DOCTORAL THESIS
DOCTOR OF SCIENCE

STUDY OF REACTION-DIFFUSION SYSTEMS MODELING CHEMOTAXIS

(走化性をモデルとする反応拡散系についての研究)

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Chapter 1

Introduction

In this thesis we consider reaction-diffusion systems modeling “chemotaxis”. The biological phenomenon “chemotaxis” is the directed movement of cells as response to gradients of a chemical substance. It is pointed out that such chemotaxis processes play an essential role in various biological contexts ([**18**]). In particular this thesis concentrates on the case that the chemotaxis process is dominated by the Weber–Fechner law, which means we introduce signal-dependent sensitivity function (this function is usually nonlinear).

In this thesis we deal with the following systems:

Parabolic-elliptic Keller–Segel system:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla \chi(v)), \\ 0 = \Delta v - v + u. \end{cases}$$

Parabolic-parabolic Keller–Segel system:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla \chi(v)), \\ v_t = \Delta v - v + u. \end{cases}$$

Chemotaxis system for tumor invasion:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), \\ v_t = \Delta v + wz, \\ w_t = -wz, \\ z_t = \Delta z - z + u. \end{cases}$$

PART I and PART II are organized to analyze the parabolic-elliptic and parabolic-parabolic Keller–Segel systems, respectively.

In the other context, quite recently, mathematical analysis has been expected to play an important role in medical sciences. The target of PART III is a mathematical model for tumor invasion. We propose some chemotaxis model for tumor invasion and apply mathematical analysis to the system.

1.1. The Keller–Segel system

1.1.1. Mathematical model

In this subsection we explain variants of the Keller–Segel system from historical and mathematical view points.

In 1970 Keller and Segel proposed a mathematical model concerning about cell's life circle, especially an aggregation process. In this model "chemotaxis" plays an essential role to induce the aggregation process of the cell. When cells are starving, cells move towards increasing concentrations of the signal substance which is produced by cells ([**54**]). They start the discussion with the reaction-diffusion system consisting of four equations, which describe cellular slime molds, chemical substance, enzymes and complexes accounting for diffusion processes of them, a chemotaxis process between cells and chemical substance, production processes of chemical substance and enzyme, and a chemical reaction that chemical substance combines with enzyme and produces complexes. By applying the Michaelis–Menten reduction law, they proposed the following reaction-diffusion system:

$$\begin{cases} u_t = \nabla \cdot (D_1(u,v)\nabla u) - \nabla \cdot (D_2(u)\nabla \chi(v)), \\ v_t = d_v \Delta v - \frac{k_1 v}{k_2 + v} + f(v)u, \end{cases}$$

with nonnegative constants d_v, k_1 and k_2, and some functions D_1, D_2, χ and f. Here $u(x,t)$ represents the population of cell and $v(x,t)$ denotes the concentration of signal substance at place x and time t. The cross-diffusion term $-\nabla \cdot (D_2(u)\nabla \chi(v))$ describes "chemotaxis". Keller and Segel deduced some instability of constant solutions of the system and asserted a validity of this system.

After that, Nanjundiah [**73**] focused on the simplified model:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\nabla \chi(v)), \\ v_t = \Delta v - v + u, \end{cases}$$

and gave the conjecture about blow-up phenomenon by the nonlinear stability analysis. Nowadays in particular, the above system with $\chi(v) = v$ is usually called "classical Keller–Segel system" or "minimal Keller–Segel system".

Next Jäger and Luckhous [**46**] considered the case that the second component of solutions is a steady state:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), \\ 0 = \Delta v - u + 1. \end{cases}$$

This assumption is based on the experimental facts. This system is called "Jäger–Luckhous system".

Nagai [**66**] also proposed the simplified model in which the second equation is elliptic:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla \chi(v)), \\ 0 = \Delta v - v + u. \end{cases}$$

This system is called "Nagai model" or "parabolic-elliptic Keller–Segel system".

In a different context, Mimura and Tsujikawa [**63**] introduced the following chemotaxis model with growth:

$$\begin{cases} u_t = d_u \Delta u - \nabla \cdot (u \nabla \chi(v)) + f(u), \\ v_t = \Delta v - \beta v + \gamma u, \end{cases}$$

with positive constants d_u, d_v, β and γ, and functions χ and f. From a view point of pattern formation, they considered aggregating pattern-dynamics arising this system. Different from Turing's diffusion-induced instability ([**101**]), they pointed out the new mechanism "chemotaxis-induced" instability. This system is usually called "Mimura–Tsujikawa system".

On the other hand, Biler and Naziej, and Wolansky [**7, 109**] proposed a mathematical model describing gravitational interactions of particles:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla \chi(v)), \\ 0 = \Delta v + u. \end{cases}$$

This system looks as the simplified system of the Keller–Segel system. Furthermore a similar system appears in modeling of burglary in residential areas ([**79, 82**]).

Currently, the following generalized system is called as "Keller–Segel system" or "chemotaxis system":

$$\begin{cases} u_t = \nabla \cdot (A(u,v)\nabla u - B(u)\nabla C(v)) + D(u,v), \\ \tau v_t = d_v \Delta v + E(u,v), \end{cases}$$

with constants $\tau \geq 0$ and $d_v \geq 0$, and functions A, B, C, D and E. These above systems are regarded as a variant of the Keller–Segel system. Especially in this thesis, our interest is in the case of nonlinear signal-dependent sensitivity, which means the function C in the above is a nonlinear function.

1.1.2. Background

Since the Keller–Segel systems were introduced, considerable attention has been devoted to studying them mathematically, especially to analyzing behavior of solutions ([**35, 37, 3**]). In this subsection we recall important results on the minimal Keller–Segel system:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\nabla v), \\ \tau v_t = \Delta v - v + u, \end{cases} \tag{1.1}$$

and give fundamental observations briefly. We remark that these facts will be compared with the main results in this thesis.

At first Nanjundiah [**73**] pointed out that the cross-diffusion term $-\nabla\cdot(u\nabla v)$ brings the possibility of finite-time blow-up in the sense that the first component u of solutions blows up in finite time with respect to the norm in $L^\infty(\Omega)$. More precisely, Nanjundiah considered the system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\nabla \chi(v)), \\ v_t = \Delta v - v + u, \end{cases}$$

and gave a conjecture that the first component u of solutions blows up in finite time with forming δ-function singularity in both cases

$$\chi(v) = \chi_0 v \quad \text{and} \quad \chi(v) = \chi_0 \log v \qquad (\chi_0 : \text{ constant})$$

for arbitrary spacial dimension $n \in \mathbb{N}$. After that, Childress and Percus [**15**] focused on the minimal case (1.1) and claimed that the blow-up phenomenon may be dominated by the spacial dimension n. They gave the following conjecture:

1. If $n = 1$ then a solution of (1.1) exists globally.

2. If $n = 2$ then there exists some constant $c > 0$ satisfying that a solution exists globally when $\int_\Omega u_0 < 2\pi c$; some solution of (1.1) blows up in finite time with forming δ-function singularity when $\int_\Omega u_0 > 2\pi c$.

3. If $n \geq 3$ then some solution blows up in finite time independently of the size of the mass $\int_\Omega u_0$.

Here we remark that the mass $\int_\Omega u$ is preserved in the Neumann boundary value problem or Cauchy problem of (1.1). The study of the Keller–Segel system has developed along the above conjecture.

As to the one dimensional setting, Osaki and Yagi [**76**] proved that a solution of generalized system including (1.1) exists globally. As to the two dimensional setting, a solution of (1.1) is global and bounded when $\int_\Omega u_0 < 4\pi$ and Ω is bounded, or $\int_\Omega u_0 < 8\pi$ and $\Omega = \mathbb{R}^2$ or Ω is ball. Moreover the solution converges to the self-similar solution (for the parabolic-elliptic case, see [**46, 66, 6, 9, 68, 69**]; for the parabolic-parabolic case, see [**72**]). Whereas some blow-up solution with the large initial data is constructed ([**46, 7, 66**]) and the profile of blow-up solutions is precisely analyzed ([**34, 81, 90**]). Nagai [**67**] showed that there are many nonradial blow-up solutions when $\tau = 0$. When the initial data has the critical mass, a solution exists globally and approaches δ-function ([**80, 8, 70**]). As to the higher dimensional case, Winkler [**108**] established existence of blow-up solutions with arbitrary mass $\int_\Omega u$ when $n \geq 3$.

From a mathematical point of view, the analysis of (1.1) is based on the Lyapunov functional. In the parabolic-elliptic case, setting

$$\mathcal{F}(u,v) := \int_\Omega u \log u - \int_\Omega uv + \frac{1}{2}\int_\Omega |\nabla v|^2 + \int_\Omega v^2,$$

by a simple calculation we see that

$$\frac{d}{dt}\mathcal{F}(u,v)(t) \leq 0 \qquad \text{for all } t > 0. \tag{1.2}$$

In the analysis of (1.1), the properties (1.2) and the mass conservation law play an essential role (for the parabolic-elliptic case, this situation is same). Indeed, global existence and boundedness of solutions are established by combining the above properties and the Trudinger–Moser inequality in [**72**]. Otherwise, based on the observation of the Lyapunov functional, existence of bolwup solutions is established in [**108**].

Finally we give some comments about the study of variants of the Keller–Segel system. The following Keller–Segel system with nonlinear diffusion:

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v), \\ v_t = \Delta v - v + u, \end{cases}$$

with some given functions D and S, has been studied widely. The typical choice of the functions D and S is

$$D(u) = u^m, \quad S(u) = u^{q-1} \qquad \text{with } m \geq 1,\ q \geq 2.$$

From a view point of the competition between the spreading effect of the diffusion term and the concentrating effect of the cross-diffusion term, a behavior of solutions is classified by the parameters m and q ([**86, 87, 89, 40, 41, 96, 39**]). Moreover since the diffusion term degenerates in the above case, it is known that uniqueness of solutions is difficult. Recently Miura and Sugiyama [**64**] made a progress in this field.

1.1.3. Motivation

In this thesis we especially focus on a model of chemotaxis processes which the movement towards higher signal concentrations is inhibited at points where these concentrations are high. Such saturation effects are usually accounted for by introducing a signal-dependent sensitivity function $\chi(v)$,

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\nabla\chi(v)), \\ \tau v_t = \Delta v - v + u. \end{cases} \tag{1.3}$$

The sensitivity function was proposed in an original work by Keller and Segel [**54, 55**]. The prototypical choice of $\chi(v)$ is

$$\chi(v) = \chi_0 \log v, \qquad \chi_0 > 0$$

based on the Weber–Fechner law of stimulus perception in the process of chemotactic response. For an independent derivation thereof, and for several examples of systems with related signal-dependent chemotactic sensitivity functions, we refer to [**35, 77, 83**].

Our motivation is that weakening the cross-diffusion term by sensitivity function $\chi(v)$ enables us to expect that the system has a global and bounded solution independently of the size of initial data. Here we remark that the logarithmic case is absolutely different from the classical case ($\chi(v) = v$). Indeed, χ_0 is invariant of the scaling $(u, v) \mapsto (\lambda u, \lambda v)$ ($\lambda > 0$) in the logarithmic case: Let (u, v) be a solution of (1.3) and $\lambda > 0$. Setting $U(x, t) := \lambda u(x, t)$ and $v(x, t) := \lambda v(x, t)$, we multiply the first equation of (1.3) by λ and deduce that

$$U_t = \Delta U - \chi_0 \nabla \cdot (U \nabla \log v).$$

Since

$$\nabla \log v = \nabla(\log v + \log \lambda) = \nabla \log V,$$

it follows that

$$U_t = \Delta U - \chi_0 \nabla \cdot (U \nabla \log V).$$

The second equation is naturally derived as

$$\tau V_t = \Delta V - V + U.$$

Hence, as compared with the classical case, we will see below that the solution exists globally, independently of the size of initial data. We recall some results in this context. Here let n be the spacial dimension.

As to the parabolic-elliptic case ($\tau = 0$), in the radially symmetric setting, we can find a nice picture about the system with $\chi(v) = \chi_0 \log v, \chi_0 > 0$ as follows ([**71**]):

- If $n = 2$, or $\chi_0 < \frac{2}{n-2}$ and $n \geq 3$ then a *radial* solution is global and bounded.
- If $\chi_0 > \frac{2n}{n-2}$ and $n \geq 3$ then there exists some initial data u_0 such that a *radial* solution blows up in finite time.

In [**71**] Nagai and Senba also considered the case $\chi(v) = v^p$ ($p > 0$). When $n = 2$ and $0 < p < 1$ then a solution of (1.3) is global and bounded; if $n \geq 3$ and $p > 0$ there exists a finite time blow-up solution. Without requiring such a symmetry hypothesis, Biler [**5**] showed that in the system with $\chi(v) = \chi_0 \log v, \chi_0 > 0$,

- If $\chi_0 \leq 1$ and $n = 2$, or $\chi_0 < \frac{2}{n}$ and $n \geq 3$ then a solution is global.

Consequently, in the parabolic-elliptic system we can find the gap between *radial* case and *nonradial* case when $\chi(v) = \chi_0 \log v$. Especially in the two dimensional setting, for all $\chi_0 > 0$ a *radial* solution is global and bounded; on the other hand, the large time behavior of *nonradial* solutions of the system with large $\chi_0 > 0$ has been posted as an open problem. In [**5**, Remark 4] Biler and Velázquez gave the following conjecture:

- The optimal range of the coefficient $\chi_0 > 0$ guaranteeing the global in time existence is $\chi_0 < \frac{n+2}{n-2}$.

Here we give a remark about the stationary solution. The stationary problem of the system (1.3) with $\chi(v) = \chi_0 \log v$ is rewritten as the following elliptic equation,

$$0 = \Delta w - w + w^{\chi_0}.$$

It is known that the constant $\chi_0 = \frac{n+2}{n-2}$ is critical in some sense ([**74, 59**]).

As to the parabolic-parabolic case ($\tau = 1$), the picture seems to be more involved. Winkler [**107**] proved that if $\chi_0 < \sqrt{\frac{2}{n}}$, then (1.3) with $\chi(v) = \chi_0 \log v$ possesses a global classical solution. Boundedness of the above solution has been left as an open problem and Winkler conjectured boundedness of solutions under the same conditions on $\chi_0 > 0$. Also there were other approaches by considering certain weak solutions. As to the problem (1.3) with $\chi(v) = \chi_0 \log v$, global existence of weak solutions was established when $\chi_0 < \sqrt{\frac{n+2}{3n-4}}$ ([**107**]). In the radially symmetric setting, Stinner and Winkler [**85**] constructed certain weak solutions under the condition $\chi_0 < \sqrt{\frac{n}{n-2}}$.

We summarize the above as follows:

- At least in the logarithmic case $\chi(v) = \chi_0 \log v$, it is expected that behavior (global existence, boundedness and blow-up) of the corresponding solutions is classified by the value of χ_0 in both parabolic-elliptic and parabolic-parabolic cases.
- The precise condition on χ_0 in the above question has not been established until now.
- There also remains the same question for other sensitivity functions.

1.1.4. Overview

We give an overview of PART I and PART II in this thesis. In PART I we consider the following Neumann boundary value problem for parabolic-elliptic Keller–Segel systems:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla \chi(v)) + f(u), & x \in \Omega, \quad t > 0, \\ 0 = \Delta v - v + u, & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with smooth boundary $\partial\Omega$, with suitably regular and nonnegative initial data u_0. At first we consider the problem (1.4) when the sensitivity function χ satisfies

$$0 < \chi'(v) \leq \frac{\chi_0}{v^k}$$

with $k \geq 1$ and sufficiently small $\chi_0 > 0$, and $f \equiv 0$. Global existence and boundedness of solutions to (1.4) will be established in Chapter 3 (which is based on Fujie–Winkler–Yokota [**29**]). The cornerstone of this work is *uniform-in-time* lower bound for v, which will be established in Section 2.2. In virtue of this estimate, we derive L^p-estimate for u directly apart from using the Lyapunov functional. In Chapter 4 (which is originated from Fujie–Yokota [**30**]) we apply the above method to the generalized system (1.4) with growth term $f(u)$ satisfying

$$\lambda_1 - \mu_1 u \leq f(u) \leq \lambda_2 - \mu_2 u$$

with positive constants λ_1, λ_2, μ_1 and μ_2. Moreover Chapter 5 (which is grounded on Fujie–Winkler–Yokota [**28**]) is devoted to analyzing the system with logistic source. In this chapter we focus on the case

$$\chi(v) = \chi_0 \log v \quad \text{and} \quad f(u) = ru - \mu u^2$$

and by the dampening effect of the logistic source we establish global existence and boundedness of solutions under some conditions on the parameters. The mathematical challenge is on excluding mass loss.

Finally, we have another point of view in Chapter 6 (which is based on Fujie–Senba [**26**]). We restrict our eyes to *local-in-space* estimates in the two dimensional setting. In light of localizing, we succeed in establishing a local-in-space energy estimate instead of the Lyapunov functional. Due to this estimate, global existence and boundedness in (1.4) will be established when the sensitivity function is sufficiently smooth and satisfies

$$\chi' > 0 \quad \text{and} \quad \chi'(s) \to 0 \quad \text{as} \quad s \to \infty.$$

Especially, this result gives an answer to the Biler–Velázquez conjecture in the two dimensional setting. Moreover we note that in the radial setting the above decaying condition is the essential condition for global existence. This strategy can be applied to the fully parabolic case. In Fujie–Senba [**27**] global existence and boundedness in the parabolic-parabolic Keller–Segel system are established under some conditions.

PART II deals with the following Neumann boundary value problem of parabolic-parabolic Keller–Segel system:

$$
\begin{cases}
u_t = \Delta u - \nabla \cdot (u \nabla \chi(v)), & x \in \Omega, \quad t > 0, \\
v_t = \Delta v - v + u, & x \in \Omega, \quad t > 0, \\
\dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\
u(x,0) = u_0(x), \ v(x,0) = v_0(x), & x \in \Omega,
\end{cases} \tag{1.5}
$$

where Ω is a bounded domain in $\mathbb{R}^n$ ($n \geq 2$) with smooth boundary $\partial\Omega$, with suitably regular and nonnegative initial data u_0 and v_0. In Chapter 7 (which is originated from Fujie [**21**]) we consider the case $\chi(v) = \chi_0 \log v$. This chapter solves the open problem of uniform-in-time boundedness of solutions for $\chi_0 < \sqrt{\frac{n}{2}}$, which was conjectured by Winkler [**107**]. The *uniform-in-time* lower bound for v (Section 2.3) is the key. Next, we consider the strongly singular sensitivity case in Chapter 8 (which is grounded on Fujie–Yokota [**31**]). Global existence and boundedness will be established when the sensitivity function satisfies that

$$
0 < \chi'(s) \leq \frac{\chi_0}{v^k}
$$

with $\chi_0 > 0$ and $k > 1$. Furthermore, in Chapter 9 (This work is based on Fujie–Nishiyama–Yokota [**25**]) we consider a quasilinear parabolic-parabolic Keller–Segel system, in which the first equation is

$$
u_t = \nabla \cdot (D(u)\nabla u - S(u)\nabla \log v)
$$

with given functions D and S. We will establish global existence and boundedness in the above problem under some conditions on the functions D and S.

1.2. Mathematical model for tumor invasion

It is well known that solid tumor brings about various phenomena, for example, angiogenesis, invasion and metastasis. Recently, it becomes much more important to make the mechanisms of such life phenomena clear by utilizing the mathematical theory because it gives us one method to control them from a mathematical viewpoint.

In the past two decades, a large variety of mathematical models describing tumor invasion phenomena has been developed by focusing on different aspects. Besides models purely based on reaction-diffusion equations ([**32**]), most of these models at their core assume taxis mechanisms which are of *haptotaxis* type, meaning that the respective attractant is non-diffusible (see e.g. [**13**] and [**2**] or also the discussion in [**24**]).

In [**13**] Chaplain and Anderson proposed the following mathematical model:

$$\begin{cases} n_t = \nabla \cdot (D_n(n,f)\nabla n) - \lambda \nabla \cdot (n \nabla f), \\ f_t = -R(m,f), \\ m_t = D_m \Delta m + P(n,f) - G(n,f,m), \end{cases} \tag{1.6}$$

where n, f and m describe the densities of tumor cells, the extracellular matrix, denoted by ECM from now on, and the matrix degrading enzyme, denoted by MDE from now on, respectively. We explain each term in the following:

(1) The coefficient $D_n(n,f)$ of the random motility of tumor cells is given by the function of n and f in general. As the typical example of D_n, we give

$$D_n(n,f) = D_1 + \frac{D_2 e^n}{1+e^n} + \frac{D_3 e^{-f}}{1+e^{-f}} > D_1 \quad (D_1,\ D_2,\ D_3 > 0\colon \text{ constants}),$$

which is increasing with respect to n but decreasing with respect to f. Phenomenologically, tumor cells cannot move freely when its density is small or the density of ECM is large. Actually, we are able to consider f as the order parameter describing the state of ECM. When the value of f is large, the state of ECM is complete, so, tumor cells cannot move freely. Conversely, the state of ECM with small value of f means that ECM is resolved by MDE and as a result tumor cells can move freely.

(2) $\lambda > 0$ is a sensibility coefficient of the haptotaxis of tumor cells.

(3) ECM is resolved by the following irreversible biochemical reaction with MDE:

$$\text{ECM} + \text{MDE} \quad \longrightarrow \quad \text{C}_1 + \text{C}_2 + \cdots + \text{C}_k + \text{MDE}, \tag{1.7}$$

where C_j, $j = 1, 2, \ldots, k$, are some substances. Hence $R(m,f) = \alpha m f$ is given as the typical example of R, where $\alpha > 0$ is the biochemical reaction velocity in (1.7).

(4) $P(n,f)$ implies the production of MDE. Since MDE is secreted by tumor cells, $P(n,f) = \beta n$ is given as the typical example of P, where $\beta > 0$ is the production rate of MDE per a tumor cell.

(5) $G(n, f, m)$ implies the decay of MDE. Since MDE is an enzyme, it does not decrease by the biochemical reaction with ECM in view of (1.7). However it has the natural decay property. Hence $G(n, f, m) = \gamma m$ is given as the typical example of G, where $\gamma > 0$ is the natural decay rate of MDE.

1.2.1. Background and motivation

The model (1.6) gives us the idea of the control method of tumor invasion from a mathematical viewpoint. Since ECM has an influence on the random motility and the haptotaxis of tumor cells and is resolved by the biochemical reaction with MDE, we can control the behavior of tumor cells by controlling its reaction. Roughly speaking, if an inhibitor to the biochemical reaction between ECM and MDE is developed, we can control the degradation of ECM by the reaction between the inhibitor and MDE from a mathematical viewpoint. Actually, in [**13**] they introduced the density of inhibitor denoted by u and proposed the other model, in which the following equation is added to (1.6):

$$u_t = D_u \Delta u - H(m, u),$$

as well as the term $-H(m, u)$ is added in the kinetic equation of m, where $H(m, u)$ is a nonnegative function coming from chemical reaction between m and u.

