x_0}{\sqrt{1+{x_0}^2}} + \frac{\psi(x_0)}{2}\log(1+{x_0}^2) =: h_6(x).\end{aligned}$$

and hence

$$\psi(x) > \tan(-h_6(x) + \arctan\psi(x_0)).$$

On the other hand, h_6 is increasing on $(0, x_0)$ and the following relations hold:

$$h_6(x_0) = 0 \quad \text{and} \quad \lim_{x\downarrow 0} h_6(x) = -\infty.$$

Therefore, there exists $\bar{x}_1 \in (0, x_0)$ such that

$$\psi(x) > \tan(-h_6(x) + \arctan\psi(x_0)) \;\to\; \infty \quad (x \to \bar{x}_1).$$

Thus the fact ($*_1$) is shown.

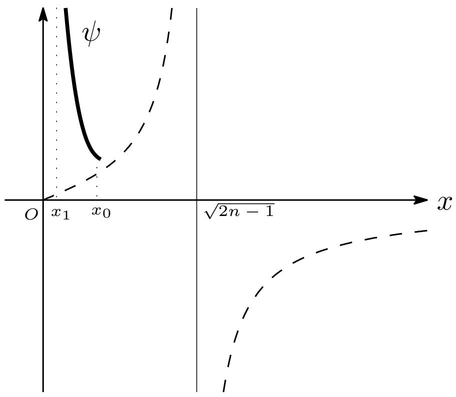

Figure 3.2.2: The behavior I of the graph of ψ

Similarly, by using (i-b), we can show the following facts for ψ;

($*_2$) *If there exists* $x_0 \in (0, \sqrt{2n-1})$ *with* $\psi(x_0) < 0$, $\lim_{x \downarrow x_1} \psi(x) = -\infty$ *holds for some* $x_1 \in (0, x_0)$ *(see Figure 3.2.3);*

Also, by using (ii-a), we can show the following facts for ψ;

($*_3$) If there exists $x_0 \in (\sqrt{2n-1}, \infty)$ with $\psi(x_0) > 0$, $\lim_{x \uparrow x_1} \psi(x) = \infty$ holds for some $x_1 \in (x_0, \infty)$ (see Figure 3.2.4);

Also, by using (ii-b), we can show the following facts for ψ;

($*_4$) *If there exists* $x_0 \in (\sqrt{2n-1}, \infty)$ *with* $\psi(x_0) < \eta(x_0)$, $\lim_{x \uparrow x_1} \psi(x) = -\infty$ *holds for some* $x_1 \in (x_0, \infty)$ *(see Figure 3.2.5).*

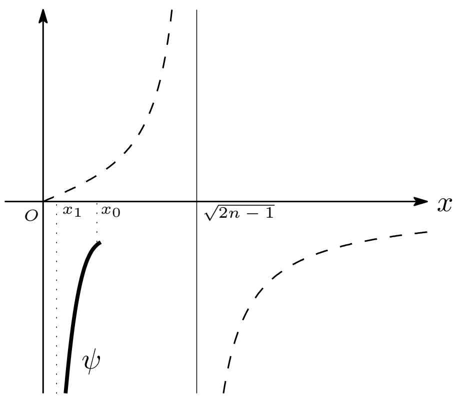

Figure 3.2.3: The behavior II of the graph of ψ

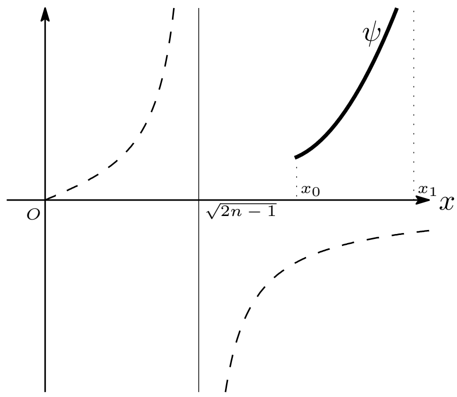

Figure 3.2.4: The behavior III of the graph of ψ

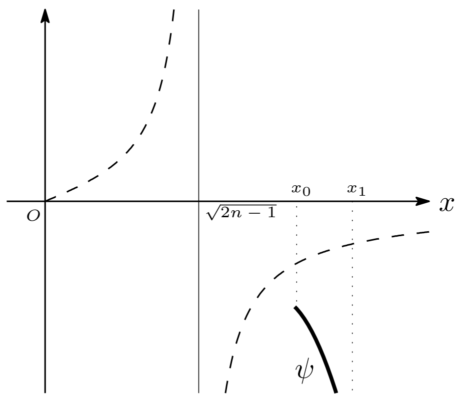

Figure 3.2.5: The behavior IV of the graph of ψ

From the facts $(*_1) - (*_4)$, the shape of the graph of the solution ψ is one of the curves as in Figures 3.2.6-3.2.10 in the case of $G/K = SU/S(U(1) \times U(n))$.