On the other hand, in [**17, 43, 92**] the authors proposed another control method where the role of a heat shock protein, denoted by HSP from now on, is taken into consideration. In [**92**] Szymańska et al. pointed out the fact that a certain HSP has an influence on $D_n(n, f)$ and λ. Then they denoted by $I = I(t)$ the quantity of the HSP at time t and proposed the kinetic equation of n, in which $D_n(n, f)$ and λ are replaced by $D_n I$ for some constant $D_n > 0$ and λI, respectively. Using their model, we can control the behavior of tumor cells by controlling the quantity of such HSP.

As to a mathematical analysis for (1.6), C. Morales-Rodrigo [**65**] showed local existence and uniqueness of solutions, while in [**49, 50, 51, 52**] the authors established global existence in a more general setting, but they modified the equation by adding the subdifferential of the indicator function and so they required the constraint condition that $n + f \leq 1$. Especially, in [**51**] Kano and Ito showed existence of global-in-time solutions for the case that $D_n(n, f)$ and $\lambda \nabla \cdot (n \nabla f)$ are replaced with $D_n(x, t, f)$ and $\nabla \cdot (\lambda(x, t) n \nabla f)$, respectively. However, they did not succeed in showing uniqueness of solutions to (1.6).

Hence one of the main purpose in this thesis is to prove not only existence but also uniqueness of solutions to (1.6) with another modification.

Analytical results on the Chaplain–Anderson model (1.6), essentially containing certain memory-type evolution problems as subsystems, are yet quite fragmentary, so far mainly concentrating on issues such as global existence and boundedness ([**62**], [**65**], [**84**], [**91**], [**94**], [**102**]); more detailed answers have been given only in certain special cases ([**20**], [**36**], [**48**], [**60**]). After all, certain global existence results can be achieved for such haptotaxis systems even when expanded to more realistic models ([**14**]) by including additional mechanisms ([**93**], [**95**], [**97**], [**98**], [**99**]). So one of the main purpose in this thesis is also to establish asymptotic stability in some modified Chaplain–Anderson model.

1.2.2. Overview

PART III is devoted to analyzing the chemotaxis model for tumor invasion model:

$$
(1.8) \qquad \begin{cases}
u_t = \Delta u - \nabla \cdot (u \nabla v), & x \in \Omega, \quad t > 0, \\
v_t = \Delta v + wz, & x \in \Omega, \quad t > 0, \\
w_t = -wz, & x \in \Omega, \quad t > 0, \\
z_t = \Delta z - z + u, & x \in \Omega, \quad t > 0, \\
\dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = \dfrac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\
u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \\
w(x,0) = w_0(x), \quad z(x,0) = z_0(x), & x \in \Omega,
\end{cases}
$$

where Ω is a bounded domain in $\mathbb{R}^n$ ($n \leq 3$) with smooth boundary $\partial\Omega$, with suitably regular and nonnegative initial data. In Chapter 10 (which is based on Fujie–Ito–Yokota [**24**]), we propose the above chemotaxis model and establish existence and uniqueness of local solutions to this model. In this chapter we prove existence of solutions by applying the Banach fixed point theorem to the corresponding integral equations.

In Chapter 11 (which is grounded on Fujie–Ito–Winkler–Yokota [**23**]) it is shown that for any choice of nonnegative and suitably regular initial data, the problem (1.8) possesses a global solution which is bounded. Moreover it is proved that these solutions approach a certain spatially homogeneous equilibrium.

Chapter 2

Tool box

2.1. Properties of the Neumann heat semigroup

In this section we collect some known facts concerning the Laplacian in Ω supplemented with homogeneous Neumann boundary condition.

The following lemma is proved in the same way as in [**33**, pp.32–40].

Lemma 2.1 (Neumann Laplacian)**.** *For $q \in (1,\infty)$ let Δ denote the realization of the Laplacian in $L^q(\Omega)$ with domain*

$$D(\Delta) := \Big\{ w \in W^{2,q}(\Omega) \,\Big|\, \frac{\partial w}{\partial \nu} = 0 \ \text{on } \partial\Omega \Big\}.$$

Then the operator $-\Delta+1$ is sectorial and possesses closed fractional powers $(-\Delta+1)^\theta$, $\theta \in (0,1)$, with dense domain $D((-\Delta+1)^\theta)$. Moreover, if $m \in \{0,1\}$, $p \in [1,\infty]$ and $q \in (1,\infty)$, then there exists a constant $c_{m,p} > 0$ such that for all $w \in D((-\Delta+1)^\theta)$,

$$\|w\|_{W^{m,p}(\Omega)} \le c_{m,p} \|(-\Delta+1)^\theta w\|_{L^q(\Omega)}, \tag{2.1}$$

provided that $m < 2\theta$ and $m - \frac{n}{p} < 2\theta - \frac{n}{q}$.

Next we give important estimates for the Neumann heat semigroup. The estimates **(i)** and **(ii)** are obtained by a general result for sectorial operators (see [**33**, Theorem 1.4.3]). The estimate **(iii)** is a special case of [**104**, Lemma 1.3 (iv)]. The estimate **(v)** is also established in [**38**, Lemma 2.1]. We give a proof of **(iv)**.

Lemma 2.2 (L^p-estimate for the Neumann heat semigroup (with divergence))**.** *Let $p \in (1,\infty)$. Denote by Δ the Laplacian in $L^p(\Omega)$ as in* Lemma 2.1*. Let $\theta \in (0,1)$. Then the following* **(i)**-**(v)** *hold:*

(i) *There exists a constant $C_1 > 0$ such that for all $\varphi \in L^p(\Omega)$,*

$$\|(-\Delta+1)^\theta e^{t\Delta}\varphi\|_{L^p(\Omega)} \le C_1\, t^{-\theta}\|\varphi\|_{L^p(\Omega)} \qquad \text{for all } t>0.$$

(ii) *There exist $C_2 > 0$ and $\nu_1 > 0$ such that for all $\varphi \in L^p(\Omega)$,*

$$\|(-\Delta+1)^\theta e^{t(\Delta-1)}\varphi\|_{L^p(\Omega)} \le C_2\, t^{-\theta}e^{-\nu_1 t}\|\varphi\|_{L^p(\Omega)} \qquad \text{for all } t>0.$$

(iii) *There exist constants $C_3 > 0$ and $\nu_2 > 0$ such that for all $\varphi \in (C_0^\infty(\Omega))^n$,*

$$\|e^{t\Delta}\nabla\cdot\varphi\|_{L^p(\Omega)} \le C_3(1+t^{-\frac{1}{2}})e^{-\nu_2 t}\|\varphi\|_{L^p(\Omega)} \qquad \text{for all } t>0.$$

Accordingly, for all $t > 0$ the operator $e^{t\Delta}\nabla\cdot$ admits a unique extension to all of $(L^p(\Omega))^n$ which, again denoted by $e^{t\Delta}\nabla\cdot$, satisfies the above estimate for all $\varphi \in (L^p(\Omega))^n$. In particular, if $0 < t < 1$, then for all $\varphi \in (L^p(\Omega))^n$,

$$\|e^{t\Delta}\nabla\cdot\varphi\|_{L^p(\Omega)} \le C_4 t^{-\frac{1}{2}}e^{-\nu_2 t}\|\varphi\|_{L^p(\Omega)},$$

where $C_4 := 2C_3$.

(iv) *Let $r \in (1,\infty]$. Then there exists $C_5 > 0$ such that for all $\varphi \in (C^1(\overline{\Omega}))^n$ fulfilling $\varphi\cdot\nu = 0$ on $\partial\Omega$ it holds that*

$$\|e^{t\Delta}\nabla\cdot\varphi\|_{L^\infty(\Omega)} \le C_5 t^{-\frac{1}{2}-\frac{n}{2r}}\|\varphi\|_{L^r(\Omega)} \qquad \text{for all } t>0.$$

(v) *There exists $\nu_3 > 0$ such that for $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that for all $\varphi \in (C_0^\infty(\Omega))^n$,*

$$\|(-\Delta+1)^\theta e^{t\Delta}\nabla\cdot\varphi\|_{L^p(\Omega)} \le c_\varepsilon t^{-\theta-\frac{1}{2}-\varepsilon}e^{-\nu_3 t}\|\varphi\|_{L^p(\Omega)} \qquad \text{for all } t>0.$$

Accordingly, for all $t > 0$ the operator $(-\Delta+1)^\theta e^{t\Delta}\nabla\cdot$ admits a unique extension to all of $(L^p(\Omega))^n$ which, again denoted by $(-\Delta+1)^\theta e^{t\Delta}\nabla\cdot$, satisfies the above estimate for all $\varphi \in (L^p(\Omega))^n$.

Proof. We give a proof to **(iv)**. By known smoothing properties of $(e^{t\Delta})_{t\ge 0}$, there exists $c_1 > 0$ such that for all $\psi \in C_c^\infty(\Omega)$,

$$\|\nabla e^{t\Delta}\psi\|_{L^{r'}(\Omega)} \le c_1 t^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r'})}\|\psi\|_{L^1(\Omega)} \qquad \text{for all } t>0,$$

where $r' \in [1,\infty)$ is such that $\frac{1}{r}+\frac{1}{r'}=1$. By the duality characterization of the norm in $L^\infty(\Omega) \simeq (L^1(\Omega))^\star$, and by density of $C_c^\infty(\Omega)$ in $L^1(\Omega)$, we thus obtain on integrating by parts and using the self-adjointness of $e^{t\Delta}$ in $L^2(\Omega)$ that

$$\begin{aligned}
\|e^{t\Delta}\nabla\cdot\varphi\|_{L^\infty(\Omega)} &= \sup_{\substack{\psi\in C_c^\infty(\Omega)\\ \|\psi\|_{L^1(\Omega)}\le 1}} \left|\int_\Omega (e^{t\Delta}\nabla\cdot\varphi)\cdot\psi\right| \\
&= \sup_{\substack{\psi\in C_c^\infty(\Omega)\\ \|\psi\|_{L^1(\Omega)}\le 1}} \left|\int_\Omega \varphi\cdot\nabla e^{t\Delta}\psi\right| \\
&\le \|\varphi\|_{L^r(\Omega)}\cdot \sup_{\substack{\psi\in C_c^\infty(\Omega)\\ \|\psi\|_{L^1(\Omega)}\le 1}} \|\nabla e^{t\Delta}\psi\|_{L^{r'}(\Omega)} \\
&\le \|\varphi\|_{L^r(\Omega)}\cdot c_1 t^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r'})} \qquad \text{for all } t>0.
\end{aligned}$$

Since $1-\frac{1}{r'}=\frac{1}{r}$, this proves the estimate **(iv)**. □

2.2. Lower bound in an elliptic equation

In this section we will establish a pointwise lower bound in the Neumann problem for the Helmholtz equation, and this estimate gives a lower bound for the second component of solutions to the parabolic-elliptic Keller–Segel system. The following lemma provides a quantitative estimate on positivity of solutions to the Neumann problem for the Helmholtz equation with nonnegative inhomogeneity having given norm in $L^1(\Omega)$.

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) be a bounded domain. Let $w \in C^0(\overline{\Omega})$ be a nonnegative function such that $w \not\equiv 0$. If $z \in C^2(\overline{\Omega})$ is a solution of*

$$\begin{cases} -\Delta z + z = w, & x\in\Omega, \\ \dfrac{\partial z}{\partial \nu} = 0, & x\in\partial\Omega, \end{cases}$$

then there exists some constant $c>0$ such that

$$z \ge c\int_\Omega w > 0 \qquad \text{in } \Omega.$$

Proof. Due to the positivity of the Green function to the Helmholtz equation (see [**42**, Theorem 18.2]), the proof is completed. □

We will apply the above lemma to the Keller–Segel system. Let (u, v) be a classical solution in $\Omega \times (0, T)$ of the problem

$$
\begin{cases}
u_t = \Delta u - \nabla \cdot (u \nabla \chi(v)), & x \in \Omega,\ t > 0, \\
0 = \Delta v - v + u, & x \in \Omega,\ t > 0, \\
\dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\
u(x,0) = u_0(x), & x \in \Omega.
\end{cases}
$$

Then this system evidently preserves the norm of the first solution component u in $L^1(\Omega)$. By Lemma 2.3, we can thereby estimate v from below according to

$$
\begin{aligned}
\inf_{x \in \Omega} v(x,t) &\geq c \int_\Omega u(x,t)\, dx \\
&= c\|u_0\|_{L^1(\Omega)} \\
&=: \gamma \qquad \text{for all } t \geq 0.
\end{aligned}
\tag{2.2}
$$

This lower bound is a cornerstone of analyzing the Keller–Segel system with sensitivity function $\chi(v)$, especially the case that the function $\chi'(v)$ has singularity at $v = 0$. The constant $\gamma > 0$ above appears in the condition for global existence of solutions to the Keller–Segel system (Chapter 3).

Moreover if we assume the convexity of the domain Ω, we can represent the constant explicitly. As a preparation for Lemma 2.5, we establish a pointwise estimate for the Neumann heat semigroup. We will generalize [**36**, Lemma 3.1] to multi-dimension. The proof builds on [**36**, Lemma 3.1] but needs some modification. We give a rigorous proof.

Lemma 2.4. *Let $(e^{t\Delta})_{t\geq 0}$ be the Neumann heat semigroup in a convex bounded domain $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N})$. Then the following inequality holds for all nonnegative $z \in C^0(\overline{\Omega})$:*

$$
e^{t\Delta} z \geq \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{(\operatorname{diam}\Omega)^2}{4t}} \int_\Omega z \qquad \text{in } \Omega \qquad \text{for all } t > 0,
\tag{2.3}
$$

where $\operatorname{diam}\Omega := \max_{x,y \in \overline{\Omega}} |x - y|$.

Proof. We first prepare a set of functions approximating $z \in C^0(\overline{\Omega})$. For $\lambda > 0$ we define

$$
\Omega + \lambda := \{x + \lambda y \,|\, x \in \Omega,\ y \in \mathbb{R}^n,\ |y| < 1\},
$$

and we define the following outward nonincreasing radial symmetric function set,

$$S := \left\{ \varphi \in C_c^\infty(\mathbb{R}^n) \,\middle|\, \varphi \geq 0, \quad \operatorname{supp}\varphi \subset \overline{\Omega + \lambda}, \quad \text{there exists } x_0 \in \Omega \text{ such that} \right.$$
$$\varphi(x_0 + y) = \varphi(x_0 + |y|z) \qquad \text{for all } y \in \mathbb{R}^n \text{ and } z \in \mathbb{R}^n,\ |z| = 1,$$
$$\left. \text{and} \quad \frac{\partial \varphi}{\partial x_1}(x_0 + \mu e_1) \leq 0 \qquad \text{for all } \mu > 0 \right\},$$

where e_1 is a unit vector $(1, 0, ..., 0)$ in $\mathbb{R}^n$. Note that for all $\varphi \in S$ there exists $x_0 \in \Omega$ satisfying

$$\frac{\partial \varphi}{\partial x_j}(x_0 + \mu e_j) \leq 0 \qquad \text{for all } \mu > 0,\ j = 1, ..., n.$$

Since φ is an outward nonincreasing radial function and Ω is convex, the maximum principle yields that

$$e^{t\Delta}\varphi \geq e^{t\Delta_C}\varphi \qquad \text{in } \Omega \qquad \text{for all } t \geq 0, \tag{2.4}$$

where $u = e^{t\Delta_C}\varphi$ is the solution of the following Cauchy problem:

$$\begin{cases} u_t = \Delta u & x \in \mathbb{R}^n,\ t > 0, \\ u(x, 0) = \varphi(x) & x \in \mathbb{R}^n. \end{cases}$$

(Step 1) In this step we prove that all $\varphi \in S$ satisfies a modification of (2.3). Using the explicit representation formula for $e^{t\Delta_C}\varphi$, we can estimate that

$$\begin{aligned} (e^{t\Delta_C}\varphi)(x) &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\Omega+\lambda} e^{-\frac{(x-y)^2}{4t}} \cdot \varphi(y)\, dy \\ &\geq \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{(\operatorname{diam}(\Omega+\lambda))^2}{4t}} \int_{\Omega+\lambda} \varphi(y)\, dy \qquad \text{for all } x \in \Omega \end{aligned}$$

due to the fact $\operatorname{supp}\varphi \subset \overline{\Omega + \lambda}$. In virtue of (2.4) we see that for all $\varphi \in S$,

$$(e^{t\Delta}\varphi)(x) \geq \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{(\operatorname{diam}(\Omega+\lambda))^2}{4t}} \int_{\Omega+\lambda} \varphi(y)\, dy \qquad \text{for all } x \in \Omega. \tag{2.5}$$

(Step 2) We approximate arbitrary nontrivial and nonnegative $z \in C^0(\overline{\Omega})$. For $N \in \mathbb{N}$ $(N \geq 2)$ fixed, we can construct $\varphi_{x,\delta} \in S$ $(x \in \Omega,\ \delta > 0)$ such as

$$\varphi_{x,\delta} = \begin{cases} 1 & \text{in } B_{\delta - \frac{\delta}{N}}(x), \\ 0 & \text{in } \mathbb{R}^n \setminus B_\delta(x). \end{cases}$$

Since z is a uniformly continuous function in $\overline{\Omega}$, for all $\varepsilon > 0$ we can choose sufficiently small $\delta > 0$ satisfying

$$\text{if } |x-y| < \delta \quad \text{then } |z(x) - z(y)| < \varepsilon \qquad \text{for all } x, y \in \Omega. \tag{2.6}$$

Firstly we fix some point $x_1 \in \Omega$ and then we confirm $B_\delta(x_1) \subset \Omega + \lambda$ by assuming $\delta > 0$ is sufficiently small. Now we can pick up points $x_i \in \Omega$ and $\delta_i = \delta$ $(i = 2, ..., m_1)$ satisfying the following property: For all $i, j = 1, ..., m_1$,

$$B_{\delta_i}(x_i) \subset \Omega + \lambda, \qquad B_{\delta_i}(x_i) \cap B_{\delta_j}(x_j) = \emptyset \quad (i \neq j) \tag{2.7}$$

and

$$B_\delta(y) \cap \left(\bigcup_{i=1}^{m_1} B_{\delta_i}(x_i) \right) \neq \emptyset \qquad \text{for all } y \in \Omega \setminus \{x_1, ..., x_{m_1}\}.$$

Next we pick up

$$x_{m_1+1} \in \Omega \setminus K_1 := \bigcup_{i=1}^{m_1} (B_{\delta_i}(x_i) \cap \Omega) \quad \text{such that} \quad B_{\frac{\delta}{2}}(x_{m_1+1}) \subset \Omega + \lambda.$$

Proceeding similarly as above, we can choose points $x_i \in \Omega \setminus K_1$ and $\delta_i = \frac{\delta}{2}$ $(i = m_1 + 1, ..., m_2)$ satisfying (2.7) and

$$B_{\frac{\delta}{2}}(y) \cap \left(\bigcup_{i=1}^{m_2} B_{\delta_i}(x_i) \right) \neq \emptyset \qquad \text{for all } y \in (\Omega \setminus K_1) \setminus \{x_{m_1+1}, ..., x_{m_2}\}.$$

Here let us define

$$K_2 := \bigcup_{i=1}^{m_2} (B_{\delta_i}(x_i) \cap \Omega).$$

Inductively we can define the points $\{x_i\}$ and the sequence $\{K_\ell\}_{\ell \in \mathbb{N}}$ such as

$$K_\ell := \bigcup_{i=1}^{m_\ell} (B_{\delta_i}(x_i) \cap \Omega), \quad K_1 \subset K_2 \subset ... \subset K_\ell \subset K_{\ell+1} \subset ..., \qquad \text{and}$$

$$|\Omega \setminus K_\ell| \to 0 \qquad \text{as} \quad \ell \to \infty.$$

Moreover we also define

$$L_{\ell,N} := \bigcup_{i=1}^{m_\ell} (B_{\delta_i - \frac{\delta_i}{N}}(x_i) \cap \Omega)$$

and we remark that for each $\ell \in \mathbb{N}$ it follows that

$$|K_\ell \setminus L_{\ell,N}| \to 0 \qquad \text{as} \quad N \to \infty.$$

By (2.6) we have for all $\ell \in \mathbb{N}$,

$$z + \varepsilon \geq \sum_{i=1}^{m_\ell} z(x_i)\varphi_{x_i,\delta_i} \geq z - \varepsilon \qquad \text{in } L_{\ell,N}.$$

Thus by linearity (2.5) yields that

$$\begin{aligned}
e^{t\Delta} z &\geq e^{t\Delta}\left(\sum_{i=1}^{m_\ell} z(x_i)\varphi_{x_i,\delta_i} - \varepsilon\right) \\
&= \sum_{i=1}^{m_\ell} z(x_i) e^{t\Delta}(\varphi_{x_i,\delta_i}) - \varepsilon \\
&\geq \sum_{i=1}^{m_\ell} z(x_i) \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{(\operatorname{diam}(\Omega+\lambda))^2}{4t}} \int_{\Omega+\lambda} \varphi_{x_i,\delta_i}(y)\, dy - \varepsilon \\
&\geq \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{(\operatorname{diam}(\Omega+\lambda))^2}{4t}} \int_{\Omega+\lambda} z - \varepsilon|\Omega+\lambda| - \varepsilon \qquad \text{in } L_{\ell,N}.
\end{aligned} \tag{2.8}$$

Here we fix $t > 0$ and $x \in \Omega$. Since the function $e^{t\Delta}z$ is uniformly continuous, then for all $\theta > 0$ there exists some $\kappa > 0$ such that

$$|e^{t\Delta}z(y) - e^{t\Delta}z(x)| < \theta \qquad \text{for all } y \in B_\kappa(x_0).$$

Moreover in view of the construction of $L_{\ell,N}$, it follows that for all $y \in \Omega$ and $\eta > 0$ there exist some $\ell_0 \in \mathbb{N}$ and $N_0 \in \mathbb{N}$ such that

$$B_\eta(y) \cap L_{\ell_0,N_0} \neq \emptyset.$$

Indeed, if $\eta > \frac{\delta}{2^{\ell_0 - 1}}$ then this situation contradicts the construction of $L_{\ell,N}$. Therefore for all $\theta > 0$ there exist some $\ell \in \mathbb{N}$, $N \in \mathbb{N}$ and $x_0 \in L_{\ell,N}$ such that

$$|e^{t\Delta}z(x_0) - e^{t\Delta}z(x)| < \theta.$$

After that, combining (2.8) and the above implies that for all $t > 0$,

$$e^{t\Delta}z \geq \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{(\operatorname{diam}(\Omega+\lambda))^2}{4t}} \int_{\Omega+\lambda} z - \varepsilon|\Omega+\lambda| - \varepsilon \qquad \text{in } \Omega.$$

Finally we can pick up sufficiently small λ and ε, and thus the desired inequality is established. □

We will give an improvement of Lemma 2.3 under the assumption of the convexity of the domain Ω.

Lemma 2.5. *Let that $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ be a convex bounded domain. Let $w \in C^0(\overline{\Omega})$ be a nonnegative function such that $w \not\equiv 0$. If $z \in C^2(\overline{\Omega})$ is a solution of*

$$\begin{cases} -\Delta z + z = w, & x \in \Omega, \\ \dfrac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases}$$

then

$$z \ge \left(\int_0^\infty \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-(t + \frac{(\operatorname{diam}\Omega)^2}{4t})}\, dt \right) \cdot \int_\Omega w > 0 \qquad \text{in } \Omega.$$

Proof. By the representation of resolvents via semigroups and Lemma 2.4, we have

$$\begin{aligned} (I-\Delta)^{-1} w &= \int_0^\infty e^{-t} e^{t\Delta} w\, dt \\ &\ge \left(\int_0^\infty \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-(t + \frac{(\operatorname{diam}\Omega)^2}{4t})}\, dt \right) \cdot \int_\Omega w. \end{aligned}$$

This completes the proof. □

Therefore under the convexity of Ω, we can choose the constant γ explicitly as follows:

$$\begin{aligned} \inf_{x\in\Omega} v(x,t) \ge \|u_0\|_{L^1(\Omega)} \int_0^\infty \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-(t + \frac{(\operatorname{diam}\Omega)^2}{4t})}\, dt \\ =: \gamma \qquad \text{for all } t \ge 0. \end{aligned} \tag{2.9}$$

2.3. Lower bound in a parabolic equation

In this section we give a quantitative lower estimate for solutions to the fully parabolic Keller–Segel system. As a preparation for this estimate, let us derive a pointwise lower bound in the Neumann problem for some parabolic equation. The mass conservation plays a key role in the proof.

Lemma 2.6. *Let $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ be a bounded domain and let $w \in C^0(\overline{\Omega} \times [0,T))$ be a nonnegative function such that $\int_\Omega w(\cdot,t) = \int_\Omega w(\cdot,0)$ $(t \in [0,T))$ and $z_0 \in C^0(\overline{\Omega})$ is positive in $\overline{\Omega}$. If $z \in C^{2,1}(\overline{\Omega} \times (0,T)) \cap C^0(\overline{\Omega} \times [0,T))$ is a classical solution to*

$$\begin{cases} z_t = \Delta z - z + w & \text{in } \Omega \times (0,T), \\ \dfrac{\partial z}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T), \\ z(\cdot,0) = z_0 & \text{in } \Omega, \end{cases}$$

then there exists $\eta > 0$ *such that*

$$\inf_{x\in\Omega} z(x,t) \geq \eta > 0 \qquad \textit{for all } t \in (0,T),$$

where η *depends only on* z_0, $\|w(0)\|_{L^1(\Omega)}$ *and* Ω.

Proof. First by the maximum principle and the positivity of $z_0 > 0$ in $\overline{\Omega}$ we have

$$z(t) \geq \min_{x\in\overline{\Omega}} z_0(x) \cdot e^{-t} > 0 \qquad \text{for all } t \geq 0.$$

Now fix $\tau > 0$ such that $2\tau \in [0,T]$. Then it follows that

$$z(t) \geq \min_{x\in\overline{\Omega}} z_0(x) \cdot e^{-2\tau} =: \eta_1 > 0 \qquad \text{for all } t \in [0,2\tau].$$

Next, we denote the fundamental solution $U(t,x;s,y)$ to the following boundary problem:

$$\begin{cases} z_t = \Delta z - z & \text{in } \Omega \times (0,T), \\ \dfrac{\partial z}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{2.10}$$

Due to the positivity of the fundamental solution (see [**42**, Theorem 10.1]), there exists some constant c_0 such that

$$U(s+\tau,x;s,y) = U(\tau,x;0,y) \geq c_0 > 0 \qquad \text{for all } x,y \in \overline{\Omega},\ s > 0,$$

where we used the semigroup property (see [**42**, Theorem 8.1 and (8.6)]). Here we remark that the constant $c_0 > 0$ is independent of $s > 0$. Moreover, we recall that the fundamental solution $(t,x) \to U(t,x;s,y)$ is a solution to the problem (2.10). By regarding $U(s+\tau,x;s,y)$ as an initial data, the maximal principle implies that

$$\begin{aligned} U(t,x;s,y) &\geq \left(\min_{x\in\overline{\Omega}} U(s+\tau,x;s,y)\right) e^{-(t-(s+\tau))} \\ &\geq c_0 e^{-(t-(s+\tau))} \qquad \text{for all } x,y \in \overline{\Omega},\ t > s+\tau. \end{aligned} \tag{2.11}$$

By using the fundamental solution $U(t,x;s,y)$, z is represented as follows (see [**42**, Theorem 9.1]):

$$z(x,t) = \int_\Omega U(t,x;0,y) z_0\, dy + \int_0^t \left(\int_\Omega U(t,x;s,y) w(s,y)\, dy\right) ds.$$

Due to the nonnegativity of $U(t,x;s,y)$ (see [**42**, Theorem 8.3]), it follows that

$$\int_\Omega U(t,x;0,y)z_0\,dy \ge 0 \qquad \text{for all } x,y\in\overline{\Omega},\ t>0.$$

We will calculate the integral term. Using (2.11), we have

$$\begin{aligned}\int_0^t \left(\int_\Omega U(t,x;s,y)w(s,y)\,dy\right) ds &\ge \int_0^{t-\tau} \left(\int_\Omega U(t,x;s,y)w(s,y)\,dy\right) ds \\ &\ge \int_0^{t-\tau} \left(\int_\Omega c_0 e^{-(t-(s+\tau))} w(s,y)\,dy\right) ds \\ &= c_0\|w_0\|_{L^1(\Omega)} \int_0^{t-\tau} e^{-(t-(s+\tau))}\,ds.\end{aligned}$$

Changing variables with $\sigma = t-s$ yields that

$$\begin{aligned}\int_0^t \left(\int_\Omega U(t,x;s,y)w(s,y)\,dy\right) ds &\ge c_0\|w_0\|_{L^1(\Omega)} \int_\tau^t e^{\tau-\sigma}\,d\sigma \\ &\ge c_0\|w_0\|_{L^1(\Omega)}(1-e^{\tau-t}) \\ &\ge c_0\|w_0\|_{L^1(\Omega)}(1-e^{-\tau}) \qquad \text{for all } t>2\tau.\end{aligned}$$

Hence we see that

$$z(t) \ge c_0\|w_0\|_{L^1(\Omega)}(1-e^{-\tau}) =: \eta_2 > 0 \qquad \text{for all } t>2\tau.$$

Therefore we have $z(t)\ge \min\{\eta_1,\eta_2\} =: \eta$ for all $t\ge 0$. This completes the proof. □

Let (u,v) be a classical solution in $\Omega\times(0,T)$ to the problem

$$\begin{cases} u_t = \Delta u - \nabla\cdot(u\nabla\chi(v)), & x\in\Omega,\ t>0, \\ v_t = \Delta v - v + u, & x\in\Omega,\ t>0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x\in\partial\Omega,\ t>0, \\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), & x\in\Omega. \end{cases}$$

Then by Lemma 2.6 there exists $\eta>0$ such that

$$\inf_{x\in\Omega} v(x,t) \ge \eta > 0 \qquad \text{for all } t\ge 0, \tag{2.12}$$

where η depends only on v_0, $\|u_0\|_{L^1(\Omega)}$ and Ω.

We remark that if we assume convexity of the domain Ω then the latter half of the proof of Lemma 2.6 will be simplified. Indeed, the representation formula of z, the maximal principle, Lemma 2.4 and the assumption $\int_\Omega w(t) = \int_\Omega w(0)$ imply that

$$\begin{aligned}
z(t) &= e^{t(\Delta-1)}z_0 + \int_0^t e^{(t-s)(\Delta-1)}w(s)\,ds \\
&\geq \int_0^t \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} e^{-\left((t-s)+\frac{(\operatorname{diam}\Omega)^2}{4(t-s)}\right)} \cdot \left(\int_\Omega w(x,s)\,dx\right) ds \\
&= \|w(0)\|_{L^1(\Omega)} \cdot \int_0^t \frac{1}{(4\pi r)^{\frac{n}{2}}} e^{-\left(r+\frac{(\operatorname{diam}\Omega)^2}{4r}\right)}\,dr \\
&\geq \|w(0)\|_{L^1(\Omega)} \cdot \int_0^\tau \frac{1}{(4\pi r)^{\frac{n}{2}}} e^{-\left(r+\frac{(\operatorname{diam}\Omega)^2}{4r}\right)}\,dr =: \eta_2 > 0 \qquad \text{for all } t \in [\tau,\infty).
\end{aligned}$$

2.4. Short course on parabolic equations

We give some definitions and recall several standard results of parabolic equations. Throughout this section, we assume that the domain $\Omega \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) is bounded and has a smooth boundary $\partial\Omega$. We use the following notation:

$$\begin{aligned}
&Q_T := \Omega \times (0,T), \\
&\Gamma_T := \{(x,t) \,|\, x \in \partial\Omega, t \in [0,T]\} \cup \{(x,t) \,|\, x \in \Omega, t = 0\}.
\end{aligned}$$

2.4.1. The Schauder estimates in linear parabolic equations

In this subsection we recall the interior Schauder estimate and the global Schauder estimate.

Throughout this subsection, we assume that there exists some $\nu > 0$ such that

$$\nu\xi^2 \leq \sum_{i,j=1}^n a_{ij}(x,t,y)\xi_j\xi_j \leq \nu\xi^2$$

for all $\xi = (\xi_1,\cdots,\xi_n) \in \mathbb{R}^n$.

Lemma 2.7 (c.f. [**58**, Theorem 4.9])**.** *Suppose that $\ell > 0$ is a nonintegral number. Let $a_{i,j}$, a_i, a, $f \in C^{\ell,\frac{\ell}{2}}(\overline{\Omega}\times[0,T])$. Let $u \in C^{\ell,\frac{\ell}{2}}(\overline{\Omega}\times[0,T])$ be a solution of the following parabolic equation,*

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^n a_{ij}(x,t)u_{x_ix_j} + \sum_{i=1}^n a_i(x,t)u_{x_i} + a(x,t)u + f(x,t).$$

Then for all $\Omega' \subset\subset \Omega$ and $\eta > 0$, the solution u belongs to $C^{\ell+2,\frac{\ell}{2}+1}(\overline{\Omega} \times [0,T])$ and there exists some constant $c > 0$ such that

$$\|u\|_{C^{\ell+2,\frac{\ell}{2}+1}(\overline{\Omega'}\times[\eta,T])} \leq c\left(\|f\|_{C^{\ell,\frac{\ell}{2}}(\overline{\Omega'}\times[0,T])} + \|u\|_{C(\overline{\Omega'}\times[0,T])}\right).$$

Next we proceed to the global Schauder estimate in the following problem:

$$\tag{2.13} \begin{cases} \dfrac{\partial u}{\partial t} = \displaystyle\sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_ix_j} + \sum_{i=1}^{n} a_i(x,t)u_{x_i} \\ \qquad\qquad\qquad\qquad +a(x,t)u + f(x,t), & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

Lemma 2.8 ([**57**, Theorem IV.5.3])**.** *Suppose that $\ell > 0$ is a nonintegral number. Let $a_{i,j}$, a_i, $a \in C^{\ell,\frac{\ell}{2}}(\overline{\Omega} \times [0,T])$. Then for any $f \in C^{\ell,\frac{\ell}{2}}(\overline{\Omega} \times [0,T])$ and $u_0 \in C^{\ell+2}(\overline{\Omega})$ satisfying the compatibility condition of order $[\frac{\ell+1}{2}]$:*

$$\frac{\partial^k u(x,t)}{\partial t^k}|_{t=0} = 0 \qquad \text{for all } k = 1, 2, ..., \left[\frac{\ell+1}{2}\right],$$

then the problem (2.13) has a unique solution belonging to the class $C^{\ell+2,\frac{\ell}{2}+1}(\overline{\Omega} \times [0,T])$ and there exists some constant $c > 0$ such that

$$\|u\|_{C^{\ell+2,\frac{\ell}{2}+1}(\overline{\Omega}\times[0,T])} \leq c\left(\|f\|_{C^{\ell,\frac{\ell}{2}}(\overline{\Omega}\times[0,T])} + \|u_0\|_{C^{\ell+2}(\overline{\Omega})}\right).$$

2.4.2. Regularity properties in quasi-linear parabolic equations

In this subsection we consider regularity of solutions to the following problem:

$$\tag{2.14} \begin{cases} \dfrac{\partial u}{\partial t} = \displaystyle\sum_{i,j=1}^{n} a_{ij}(x,t,u)u_{x_ix_j} - b(x,t,u,u_x), & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0. \end{cases}$$

We obtain an estimate of ∇u, assuming that

$$\max_{Q_T} |u| \leq M$$

with some constant $M > 0$.

Throughout this subsection, we will suppose that a_{ij} and b are sufficiently smooth to satisfy the following conditions: for all $(x,t) \in \overline{Q}_T$ and $|y| \le M$ there exist some positive constants ν and μ such that

$$\nu\xi^2 \le \sum_{i,j=1}^{n} a_{ij}(x,t,y)\xi_j\xi_j \le \nu\xi^2, \tag{2.15}$$

$$\left|\frac{\partial a_{ij}(x,t,y)}{\partial y}\right|, \left|\frac{\partial a_{ij}}{\partial x}\right| \le \mu, \tag{2.16}$$

$$|b(x,t,y,p)| \le \mu(1+p^2), \tag{2.17}$$

$$|b_p|(1+|p|) + |b_y| + |b_t| \le \mu(1+p^2), \tag{2.18}$$

$$|a_{ijt}|,\ |a_{ijyy}|,\ |a_{ijyt}|,\ |a_{ijyx}|,\ |a_{ijxt}| \le \mu. \tag{2.19}$$

We have the following estimate.

Lemma 2.9 ([**57**, Theorem V.7.2])**.** *Let $u_0 \in C^2(\overline{\Omega})$. Suppose that $a_{ij}(x,t,u)$ and $b(x,t,u,p)$ satisfy* (2.15)-(2.19)*. Let $u \in C^{2,1}(\overline{Q}_T)$ be a classical solution of* (2.14) *with* $\max_{Q_T}|u| \le M$*. Then u has the following estimate*:

$$\max_{Q_T} |\nabla u| \le M_1,$$

where $M_1 > 0$ depends only on $\|u(x,0)\|_{C^2(\overline{\Omega})}$, M, ν and μ from (2.15)-(2.19)*.*

Finally we recall the regularity result. To state the result we add the following condition:

$$-yb(x,t,y,p) \le c_0p^2 + c_1y^2 + c_2, \quad (x,t) \in \overline{Q}_T \setminus \Gamma_T \tag{2.20}$$

with some positive constants c_0, c_1 and c_2.

Lemma 2.10 (c.f. [**57**, Theorem V.7.4])**.** *Let $u \in C^{2,1}(\overline{Q}_T)$ be a classical solution of* (2.14)*. Assume that u satisfies* $\max_{Q_T}|u| \le M$ *with some $M > 0$. Suppose that the following conditions (a) and (b) are fulfilled*:

- (a) *a_{ij} and b satisfy* (2.15)-(2.20)*.*
- (b) *For all $(x,t) \in \overline{Q}_T$, $|y| \le M$, and $|p| \le M_1$ where M_1 is from* Theorem 2.9*, the functions a_{ij} are Hölder continuous with some exponent $\beta > 0$ in the variable x and b is Hölder continuous with some exponent $\beta > 0$ in the variable x, respectively.*

Then u belongs to $C^{2+\beta,1+\frac{\beta}{2}}(\overline{\Omega} \times [0,T])$.

PART I:

PARABOLIC-ELLIPTIC KELLER–SEGEL SYSTEM

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla \chi(v)), \\ 0 = \Delta v - v + u. \end{cases}$$

Chapter 3

Global existence and boundedness in a parabolic-elliptic Keller–Segel system with signal-dependent sensitivity

3.1. Problem and results

In this chapter we concern with the questions of global existence and boundedness in the parabolic-elliptic Keller–Segel system

$$
\begin{cases}
u_t = \Delta u - \nabla \cdot (u \nabla \chi(v)), & x \in \Omega, \quad t > 0, \\
0 = \Delta v - v + u, & x \in \Omega, \quad t > 0, \\
\dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\
u(x,0) = u_0(x), & x \in \Omega,
\end{cases}
\tag{3.1}
$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with smooth boundary $\partial\Omega$. We assume that the initial data u_0 satisfies

$$
u_0 \in C^0(\overline{\Omega}), \quad u_0 \geq 0 \quad \text{and} \quad u_0 \not\equiv 0. \tag{3.2}
$$

As for the chemotactic sensitivity function, we assume that

$$
\chi \in C^{2+\omega}_{\mathrm{loc}}((0,\infty)) \quad \text{with some } \omega \in (0,1), \quad \text{and} \quad \chi' > 0. \tag{3.3}
$$

From a mathematical point of view, in this context the boundedness topic appears to be quite challenging. We recall some known results related to this problem. In

[**103, 107**], the corresponding Neumann problem for the fully parabolic Keller–Segel system

(3.4)
$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla \chi(v)), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases}$$

was studied. In [**103**], global existence and boundedness of classical solutions to (3.4) with $0 < \chi'(v) \le \frac{\chi_0}{(1+\alpha v)^k}$ with some $\alpha > 0$ and $k > 1$, were proved for any $\chi_0 > 0$. On the other hand, in [**107**], global existence of classical solutions to (3.4) with $\chi(v) = \chi_0 \log v$ was proved when $\chi_0 < \sqrt{\frac{2}{n}}$; moreover, if $\chi_0 < \sqrt{\frac{n+2}{3n-4}}$, global existence of weak solutions is established. Later, in [**85**] certain weak solutions of (3.4) in the radially symmetric setting have been constructed with $\chi(v) = \chi_0 \log v$ when $\chi_0 < \sqrt{\frac{n}{n-2}}$, in particular allowing for arbitrarily large χ_0 when $n = 2$. Recently, Manásevich-Phan-Souplet [**61**] studied global existence and boundedness in a related system with additional dampening kinetic terms in the case $n = 2$ for any $\chi_0 \in \mathbb{R}$, but only for initial data for which the distribution of the attractant is sufficiently close to some explicit homogeneous state. As for the parabolic-elliptic system (3.1) with this particular choice of χ, that is, for

(3.5)
$$\begin{cases} u_t = \Delta u - \nabla \cdot \Big(u \nabla (\chi_0 \log v) \Big), & x \in \Omega,\ t > 0, \\ 0 = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases}$$

with homogeneous Neumann boundary conditions, Biler [**5**] proved global existence of weak solutions under the condition $n = 2$ and $\chi_0 \le 1$, or $n \ge 3$ and $\chi_0 < \frac{2}{n}$; their boundedness, however, is left as an open problem. Moreover, as noted in [**5**, Proof of Theorem 2], the proof given there "cannot be applied to other sublinear sensitivity functions, it heavily depends on a particular structure of the system (3.5), e.g., on the relation $\Delta(\log v) = v^{-1}\Delta v - v^{-2}|\nabla v|^2$." Independently, Nagai and Senba [**71**] studied radially symmetric solutions of (3.5), and they showed that solutions are global and remain bounded when either $n \ge 3$ and $\chi_0 < \frac{n}{n-2}$, or $n = 2$ and $\chi_0 > 0$ is arbitrary, whereas if $n \ge 3$ and $\chi_0 > \frac{2n}{n-2}$ and $\int_\Omega u_0 |x|^2 \, dx$ is sufficiently small, the solution of (3.5) will blow up in finite time. Concerning nonradial solutions, the boundedness question even for the particular system (3.5) appears to be an open problem.

Correspondingly, it is the purpose of the present chapter to derive a rather general condition on χ which ensures global existence and boundedness of solutions to (3.1). Assuming that

(3.6)
$$\chi'(s) \le \frac{\chi_0}{s^k}, \qquad s \in [\gamma, \infty),$$

we will obtain the following (cf. Theorem 3.7):

- If $k = 1$ and $\chi_0 < \frac{2}{n}$, then (3.1) possesses a unique global bounded classical solution.
- If $k > 1$ and $\chi_0 < \frac{2}{n} \cdot \frac{k^k}{(k-1)^{k-1}} \gamma^{k-1}$, then (3.1) possesses a unique global bounded classical solution.

Here the constant $\gamma > 0$ is defined in (2.2); especially under the assumption of convexity of the domain Ω, the constant $\gamma > 0$ is given in (2.9):

$$\gamma = \|u_0\|_{L^1(\Omega)} \int_0^\infty \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-(t+\frac{(\operatorname{diam}\Omega)^2}{4t})}\, dt > 0,$$

where $\operatorname{diam}\Omega := \max_{x,y\in\overline{\Omega}} |x-y|$. We firstly remark that our result for $k = 1$ goes somewhat beyond that given in [**5**] in that it provides classical solutions, rather than weak solutions, and moreover it asserts their boundedness, thus ruling out any blow-up phenomenon in infinite time. Secondly, unlike in [**5**] our proof does not depend on any particular structure of the system (3.1) with $\chi(v) = \chi_0 \log v$. Finally, if we assume the convexity of the domain Ω we observe that γ depends on $\operatorname{diam}\Omega$ in such a way that $\gamma \to \infty$ as $\operatorname{diam}\Omega \to 0$ (see (2.9)); in particular, in the case $k > 1$ for each $\chi_0 > 0$ and any choice of the mass $m > 0$, our above condition will be satisfied for any Ω with sufficiently small diameter and all nonnegative $u_0 \in C^0(\overline{\Omega})$ having mass $\int_\Omega u_0 = m$.