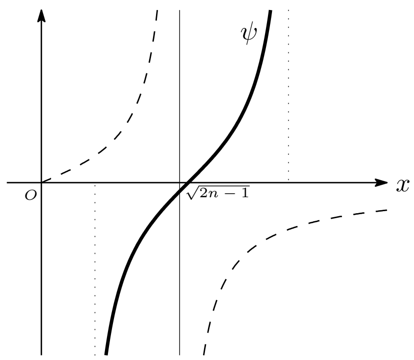

Figure 3.2.6: The graph of ψ (Type I)

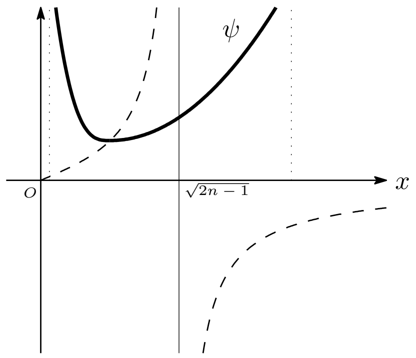

Figure 3.2.7: The graph of ψ (Type II)

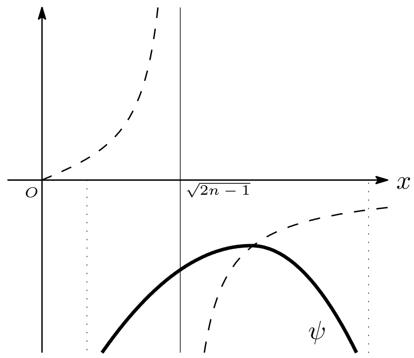

Figure 3.2.8: The graph of ψ (Type III)

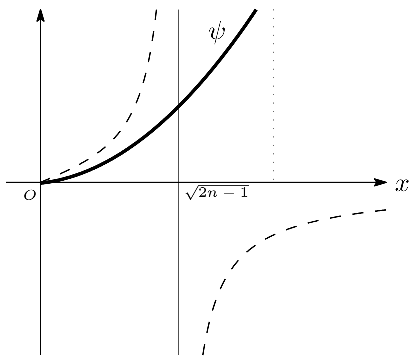

Figure 3.2.9: The graph of ψ (Type IV)

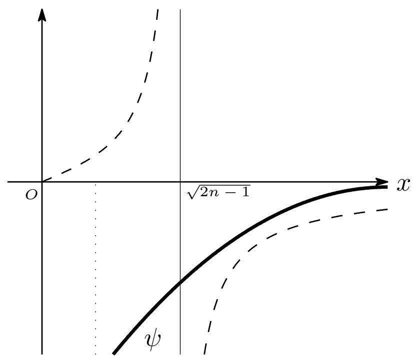

Figure 3.2.10: The graph of ψ (Type V)

From (3.2.5), we see that the domain V is included by $(0, \sqrt{n+1}\pi)$, that is, the value α in the statement of Theorem 3.1.1 is equal to $\sqrt{n+1}\pi$. Hence, from the above classification of the shape of ψ, we can classify the shape of V as in Theorem 3.1.1.

Similarly, in the case where $G/K = SO(n+1)/SO(n)$, $Sp(n+1)/(Sp(1) \times Sp(n))$ and $F_4/\mathrm{Spin}(9)$, we can classify the shape of the graph of V. In these cases, by (3.2.5), the value α in the statement of Theorem 3.1.1 is given by

$$\alpha = \begin{cases} \sqrt{\dfrac{n-1}{2}}\pi & (\text{when } G/K = SO(n+1)/SO(n)) \\ \sqrt{2(n+2)}\pi & (\text{when } G/K = Sp(n+1)/(Sp(1) \times Sp(n))) \\ \dfrac{\pi}{4a} & (\text{when } G/K = F_4/\mathrm{Spin}(9)). \end{cases}$$

□

3.3 The case of cohomogenity two

In this section, we consider the case where G/K is a higher rank irreducible symmetric space of compact type and a translator given by a graph of a function u on a H-invariant domian M of G/K which is invariant under a Hermann action $H \curvearrowright G/K$ of cohomegeneity two. Assume that $H \curvearrowright G/K$ is commutative, that is, $\theta_K \circ \theta_H = \theta_H \circ \theta_K$ holds for the involutions θ_K and θ_H of G satisfying $(\mathrm{Fix}\theta_K)_0 \subset K \subset \mathrm{Fix}\theta_K$ and $(\mathrm{Fix}\theta_H)_0 \subset H \subset \mathrm{Fix}\theta_H$. Let $r = (r_1, r_2) : G/K \to \mathbb{R}^2$ be a map on G/K with $g(\nabla r_i, \nabla r_j) = \delta_{ij}$ $(i, j \in \{1, 2\})$ whose level sets give the orbits of the action $H \curvearrowright G/K$. Then, a function u is described as $u = V \circ r$ for some function V on $r(M)$.