Before going into details, let us emphasize the main idea underlying our proof. First, testing the first equation in (3.1) by $u^{p-1}, p > 1$, and applying Young's inequality in a standard manner, we obtain the basic inequality

$$\frac{d}{dt}\int_\Omega u^p \le -\frac{p(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \frac{p(p-1)}{2}\int_\Omega u^p (\chi'(v))^2 |\nabla v|^2.$$

In order to control the rightmost term here appropriately, we shall multiply the second equation in (3.1) by $u^{p-1}\varphi(v)$ for arbitrary $\varphi \in C^1((0,\infty))$ to see upon integration that

$$\int_\Omega u^p \Big(-\varphi'(v) - C_1\varphi^2(v)\Big)|\nabla v|^2 \le C_2 \int_\Omega u^{p-2}|\nabla u|^2 + C_3 \int_\Omega u^p$$

holds with certain positive constants C_1, C_2 and C_3. Now if φ is such that the Riccati inequality

$$\frac{p(p-1)}{2}(\chi'(v))^2 \le -\varphi'(v) - C_1\varphi^2(v)$$

holds, then combining the above two inequalities will yield a uniform-in-time estimate for $\int_\Omega u^p(x,t)\,dx$ for any finite $p > 1$. Applying this to sufficiently large p will finally allow us to derive a corresponding estimate with respect to the norm in $L^\infty(\Omega)$ and conclude.

This chapter is organized as follows. Local existence of solution will be asserted in Section 3.2, whereas Section 3.3 is devoted to L^p-boundedness of solutions to (3.1) and thereby forms the main part of this chapter. Finally, the statement and the proof of the main result in Theorem 3.7 will be given in Section 3.4.

3.2. Local existence

In this section we prove local existence of classical solutions to (3.1). The arguments used here are based on [**103**]. The key point of the proof is the lower bound for v (2.2). We use the Banach fixed point theorem in a suitable space in which functions preserve L^1-norm.

Proposition 3.1. *Let u_0 and χ be as in* (3.2) *and* (3.3), *respectively. Then there exist $T_{\max} \le \infty$ (depending only on $\|u_0\|_{L^\infty(\Omega)}$) and exactly one pair (u,v) of nonnegative functions*

$$u \in C^{2,1}(\overline{\Omega} \times (0,T_{\max})) \cap C^0([0,T_{\max}); C^0(\overline{\Omega})),$$
$$v \in C^{2,0}(\overline{\Omega} \times (0,T_{\max})) \cap C^0((0,T_{\max}); C^0(\overline{\Omega}))$$

that solves (3.1) *in the classical sense. Also, the solution (u,v) satisfies the mass identities*

$$\int_\Omega u(x,t)\,dx = \int_\Omega u_0 \qquad \text{for all } t \in (0,T_{\max}) \tag{3.7}$$

and

$$\int_\Omega v(x,t)\,dx = \int_\Omega u_0 \qquad \text{for all } t \in (0,T_{\max}). \tag{3.8}$$

Moreover, if $T_{\max} < \infty$, then

$$\lim_{t \to T_{\max}} \|u(t)\|_{L^\infty(\Omega)} = \infty. \tag{3.9}$$

Proof. Existence. The existence proof follows a standard contraction argument. With $R := \|u_0\|_{L^\infty(\Omega)} + 1$ and $T \in (0,1)$ to be fixed below, we let X be the Banach space defined as

$$X := C^0([0,T]; C^0(\overline{\Omega}))$$

with norm $\|u\|_X := \|u\|_{L^\infty(\overline{\Omega}\times[0,T])}$. We claim that if T is sufficiently small, then on the closed set

$$S := \left\{ u \in X \,\middle|\, \|u\|_X \le R \quad \text{and} \quad \int_\Omega u(t)\,dx = \int_\Omega u_0 \qquad \text{for all}\, t \in [0,T] \right\},$$

the mapping

$$\Psi(u)(t) := e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\nabla\cdot(u(s)\nabla\chi(v(s)))\,ds, \qquad t\in[0,T], \tag{3.10}$$

where

$$v(s) := (I-\Delta)^{-1}u(s),$$

acts as a contraction from S into itself.

To see this, we let $u \in S$ and $v := (I-\Delta)^{-1}u$, and first deduce that $\Psi(u) \in S$ for $u \in S$. Indeed, from elementary properties of the heat semigroup we have $e^{t\Delta}u_0 \in C^0([0,T];C^0(\overline{\Omega}))$, and by Lemmas 2.1 and 2.2 we obtain

$$\int_0^t e^{(t-s)\Delta}\nabla\cdot(u(s)\nabla\chi(v(s)))\,ds \in C^0([0,T];C^0(\overline{\Omega})),$$

whence it follows that $\Psi(u) \in X$. Next, using the known property of the Neumann heat semigroup

$$\int_\Omega e^{\tau\Delta}z = \int_\Omega z \qquad \text{for all } z \in C_0^\infty(\Omega),$$

upon a completion argument it is immediate from (3.10) that

$$\int_\Omega \Psi(u)\,dx = \int_\Omega u_0 \qquad \text{for all } t\in[0,T]. \tag{3.11}$$

In order to prove that $\Psi(u) \in S$ if T is appropriately small, we let $q > n$ and choose $\theta \in (\frac{n}{2q}, \frac{1}{2})$. Moreover, fix $\varepsilon \in (0, \frac{1}{2}-\theta)$. By virtue of (2.1) with $m = 0$ and $p = \infty$, and Lemma 2.2 **(v)**, we see that for all $t \in [0,T]$,

$$\begin{aligned}
&\|\Psi(u)(t)\|_{L^\infty(\Omega)}\\
&\le \|e^{t\Delta}u_0\|_{L^\infty(\Omega)} + c_{0,\infty}\int_0^t \|(-\Delta+1)^\theta e^{(t-s)\Delta}\nabla\cdot(u(s)\chi'(v(s))\nabla v(s))\|_{L^q(\Omega)}\,ds\\
&\le \|u_0\|_{L^\infty(\Omega)} + c_{0,\infty}c_\varepsilon\int_0^t (t-s)^{-\theta-\frac{1}{2}-\varepsilon}e^{-\nu(t-s)}\|(u(s)\chi'(v(s))\nabla v(s))\|_{L^q(\Omega)}\,ds.
\end{aligned}$$

By Lemma 2.3 (also see Lemma 2.5) and the fact that $u \in S$ preserves the L^1-norm, we have

$$\begin{aligned} v(x,t) &= (I-\Delta)^{-1}u(x,t) \\ &\geq \gamma \qquad \text{for all } x \in \Omega,\ t \in [0,T]. \end{aligned}$$

On the other hand, the maximal principle implies $v(x,t) \leq R$, so that

$$\gamma \leq v(x,t) \leq R.$$

By the assumption (3.3) we obtain $\chi'(v) \in L^\infty(\Omega \times (0,T))$ and

$$\|\chi'(v)\|_X \leq \|\chi'\|_{L^\infty((\gamma,R))}.$$

Then standard elliptic regularity theory ([**19**]) implies that for all $s \in [0,T]$,

$$\begin{aligned} \|u(s)\chi'(v(s))\nabla v(s)\|_{L^q(\Omega)} &\leq \|u(s)\|_{L^\infty(\Omega)}\|\chi'\|_{L^\infty((\gamma,R))}\|\nabla v(s)\|_{L^q(\Omega)} \\ &\leq c'\|u(s)\|_{L^\infty(\Omega)}\|\chi'\|_{L^\infty((\gamma,R))}\|u(s)\|_{L^\infty(\Omega)} \\ &\leq c'\|\chi'\|_{L^\infty((\gamma,R))}R^2 \end{aligned}$$

for some positive constant $c' > 0$. Hence, we have

$$\begin{aligned} \|\Psi(u)(t)\|_{L^\infty(\Omega)} &\leq \|u_0\|_{L^\infty(\Omega)} + c_{0,\infty}c_\varepsilon c'\|\chi'\|_{L^\infty((\gamma,R))}R^2\int_0^t (t-s)^{-\theta-\frac{1}{2}-\varepsilon}\,ds \\ &\leq \|u_0\|_{L^\infty(\Omega)} + c_{0,\infty}c_\varepsilon c'\|\chi'\|_{L^\infty((\gamma,R))}R^2 \cdot T^{\frac{1}{2}-\theta-\varepsilon} \qquad \text{for all } t \in [0,T]. \end{aligned}$$

Therefore, according to our definition of R it follows that $\Psi(S) \subset S$ if we choose $T \in (0,1)$ such that

$$T \leq \left[\frac{1}{c_{0,\infty}c_\varepsilon c'\|\chi'\|_{L^\infty((\gamma,R))}R^2}\right]^{\frac{1}{\frac{1}{2}-\theta-\varepsilon}}.$$

We proceed to check that on further diminishing T if necessary we obtain that Ψ is a contraction mapping. To see this, we let $u, \overline{u} \in S$ and

$$v := (I-\Delta)^{-1}u, \qquad \overline{v} := (I-\Delta)^{-1}\overline{u}.$$

Then

$$\begin{aligned} &\|\Psi(u)(t) - \Psi(\overline{u})(t)\|_{L^\infty(\Omega)} \\ &\leq c_{0,\infty}\int_0^t \|(-\Delta+1)^\theta e^{(t-s)\Delta}\nabla\cdot\big(u(s)\chi'(v(s))\nabla v(s) - \overline{u}(s)\chi'(\overline{v}(s))\nabla\overline{v}(s)\big)\|_{L^q(\Omega)}\,ds \\ &\leq c_{0,\infty}c_\varepsilon\int_0^t (t-s)^{-\theta-\frac{1}{2}-\varepsilon}\|u(s)\chi'(v(s))\nabla v(s) - \overline{u}(s)\chi'(\overline{v}(s))\nabla\overline{v}(s)\|_{L^q(\Omega)}\,ds \end{aligned}$$

for all $t \in [0,T]$. Since $u, \overline{u} \in S$, by elliptic regularity theory we have

$$
\begin{aligned}
(3.12) \quad & \|u(s)\chi'(v(s))\nabla v(s) - \overline{u}(s)\chi'(\overline{v}(s))\nabla\overline{v}(s)\|_{L^q(\Omega)} \\
&\le \|u(s)\|_{L^\infty(\Omega)}\|\chi'\|_{L^\infty((\gamma,R))}\|\nabla(v(s)-\overline{v}(s))\|_{L^q(\Omega)} \\
&\quad + \|u(s)\|_{L^\infty(\Omega)}\|\chi'(v(s))-\chi'(\overline{v}(s))\|_{L^\infty(\Omega)}\|\nabla\overline{v}(s)\|_{L^q(\Omega)} \\
&\quad + \|u(s)-\overline{u}(s)\|_{L^\infty(\Omega)}\|\chi'\|_{L^\infty((\gamma,R))}\|\nabla\overline{v}(s)\|_{L^q(\Omega)} \\
&\le c'(\|\chi'\|_{L^\infty((\gamma,R))}R + \|\chi''\|_{L^\infty((\gamma,R))}R^2 + \|\chi'\|_{L^\infty((\gamma,R))}R)\|u(s)-\overline{u}(s)\|_{L^\infty(\Omega)} \\
&= c'(2\|\chi'\|_{L^\infty((\gamma,R))}R + \|\chi''\|_{L^\infty((\gamma,R))}R^2)\|u-\overline{u}\|_X \qquad \text{for all } s \in [0,T].
\end{aligned}
$$

Therefore we obtain

$$\|\Psi(u)-\Psi(\overline{u})\|_X \le c_{0,\infty}c_\varepsilon c'(2\|\chi'\|_{L^\infty((\gamma,R))}R + \|\chi''\|_{L^\infty((\gamma,R))}R^2)T^{\frac{1}{2}-\theta-\varepsilon}\|u-\overline{u}\|_X,$$

so that Ψ is shown to be a contraction if T is sufficiently small satisfying

$$T < \left[\frac{1}{c_{0,\infty}c_\varepsilon c'(2\|\chi'\|_{L^\infty((\gamma,R))}R + \|\chi''\|_{L^\infty((\gamma,R))}R^2)}\right]^{\frac{1}{\frac{1}{2}-\theta-\varepsilon}}.$$

From the Banach fixed point theorem we thus obtain the existence of $u \in X$ satisfying $u = \Psi(u)$.

Since the above choice of T depends only on $\|u_0\|_{L^\infty(\Omega)}$, it is clear by a standard argument that u can be extended up to some $T_{\max} \le \infty$, where necessarily (3.9) holds in case where $T_{\max} < \infty$.

Regularity. Since $u \in C^0([0,T_{\max}); C^0(\overline{\Omega}))$, the relation $v(t) = (I-\Delta)^{-1}u(t)$ shows that $v \in C^0((0,T_{\max}); C^0(\overline{\Omega}))$. Then due to standard parabolic regularity arguments (Lemma 2.8), using $u = \Psi(u)$ and semigroup techniques, we can observe that $u \in C^{2,1}(\overline{\Omega}\times(0,T_{\max}))$, $v \in C^{2,0}(\overline{\Omega}\times(0,T_{\max}))$ and (u,v) solves (3.1) in the classical sense.

It is then clear upon applying Lemma 2.3 and then the maximum principle to the first equation in (3.1) that both v and u are nonnegative.

The properties (3.7) and (3.8) then easily follow by integrating the equations in (3.1) in space.

Uniqueness. To prove uniqueness of solutions in the indicated class, let us assume that (u,v) and $(\overline{u},\overline{v})$ are the solutions on some interval $[0,T]$. Setting

$$w := u - \overline{u} \quad \text{and} \quad z := v - \overline{v},$$

by subtracting the respective equations in (3.1), multiplying by w and z, respectively, and integrating in space we obtain

$$
\begin{aligned}
(3.13)\qquad \frac{1}{2}\frac{d}{dt}\int_\Omega w^2 + \int_\Omega |\nabla w|^2 &= \int_\Omega \big(u\chi'(v)\nabla v - \overline{u}\chi'(\overline{v})\nabla\overline{v}\big)\cdot\nabla w \\
&\le \frac{1}{2}\int_\Omega |u\chi'(v)\nabla v - \overline{u}\chi'(\overline{v})\nabla\overline{v}|^2 + \frac{1}{2}\int_\Omega |\nabla w|^2,
\end{aligned}
$$

$$
\begin{aligned}
(3.14)\qquad \int_\Omega |\nabla z|^2 &= -\int_\Omega z^2 + \int_\Omega wz \\
&\le -\int_\Omega z^2 + \frac{1}{2}\int_\Omega w^2 + \frac{1}{2}\int_\Omega z^2 \\
&\le \frac{1}{2}\int_\Omega w^2 - \frac{1}{2}\int_\Omega z^2
\end{aligned}
$$

for all $t \in (0,T)$. Because (u,v) and $(\overline{u},\overline{v})$ are classical solutions, it follows that $\|u(t)\|_{L^\infty(\Omega)}$, $\|v(t)\|_{L^\infty(\Omega)}$, $\|\overline{v}(t)\|_{L^\infty(\Omega)}$, $\|\nabla v(t)\|_{L^\infty(\Omega)}$ and $\|\nabla\overline{v}(t)\|_{L^\infty(\Omega)}$ are bounded on $(0,T)$. Using a similar reasoning as in (3.12), we therefore obtain

$$
\begin{aligned}
&|u(t)\chi'(v(t))\nabla v(t) - \overline{u}(t)\chi'(\overline{v}(t))\nabla\overline{v}(t)|^2 \\
&\le \Big(\|u(t)\|_{L^\infty(\Omega)}\|\chi'\|_{L^\infty((\gamma,R'))}|\nabla(v(t)-\overline{v}(t))| \\
&\qquad + \|u(t)\|_{L^\infty(\Omega)}\|\chi''\|_{L^\infty((\gamma,R'))}|v(t)-\overline{v}(t)|\|\nabla\overline{v}(t)\|_{L^\infty(\Omega)} \\
&\qquad\qquad + |u(t)-\overline{u}(t)|\|\chi'\|_{L^\infty((\gamma,R'))}\|\nabla\overline{v}(t)\|_{L^\infty(\Omega)}\Big)^2 \\
&\le C\Big(|\nabla z(t)|^2 + \frac{1}{2}|z(t)|^2 + \frac{1}{2}|w(t)|^2\Big) \qquad \text{for all } t\in(0,T),
\end{aligned}
$$

where

$$
R' := \max\{\|v\|_X, \|\overline{v}\|_X\}
$$

and C is a positive constant. Therefore, by (3.13) we find that

$$
\frac{d}{dt}\int_\Omega w^2 \le C\Big(\int_\Omega |\nabla z|^2 + \frac{1}{2}\int_\Omega z^2 + \frac{1}{2}\int_\Omega w^2\Big) \qquad \text{for all } t\in(0,T).
$$

Combining this with (3.14) yields

$$
\frac{d}{dt}\int_\Omega w^2 \le C\int_\Omega w^2 \qquad \text{for all } t\in(0,T),
$$

which upon integration shows that $w \equiv 0$ and thereby completes the proof. □

3.3. L^p-boundedness

This section is the main part in this chapter.

Lemma 3.2. *Let $p > 1$, and let (u, v) be a classical solution of* (3.1) *in* $\Omega \times (0, T)$ *for some $T > 0$. Then for all $t \in (0, T)$ it follows that*

$$\frac{d}{dt}\int_\Omega u^p \le -\frac{p(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \frac{p(p-1)}{2}\int_\Omega u^p(\chi'(v))^2|\nabla v|^2.$$

Proof. By virtue of the first equation in (3.1), we have

$$\frac{d}{dt}\int_\Omega u^p = -p(p-1)\int_\Omega u^{p-2}|\nabla u|^2 + p(p-1)\int_\Omega u^{p-1}\nabla u \cdot \nabla\chi(v).$$

Noting that by Young's inequality,

$$|u^{p-1}\nabla u \cdot \nabla\chi(v)| \le \frac{1}{2}u^{p-2}|\nabla u|^2 + \frac{1}{2}u^p(\chi'(v))^2|\nabla v|^2,$$

we obtain the desired inequality. □

Lemma 3.3. *Let $p > 1$, and suppose that (u, v) is a classical solution of* (3.1) *in* $\Omega \times (0, T)$ *for some $T > 0$. Moreover, with $\gamma > 0$ as in* (2.2) *(see also* (2.9)*), let $\varphi \in C^1([\gamma, \infty))$ be nonnegative and such that there exists a constant $M > 0$ satisfying*

$$s\varphi(s) \le M \qquad \text{for all } s \in [\gamma, \infty).$$

Then for all $t \in (0, T)$,

$$\int_\Omega u^p\Big(-\varphi'(v) - \frac{B^2}{2}\varphi^2(v)\Big)|\nabla v|^2 \le \frac{A^2}{2}\int_\Omega u^{p-2}|\nabla u|^2 + M\int_\Omega u^p,$$

where A and B are positive constants such that $AB = p$.

Proof. Using the second equation in (3.1), we see that

$$\int_\Omega u^p\varphi(v)(\Delta v - v + u) = 0.$$

Here from the Neumann boundary condition it follows that

$$-p\int_\Omega u^{p-1}\varphi(v)\nabla u \cdot \nabla v - \int_\Omega u^p\varphi'(v)|\nabla v|^2 - \int_\Omega u^p\varphi(v)v + \int_\Omega u^{p+1}\varphi(v) = 0.$$

Noting that $u \ge 0$ and $\varphi(v) \ge 0$ imply that $\int_\Omega u^{p+1}\varphi(v) \ge 0$, we thus find that

$$\begin{aligned}-\int_\Omega u^p\varphi'(v)|\nabla v|^2 &\le p\int_\Omega u^{p-1}\varphi(v)\nabla u \cdot \nabla v + \int_\Omega u^p\varphi(v)v\\ &\le \frac{A^2}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \frac{B^2}{2}\int_\Omega u^p\varphi^2(v)|\nabla v|^2 + M\int_\Omega u^p,\end{aligned}$$

where $AB = p$. This proves the desired inequality. □

Proposition 3.4. *Suppose that $n \geq 2$, and that u_0 and χ satisfy* (3.2) *and* (3.3), *respectively. Moreover, let $\gamma > 0$ be as in* (2.2) *(see also* (2.9)*), and let (u,v) denote the classical solution of* (3.1) *in $\Omega \times (0, T_{\max})$ as constructed in* Proposition 3.1.

(i) *Assume that with some $\chi_0 > 0$ we have*

$$\chi'(s) \leq \frac{\chi_0}{s}, \qquad s \in [\gamma, \infty).$$

Then for all $p \in [1, \frac{1}{\chi_0})$ there exists $M_p > 0$ such that

$$\|u(\cdot,t)\|_{L^p} \leq M_p \qquad \textit{for all } t \in [0, T_{\max}).$$

(ii) *Suppose that there exist $k > 1$ and $\chi_0 > 0$ such that*

$$\chi'(s) \leq \frac{\chi_0}{s^k}, \qquad s \in [\gamma, \infty).$$

Then for any $p \in \left[1, \frac{1}{\chi_0} \cdot \frac{k^k}{(k-1)^{k-1}} \gamma^{k-1}\right)$ we can find $M_p > 0$ fulfilling

$$\|u(\cdot,t)\|_{L^p} \leq M_p \qquad \textit{for all } t \in [0, T_{\max}).$$

Proof. **(i)** Let $p \in (1, \frac{1}{\chi_0})$, so that $\chi_0 < \frac{1}{p}$. By continuity, we can then pick some $\varepsilon > 0$ such that

$$\varepsilon < p(p-1) \qquad \text{and} \qquad \chi_0 \leq \frac{1}{p} \cdot \sqrt{\frac{p(p-1)-\varepsilon}{p(p-1)}}.$$

Applying Lemma 3.3 to

$$\begin{aligned} &\varphi(s) := \frac{1}{B^2 s}, \qquad s > 0, \\ &A := \sqrt{p(p-1)-\varepsilon}, \qquad \text{and} \qquad B := \frac{p}{\sqrt{p(p-1)-\varepsilon}}, \end{aligned}$$

we obtain

$$\int_\Omega u^p \Big(-\varphi'(v) - \frac{B^2}{2}\varphi^2(v) \Big) |\nabla v|^2 \leq \frac{p(p-1)-\varepsilon}{2} \int_\Omega u^{p-2}|\nabla u|^2 + M \int_\Omega u^p. \tag{3.15}$$

Moreover, computing

$$-\varphi'(s) - \frac{B^2}{2}\varphi^2(s) = \frac{1}{2B^2 s^2}, \qquad s > 0,$$

we find that

$$(3.16)\qquad \begin{aligned} \frac{p(p-1)}{2}(\chi'(s))^2 &\le \frac{p(p-1)}{2}\cdot\frac{{\chi_0}^2}{s^2} \\ &\le \frac{p(p-1)}{2}\cdot\frac{1}{p^2}\cdot\frac{p(p-1)-\varepsilon}{p(p-1)}\cdot\frac{1}{s^2} \\ &= \frac{1}{2B^2s^2} \\ &= -\varphi'(s) - \frac{B^2}{2}\varphi^2(s) \qquad \text{for all } s>0. \end{aligned}$$

By virtue of (3.16), we can now combine (3.15) with Lemma 3.2 to achieve the inequality

$$(3.17)\qquad \begin{aligned} \frac{d}{dt}\int_\Omega u^p &\le -\frac{p(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \frac{p(p-1)}{2}\int_\Omega u^p(\chi'(v))^2|\nabla v|^2 \\ &\le -\frac{p(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \int_\Omega u^p\Big(-\varphi'(v) - \frac{B^2}{2}\varphi^2(v)\Big)|\nabla v|^2 \\ &\le -\frac{p(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \frac{p(p-1)-\varepsilon}{2}\int_\Omega u^{p-2}|\nabla u|^2 + M\int_\Omega u^p \\ &= -\frac{\varepsilon}{2}\int_\Omega u^{p-2}|\nabla u|^2 + M\int_\Omega u^p \end{aligned}$$

for all $t\in(0,T_{\max})$. Now invoking the Gagliardo–Nirenberg inequality, we see that

$$(3.18)\qquad \int_\Omega u^p = \|u^{\frac{p}{2}}\|^2_{L^2(\Omega)} \le C_{\rm GN}\Big(\|\nabla u^{\frac{p}{2}}\|_{L^2(\Omega)} + \|u^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}\Big)^{2a}\|u^{\frac{p}{2}}\|^{2(1-a)}_{L^{\frac{2}{p}}(\Omega)},$$

where $C_{\rm GN}$ is a positive constant and

$$(3.19)\qquad a := \frac{\frac{p}{2}-\frac{1}{2}}{\frac{p}{2}+\frac{1}{n}-\frac{1}{2}} \in (0,1).$$

Since according to the mass conservation property (3.7) we have

$$(3.20)\qquad \|u^{\frac{p}{2}}(\cdot,t)\|^{\frac{2}{p}}_{L^{\frac{2}{p}}(\Omega)} = \int_\Omega u(x,t)\,dx = \int_\Omega u_0(x) \qquad \text{for all}\, t\in(0,T_{\max}),$$

from (3.18) and (3.18) we infer the existence of some $K>0$ such that

$$\int_\Omega u^p \le K\Big(\|\nabla u^{\frac{p}{2}}\|^2_{L^2(\Omega)} + 1\Big)^a,$$

so that we have

$$\int_\Omega u^{p-2}|\nabla u|^2 = \frac{4}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 \tag{3.21}$$
$$\geq \frac{4}{K^{\frac{1}{a}}p^2}\Big(\int_\Omega u^p\Big)^{\frac{1}{a}} - \frac{4}{p^2}.$$

Inserting (3.21) into (3.17), we obtain

$$\frac{d}{dt}\int_\Omega u^p \leq -\frac{2\varepsilon}{K^{\frac{1}{a}}p^2}\Big(\int_\Omega u^p\Big)^{\frac{1}{a}} + M\int_\Omega u^p + \frac{2\varepsilon}{p^2}$$

for all $t \in (0, T_{\max})$. Consequently, $y(t) := \int_\Omega u^p(x,t)\,dx$ satisfies

$$y'(t) \leq -C_1 y^{\frac{1}{a}}(t) + C_2 y(t) + C_3$$

with certain positive constants $C_1,\ C_2$ and C_3. In view of (3.19), we have $\frac{1}{a} > 1$ and thus a standard ODE comparison argument implies the boundedness of y on $(0, T_{\max})$. Thus we conclude that $\|u(\cdot,t)\|_{L^p(\Omega)} \leq M_p < \infty$ holds for all $t \in (0, T_{\max})$ and some $M_p > 0$.

(ii) Taking any $p \in \Big[1, \frac{1}{\chi_0}\cdot\frac{k^k}{(k-1)^{k-1}}\gamma^{k-1}\Big)$, we have $\chi_0 < \frac{1}{p}\cdot\frac{k^k}{(k-1)^{k-1}}\gamma^{k-1}$. We now take $\varepsilon > 0$ and $L > 0$ such that

$$\varepsilon < p(p-1), \quad L < \gamma < \frac{k}{k-1}L \quad \text{and} \quad \chi_0 \leq \frac{1}{p}\cdot\sqrt{\frac{p(p-1)-\varepsilon}{p(p-1)}}\cdot\frac{k^k}{(k-1)^{k-1}}L^{k-1}.$$

Applying Lemma 3.3 to

$$\varphi(s) := \frac{1}{B^2(s-L)}, \qquad s \geq \gamma,$$
$$A := \sqrt{p(p-1)-\varepsilon} \qquad \text{and} \qquad B := \frac{p}{\sqrt{p(p-1)-\varepsilon}},$$

we infer that

$$\int_\Omega u^p\Big(-\varphi'(v) - \frac{B^2}{2}\varphi^2(v)\Big)|\nabla v|^2 \leq \frac{p(p-1)-\varepsilon}{2}\int_\Omega u^{p-2}|\nabla u|^2 + M\int_\Omega u^p. \tag{3.22}$$

Moreover, we have

$$\Big(-\varphi'(s) - \frac{B^2}{2}\varphi^2(s)\Big)\cdot s^{2k} = \frac{1}{2B^2}\cdot\frac{s^{2k}}{(s-L)^2}.$$

Since the minimum value of $[\gamma, \infty) \ni s \mapsto \frac{s^{2k}}{(s-L)^2}$ is attained at $s = \frac{k}{k-1}L$, we have

$$\Big(-\varphi'(s) - \frac{B^2}{2}\varphi^2(s)\Big) \cdot s^{2k} \geq \frac{1}{2B^2} \cdot \frac{\left(\frac{k}{k-1}L\right)^{2k}}{\left(\frac{k}{k-1}L - L\right)^2} = \frac{1}{2B^2}\left[\frac{k^k}{(k-1)^{k-1}}L^{k-1}\right]^2$$

for all $s \in [\gamma, \infty)$. Hence it follows that

$$\begin{aligned} \frac{p(p-1)}{2}(\chi'(s))^2 &\leq \frac{p(p-1)}{2} \cdot \frac{{\chi_0}^2}{s^{2k}} \\ &\leq \frac{p(p-1)}{2} \cdot \frac{1}{p^2} \cdot \frac{p(p-1)-\varepsilon}{p(p-1)} \cdot \left[\frac{k^k}{(k-1)^{k-1}}L^{k-1}\right]^2 \cdot \frac{1}{s^{2k}} \\ &= \frac{1}{2B^2}\left[\frac{k^k}{(k-1)^{k-1}}L^{k-1}\right]^2 \cdot \frac{1}{s^{2k}} \\ &\leq -\varphi'(s) - \frac{B^2}{2}\varphi^2(s) \end{aligned} \tag{3.23}$$

for any such s. Now by (3.23), we can combine (3.22) with Lemma 3.2 to see that

$$\begin{aligned} \frac{d}{dt}\int_\Omega u^p &\leq -\frac{p(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \frac{p(p-1)}{2}\int_\Omega u^p(\chi'(v))^2|\nabla v|^2 \\ &\leq -\frac{p(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \int_\Omega u^p\Big(-\varphi'(v) - \frac{B^2}{2}\varphi^2(v)\Big)|\nabla v|^2 \\ &\leq -\frac{p(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \frac{p(p-1)-\varepsilon}{2}\int_\Omega u^{p-2}|\nabla u|^2 + M\int_\Omega u^p \\ &= -\frac{\varepsilon}{2}\int_\Omega u^{p-2}|\nabla u|^2 + M\int_\Omega u^p \end{aligned}$$

for all $t \in (0, T_{\max})$, from which proceeding similarly as in the case **(i)** we conclude that

$$\|u(\cdot, t)\|_{L^p} \leq M_p < \infty$$

holds for all $t \in (0, T_{\max})$ and some $M_p > 0$. □

3.4. L^∞-boundedness

Let us first show that when $p > \frac{n}{2}$, L^p-boundedness in time implies L^∞-boundedness in time. Combining this result with Proposition 3.4 will prove our main theorem.

Lemma 3.5. *Let $p \in (1, n)$, and let (u, v) be a classical solution of* (3.1) *in $\Omega \times (0, T)$ for some $T > 0$. Then there exists $C > 0$ such that*

$$\|\nabla v(\cdot, t)\|_{L^{\frac{np}{n-p}}(\Omega)} \le C\|u(\cdot, t)\|_{L^p(\Omega)} \qquad \text{for all } t \in (0, T).$$

Proof. This follows from standard elliptic regularity theory and the Sobolev embedding theorem ([**19**]). □

Proposition 3.6. *Let $n \ge 2$, and suppose that u_0 and χ are as in* (3.2) *and* (3.3), *respectively. Let (u, v) be the classical solution of* (3.1) *in $\Omega \times (0, T_{\max})$, and assume further that with $\gamma > 0$ given by* (2.2) *(see also* (2.9)*), which leads $\chi' \in L^\infty((\gamma, \infty))$. Then if for some $p > \frac{n}{2}$ and $M_p > 0$,*

$$\|u(\cdot, t)\|_{L^p} \le M_p \qquad \text{for all } t \in (0, T_{\max}), \tag{3.24}$$

then (u, v) actually is global in time, that is, $T_{max} = \infty$, and moreover there exists $M_\infty > 0$ such that

$$\|u(\cdot, t)\|_{L^\infty} \le M_\infty \qquad \text{for all } t \in (0, \infty). \tag{3.25}$$

Proof. We may assume that $p < n$. Since $p > \frac{n}{2}$, there exists $q > n$ such that

$$1 - \frac{(n-p)q}{np} > 0,$$

which enables us to pick $\lambda \in (1, \infty)$ fulfilling

$$\frac{1}{\lambda} < 1 - \frac{(n-p)q}{np}.$$

Then by the Hölder inequality and the assumption

$$\chi' \in L^\infty((\gamma, \infty))$$

we can estimate

$$\begin{aligned}\|u(\cdot, t)\nabla\chi(v(\cdot, t))\|_{L^q(\Omega)} &\le \|\chi'\|_{L^\infty((\gamma,\infty))}\|u(\cdot, t)\nabla v(\cdot, t)\|_{L^q(\Omega)} \\ &\le \|\chi'\|_{L^\infty((\gamma,\infty))}\|u(\cdot, t)\|_{L^{q\lambda}(\Omega)}\|\nabla v(\cdot, t)\|_{L^{q\lambda'}(\Omega)}\end{aligned}$$

for all $t \in (0, T_{\max})$, where

$$\lambda' := \frac{\lambda}{\lambda - 1},$$

and hence conclude using (3.24) and Lemma 3.5 that

$$\|u(\cdot,t)\nabla\chi(v(\cdot,t))\|_{L^q(\Omega)} \le c_1\|u(\cdot,t)\|_{L^{q\lambda}(\Omega)} \qquad \text{for all } t\in(0,T_{\max})$$

with some $c_1>0$. Again by the Hölder inequality, we thus find that

$$\begin{aligned}\|u(\cdot,t)\nabla\chi(v(\cdot,t))\|_{L^q(\Omega)} &\le c_1\|u(\cdot,t)\|^{\beta}_{L^\infty(\Omega)}\|u(\cdot,t)\|^{1-\beta}_{L^1(\Omega)} \\ &\le c_2\|u(\cdot,t)\|^{\beta}_{L^\infty(\Omega)} \qquad \text{for all } t\in(0,T_{\max})\end{aligned} \tag{3.26}$$

with a certain $c_2>0$ and some $\beta\in(0,1)$.

We now let $\theta\in(\frac{n}{2q},\frac12)$ and $\varepsilon\in(0,\frac12-\theta)$ and fix any $T\in(0,T_{\max})$. In view of (2.1) applied to $m=0$ and $p=\infty$ and the representation formula

$$u(\cdot,t)=e^{t\Delta}u_0-\int_0^t e^{(t-s)\Delta}\nabla\cdot(u(s)\nabla\chi(v(s)))\,ds \qquad \text{for } t\in(0,T_{\max}),$$

Lemma 2.2 **(v)** yields that

$$\begin{aligned}&\|u(\cdot,t)\|_{L^\infty(\Omega)} \\ &\le \|u_0\|_{L^\infty(\Omega)}+c_{0,\infty}\int_0^t \|(-\Delta+1)^{\theta}e^{(t-s)\Delta}\nabla\cdot(u(s)\nabla\chi(v(s)))\|_{L^q(\Omega)}\,ds \\ &\le \|u_0\|_{L^\infty(\Omega)}+c_{0,\infty}c_\varepsilon\int_0^t (t-s)^{-\theta-\frac12-\varepsilon}e^{-\nu(t-s)}\|u(s)\nabla\chi(v(s))\|_{L^q(\Omega)}\,ds\end{aligned} \tag{3.27}$$

for all $t\in(0,T)$. Combining (3.27) with (3.26), we have

$$\begin{aligned}\|u(\cdot,t)\|_{L^\infty(\Omega)} &\le \|u_0\|_{L^\infty(\Omega)}+c_{0,\infty}c_\varepsilon c_2\int_0^t (t-s)^{-\theta-\frac12-\varepsilon}e^{-\nu(t-s)}\|u(\cdot,s)\|^{\beta}_{L^\infty(\Omega)}\,ds \\ &\le \|u_0\|_{L^\infty(\Omega)}+\frac{c_{0,\infty}c_\varepsilon c_2}{\nu^{\frac12-\theta-\varepsilon}}\Big(\int_0^\infty r^{-\theta-\frac12-\varepsilon}e^{-r}\,dr\Big)\cdot\sup_{t\in[0,T]}\|u(\cdot,t)\|^{\beta}_{L^\infty(\Omega)}\end{aligned}$$

for all $t\in(0,T)$, where constant $K':=\frac{c_{0,\infty}c_\varepsilon c_2}{\nu^{\frac12-\theta-\varepsilon}}\int_0^\infty r^{-\theta-\frac12-\varepsilon}e^{-r}\,dr>0$ is independent of T. Therefore we obtain

$$\sup_{t\in[0,T]}\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)}+K'\Big(\sup_{t\in[0,T]}\|u(\cdot,t)\|_{L^\infty(\Omega)}\Big)^{\beta}$$

for all $T\in(0,T_{\max})$. Consequently, (u,v) is a global and bounded solution since $\beta\in(0,1)$. □

Theorem 3.7. *Let $n \geq 2$, and suppose that u_0 and χ satisfy* (3.2) *and* (3.3), *respectively. Moreover, assume that χ satisfies*

$$\chi'(s) \leq \frac{\chi_0}{s^k} \qquad \text{for all } s \in [\gamma, \infty),$$

with some $k \geq 1$ and some $\chi_0 > 0$ fulfilling

$$\chi_0 < \begin{cases} \dfrac{2}{n} & \text{if } k = 1, \\[2ex] \dfrac{2}{n} \cdot \dfrac{k^k}{(k-1)^{k-1}} \gamma^{k-1} & \text{if } k > 1. \end{cases}$$

Then (3.1) *possesses a unique global classical solution which satisfies*

$$\|u(\cdot, t)\|_{L^\infty} \leq M_\infty \qquad \text{for all } t \in [0, \infty)$$

with some constant $M_\infty > 0$.

Proof. According to the hypothesis on $\chi(v)$, by Proposition 3.4 we can find some $p > \frac{n}{2}$ and $M_p > 0$ such that

$$\|u(\cdot, t)\|_{L^p} \leq M_p \qquad \text{for all } t \in (0, T_{\max});$$

moreover, we have

$$\chi'(s) \leq \frac{\chi_0}{\gamma^k}.$$

In particular, this implies that $\chi' \in L^\infty((\gamma, \infty))$, so that we can apply Proposition 3.6 to complete the proof. $\square$

Chapter 4

Global existence and boundedness in a parabolic-elliptic Keller–Segel system with signal-dependent sensitivity and linear growth source

4.1. Problem and results

In this chapter we consider global existence and boundedness in the parabolic-elliptic chemotaxis-growth system

$$
\begin{cases}
u_t = \Delta u - \nabla \cdot (u \nabla \chi(v)) + f(u), & x \in \Omega, \quad t > 0, \\
0 = \Delta v - v + u, & x \in \Omega, \quad t > 0, \\
\dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\
u(x,0) = u_0(x), & x \in \Omega,
\end{cases}
\tag{4.1}
$$

where Ω is a bounded domain in $\mathbb{R}^n$ ($n \in \mathbb{N}$) with smooth boundary $\partial\Omega$. We assume that the initial data u_0 satisfies

$$
u_0 \in C^0(\overline{\Omega}), \quad u_0 \geq 0 \quad \text{and} \quad u_0 \not\equiv 0. \tag{4.2}
$$

As for the chemotactic sensitivity function, we assume that

$$
\chi \in C^{2+\omega}_{\rm loc}((0,\infty)) \quad \text{with some } \omega \in (0,1), \quad \text{and} \quad \chi' > 0. \tag{4.3}
$$

Also we assume that $f \in C^1([0,\infty))$ and there exist constants $\lambda_1, \lambda_2, \mu_1, \mu_2 > 0$ such that

$$\lambda_1 - \mu_1 s \le f(s) \le \lambda_2 - \mu_2 s \qquad \text{for all } s \in [0,\infty). \tag{4.4}$$

The present work is devoted to global existence and boundedness. We remark that existence of classical solutions of (4.1) is shown by a similar way in Proposition 3.1. Since $f(0) \ge \lambda_1 > 0$ by (4.4), the solution of (4.1) is nonnegative.

In order to formulate our main result, given a nonnegative $0 \not\equiv u_0 \in C^0(\overline{\Omega})$, let us define a constant $\gamma_f > 0$ as

$$\gamma_f := c \cdot \min\left\{ \|u_0\|_{L^1(\Omega)}, \ \frac{\lambda_1}{\mu_1}|\Omega| \right\} < \infty, \tag{4.5}$$

where $c > 0$ is a constant in (2.2). If we assume the convexity of the domain Ω, the constant $\gamma_f > 0$ is also given as follows

$$\gamma_f = \min\left\{ \|u_0\|_{L^1(\Omega)}, \ \frac{\lambda_1}{\mu_1}|\Omega| \right\} \cdot \int_0^\infty \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-(t + \frac{(\operatorname{diam}\Omega)^2}{4t})}\, dt < \infty, \tag{4.6}$$

where

$$\operatorname{diam}\Omega := \max_{x,y\in\overline{\Omega}} |x-y|.$$

The constant γ_f marks an a priori pointwise lower bound on the solution component v, as we shall see below. In what follows, when $k = 1$ we regard the value of $\frac{k^k}{(k-1)^{k-1}}$ as 1.

Theorem 4.1. *Let $n \in \mathbb{N}$, and suppose that u_0, χ and f satisfy* (4.2), (4.3) *and* (4.4)*, respectively. Moreover, assume that χ satisfies*

$$\chi'(s) \le \frac{\chi_0}{s^k} \qquad \text{for all } s \in [\gamma_f, \infty),$$

with some $k \ge 1$ and some $\chi_0 > 0$ fulfilling

$$\chi_0 < \frac{2}{n} \cdot \frac{k^k}{(k-1)^{k-1}} {\gamma_f}^{k-1}.$$

Then (4.1) *possesses a unique global classical solution (u, v) which satisfies*

$$\|u(\cdot,t)\|_{L^\infty} \le M_\infty \qquad \text{for all } t \in [0,\infty)$$

with some constant $M_\infty > 0$.

4.2. Preliminaries

We first give an a priori pointwise lower bound on the solution component v. The first equation in (4.1) and the condition (4.4) imply

$$\begin{aligned}\frac{d}{dt}\int_\Omega u &= \int_\Omega f(u) \\ &\geq \lambda_1|\Omega| - \mu_1\int_\Omega u.\end{aligned}$$

Integrating this inequality, we have

$$\int_\Omega u \geq \frac{\lambda_1}{\mu_1}|\Omega| + e^{-\mu_1 t}\Big(\|u_0\|_{L^1(\Omega)} - \frac{\lambda_1}{\mu_1}|\Omega|\Big) \qquad \text{for all } t \in (0,\infty),$$

and then

$$\int_\Omega u \geq \min\Big\{\|u_0\|_{L^1(\Omega)},\ \frac{\lambda_1}{\mu_1}|\Omega|\Big\}.$$

By virtue of Lemma 2.3 (see also Lemma 2.5) we can thereby estimate v from below as follows:

$$v(x,t) \geq \gamma_f \tag{4.7}$$

for all $x \in \Omega$ and $t \in (0,T)$, whenever (u,v) solves (4.1) in $\Omega \times (0,T)$ for some $T > 0$. Here $\gamma_f > 0$ is a constant defined as (4.5) (see also (4.6)).

4.3. Global existence and boundedness

We first deduce L^p-boundedness of solutions to (4.1). Next let us show that L^p-boundedness with sufficiently large p implies L^∞-boundedness. Combining these results will prove Theorem 4.1.

Lemma 4.2. *Let $p > 1$, and suppose that (u,v) is a classical solution of (4.1) in $\Omega \times (0,T)$ for some $T > 0$. Then there exist $C_1, C_2 > 0$ such that*

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^p \leq &-\frac{p(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \frac{p(p-1)}{2}\int_\Omega u^p(\chi'(v))^2|\nabla v|^2 \\ &+ C_1\int_\Omega u^p + C_2 \qquad \text{for all } t \in (0,T).\end{aligned}$$

Proof. By virtue of the first equation in (4.1) and Young's inequality, we have

$$\frac{d}{dt}\int_\Omega u^p \leq -\frac{p(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \frac{p(p-1)}{2}\int_\Omega u^p(\chi'(v))^2|\nabla v|^2 + \int_\Omega u^{p-1}f(u).$$

The condition (4.4) yields

$$\begin{aligned}\int_\Omega u^{p-1}f(u) &\le \lambda_2\int_\Omega u^{p-1} - \mu_2\int_\Omega u^p\\ &\le C_1\int_\Omega u^p + C_2\end{aligned}$$

for some constants $C_1, C_2 > 0$, and hence we obtain the desired inequality. □

The next lemma is obtained in Lemma 3.3. For convenience we give the sketch of the proof.

Lemma 4.3. *Let $p > 1$, and suppose that (u, v) is a classical solution of (4.1) in $\Omega \times (0, T)$ for some $T > 0$. Moreover, for $\gamma_f > 0$ given by (4.5) (see also (4.6)), let $\varphi \in C^1([\gamma_f, \infty))$ such that $\varphi \ge 0$ and there exists a constant $M > 0$ satisfying*

$$s\varphi(s) \le M \qquad \text{for all } s \ge \gamma_f.$$

Let A and B be positive constants such that $AB = p$. Then

$$\int_\Omega u^p\Big(-\varphi'(v) - \frac{B^2}{2}\varphi^2(v)\Big)|\nabla v|^2 \le \frac{A^2}{2}\int_\Omega u^{p-2}|\nabla u|^2 + M\int_\Omega u^p \qquad \text{for all } t \in (0, T).$$

Sketch of the proof. Multiplying the second equation in (4.1) by $u^p\varphi(v)$ and using integration by parts, we see that

$$-\int_\Omega u^p\varphi'(v)|\nabla v|^2 = p\int_\Omega u^{p-1}\varphi(v)\nabla u\cdot\nabla v + \int_\Omega u^p\varphi(v)v - \int_\Omega u^{p+1}\varphi(v).$$

Applying Young's inequality completes the proof. □

Now we give L^p-boundedness of solutions to (4.1).