By using Theorem 3.1.1, we prove Theorem 3.1.2.

Proof of Theorem 3.1.2. The function V is regarded as a function on $\mathfrak{a}$ through the Euclidean coordinate $(x_1, x_2) : \mathfrak{a} \to \mathbb{R}^2$ of $\mathfrak{a}$. As stated in Introduction, we may assume that

$$r|_{\exp_o(\mathcal{C})} = -(x_1, x_2) \circ (\exp_o|_{\mathcal{C}})^{-1}. \tag{3.3.1}$$

From $u = V \circ r$, we find

$$\nabla u = \sum_{i=1}^{2} \frac{\partial V}{\partial x_i} \nabla r_i, \quad \|\nabla u\|^2 = |\nabla V|^2,$$
$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + \|\nabla u\|^2}}\right) = \sum_{1=1}^{2} \Delta r_i \frac{\partial V}{\partial x_i} + \Delta V - \frac{1}{1 + |\nabla V|^2} \sum_{i,j=1}^{2} \frac{\partial V}{\partial x_i} \frac{\partial V}{\partial x_j} \frac{\partial^2 V}{\partial x_i \partial x_j}.$$

From these relations, we can show that the PDE (3.2.1) is reduced to

$$\sum_{i,j=1}^{2} \frac{\partial V}{\partial x_i} \frac{\partial V}{\partial x_j} \frac{\partial^2 V}{\partial x_i \partial x_j} - \left(1 + |\nabla V|^2\right) \left(\sum_{i=1}^{2} \Delta r_i \frac{\partial V}{\partial x_i} + \Delta V - 1\right) = 0. \tag{3.3.2}$$

Since the tangent vector $\boldsymbol{X}$ on $\exp_o(\mathcal{C})$ is defined by assigning the mean curvature vector of the orbit $H \cdot w$ at w to each $w \in \exp_o(\mathcal{C})$, we have

$$\langle \boldsymbol{X}, \nabla r_i \rangle = -\Delta r_i.$$

From this relation and (3.3.1), we find

$$\boldsymbol{X} = \sum_{i=1}^{2} \Delta r_i \left(-\nabla r_i\right).$$

Therefore, from (3.3.1), we obtain $X_i = \Delta r_i$. $\square$

By using Theorem 3.1.2, we prove Theorem 3.1.3.

Proof of Theorem 3.1.3. Assume that V is constant along each level set of ρ, where ρ is the convex function with $\nabla \rho = \boldsymbol{X}$. Then, ∇V is described as $\nabla V = F\boldsymbol{X}$ for some function F on $\mathcal{C}$. Clearly we have

$$\frac{\partial V}{\partial x_i} = F X_i, \quad \frac{\partial^2 V}{\partial x_i \partial x_j} = \frac{\partial F}{\partial x_i} X_j + F \frac{\partial X_j}{\partial x_i} \quad (i, j \in \{1, 2\}).$$

Also, we can derive

$$|\nabla V|^2 = |\boldsymbol{X}|^2 F^2, \quad \Delta V = \boldsymbol{X}(F) + (\operatorname{div}\boldsymbol{X}) F, \quad \sum_{i=1}^{2} X_i \frac{\partial V}{\partial x_i} = |\boldsymbol{X}|^2 F,$$
$$\sum_{i,j=1}^{2} \frac{\partial^2 V}{\partial x_i \partial x_j} \frac{\partial V}{\partial x_i} \frac{\partial V}{\partial x_j} = |\boldsymbol{X}|^2 F^2 \boldsymbol{X}(F) + \frac{1}{2} \boldsymbol{X}(|\boldsymbol{X}|^2) F^3.$$

Hence we obtain

$$
\begin{aligned}
&\sum_{i,j=1}^{2}\frac{\partial^2 V}{\partial x_i\partial x_j}\frac{\partial V}{\partial x_i}\frac{\partial V}{\partial x_j}-\left(1+|\nabla V|^2\right)\left(\sum_{i=1}^{2}X_i\frac{\partial V}{\partial x_i}+\Delta V-1\right)\\
=&|\boldsymbol{X}|^2F^2\boldsymbol{X}(F)+\frac{1}{2}\boldsymbol{X}(|\boldsymbol{X}|^2)F^3-\left(1+|\boldsymbol{X}|^2F^2\right)\left(|\boldsymbol{X}|^2F+\boldsymbol{X}(F)+(\mathrm{div}\boldsymbol{X})F-1\right)\\
=&-\boldsymbol{X}(F)+\frac{1}{2}\boldsymbol{X}(|\boldsymbol{X}|^2)F^3-\left(1+|\boldsymbol{X}|^2F^2\right)\left(\left(|\boldsymbol{X}|^2+(\mathrm{div}\boldsymbol{X})\right)F-1\right).
\end{aligned}
$$