Proposition 4.4. *Suppose that $n \in \mathbb{N}$, and that u_0, χ and f satisfy (4.2), (4.3) and (4.4), respectively. Let (u, v) be a classical solution of (4.1) in $\Omega \times (0, T)$ for some $T > 0$. Moreover, let $\gamma_f > 0$ be as in (4.5) (see also (4.6)). Suppose that there exist $k \ge 1$ and $\chi_0 > 0$ such that*

$$\chi'(s) \le \frac{\chi_0}{s^k} \qquad \text{for all } s \ge \gamma_f.$$

Then for any $p \in \Big[1, \frac{1}{\chi_0}\cdot\frac{k^k}{(k-1)^{k-1}}{\gamma_f}^{k-1}\Big)$ there exists a constant $M_p > 0$ fulfilling

$$\|u(\cdot, t)\|_{L^p} \le M_p \qquad \text{for all } t \in [0, T).$$

Proof. Taking any $p \in \Big[1, \frac{1}{\chi_0} \cdot \frac{k^k}{(k-1)^{k-1}} {\gamma_f}^{k-1}\Big)$, we have

$$\chi_0 < \frac{1}{p} \cdot \frac{k^k}{(k-1)^{k-1}} {\gamma_f}^{k-1}.$$

Now we take $\varepsilon > 0$ and $L > 0$ such that

$$\varepsilon < p(p-1), \quad L < \gamma_f < \frac{k}{k-1} L \quad \text{and} \quad \chi_0 \le \frac{1}{p} \cdot \sqrt{\frac{p(p-1)-\varepsilon}{p(p-1)}} \cdot \frac{k^k}{(k-1)^{k-1}} L^{k-1}.$$

Applying Lemma 4.3 to

$$\varphi(s) := \frac{1}{B^2(s-L)}, \qquad s > \gamma_f,$$

$$A := \sqrt{p(p-1)-\varepsilon} \qquad \text{and} \qquad B := \frac{p}{\sqrt{p(p-1)-\varepsilon}},$$

we infer that

$$\int_\Omega u^p \Big(-\varphi'(v) - \frac{B^2}{2} \varphi^2(v) \Big) |\nabla v|^2 \le \frac{p(p-1)-\varepsilon}{2} \int_\Omega u^{p-2} |\nabla u|^2 + M \int_\Omega u^p \tag{4.8}$$

and

$$\frac{p(p-1)}{2} (\chi'(v))^2 \le -\varphi'(s) - \frac{B^2}{2} \varphi^2(s) \qquad \text{for all } s \ge \gamma_f. \tag{4.9}$$

Now by (4.9), we can combine (4.8) with Lemma 4.2 to see that

$$\begin{aligned} \frac{d}{dt} \int_\Omega u^p &\le -\frac{p(p-1)}{2} \int_\Omega u^{p-2} |\nabla u|^2 + \frac{p(p-1)-\varepsilon}{2} \int_\Omega u^{p-2} |\nabla u|^2 \\ &\quad + (M + C_1) \int_\Omega u^p + C_2 \\ &= -\frac{\varepsilon}{2} \int_\Omega u^{p-2} |\nabla u|^2 + (M + C_1) \int_\Omega u^p + C_2 \end{aligned} \tag{4.10}$$

for all $t \in (0, T)$. Since the first equation in (4.1) and the condition (4.4) yield

$$\begin{aligned} \frac{d}{dt} \int_\Omega u &= \int_\Omega f(u) \\ &\le \lambda_2 |\Omega| - \mu_2 \int_\Omega u, \end{aligned}$$

we see that for all $t \in (0, \infty)$,

$$\int_\Omega u \le \frac{\lambda_2}{\mu_2}|\Omega| + e^{-\mu_2 t}\Big(\|u_0\|_{L^1(\Omega)} - \frac{\lambda_2}{\mu_2}|\Omega|\Big)$$
$$\le \max\Big\{\|u_0\|_{L^1(\Omega)},\ \frac{\lambda_2}{\mu_2}|\Omega|\Big\}.$$

By virtue of this estimate, proceeding similarly as in Proposition 3.4, we can complete the proof from (4.10). □

Next, assuming L^p-boundedness, we derive L^∞-boundedness.

Proposition 4.5. *Let $n \in \mathbb{N}$, and assume that u_0, χ and f satisfy* (4.2), (4.3) *and* (4.4), *respectively. Let (u, v) be the classical solution of* (4.1) *in $\Omega \times (0, T)$, and assume further that $\chi' \in L^\infty((\gamma_f, \infty))$ with $\gamma_f > 0$ given by* (4.5) (*see also* (4.6)). *Then if there exist some $p > \frac{n}{2}$ and a constant $M_p > 0$ such that $\|u(\cdot, t)\|_{L^p} \le M_p$ for all $t \in (0, T)$, then there exists a constant $M_\infty > 0$ independent of T such that*

$$\|u(\cdot, t)\|_{L^\infty} \le M_\infty \qquad \textit{for all } t \in (0, T).$$

Proof. Let $p > \frac{n}{2}$. We may assume that $p < n$. We see from (4.4) that $f(s) + s \le C(1 + s)$ for some $C > 0$. We can take $q > n$ so that $q > p$. Then we have

$$\begin{aligned}\|f(u) + u\|_{L^q(\Omega)} &\le C\|1 + u\|^{\frac{p}{q}}_{L^p(\Omega)}\|1 + u\|^{1-\frac{p}{q}}_{L^\infty(\Omega)} \\ &\le C'_p\|1 + u\|^{1-\frac{p}{q}}_{L^\infty(\Omega)} \\ &\le C''_p + C''_p\|u\|^{1-\frac{p}{q}}_{L^\infty(\Omega)},\end{aligned} \tag{4.11}$$

where C'_p, C''_p are some positive constants. Recalling the choice of q, we see that $1 - \frac{p}{q} \in (0, 1)$. Moreover, we choose $q > n$ satisfying further that $1 - \frac{(n-p)q}{np} > 0$, which enables us to pick $\lambda \in (1, \infty)$ fulfilling $\frac{1}{\lambda} < 1 - \frac{(n-p)q}{np}$. The elliptic regularity ($\|\nabla v\|_{L^{\frac{np}{n-p}}(\Omega)} \le k_p\|u\|_{L^p(\Omega)}$) and Hölder's inequality yield

$$\begin{aligned}\|u\nabla\chi(v)\|_{L^q(\Omega)} &\le \|\chi'\|_{L^\infty((\gamma,\infty))}\|\nabla v\|_{L^{q\lambda'}(\Omega)}\|u\|_{L^{q\lambda}(\Omega)} \\ &\le \|\chi'\|_{L^\infty((\gamma,\infty))}|\Omega|^{\frac{1}{q\lambda'} - \frac{n-p}{np}}\|\nabla v\|_{L^{\frac{np}{n-p}}(\Omega)}\|u\|_{L^{q\lambda}(\Omega)} \\ &\le \|\chi'\|_{L^\infty((\gamma,\infty))}|\Omega|^{\frac{1}{q\lambda'} - \frac{n-p}{np}} k_p M_p\|u\|^{1-\beta}_{L^1(\Omega)}\|u\|^\beta_{L^\infty(\Omega)} \\ &\le K_p\|u\|^\beta_{L^\infty(\Omega)},\end{aligned} \tag{4.12}$$

where $\lambda' := \frac{\lambda}{\lambda-1}$, for some $\beta \in (0,1)$ and $K_p > 0$. Now let $t \in (0,T)$. Then we have

$$u(\cdot,t) = e^{t(\Delta-1)}u_0 - \int_0^t e^{(t-s)(\Delta-1)}\Big(\nabla\cdot(u(s)\nabla\chi(v(s))) + (f(u(s)) + u(s))\Big)\,ds.$$

Let $\theta \in (\frac{n}{2q}, \frac{1}{2})$ and $\varepsilon \in (0, \frac{1}{2} - \theta)$. Using Lemma 2.1 and Lemma 2.2, we see that

$$\begin{aligned}\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} &+ c_{0,\infty}\, c\int_0^t (t-s)^{-\theta}e^{-\nu_1(t-s)}\|(f(u(s)) + u(s))\|_{L^q(\Omega)}\,ds\\ &+ c_{0,\infty}c_\varepsilon\int_0^t (t-s)^{-\theta-\frac{1}{2}-\varepsilon}e^{-\nu_2(t-s)}\|u(s)\nabla\chi(v(s))\|_{L^q(\Omega)}\,ds.\end{aligned}$$

Combining (4.11) and (4.12) with the above inequality implies the uniform estimate:

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le K_0 + K_1\Big(\sup_{t\in[0,T]}\|u(\cdot,t)\|_{L^\infty(\Omega)}\Big)^{\beta} + K_2\Big(\sup_{t\in[0,T]}\|u(\cdot,t)\|_{L^\infty(\Omega)}\Big)^{1-\frac{p}{q}}$$

for some $K_0, K_1, K_2 > 0$. Since $\beta, 1 - \frac{p}{q} \in (0,1)$, we obtain the desired inequality. □

We are now in a position to prove the main result.

Proof of Theorem 4.1. As stated in Section 4.1, by a similar way in Proposition 3.1 we can show that there exist $T_{\max} \le \infty$ (depending only on $\|u_0\|_{L^\infty(\Omega)}$) and exactly one pair (u,v) of nonnegative functions $u \in C^{2,1}(\overline{\Omega}\times(0,T_{\max}))\cap C^0([0,T_{\max});C^0(\overline{\Omega}))$, and $v \in C^{2,0}(\overline{\Omega}\times(0,T_{\max}))\cap C^0((0,T_{\max});C^0(\overline{\Omega}))$ that solves (4.1) in the classical sense. According to the condition for k and χ_0, by Proposition 4.4 we can find some $p > \frac{n}{2}$ and $M_p > 0$ such that $\|u(\cdot,t)\|_{L^p} \le M_p$ for all $t \in (0,T_{\max})$. Therefore Proposition 4.5 completes the proof. □

Remark 4.1. Local-in-time existence of classical solutions to (4.1) can be provided under the only lower condition: $\lambda_1 - \mu_1 s \le f(s)$. Moreover, if the growth term f satisfies the relaxed condition: $\lambda_1 - \mu_1 s \le f(s) \le \lambda_2 + \mu_2 s$, then we have the upper mass estimate depending on time t similarly, and so global existence of solutions without uniform boundedness is proved.

Chapter 5

Global existence and boundedness in a parabolic-elliptic Keller–Segel system with signal-dependent sensitivity and logistic source

5.1. Background and motivation

In this chapter we consider couples of nonnegative solutions to the parabolic-elliptic system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla \chi(v)) + f(u), & x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases} \tag{5.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, where our primary interest is in the case when $n = 2$, $\tau = 0$,

$$\chi(v) := \chi_0 \log v \quad \text{for } v > 0 \qquad \text{and} \qquad f(u) = ru - \mu u^2 \quad \text{for } u \geq 0 \tag{5.2}$$

with constants $\chi_0 > 0, r \in \mathbb{R}$ and $\mu > 0$.

In the past few years, a growing literature is concerned with generalizations of the Keller–Segel system which account for additional effects relevant in various applications. For instance, the particular choice of the sensitivity function χ made in (5.2) was proposed already in an original work by Keller and Segel in order to incorporate the so-called Weber–Fechner law of stimulus perception in the process of chemotactic response ([**54**], cf. also [**35**] and the recent modeling approach in [**110**]). With regard to the phenomenon of blow-up, this mechanism has a certain dampening effect: In the boundary-value problem for (5.1) with $n \geq 2$, $f \equiv 0$, $\tau = 1$ and χ as in (5.2), global classical solutions exist when $\chi_0 < \sqrt{\frac{2}{n}}$, whereas certain global weak solutions can be constructed when either $\chi_0 < \sqrt{\frac{n+2}{3n-4}}$, or $\chi_0 > 0$ is arbitrary, Ω is a ball and the initial data are radially symmetric ([**107, 85**]). The question whether or not blow-up may occur seems open in this context.

In the corresponding parabolic-elliptic system obtained for $\tau = 0$, a somewhat more complete knowledge is available, at least in the radial case: Indeed, in this setting classical solutions are known to exist globally and to be bounded if $n \geq 2$ and $\chi_0 < \frac{n}{n-2}$, while if $n \geq 3$ and $\chi_0 > \frac{2n}{n-2}$, then some exploding solutions can be found ([**71**]). Without requiring such a symmetry hypothesis, global bounded solutions exist when $n \geq 2$ and $\chi_0 < \frac{2}{n}$ (Chapter 3); cf. also the precedent [**5**], where global weak solutions were constructed under the same assumption).

In [**61**], global classical solutions near homogeneous steady states are constructed for a parabolic system related to (5.1) with χ as in (5.2) but with different zero-order sources in both equations.

In processes where cell migration occurs at time scales comparable to those of cell kinetic mechanisms such as proliferation and death, nontrivial choices of f in (5.1) seem appropriate, especially with f as given by (5.2). When considered along with the linear sensitivity $\chi(v) = \chi_0 v$ ($\chi_0 > 0$), such logistic sources also inhibit the tendency toward explosions: For any choice of $r \in \mathbb{R}$, the corresponding versions of (5.1) then possess global bounded solutions when either $n = 2$ and $\mu > 0$ is arbitrary, or when $n \geq 3$ and $\mu > 0$ is suitably large (see [**75, 105**] for the case $\tau = 1$ and [**100**] for the case $\tau = 0$). Again, it remains an open question whether blow-up solutions exist e.g. if $n = 3$ and $\mu > 0$ is suitably small.

The purpose of the present chapter is to take into account the latter two effects simultaneously by choosing both χ and f as in (5.2), concentrating henceforth on the two-dimensional situation. To underline the particular mathematical challenge going along with this coupling, we note that in this case the sensitivity $\chi(v)$, the derivative χ' decays as $v \to +\infty$ and becomes unbounded near $v = 0$, indicating a strong influence of chemotaxis near small signal concentrations. Accordingly, known results on the parabolic version of (5.1) for $\tau = 1$ and $n = 2$, under assumptions generalizing (5.2), only assert global existence of classical solutions, leaving open the question whether or not they are bounded ([**1**]). Here we note that in [**1**] the conxexity of Ω is not assumed.

As compared to this, the parabolic-elliptic case $\tau = 0$ even seems significantly more delicate: Whereas in the case $\tau = 1$ a simple parabolic comparison argument can be applied to assert a positive a priori bound for v, locally uniformly in $\overline{\Omega} \times [0, \infty)$, a similar reasoning is no longer available when $\tau = 0$. In light of the second equation in (5.1), deriving useful positivity properties of v in that case apparently amounts to estimating the total mass $\int_\Omega u$ of cells from Lemma 2.3 (see also (2.2)). However, unlike the situation when $f \equiv 0$, the system (5.1) now does no longer preserve the total mass $\int_\Omega u$. Albeit an upper estimate for this quantity can be gained in quite a trivial manner (cf. Lemma 5.4), it is a priori not clear whether one can conversely also derive a lower bound for the mass, and thereby rule out phenomena of mass loss, or of extinction. For source terms of the form $f(u) = a + ru$, $a > 0, r \in \mathbb{R}$, such a lower bound can be achieved in quite a straightforward manner, thus leading to a global existence result in that case (Chapter 4); as for logistic sources as in (5.2) with quadratic absorption, however, nothing seems known in this direction so far.

The goal of this chapter is to derive global existence and boundedness results for (5.1) in the parabolic-elliptic with $n = 2$ and χ and f given by (5.2). More specifically, we shall consider the problem

$$(5.3) \qquad \begin{cases} u_t = \Delta u - \chi_0 \nabla \cdot (u \nabla \log v) + ru - \mu u^2, & x \in \Omega, \quad t > 0, \\ 0 = \Delta v - v + u, & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases}$$

in a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary, where $\chi_0 > 0, r \in \mathbb{R}, \mu > 0$ and

$$(5.4) \qquad u_0 \in C^0(\overline{\Omega}) \quad \text{is nonnegative with } u_0 \not\equiv 0.$$

In this framework, we shall first assert global existence of classical solutions without any further restriction on the parameters.

Theorem 5.1. *Let $\chi_0 > 0, r \in \mathbb{R}$ and $\mu > 0$, and suppose that* (5.4) *holds. Then the problem* (5.3) *possesses a uniquely determined global classical solution* (u, v) *such that*

$$u \in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)) \qquad \text{and}$$
$$v \in C^{2,0}(\overline{\Omega} \times (0, \infty)),$$

and such that both u and v are positive in $\overline{\Omega} \times (0, \infty)$.

Secondly, we shall see that if the reproduction rate r is conveniently large, then the above solutions are even bounded.

Theorem 5.2. *Let $\chi_0 > 0$ and $\mu > 0$, and suppose that*

$$r > \begin{cases} \frac{{\chi_0}^2}{4} & \text{if } \chi_0 \le 2, \\ \chi_0 - 1 & \text{if } \chi_0 > 2. \end{cases} \tag{5.5}$$

Then for any choice of u_0 complying with (5.4)*, the solution (u, v) of* (5.3) *is bounded in the sense that there exists $C > 0$ such that*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \le C \qquad \text{for all } t > 0 \tag{5.6}$$

and

$$\|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \le C \qquad \text{for all } t > 0. \tag{5.7}$$

The above statement goes significantly beyond the results obtained in [**1**] for $\tau = 1$, where only global existence was established. Theorem 5.2 may be viewed as a starting point for a more detailed examination of the global dynamical properties of (5.3). Since numerical evidence suggests that the interplay between logistic sources and chemotactic cross-diffusion may lead to chaotic behavior already in the case $\chi(v) = \chi_0 v$ ($\chi_0 > 0$) ([**78**]), we expect that even more colorful dynamics due to the possibility of wave-like solution behavior facilitated by the special form of the sensitivity in (5.2) ([**55, 47**]).

After proving a basic result on local existence and extensibility of classical solutions in Section 5.2, we shall, given $T > 0$, derive a T-dependent positive a priori lower bound on the mass functional $\int_\Omega u(x,t)dx$ for $t \in (0, T)$ in Section 5.3. This will be achieved on the basis of the identity

$$\frac{d}{dt}\int_\Omega \log u = \int_\Omega \frac{|\nabla u|^2}{u^2} - \chi_0 \int_\Omega \frac{\nabla u}{u} \cdot \frac{\nabla v}{v} + r|\Omega| - \mu \int_\Omega u,$$

in which the action of the death term can be controlled using a previously gained upper bound for $\int_\Omega u$ (Lemma 5.6).

In Section 5.4 we shall essentially use the assumed spatial two-dimensionality in showing that the assumption that v be uniformly bounded from below by a positive constant can be turned into a bound for u with respect to the norm in $L^\infty(\Omega)$ in a quantitative manner, independent of the length of the time interval in question. Whereas in the first step toward this we make use of a well-established approach to estimate u in $L \log L(\Omega)$ (Lemma 5.8), the second step, to be accomplished in Lemma 5.12, seems to be original in this context in that it tracks the evolution of the functional $\int_\Omega u^p(x,t)dx$ for $t \ge \tau > 0$ and some $p > 1$ sufficiently close to 1 which is not, as in related studies, determined by the system parameters only, but will moreover strongly dependent on both τ and the assumed value of $\inf v$.

The outcome of Section 5.4 will first be combined with that of Section 5.3 to establish the proof of Theorem 5.1 in Section 5.5, and then once more be applied in Section 5.6 to verify Theorem 5.2. To obtain, as a preparation therefor, a time-independent positive lower bound for $\int_\Omega u(x,t)$, and hence of $\inf v$, in Lemma 5.15 we shall assert that the largeness assumption (5.5) on r is sufficient to guarantee that for some $\lambda > 0$ depending on χ_0, r and μ (but not necessarily small or large), the functional $\int_\Omega u^{-\lambda}(x,t)dx$ is uniformly bounded for $t > 1$.

5.2. Local existence

The derivation of the following local existence and uniqueness result can be achieved by modifying the proof of Proposition 3.1. In particular, the a priori lower estimate for v and the extensibility criterion are modified points.

Lemma 5.3. *Let $\chi_0 > 0, r \in \mathbb{R}$ and $\mu > 0$, and let* (5.4) *hold. Then there exist $T_{\max} \in (0,\infty]$ and a uniquely determined pair (u,v) of functions*

$$u \in C^0(\overline{\Omega}\times[0,T_{\max}))\cap C^{2,1}(\overline{\Omega}\times(0,T_{\max})) \qquad \textit{and}$$
$$v \in C^{2,0}(\overline{\Omega}\times(0,T_{\max})),$$

such that both $u > 0$ and $v > 0$ in $\overline{\Omega}\times(0,T_{\max})$, that (u,v) solves (5.3) *classically in $\Omega\times(0,T_{\max})$, and such that*

$$\text{(5.8)} \qquad \textit{if } T_{\max} < \infty \textit{ then}$$
$$\textit{either} \quad \limsup_{t\nearrow T_{\max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty \quad \textit{or} \quad \liminf_{t\nearrow T_{\max}} \inf_{x\in\Omega} v(x,t) = 0.$$

Proof. We let $R := \|u_0\|_{L^\infty(\Omega)} + 1$, and we set $\varepsilon := \frac{\gamma}{2} > 0$ taken from (2.2). For $T > 0$, we then let

$$X := C^0([0,T];C^0(\overline{\Omega}))$$

with norm $\|u\|_X := \|u\|_{L^\infty(\overline{\Omega}\times[0,T])}$, and we claim that if T is sufficiently small, then on the closed set

$$S := \Big\{u \in X \Big| \; \|u\|_X \le R \quad \text{and} \quad (I-\Delta)^{-1}u(\cdot,t) \ge \varepsilon \quad \text{for all } t\in[0,T]\Big\},$$

for all $t \in [0,T]$, the mapping Ψ defined by

$$\Psi(u)(\cdot,t) := e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\Big\{\chi_0\nabla\cdot\Big(\frac{u(\cdot,s)}{v(\cdot,s)}\nabla v(s\cdot,)\Big) + ru(\cdot,s) - \mu u^2(\cdot,s)\Big\}\,ds,$$

where $v(\cdot,s) := (I-\Delta)^{-1}u(\cdot,s)$ for $s \in [0,T]$, acts as a contraction from S into itself. To see this, we let $u \in S$, and invoke the lower estimate $v \geq \varepsilon$ to see that for all $t \in [0,T]$,

$$\begin{aligned}
\|\Psi(u)(\cdot,t)\|_{L^\infty(\Omega)} &\leq \|u_0\|_{L^\infty(\Omega)} + \frac{c_1\chi_0}{\varepsilon}R^2T^\alpha + \int_0^t \|ru(\cdot,s) - \mu u^2(\cdot,s)\|_{L^\infty(\Omega)}\,ds \\
&\leq \|u_0\|_{L^\infty(\Omega)} + \frac{c_1\chi_0}{\varepsilon}R^2T^\alpha + \frac{r^2}{4\mu}T,
\end{aligned}$$

where α and c_1 are certain positive constants. Therefore, choosing T sufficiently small ensures that $\|\Psi(u)\|_X \leq R$.

Next, in order to show that $(I-\Delta)^{-1}\Psi(u)(\cdot,t) \geq \varepsilon$ for all $t \in [0,T]$, we first observe

$$\begin{aligned}
\int_\Omega \Psi(u)(\cdot,t) &\geq \int_\Omega e^{t\Delta}u_0 + \int_\Omega \left(\int_0^t e^{(t-s)\Delta}(ru(\cdot,s) - \mu u^2(\cdot,s))\,ds \right) \\
&= \int_\Omega u_0 + \int_0^t \left(\int_\Omega (ru(\cdot,s) - \mu u^2(\cdot,s)) \right) ds \\
&\geq \int_\Omega u_0 - \mu R^2|\Omega|T.
\end{aligned}$$

Thus, if T is suitably small then we have $\int_\Omega \Psi(u)(\cdot,t) \geq \frac{1}{2}\int_\Omega u_0$ for all $t \in [0,T]$. Since moreover $\Psi(u)(\cdot,t)$ belongs to $C^1(\overline{\Omega})$ for any such t by the regularizing properties of the heat semigroup, Lemma 2.3 (see also (2.2)) yields the inequality

$$(I-\Delta)^{-1}\Psi(u)(\cdot,t) \geq \frac{\gamma}{2} = \varepsilon \qquad \text{for all } t \in [0,T].$$

To achieve the desired contractivity property of Ψ, we let $u \in S$ and $\overline{u} \in S$ be arbitrary. Then there exists $c_2 > 0$ such that

$$\begin{aligned}
&\|\Psi(u)(\cdot,t) - \Psi(\overline{u})(\cdot,t)\|_{L^\infty(\Omega)} \\
&\leq c_2\chi_0\Big(\frac{1}{\varepsilon}R + \frac{1}{\varepsilon^2}R^2\Big)T^\alpha\|u-\overline{u}\|_X \\
&\quad + \int_0^t \|(ru(\cdot,s) - \mu u^2(\cdot,s)) - (r\overline{u}(\cdot,s) - \mu\overline{u}^2(\cdot,s))\|_{L^\infty(\Omega)}\,ds \\
&\leq c_2\chi_0\Big(\frac{1}{\varepsilon}R + \frac{1}{\varepsilon^2}R^2\Big)T^\alpha\|u-\overline{u}\|_X + (r+2\mu R)T\|u-\overline{u}\|_X
\end{aligned}$$

for all $t \in [0,T]$. Hence, on further diminishing T if necessary, we obtain that Ψ indeed becomes a contraction on S, whence the Banach fixed point theorem asserts the existence of $u \in S$ fulfilling $u = \Psi(u)$.

Moreover u can be extended up to some $T_{\max} \in (0, \infty]$ by standard argument. Since the above choice of T depends on $\|u_0\|_{L^\infty(\Omega)}$ and $\varepsilon = \frac{\gamma}{2}$ only, a standard extensibility argument warrants that u can be extended up to some maximally chosen $T_{\max} \in (0, \infty]$. Here if $T_{\max} < \infty$ and there exist constants $M > 0$ and $\varepsilon_0 > 0$ satisfying

$$\limsup_{t \nearrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} < M \quad \text{and} \quad \liminf_{t \nearrow T_{\max}} \int_\Omega u(\cdot, t) > \varepsilon_0,$$

then we can find $T_1 \in (0, T_{\max})$ fulfilling

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} < M \quad \text{for all } t \in [T_1, T_{\max})$$

and for sufficient small $\delta > 0$ there exists $t_\delta \in [T_{\max} - \delta, T_{\max})$ such that

$$\int_\Omega u(\cdot, t_\delta) > \varepsilon_0.$$

Thus we can extend u beyond $T_{\max}$, which contradicts the definition of $T_{\max}$. Since in light of Lemma 2.3 we know that

$$\liminf_{t \nearrow T_{\max}} \inf_{x \in \Omega} v(x, t) = 0 \quad \text{is equivalent to} \quad \liminf_{t \nearrow T_{\max}} \int_\Omega u(\cdot, t) = 0,$$

this proves that $T_{\max}$ satisfies (5.8).

Invoking standard parabolic regularity theory (Lemma 2.8), we see that (u, v) solves (5.3) in the classical sense. Using the parabolic and elliptic strong maximal principles, we also see that both $u > 0$ and $v > 0$ in $\overline{\Omega} \times (0, T_{\max})$. Finally, uniqueness of solutions can be derived in a way quite similar to that presented in Chapter 3, so that we may refrain from giving details here. □

Although the total mass $\int_\Omega u$ of cells is not necessarily conserved due to the kinetic term in (5.3), an upper bound for this quantity can be found quite immediately.

Lemma 5.4. *Let* $\chi_0 > 0, r \in \mathbb{R}$ *and* $\mu > 0$, *and assume* (5.4). *Then*

$$\int_\Omega u(x, t)dx \le m := \max\left\{ \int_\Omega u_0 \,, \, \frac{r_+ \cdot |\Omega|}{\mu} \right\} \qquad \text{for all } t \in (0, T_{\max}), \tag{5.9}$$

where $r_+ = \max\{r, 0\}$.

Proof. We integrate the first equation in (5.3) and use the Cauchy–Schwarz inequality to see that for all $t \in (0, T_{\max})$,

$$\frac{d}{dt} \int_\Omega u = r \int_\Omega u - \mu \int_\Omega u^2 \le r \int_\Omega u - \frac{\mu}{|\Omega|} \Big(\int_\Omega u \Big)^2 \le r_+ \int_\Omega u - \frac{\mu}{|\Omega|} \Big(\int_\Omega u \Big)^2$$

Therefore, (5.9) results by invoking a straightforward ODE comparison argument. □

5.3. Excluding mass loss in finite time for arbitrary $r \in \mathbb{R}$

As a preparation for Lemma 5.6, let us derive from the second equation in (5.3) a bound for a weighted H^1 norm of v.

Lemma 5.5. *For arbitrary $\chi_0 > 0, r \in \mathbb{R}$ and $\mu > 0$ and each u_0 satisfying* (5.4), *the solution component v satisfies*

$$\int_\Omega \frac{|\nabla v|^2}{v^2} \le |\Omega| \qquad \textit{for all } t \in (0, T_{\max}). \tag{5.10}$$

Proof. Since v is positive in $\overline{\Omega} \times (0, T_{\max})$, we may test the second equation in (5.3) by $\frac{1}{v}$ to gain the identity

$$\begin{aligned} 0 &= \int_\Omega \frac{1}{v} \cdot \Big(\Delta v - v + u\Big) \\ &= \int_\Omega \frac{|\nabla v|^2}{v^2} - |\Omega| + \int_\Omega \frac{u}{v} \qquad \text{for all } t \in (0, T_{\max}). \end{aligned}$$

As moreover u is nonnegative, this directly entails (5.10). □

With this estimate at hand, we can achieve the main step toward the mass persistence statement in Corollary 5.7 below.

Lemma 5.6. *Let $\chi_0 > 0, r \in \mathbb{R}$ and $\mu > 0$, and assume* (5.4). *Then there exists $C > 0$ such that*

$$\frac{d}{dt} \int_\Omega \log u(x,t)dx \ge -C \qquad \textit{for all } t \in (0, T_{\max}). \tag{5.11}$$

Proof. Recalling that u is positive, we may multiply the first equation in (5.3) by $\frac{1}{u}$ and integrate by parts over Ω to see that

$$\begin{aligned} \frac{d}{dt} \int_\Omega \log u &= \int_\Omega \frac{1}{u} \cdot \Big\{\Delta u - \chi_0 \nabla \cdot \Big(\frac{u}{v} \nabla v\Big) + ru - \mu u^2\Big\} \\ &= \int_\Omega \frac{|\nabla u|^2}{u^2} - \chi_0 \int_\Omega \frac{\nabla u}{u} \cdot \frac{\nabla v}{v} + r|\Omega| - \mu \int_\Omega u \end{aligned} \tag{5.12}$$

for all $t \in (0, T_{\max})$. Here by Young's inequality,

$$\Big| - \chi_0 \int_\Omega \frac{\nabla u}{u} \cdot \frac{\nabla v}{v} \Big| \le \int_\Omega \frac{|\nabla u|^2}{u^2} + \frac{{\chi_0}^2}{4} \int_\Omega \frac{|\nabla v|^2}{v^2},$$

and Lemma 5.5 asserts that

$$\frac{{\chi_0}^2}{4} \int_\Omega \frac{|\nabla v|^2}{v^2} \le \frac{{\chi_0}^2}{4} \cdot |\Omega|$$

for all $t \in (0, T_{\max})$. Since moreover

$$\int_\Omega u \le m \qquad \text{for all } t \in (0, T_{\max})$$

according to Lemma 5.4, (5.12) implies the inequality

$$\frac{d}{dt} \int_\Omega \log u \ge -\frac{{\chi_0}^2}{4}|\Omega| + r|\Omega| - \mu m \qquad \text{for all } t \in (0, T_{\max}),$$

from which (5.11) immediately follows. □

Along with a straightforward use of Jensen's inequality, the above bound for $\int_\Omega \log u$ now ensures that for arbitrary $r \in \mathbb{R}$ and $\mu > 0$, finite-time extinction cannot occur.

Corollary 5.7. *Let $\chi_0 > 0, r \in \mathbb{R}$ and $\mu > 0$, and suppose that* (5.4) *holds. Then for all $\tau \in (0, T_{\max})$ and each $T > 0$ there exists $C = C(\tau, T) > 0$ with the property that*

$$\int_\Omega u(x,t)dx \ge C(\tau, T) \qquad \textit{for all } t \in (\tau, \widehat{T}), \tag{5.13}$$

where $\widehat{T} := \min\{T, T_{\max}\}$.

Proof. Again by strict positivity of u in $\overline{\Omega} \times (0, T_{\max})$, the number

$$c_1(\tau) := \int_\Omega \log u(x, \tau)dx$$

is finite. Thus, if according to Lemma 5.6 we take $c_2 > 0$ large enough such that $\frac{d}{dt}\int_\Omega \log u \ge -c_2$ for all $t \in (0, T_{\max})$, then upon integration thereof we obtain that

$$\begin{aligned} \int_\Omega \log u(x,t)dx &\ge \int_\Omega \log u(x,\tau)dx - c_2 \cdot (t - \tau) \\ &\ge c_3(\tau, T) := c_1(\tau) - c_2 \cdot (T - \tau) \qquad \text{for all } t \in (\tau, \widehat{T}). \end{aligned}$$

Since from Jensen's inequality we know that for all $t \in (0, T_{\max})$,

$$\int_\Omega \log u(x,t)dx = |\Omega| \cdot \int_\Omega \log u(x,t)\frac{dx}{|\Omega|} \le |\Omega| \cdot \log\left\{\int_\Omega u(x,t)\frac{dx}{|\Omega|}\right\},$$

this implies that

$$\begin{aligned} \int_\Omega u(x,t)dx &\ge |\Omega| \cdot \exp\left\{\frac{1}{|\Omega|} \cdot \int_\Omega \log u(x,t)dx\right\} \\ &\ge |\Omega| \cdot e^{\frac{1}{|\Omega|}\cdot c_3(\tau,T)} \qquad \text{for all } t \in (\tau, \widehat{T}) \end{aligned}$$

and thereby proves (5.13). □

5.4. Upper estimates for u in terms of lower bounds for v

The goal of this section is to establish a bound for u with respect to the norm in $L^\infty(\Omega)$ in quantitative dependence on a supposedly known pointwise lower bound for v. Our main outcome in this direction, to be provided by Lemma 5.13, will be essential to the derivation both of the global existence result for arbitrary $r \in \mathbb{R}$ in Theorem 5.1, and of the boundedness statement in Theorem 5.2.

5.4.1. An estimate for u in $L \log L(\Omega)$

We start by proving a bound for the functional $\int_\Omega u \log u$, crucially making use of the two-dimensionality in the spatial setting.

Lemma 5.8. *Let $\chi_0 > 0, r \in \mathbb{R}$ and $\mu > 0$, and suppose that u_0 complies with* (5.4). *Then for all $\delta > 0$ and each $\tau \in (0, T_{\max})$ one can find $C(\delta, \tau) > 0$ such that if for some $T \in (\tau, T_{\max}]$ we have*

$$v(x,t) \geq \delta \qquad \text{for all } x \in \Omega \text{ and } t \in (\tau, T), \tag{5.14}$$

then

$$\int_\Omega u(x,t) \log u(x,t) dx \leq C(\delta, \tau) \qquad \text{for all } t \in (\tau, T). \tag{5.15}$$

Proof. Once again by positivity of u, we may multiply the first equation in (5.3) by $\log u$ and integrate by parts to obtain

$$\begin{aligned} &\frac{d}{dt}\left\{ \int_\Omega u \log u - \int_\Omega u \right\} \\ &= \int_\Omega \log u \cdot \left\{ \Delta u - \chi_0 \nabla \cdot \left(\frac{u}{v} \nabla v \right) + ru - \mu u^2 \right\} \\ &= -\int_\Omega \frac{|\nabla u|^2}{u} + \chi_0 \int_\Omega \nabla u \cdot \frac{\nabla v}{v} + r \int_\Omega u \log u - \mu \int_\Omega u^2 \log u \end{aligned} \tag{5.16}$$

for all $t \in (0, T_{\max})$, where by Young's inequality,

$$\chi_0 \int_\Omega \nabla u \cdot \frac{\nabla v}{v} \leq \int_\Omega \frac{|\nabla u|^2}{u} + \frac{{\chi_0}^2}{4} \int_\Omega u \frac{|\nabla v|^2}{v^2}.$$

To estimate the latter integral, we make use of (5.14) and invoke the Cauchy–Schwarz inequality to see that

$$\begin{aligned} \frac{{\chi_0}^2}{4} \int_\Omega u \frac{|\nabla v|^2}{v^2} &\leq \frac{{\chi_0}^2}{4\delta^2} \int_\Omega u |\nabla v|^2 \\ &\leq \frac{{\chi_0}^2}{4\delta^2} \|u\|_{L^2(\Omega)} \|\nabla v\|_{L^4(\Omega)}^2 \qquad \text{for all } t \in (\tau, T). \end{aligned}$$

Now since in the present two-dimensional setting we know that $W^{2,\frac{4}{3}}(\Omega)$ is continuously embedded into $W^{1,4}(\Omega)$, employing standard elliptic regularity theory ([**19**]) we can find $c_1 > 0$ and $c_2 > 0$ such that

$$\begin{aligned}
\|\nabla v\|^2_{L^4(\Omega)} &\le c_1 \|v\|^2_{W^{2,\frac{4}{3}}(\Omega)} \\
&\le c_2 \| - \Delta v + v\|^2_{L^{\frac{4}{3}}(\Omega)} \\
&= c_2 \|u\|^2_{L^{\frac{4}{3}}(\Omega)} \qquad \text{for all } t \in (0, T_{\max}).
\end{aligned}$$

Here, thanks to the Hölder inequality and (5.9) we can estimate

$$\|u\|^2_{L^{\frac{4}{3}}(\Omega)} \le \|u\|_{L^2(\Omega)} \|u\|_{L^1(\Omega)} \le m \|u\|_{L^2(\Omega)} \qquad \text{for all } t \in (0, T_{\max}),$$

so that in summary we have

$$\frac{{\chi_0}^2}{4} \int_\Omega u \frac{|\nabla v|^2}{v^2} \le c_3(\delta) \int_\Omega u^2 \qquad \text{for all } t \in (\tau, T)$$

with $c_3(\delta) := \frac{c_2 {\chi_0}^2 m}{4\delta^2}$. Accordingly, (5.16) entails the inequality

$$\begin{aligned}
(5.17) \qquad & \frac{d}{dt}\Big\{ \int_\Omega u \log u - \int_\Omega u \Big\} + \Big\{ \int_\Omega u \log u - \int_\Omega u \Big\} \\
& \le c_3(\delta) \int_\Omega u^2 + (r+1) \int_\Omega u \log u - \int_\Omega u - \mu \int_\Omega u^2 \log u
\end{aligned}$$

for all $t \in (\tau, T)$. Since the function $\psi : (0, \infty) \to \mathbb{R}$ defined by

$$\psi(\xi) := c_3(\delta)\xi^2 + (r+1)\xi \log \xi - \xi - \mu \xi^2 \log \xi, \qquad \xi > 0,$$

has the properties $\psi(\xi) \to 0$ as $\xi \to 0$ and $\psi(\xi) \to -\infty$ as $\xi \to \infty$, it is evident that with some $c_4(\delta) > 0$ we have $\psi(\xi) \le c_4(\delta)$ for all $\xi > 0$. Consequently, (5.17) shows that $y(t) := \int_\Omega u(x,t) \log u(x,t) dx - \int_\Omega u(x,t) dx$, $t \in (0, T_{\max})$, satisfies

$$y'(t) + y(t) \le c_4(\delta) |\Omega| \qquad \text{for all } t \in (\tau, T).$$

By comparison, this implies that for all $t \in (\tau, T)$,

$$y(t) \le c_5(\delta, \tau) := \max \Big\{ \int_\Omega u(x,\tau) \log u(x,\tau) dx - \int_\Omega u(x,\tau) dx \, , \, c_4(\delta) |\Omega| \Big\}$$

and hence by Lemma 5.4 we conclude that (5.15) holds with $C := c_5(\delta, \tau) + m$. $\square$

5.4.2. An estimate for u in $L^p(\Omega)$ for some $p > 1$

Now the essential step toward Lemma 5.13 appears to consist of finding an estimate for u with respect to the norm in $L^p(\Omega)$ for some $p > 1$. In Lemma 5.12, this will be achieved, still under the standing assumption that $v \geq \delta > 0$, for some $p > 1$ *depending on* δ. In the derivation thereof, we shall need three auxiliary lemmas. The first contains a statement which is implied by the conclusion of Lemma 5.8 in combination with a known result on elliptic H^1 regularity in the present two-dimensional case.

Lemma 5.9. *Given* $\chi_0 > 0, r \in \mathbb{R}$ *and* $\mu > 0$, *for any* u_0 *satisfying* (5.4), *each* $\delta > 0$ *and all* $\tau \in (0, T_{\max})$ *we can find* $C(\delta, \tau) > 0$ *such that if*

$$v(x,t) \geq \delta \qquad \text{for all } x \in \Omega \text{ and } t \in (\tau, T)$$

for some $T \in (\tau, T_{\max}]$, *then we have the inequality*

$$\int_\Omega |\nabla v(x,t)|^2 dx \leq C(\delta, \tau) \qquad \text{for all } t \in (\tau, T). \tag{5.18}$$

Proof. According to a known estimate for solutions of the Neumann boundary value problem for the Helmholtz equation with inhomogeneities in $L \log L(\Omega)$, for each $L > 0$ there exists $c_1(L) > 0$ such that whenever $f \in L^2(\Omega)$ is nonnegative with

$$\int_\Omega f \log f \leq L,$$

the solution φ of $-\Delta\varphi + \varphi = f$ in Ω with $\frac{\partial \varphi}{\partial \nu} = 0$ on $\partial\Omega$ satisfies $\int_\Omega |\nabla\varphi|^2 \leq c_1(L)$ (see [**99**]). Applying this to $\varphi := v(\cdot, t)$ and using Lemma 5.8 precisely yields (5.18). □

Next, an application of the Riesz–Thorin theorem asserts a certain independence of some elliptic regularity constant on the integrability parameter. As the proof will show, the argument can easily be generalized to smoothly bounded domains in any space dimension and summation powers varying over arbitrary compact subintervals of $(1, \infty)$.

Lemma 5.10. *There exists* $C > 0$ *such that for each* $q \in (2, 3)$ *we have*

$$\|\varphi\|_{W^{2,q}(\Omega)} \leq C\| - \Delta\varphi + \varphi\|_{L^q(\Omega)} \tag{5.19}$$

for all $\varphi \in C^2(\overline{\Omega})$ *satisfying* $\frac{\partial \varphi}{\partial \nu} = 0$ *on* $\partial\Omega$.

Proof. For fixed $i, j \in \{1, 2\}$, we let $T_{ij} : L^2(\Omega) + L^3(\Omega) \to L^2(\Omega) + L^3(\Omega)$ be defined by $(T_{ij}f)(x) := (\frac{\partial^2 \varphi}{\partial_{x_i} \partial_{x_j}})(x)$ for $f \in L^2(\Omega) + L^3(\Omega)$ and $x \in \Omega$, where $-\Delta\varphi + \varphi = f$ in

Ω and $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega$. Then according to standard elliptic regularity theory ([**19**]), T_{ij} is a well-defined linear operator on both $L^2(\Omega)$ and $L^3(\Omega)$, and there exist $c_2 > 0$ and $c_3 > 0$ such that

$$\|T_{ij}f\|_{L^2(\Omega)} \le c_2 \|f\|_{L^2(\Omega)} \quad \text{for all } f \in L^2(\Omega) \qquad \text{and}$$
$$\|T_{ij}f\|_{L^3(\Omega)} \le c_3 \|f\|_{L^3(\Omega)} \quad \text{for all } f \in L^3(\Omega).$$

By the Riesz–Thorin interpolation theorem ([**4**]), for each $q \in (2,3)$ we thus have

$$\|T_{ij}f\|_{L^q(\Omega)} \le c_2^\kappa c_3^{1-\kappa} \|f\|_{L^q(\Omega)} \quad \text{for all } f \in L^q(\Omega)$$

with $\kappa := \frac{6-2q}{q} \in (0,1)$. Summation over i and j therefore yields (5.19). $\square$

As a final preparation for Lemma 5.12, let us make sure that also the constant appearing in the Gagliardo–Nirenberg inequality is conveniently independent of the involved integrability powers.