Therefore, from Theorem 3.1.2, we can derive Theorem 3.1.3. $\square$

Denote by (a_1,a_2) be the minimum point of ρ. Let F be a solution of the partial differential equation (3.1.2) and $c:(-\infty,t_0)\to\mathfrak{a}$ be an integral curve of $\boldsymbol{X}$, where we note that $\lim_{t\to-\infty}c(t)=(a_1,a_2)$ holds. Set $\widehat{F}_c:=F\circ c$. Then $\widehat{F}_c$ satisfies

$$
\widehat{F}_c'(t)=\langle c''(t),c'(t)\rangle\widehat{F}_c(t)^3-(1+|c'(t)|^2\widehat{F}_c(t)^2)((|c'(t)|^2+(\mathrm{div}\boldsymbol{X})_{c(t)})\widehat{F}_c(t)-1). \tag{3.3.3}
$$

From $\nabla V=F\boldsymbol{X}$, we have

$$
(V\circ c)'(t)=\widehat{F}_c(t)|\boldsymbol{X}_{c(t)}|^2
$$

and hence

$$
(V\circ c)(t):=\int_{t_*}^{t}\widehat{F}_c(\tau)|\boldsymbol{X}_{c(\tau)}|^2d\tau+(V\circ c)(t_*)), \tag{3.3.4}
$$

where t_* is any element of $(-\infty,t_0)$. Thus we can calculate the function V from the data of F.

We shall consider the case where the Hermann action $H\curvearrowright G/K$ is the dual action of the Hermann type action $SO_0(1,2)\curvearrowright SL(3,\mathbb{R})/SO(3)$. This action corresponds to $\rho_1(SO(3))\curvearrowright SU(3)/SO(3)$ in Table 3.1 of [14]. In this case, there exists an integral curve $c:(-\infty,t_0)\to\mathfrak{a}$ of $\boldsymbol{X}$ satisfying $x_2\circ c=0$. The component X_i of $\boldsymbol{X}$ and the domain $\mathcal{C}$ are given as in the following table.

$H\curvearrowright G/K$	$\rho_1(SO(3))\curvearrowright SU(3)/SO(3)$
X_1	$\tan(x_1+\sqrt{3}x_2)-2\cot(2x_1)+\tan(x_1-\sqrt{3}x_2)$
X_2	$\sqrt{3}\tan(x_1+\sqrt{3}x_2)-\sqrt{3}\tan(x_1-\sqrt{3}x_2)$
$\mathcal{C}$	$\{(x_1,x_2)\mid x_1>0,\ x_2>\frac{1}{\sqrt{3}}x_1-\frac{\pi}{2\sqrt{3}},\ x_2<-\frac{1}{\sqrt{3}}x_1+\frac{\pi}{2\sqrt{3}}\}$

Table 3.1: The datas of X_1,X_2 and $\mathcal{C}$ in $\rho_1(SO(3))\curvearrowright SU(3)/SO(3)$-case

In this case, (a_1,a_2) is equal to $(\frac{\pi}{6},0)$ and there exists an integral curve $c:(-\infty,t_0)\to\mathfrak{a}$ satisfying $\lim_{t\to-\infty}c(t)=(\frac{\pi}{6},0)$, $\lim_{t\to t_0}c(t)=(\frac{\pi}{2},0)$ and $x_2\circ c=0$. For the simplicity, set $c_1:=x_1\circ c$. Also, set $\widetilde{F}:=|c'|\widehat{F}_c$. Take $t_*\in(-\infty,t_0)$. Since $\boldsymbol{X}_{c(t)}=c'(t)=(c_1'(t),0)$ and hence

$\|\boldsymbol{X}_{c(t)}\| = c_1'(t)$, we have

$$\begin{aligned}
V(x_1,0) &= V(c_1(t),0) = (V\circ c)(t)\\
&= \int_{t_*}^{t} \widehat{F}_c(\tau)\|\boldsymbol{X}_{c(\tau)}\|^2 d\tau + V(c_1(t_*),0)\\
&= \int_{t_*}^{t} \widehat{F}_c(\tau)c_1'(\tau)^2 d\tau + V(c_1(t_*),0)\\
&= \int_{t_*}^{t} \widetilde{F}(\tau)c_1'(\tau)\, d\tau + V(c_1(t_*),0)\\
&= \int_{c_1(t_*)}^{x_1} (\widetilde{F}\circ c_1^{-1})(x)\, dx + V(c_1(t_*),0) \quad \left(\frac{\pi}{6}\le x_1\le\frac{\pi}{2}\right)
\end{aligned}$$

The shape of the graph of the restriction $V|_{[\frac{\pi}{6},\frac{\pi}{2}]\times\{0\}}$ of V to $\overline{c((-\infty,t_0))} = \left[\frac{\pi}{6},\frac{\pi}{2}\right]\times\{0\}$ is dominated by $\widetilde{F}\circ c_1^{-1}$. So, we shall investigate the shape of the graph of $\widetilde{F}\circ c_1^{-1}$. According to (3.3.3), $\widetilde{F}$ satisfies

$$\widetilde{F}'(t) = \left(\widetilde{F}(t)^2+1\right)\left(\left(\frac{\langle c'(t),c''(t)\rangle}{|c'(t)|^2} - \left(|c'(t)|^2 + (\mathrm{div}\boldsymbol{X})_{c(t)}\right)\right)\widetilde{F}(t) + |c'(t)|\right), \tag{3.3.5}$$

On the othe hand, we have

$$|c'(t)|^2 = 9\mathrm{tan}^2(c_1(t)) + \cot^2(c_1(t)) - 6, \tag{3.3.6}$$

$$\langle c'(t),c''(t)\rangle = \frac{27\mathrm{tan}^2(c_1(t))}{\cos^2(c_1(t))} + \frac{\cot^2(c_1(t))}{\sin^2(c_1(t))} - \frac{9}{\cos^2(c_1(t))} - \frac{3}{\sin^2(c_1(t))} \tag{3.3.7}$$

and

$$(\mathrm{div}\boldsymbol{X})_{c(t)} = \frac{9}{\cos^2(c_1(t))} + \frac{1}{\sin^2(c_1(t))}. \tag{3.3.8}$$

Define $\widetilde{\eta}(t)$ by

$$\widetilde{\eta}(t) := -\frac{|c'(t)|^3}{\langle c'(t),c''(t)\rangle - |c'(t)|^2\left(|c'(t)|^2 + (\mathrm{div}\boldsymbol{X})_{c(t)}\right)}.$$

From (3.3.6), (3.3.7) and (3.3.8), we have

$$\widetilde{\eta}(t) = \frac{3\mathrm{tan}(c_1(t)) - \frac{1}{\tan(c_1(t))}}{15\mathrm{tan}^2(c_1(t)) + \frac{1}{\tan^2(c_1(t))}}$$

Set $z(t) := \tan(c_1(t))$ and define $\sigma(z)$ by

$$\sigma(z) := \frac{3z - \frac{1}{z}}{15z^2 + \frac{1}{z^2}}.$$

From $\widetilde{\eta} = \sigma(z)$, we find

$$\widetilde{\eta}'(t) = \left(1+z(t)^2\right)\left(3z(t) - \frac{1}{z(t)}\right)\sigma'(z(t)). \tag{3.3.9}$$

Since σ satisfies

$$\sigma'(z) = \frac{-45z^6 + 45z^4 + 9z^2 - 1}{z^4 \left(15z^2 + \frac{1}{z^2}\right)^2},$$

there exists $z_0 \in (\frac{1}{\sqrt{3}}, \infty)$ with $\sigma'(z_0) = 0$ satisfying $\sigma'(z) > 0$ for all $z \in (\frac{1}{\sqrt{3}}, z_0)$ and $\sigma'(z) < 0$ for all $z \in (z_0, \infty)$. Define $t_1 \in (-\infty, t_0)$ by $\tan(c_1(t_1)) = z_0$. Then, from (3.3.9), we find $\widetilde{\eta}'(t_1) = 0$, $\widetilde{\eta}'(t) > 0$ for all $t \in (-\infty, t_1)$ and $\widetilde{\eta}'(t) < 0$ for all $t \in (t_1, t_0)$. Therefore, the graph of $\widetilde{\eta} \circ c_1^{-1}$ is as in Figure 3.3.1.

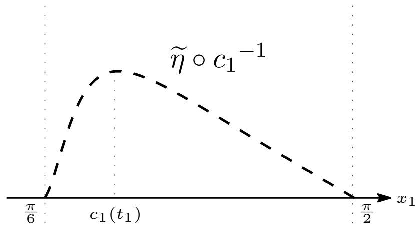

Figure 3.3.1: The graph of $\widetilde{\eta} \circ c_1^{-1}$

Here, from $\eta(t) > 0$, we find $\frac{\langle c'(t), c''(t)\rangle}{|c'(t)|^2} - \left(|c'(t)|^2 + (\mathrm{div}\boldsymbol{X})_{c(t)}\right) < 0$. According to (3.3.5), $\widetilde{F}'(t) = 0$ if and only if $\widetilde{F}(t) = \widetilde{\eta}(t)$. Also, $\widetilde{F}'(t) > 0$ if and only if $\widetilde{F}(t) < \widetilde{\eta}(t)$, and $\widetilde{F}'(t) < 0$ if and only if $\widetilde{F}(t) > \widetilde{\eta}(t)$.

Next we show the behavior of the function $\widetilde{F} \circ c_1^{-1}$ on the both sides of the domain of $\widetilde{F} \circ c_1^{-1}$. From (3.3.5), $\widetilde{F} \circ c_1^{-1}$ satisfies

$$(\widetilde{F} \circ c_1^{-1})'(x) = \left((\widetilde{F} \circ c_1^{-1})(x)^2 + 1\right) \left(1 - \left(5\tan x - \frac{1}{\tan x} + \frac{8\tan x}{3\tan^2 x - 1}\right) (\widetilde{F} \circ c_1^{-1})(x)\right). \tag{3.3.10}$$

Then we shall show that the following fact for the behavior of $\widetilde{F} \circ c_1^{-1}$ near $x_1 = \frac{\pi}{2}$ holds.

($*_5$) *If there exists* $x_0 \in (c_1(t_1), \frac{\pi}{2})$ *such that* $(\widetilde{F} \circ c_1^{-1})(x_0) > \widetilde{\eta}(x_0)$, $\lim\limits_{x \uparrow \frac{\pi}{2}} (\widetilde{F} \circ c_1^{-1})(x) = 0$ *holds.*

Assume that there exists a positive constant $M > 0$ with $(\widetilde{F} \circ c_1^{-1})(x) > M$ for all $x \in (x_0, \frac{\pi}{2})$. Take any $x \in (x_0, \frac{\pi}{2})$. Then, from $(\widetilde{F} \circ c_1^{-1})(x) < 0$, we can show

$$\begin{aligned}\frac{(\widetilde{F} \circ c_1^{-1})'(x)}{(\widetilde{F} \circ c_1^{-1})(x)^2 + 1} &= 1 - \left(5\tan x - \frac{1}{\tan x} + \frac{8\tan x}{3\tan^2 x - 1}\right) (\widetilde{F} \circ c_1^{-1})(x) \\ &< 1 - \left(5\tan x - \frac{1}{\tan x} + \frac{8\tan x}{3\tan^2 x - 1}\right) M.\end{aligned}$$

By integrating both-hand sides of this inequality from x_0 to x, we obtain

$$\begin{aligned}\arctan{}&(\widetilde{F} \circ c_1^{-1})(x) - \arctan(\widetilde{F} \circ c_1^{-1})(x_0) \\ &< x + M(5\log(\cos x) + \log(\sin x) - \log(1 - 2\cos(2x))) \\ &\quad - x_0 - M(5\log(\cos x_0) + \log(\sin x_0) - \log(1 - 2\cos(2x_0))) =: h_7(x).\end{aligned}$$

and hence

$$(\widetilde{F} \circ c_1^{-1})(x) < \tan(h_7(x) + \arctan(\widetilde{F} \circ c_1^{-1})(x_0)).$$

On the other hand, h_7 is decreasing on $(x_0, \frac{\pi}{2})$ and the following relations hold:

$$h_7(x_0) = 0 \quad \text{and} \quad \lim_{x \uparrow \frac{\pi}{2}} h_7(x) = -\infty.$$

Therefore, there exists $\bar{x}_1 \in (x_0, \frac{\pi}{2})$ such that

$$(\widetilde{F} \circ c_1^{-1})(x) < \tan(h_7(x) + \arctan(\widetilde{F} \circ c_1^{-1})(x_0)) \ \to \ -\infty \quad (x \to \bar{x}_1).$$

Then $(\widetilde{F} \circ c_1^{-1})(x) = 0 < M$ for some $x \in (x_0, \bar{x}_1)$. This is a contradiction. Thus the fact $(*_5)$ is shown.

Also, we shall show the following fact for the behavior of $\widetilde{F} \circ c_1^{-1}$ near some point $x_1 = x_0 \in (\frac{\pi}{6}, c_1(t_1))$.

$(*_6)$ *If there exists $x_0 \in (\frac{\pi}{6}, c_1(t_1))$ such that $(\widetilde{F} \circ c_1^{-1})(x_0) > \widetilde{\eta}(x_0)$, $\lim_{x \downarrow \bar{x}_0} (\widetilde{F} \circ c_1^{-1})(x) = \infty$ holds for some $\bar{x}_0 \in (\frac{\pi}{6}, x_0)$.*

Take any $x \in (\frac{\pi}{6}, x_0)$. Then, from $(\widetilde{F} \circ c_1^{-1})(x) < 0$, we can show

$$\begin{aligned}\frac{(\widetilde{F} \circ c_1^{-1})'(x)}{(\widetilde{F} \circ c_1^{-1})(x)^2 + 1} &= 1 - \left(5\tan x - \frac{1}{\tan x} + \frac{8\tan x}{3\tan^2 x - 1}\right)(\widetilde{F} \circ c_1^{-1})(x) \\ &< 1 - \left(5\tan x - \frac{1}{\tan x} + \frac{8\tan x}{3\tan^2 x - 1}\right)(\widetilde{F} \circ c_1^{-1})(x_0).\end{aligned}$$

By integrating both-hand sides of this inequality from x to x_0, we obtain

$$\begin{aligned}&\arctan(\widetilde{F} \circ c_1^{-1})(x) - \arctan(\widetilde{F} \circ c_1^{-1})(x_0) \\ &\quad > x + ((\widetilde{F} \circ c_1^{-1})(x_0))(5\log(\cos x) + \log(\sin x) - \log(1 - 2\cos(2x))) \\ &\quad\quad - x_0 - ((\widetilde{F} \circ c_1^{-1})(x_0))(5\log(\cos x_0) + \log(\sin x_0) - \log(1 - 2\cos(2x_0))) =: h_8(x).\end{aligned}$$

and hence

$$(\widetilde{F} \circ c_1^{-1})(x) > \tan(h_8(x) + \arctan(\widetilde{F} \circ c_1^{-1})(x_0)).$$

On the other hand, h_8 is decreasing on $(\frac{\pi}{6}, x_0)$ and the following relations hold:

$$h_8(x_0) = 0 \quad \text{and} \quad \lim_{x \downarrow \frac{\pi}{6}} h_8(x) = \infty.$$

Therefore, there exists $\bar{x}_1 \in (\frac{\pi}{6}, x_0)$ such that

$$(\widetilde{F} \circ c_1^{-1})(x) > \tan(h_8(x) + \arctan(\widetilde{F} \circ c_1^{-1})(x_0)) \ \to \ \infty \quad (x \to \bar{x}_1).$$

Thus the fact $(*_6)$ is shown.

Similarly, we can show the following fact for the behavior of $\widetilde{F} \circ c_1^{-1}$ near some point $x_1 = x_0 \in (\frac{\pi}{6}, c_1(t_1))$. Here, note the fact that $(\widetilde{F} \circ c_1^{-1})(x) > 0$ when $(\widetilde{F} \circ c_1^{-1})(x) < (\widetilde{\eta} \circ c_1^{-1})(x)$ for all $x \in (\frac{\pi}{6}, \frac{\pi}{2})$

$(*_7)$ *If there exists $x_0 \in (\frac{\pi}{6}, c_1(t_1))$ such that $(\widetilde{F} \circ c_1^{-1})(x_0) < 0$, $\lim_{x \downarrow \bar{x}_0} (\widetilde{F} \circ c_1^{-1})(x) = -\infty$ holds for some $\bar{x}_0 \in (\frac{\pi}{6}, x_0)$.*

From the facts $(*_5) - (*_7)$, the graph of $\widetilde{F} \circ c_1^{-1}$ is as in one of Figures 3.3.2-3.3.6.

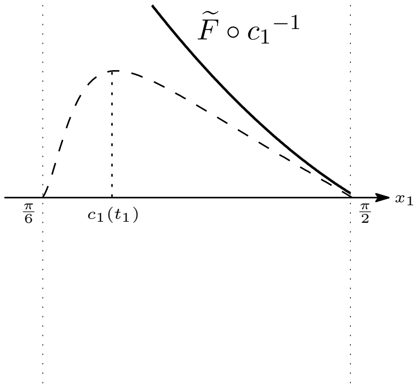

Figure 3.3.2: The graph of $\widetilde{F} \circ c_1^{-1}$ (Type I)

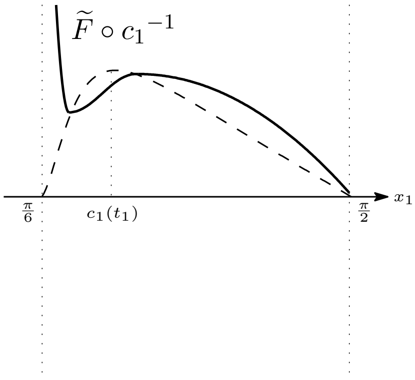

Figure 3.3.3: The graph of $\widetilde{F} \circ c_1^{-1}$ (Type II)

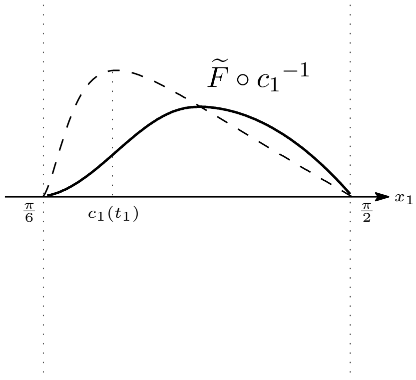

Figure 3.3.4: The graph of $\widetilde{F} \circ c_1^{-1}$ (Type III)

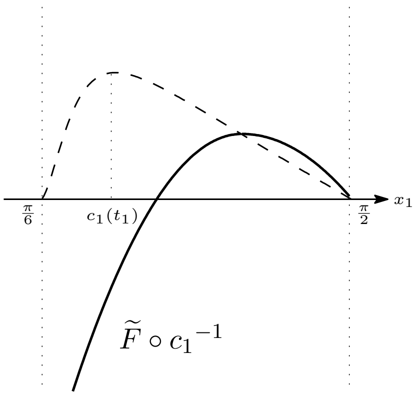

Figure 3.3.5: The graph of $\widetilde{F} \circ c_1^{-1}$ (Type IV)

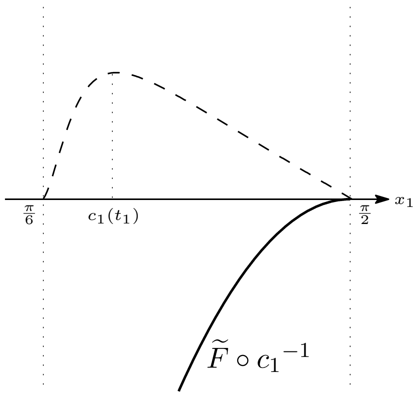

Figure 3.3.6: The graph of $\widetilde{F} \circ c_1^{-1}$ (Type V)

Hence, from the above classification of the graph of $\widetilde{F} \circ c_1^{-1}$ and $V(x_1, 0) = \int_{c_1(t_*)}^{x_1} (\widetilde{F} \circ c_1^{-1})(x)\, dx + V(c_1(t_*), 0)$, we find the shape of the graph of $V(\cdot, 0)|_{[\frac{\pi}{6}, \frac{\pi}{2}]}$ is as in one of Figures 3.3.7-3.3.11.

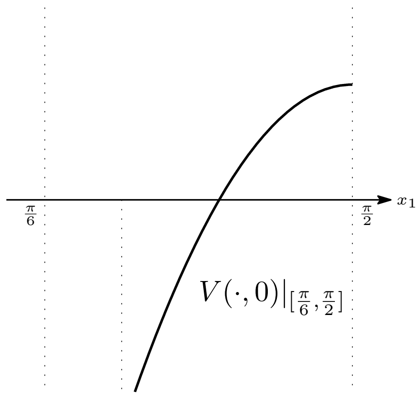

Figure 3.3.7: The graph of $V(\cdot,0)|_{[\frac{\pi}{6},\frac{\pi}{2}]}$(Type I)

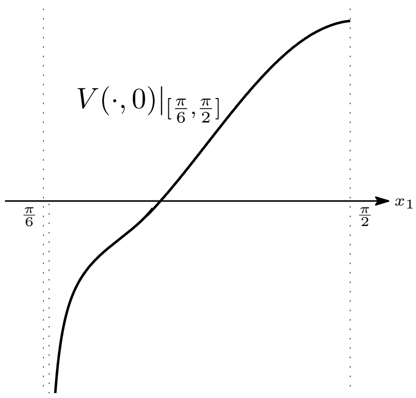

Figure 3.3.8: The graph of $V(\cdot,0)|_{[\frac{\pi}{6},\frac{\pi}{2}]}$(Type II)

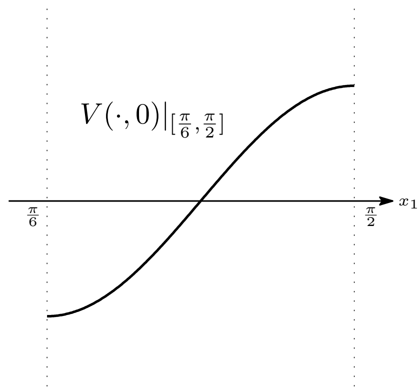

Figure 3.3.9: The graph of $V(\cdot,0)|_{[\frac{\pi}{6},\frac{\pi}{2}]}$(Type III)

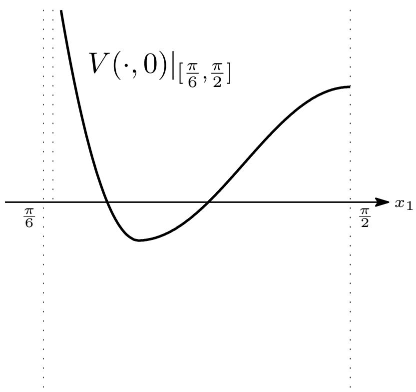

Figure 3.3.10: The graph of $V(\cdot,0)|_{[\frac{\pi}{6},\frac{\pi}{2}]}$(Type IV)

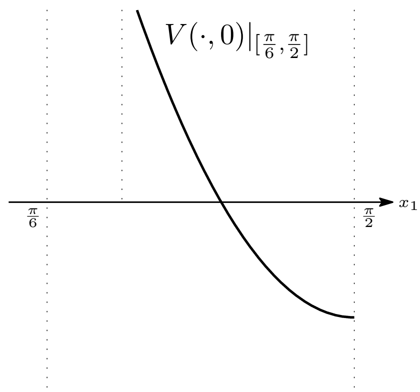

Figure 3.3.11: The graph of $V(\cdot,0)|_{[\frac{\pi}{6},\frac{\pi}{2}]}$(Type V)

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