Lemma 5.11. *There exists $C > 0$ with the property that for each $q \in (2,3)$ we have*

$$\|\psi\|_{L^{2q}(\Omega)} \le C \|\psi\|_{W^{1,q}(\Omega)}^{\frac{1}{2}} \|\psi\|_{L^2(\Omega)}^{\frac{1}{2}} \qquad \textit{for all } \psi \in C^1(\overline{\Omega}). \tag{5.20}$$

Proof. Due to the Gagliardo–Nirenberg inequality, there exists $c_1 > 0$ such that

$$\|z\|_{L^4(\Omega)}^4 \le c_1 \|z\|_{W^{1,2}(\Omega)}^2 \|z\|_{L^2(\Omega)}^2 = c_1 \|\nabla z\|_{L^2(\Omega)}^2 \|z\|_{L^2(\Omega)}^2 + c_1 \|z\|_{L^2(\Omega)}^4 \tag{5.21}$$

for all $z \in W^{1,2}(\Omega)$. Given $\psi \in C^1(\overline{\Omega})$, since $q > 2$ we know that $z := |\psi|^{\frac{q}{2}}$ belongs to $W^{1,2}(\Omega)$ with $|\nabla z| = \frac{q}{2}|\varphi|^{\frac{q-2}{2}}|\nabla\psi|$, so that (5.21) becomes

$$\int_\Omega |\psi|^{2q} \le c_1 \cdot \frac{q^2}{4} \Big(\int_\Omega |\psi|^{q-2} |\nabla\psi|^2 \Big) \cdot \Big(\int_\Omega |\psi|^q \Big) + c_1 \Big(\int_\Omega |\psi|^q \Big)^2. \tag{5.22}$$

Here we employ the Hölder inequality to estimate

$$\int_\Omega |\psi|^{q-2} |\nabla\psi|^2 \le \Big(\int_\Omega |\nabla\psi|^q \Big)^{\frac{2}{q}} \cdot \Big(\int_\Omega |\psi|^q \Big)^{\frac{q-2}{q}}$$

and decompose the last term in (5.22) so as to obtain, using that $\frac{q^2}{4} > 1$,

$$\int_\Omega |\psi|^{2q} \le c_1 \cdot \frac{q^2}{4} \cdot \Big\{ \Big(\int_\Omega |\nabla\psi|^q \Big)^{\frac{2}{q}} + \Big(\int_\Omega |\psi|^q \Big)^{\frac{2}{q}} \Big\} \cdot \Big(\int_\Omega |\psi|^q \Big)^{\frac{2q-2}{q}}. \tag{5.23}$$

Now one more application of the Hölder inequality yields

$$\Big(\int_\Omega |\psi|^q \Big)^{\frac{2q-2}{q}} \le \Big(\int_\Omega |\psi|^{2q} \Big)^{\frac{q-2}{q}} \cdot \int_\Omega |\psi|^2,$$

whereupon (5.23) implies that

$$\Big(\int_\Omega |\psi|^{2q}\Big)^{\frac{2}{q}} = \Big(\int_\Omega |\psi|^{2q}\Big)^{1-\frac{q-2}{q}} \le c_1 \cdot \frac{q^2}{4} \cdot \Big\{\Big(\int_\Omega |\nabla\psi|^q\Big)^{\frac{2}{q}} + \Big(\int_\Omega |\psi|^q\Big)^{\frac{2}{q}}\Big\} \cdot \int_\Omega |\psi|^2$$

$$\le c_1 \cdot \frac{q^2}{2} \cdot \Big\{\int_\Omega |\nabla\psi|^q + \int_\Omega |\psi|^q\Big\}^{\frac{2}{q}} \cdot \int_\Omega |\psi|^2,$$

because clearly $a^{\frac{2}{q}} + b^{\frac{2}{q}} \le 2 \cdot (a+b)^{\frac{2}{q}}$ for all $a \ge 0$ and $b \ge 0$. By recalling $q < 3$ and taking the 4th root here, we infer that (5.20) holds if we let $C := (\frac{9c_1}{2})^{\frac{1}{4}}$, for instance. □

We are now ready to derive a bound for u with respect to the norm in $L^p(\Omega)$ with some $p > 1$, assuming that v is bounded from below.

Lemma 5.12. *Let $\chi_0 > 0, r \in \mathbb{R}$ and $\mu > 0$, and assume* (5.4). *Then for all $\delta > 0$ and any $\tau \in (0, T_{\max})$ there exist $p(\delta,\tau) > 1$ and $C(\delta,\tau) > 0$ with the property that if*

$$v(x,t) \ge \delta \qquad \textit{for all } x \in \Omega \textit{ and } t \in (\tau, T) \tag{5.24}$$

for some $T \in (\tau, T_{\max}]$, then

$$\int_\Omega u^{p(\delta,\tau)}(x,t)\,dx \le C(\delta,\tau) \qquad \textit{for all } t \in (\tau, T). \tag{5.25}$$

Proof. We first invoke Lemma 5.9 to obtain $c_1 = c_1(\delta,\tau) > 0$ such that

$$\|\nabla v(\cdot,t)\|_{L^2(\Omega)} \le c_1 \qquad \text{for all } t \in (\tau, T). \tag{5.26}$$

Moreover, an application of Lemma 5.11 and of Lemma 5.10 provides $c_2 > 0$ and $c_3 > 0$ such that for any choice of $p \in (1,2)$ we have

$$\|\nabla\varphi\|_{L^{2p+2}(\Omega)} \le c_2 \|\varphi\|_{W^{2,p+1}(\Omega)}^{\frac{1}{2}} \|\nabla\varphi\|_{L^2(\Omega)}^{\frac{1}{2}} \qquad \text{for all } \varphi \in C^2(\overline{\Omega})$$

and

$$\|\varphi\|_{W^{2,p+1}(\Omega)} \le c_3 \| -\Delta\varphi + \varphi\|_{L^{p+1}(\Omega)} \qquad \text{for all } \varphi \in C^2(\overline{\Omega}) \text{ satisfying } \frac{\partial\varphi}{\partial\nu} = 0 \text{ on } \partial\Omega,$$

whence

$$\|\nabla\varphi\|_{L^{2p+2}(\Omega)} \le c_4 \| -\Delta\varphi + \varphi\|_{L^{p+1}(\Omega)}^{\frac{1}{2}} \|\nabla\varphi\|_{L^2(\Omega)}^{\frac{1}{2}} \tag{5.27}$$

for all $\varphi \in C^2(\overline{\Omega})$ satisfying $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega$ with $c_4 := c_2\sqrt{c_3}$. We finally fix $p = p(\delta,\tau) \in (1,2)$ sufficiently close to 1 such that

$$\frac{(p-1){\chi_0}^2 c_1 c_4^2}{4\delta^2} < \frac{\mu}{2}, \tag{5.28}$$

and test the first equation in (5.3) against u^{p-1} to obtain

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + (p-1)\int_\Omega u^{p-2}|\nabla u|^2 = (p-1)\chi_0\int_\Omega \frac{u^{p-1}}{v}\nabla u\cdot\nabla v + r\int_\Omega u^p - \mu\int_\Omega u^{p+1} \tag{5.29}$$

for all $t \in (0, T_{\max})$, where by Young's inequality and (5.24),

$$\begin{aligned}(p-1)\chi_0\int_\Omega \frac{u^{p-1}}{v}\nabla u\cdot\nabla v &\le (p-1)\int_\Omega u^{p-2}|\nabla u|^2 + \frac{(p-1){\chi_0}^2}{4}\int_\Omega u^p\frac{|\nabla v|^2}{v^2} \\ &\le (p-1)\int_\Omega u^{p-2}|\nabla u|^2 + \frac{(p-1){\chi_0}^2}{4\delta^2}\int_\Omega u^p|\nabla v|^2\end{aligned} \tag{5.30}$$

for all $t \in (\tau, T)$. We estimate the rightmost integral by means of the Hölder inequality according to

$$\int_\Omega u^p|\nabla v|^2 \le \|u\|^p_{L^{p+1}(\Omega)}\|\nabla v\|^2_{L^{2p+2}(\Omega)},$$

and apply (5.27) and then (5.26) to find, recalling the second equation in (5.3), that herein

$$\begin{aligned}\|\nabla v\|^2_{L^{2p+2}(\Omega)} &\le c_4^2\|-\Delta v + v\|_{L^{p+1}(\Omega)}\|\nabla v\|_{L^2(\Omega)} \\ &= c_4^2\|u\|_{L^{p+1}(\Omega)}\|\nabla v\|_{L^2(\Omega)} \\ &\le c_4^2 c_1\|u\|_{L^{p+1}(\Omega)}\end{aligned}$$

for all $t \in (\tau, T)$. In light of (5.28), (5.30) thus implies that

$$\begin{aligned}(p-1)\chi_0\int_\Omega \frac{u^{p-1}}{v}\nabla u\cdot\nabla v &\le (p-1)\int_\Omega u^{p-2}|\nabla u|^2 + \frac{(p-1){\chi_0}^2}{4\delta^2}\cdot c_4^2 c_1\int_\Omega u^{p+1} \\ &\le (p-1)\int_\Omega u^{p-2}|\nabla u|^2 + \frac{\mu}{2}\int_\Omega u^{p+1} \qquad \text{for all } t\in(\tau,T),\end{aligned}$$

so that (5.29) entails the inequality

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p \le r\int_\Omega u^p - \frac{\mu}{2}\int_\Omega u^{p+1} \le r_+\int_\Omega u^p - \frac{\mu}{2}\int_\Omega u^{p+1} \qquad \text{for all } t\in(\tau,T).$$

Since once more by the Hölder inequality we know that

$$\int_\Omega u^p \le |\Omega|^{\frac{1}{p+1}}\Big(\int_\Omega u^{p+1}\Big)^{\frac{p}{p+1}},$$

we therefore see that $y(t) := \int_\Omega u^p(x,t)dx,\ t \in (0,T_{\max})$, satisfies

$$\frac{1}{p}y'(t) \le r_+ y(t) - \frac{\mu}{2|\Omega|^{\frac{1}{p}}} y^{\frac{p+1}{p}}(t) \qquad \text{for all } t \in (\tau,T),$$

and conclude by a comparison argument that

$$y(t) \le \max\Big\{ \int_\Omega u^p(x,\tau)dx\,,\ \Big(\frac{2r_+}{\mu}\Big)^p |\Omega| \Big\} \qquad \text{for all } t \in (\tau,T).$$

This proves (5.25). □

5.4.3. An estimate for u in $L^\infty(\Omega)$

By suitably adapting a well-established regularity argument to the present setting, we can show that the above integrability property actually ensures boundedness of u with respect to the norm in $L^\infty(\Omega)$.

Lemma 5.13. *Let $\chi_0 > 0, r \in \mathbb{R}$ and $\mu > 0$, and suppose that* (5.4) *is valid. Then given $\delta > 0$ and $\tau \in (0,T_{\max})$, we can find $C(\delta,\tau) > 0$ such that whenever we know that*

(5.31) $$v(x,t) \ge \delta \qquad \textit{for all } x \in \Omega \textit{ and } t \in (\tau,T)$$

with some $T \in (\tau,T_{\max}]$, then

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C(\delta,T) \qquad \textit{for all } t \in (\tau,T).$$

Proof. We first apply Lemma 5.12 to find $p = p(\delta,\tau) > 1$ and $c_1 = c_1(\delta,\tau) > 0$ such that

(5.32) $$\|u(\cdot,t)\|_{L^p(\Omega)} \le c_1 \qquad \text{for all } t \in (\tau,T).$$

Here we clearly may assume that $p < 2$, so that it is possible to fix $q > 2$ such that

(5.33) $$q \le \frac{2p}{2-p},$$

and thereafter choose $\theta > 2$ such that $\theta < q$. Then thanks to (5.33) we know that $W^{2,p}(\Omega)$ is continuously embedded into $W^{1,q}(\Omega)$, whence there exists $c_2 > 0$ fulfilling

(5.34) $$\|\nabla\varphi\|_{L^q(\Omega)} \le c_2\|\varphi\|_{W^{2,p}(\Omega)} \qquad \text{for all } \varphi \in C^2(\overline{\Omega}).$$

Moreover, according to standard elliptic L^p estimates we can find $c_3 > 0$ satisfying

$$\|\varphi\|_{W^{2,p}(\Omega)} \le c_3 \| -\Delta\varphi + \varphi\|_{L^p(\Omega)} \tag{5.35}$$

for all $\varphi \in C^2(\overline{\Omega})$ satisfying $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega$, whereas Lemma 2.2 **(iv)** provides $c_4 > 0$ and $\lambda_1 > 0$ such that

$$\|e^{t\Delta}\nabla\cdot\varphi\|_{L^\infty(\Omega)} \le c_4\big(1 + t^{-\frac{1}{2}-\frac{1}{\theta}}\big)e^{-\lambda_1 t}\|\varphi\|_{L^\theta(\Omega)} \tag{5.36}$$
$$\text{for all } t > 0 \text{ and each } \varphi \in C^1(\overline{\Omega};\mathbb{R}^2) \text{ satisfying } \varphi\cdot\nu = 0 \text{ on } \partial\Omega,$$

where $(e^{t\Delta})_{t\ge 0}$ denotes the Neumann heat semigroup on Ω. Now given $T' \in (\tau, T)$, in order to find an appropriate upper bound for the evidently finite number

$$M(T') := \sup_{t\in(\tau,T')} \|u(\cdot,t)\|_{L^\infty(\Omega)},$$

we define $U(x,t) := u(x,t) - \frac{r_+}{\mu}$ for $x \in \overline{\Omega}$ and $t \in [0, T_{\max})$ and observe that then in both cases $r > 0$ and $r \le 0$, U satisfies the parabolic inequality

$$\begin{aligned} U_t &= \Delta u - \chi_0\nabla\cdot\Big(\frac{u}{v}\nabla v\Big) + ru - \mu u^2 \\ &\le \Delta U - \chi_0\nabla\cdot\Big(\frac{u}{v}\nabla v\Big) - r_+U - \mu U^2 \qquad \text{for all } x \in \Omega \text{ and } t \in (0, T_{\max}). \end{aligned}$$

Therefore, U can be estimated from above by means of a corresponding variation-of-constants representation according to

$$\begin{aligned} U(\cdot,t) &= e^{-r_+(t-\tau)}e^{(t-\tau)\Delta}\Big(u(\cdot,\tau) - \frac{r_+}{\mu}\Big) \\ &\quad -\chi_0\int_\tau^t e^{-r_+(t-s)}e^{(t-s)\Delta}\nabla\cdot\Big(\frac{u(\cdot,s)}{v(\cdot,s)}\nabla v(\cdot,s)\Big)ds \\ &\quad -\mu\int_\tau^t e^{-r_+(t-s)}e^{(t-s)\Delta}U^2(\cdot,s)ds \\ &=: U_1(\cdot,t) + U_2(\cdot,t) + U_3(\cdot,t) \qquad \text{for all } t \in (\tau, T_{\max}), \end{aligned} \tag{5.37}$$

where from the order preserving property of $(e^{t\Delta})_{t\ge 0}$ we know that

$$U_3(\cdot,t) \le 0 \qquad \text{for all } t \in (\tau, T_{\max}), \tag{5.38}$$

and that

$$\begin{aligned} U_1(\cdot,t) &\le e^{-r_+(t-\tau)}e^{(t-\tau)\Delta}u(\cdot,\tau) \\ &\le e^{-r_+(t-\tau)}\|u(\cdot,\tau)\|_{L^\infty(\Omega)} \\ &\le \|u(\cdot,\tau)\|_{L^\infty(\Omega)} \qquad \text{for all } t \in (\tau, T_{\max}). \end{aligned} \tag{5.39}$$

We next use (5.36) to estimate

$$(5.40) \qquad \begin{aligned} &\|U_2(\cdot,t)\|_{L^\infty(\Omega)} \\ &\le c_4\chi_0 \int_\tau^t e^{-r_+(t-s)} \cdot \Big(1+(t-s)^{-\frac{1}{2}-\frac{1}{\theta}}\Big) e^{-\lambda_1(t-s)} \Big\| \frac{u(\cdot,s)}{v(\cdot,s)} \nabla v(\cdot,s) \Big\|_{L^\theta(\Omega)} ds \end{aligned}$$

for all $t \in (\tau, T_{\max})$, where by (5.31) and the Hölder inequality we see that

$$\begin{aligned} \Big\| \frac{u(\cdot,s)}{v(\cdot,s)} \nabla v(\cdot,s) \Big\|_{L^\theta(\Omega)} &\le \frac{1}{\delta} \cdot \|u(\cdot,s)\|_{L^{\frac{q\theta}{q-\theta}}(\Omega)} \|\nabla v(\cdot,s)\|_{L^q(\Omega)} \\ &\le \frac{1}{\delta} \cdot \|u(\cdot,s)\|_{L^\infty(\Omega)}^\kappa \|u(\cdot,s)\|_{L^p(\Omega)}^{1-\kappa} \|\nabla v(\cdot,s)\|_{L^q(\Omega)} \end{aligned}$$

for all $s \in (\tau, T)$ with $\kappa := 1 - \frac{p(q-\theta)}{q\theta} \in (0,1)$, because

$$\frac{q\theta}{q-\theta} = \frac{1}{\frac{1}{\theta}-\frac{1}{q}} > \frac{1}{\frac{1}{2}-\frac{2-p}{2p}} = \frac{p}{p-1} > p$$

thanks to (5.33) and the fact that $\theta > 2$. Using (5.34), (5.35) and (5.32) and recalling the definition of $M(T')$ we thus infer that

$$\Big\| \frac{u(\cdot,s)}{v(\cdot,s)} \nabla v(\cdot,s) \Big\|_{L^\theta(\Omega)} \le \frac{c_2c_3}{\delta} \cdot M^\kappa(T') \cdot c_1^{2-\kappa} \qquad \text{for all } t \in (\tau, T').$$

Hence, (5.37)-(5.40) show that

$$\sup_{x\in\Omega} U(x,t) \le c_5 + c_6 M^\kappa(T') \qquad \text{for all } t \in (\tau, T')$$

with $c_5 \equiv c_5(\tau) := \|u(\cdot,\tau)\|_{L^\infty(\Omega)}$ and

$$c_6 \equiv c_6(\delta,\tau) := \frac{c_1^{2-\kappa} c_2 c_3 c_4 \chi_0}{\delta} \cdot \int_0^\infty \big(1+\sigma^{-\frac{1}{2}-\frac{1}{\theta}}\big) e^{-(r_+ + \lambda_1)\sigma} d\sigma$$

being finite due to the fact that $\theta > 2$. As u is nonnegative, this implies that

$$\begin{aligned} M(T') &\le \frac{r_+}{\mu} + \sup_{t\in(\tau,T')} \sup_{x\in\Omega} U(x,t) \\ &\le \frac{r_+}{\mu} + c_5 + c_6 M^\kappa(T'), \end{aligned}$$

so that since $\kappa < 1$ we can estimate

$$M(T') \le \max \left\{ \Big(\frac{\frac{r_+}{\mu} + c_5}{c_6} \Big)^{\frac{1}{\kappa}} , \ \big(2c_6\big)^{\frac{1}{1-\kappa}} \right\} \qquad \text{for all } T' \in (\tau, T)$$

and conclude upon taking $T' \nearrow T$. □

5.5. Global existence for arbitrary $r > 0$. Proof of Theorem 5.1

We can now proceed to a first application of the above in order to assert the announced global existence result.

Proof of Theorem 5.1. We only need to make sure that for the existence time $T_{\max}$ of the maximally extended local solution (u, v) from Lemma 5.3 we have $T_{\max} = \infty$. Indeed, assuming on the contrary that $T_{\max}$ be finite, we could apply Corollary 5.7 and then Lemma 2.3 (see also (2.2)) to $\tau := \frac{1}{2}T_{\max}$ and $T := T_{\max}$ to find $\delta > 0$ such that

$$v(x,t) \geq \delta \qquad \text{for all } x \in \Omega \text{ and each } t \in \left(\frac{1}{2}T_{\max}, T_{\max}\right). \tag{5.41}$$

Therefore, Lemma 5.13 would yield $c_1 > 0$ satisfying

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq c_1 \qquad \text{for all } x \in \Omega \text{ and each } t \in \left(\frac{1}{2}T_{\max}, T_{\max}\right). \tag{5.42}$$

Combining (5.41) and (5.42), however, we see that this would contradict the extensibility criterion (5.8) in Lemma 5.3. □

5.6. Boundedness for large r. Proof of Theorem 5.2

Beyond Lemma 5.13, the proof of Theorem 5.2 will require an additional preparation, yielding a lower bound for the mass functional $\int_\Omega u$ which unlike the one provided by Corollary 5.7 will be uniform with respect to $t > 1$. This will be gained in Lemma 5.15 and Corollary 5.16, which rely on the following weighted estimate.

Lemma 5.14. *Let $\chi_0 > 0, r \in \mathbb{R}$ and $\mu > 0$. Then for each $\lambda > 0$ and any initial data fulfilling* (5.4)*, the solution of* (5.3) *satisfies*

$$\int_\Omega u^{-\lambda}\frac{|\nabla v|^2}{v^2} \leq \lambda^2 \int_\Omega u^{-\lambda-2}|\nabla u|^2 + 2\int_\Omega u^{-\lambda} \qquad \textit{for all } t \in (0, T_{\max}). \tag{5.43}$$

Proof. We integrate by parts and use the second equation in (5.3) to find that

$$\begin{aligned}
\int_\Omega u^{-\lambda}\frac{|\nabla v|^2}{v^2} &= -\int_\Omega u^{-\lambda}\nabla v \cdot \nabla\frac{1}{v} \\
&= -\lambda\int_\Omega \frac{u^{-\lambda-1}}{v}\nabla u \cdot \nabla v + \int_\Omega \frac{u^{-\lambda}}{v}\Delta v \\
&= -\lambda\int_\Omega \frac{u^{-\lambda-1}}{v}\nabla u \cdot \nabla v + \int_\Omega u^{-\lambda} - \int_\Omega \frac{u^{1-\lambda}}{v} \\
&\leq -\lambda\int_\Omega \frac{u^{-\lambda-1}}{v}\nabla u \cdot \nabla v + \int_\Omega u^{-\lambda} \qquad \text{for all } t \in (0, T_{\max}).
\end{aligned}$$

Now by Young's inequality we have

$$-\lambda \int_\Omega \frac{u^{-\lambda-1}}{v} \nabla u \cdot \nabla v \le \frac{1}{2} \int_\Omega u^{-\lambda} \frac{|\nabla v|^2}{v^2} + \frac{\lambda^2}{2} \int_\Omega u^{-\lambda-2} |\nabla u|^2,$$

whence upon a straightforward rearrangement we obtain (5.43). □

By tracking the evolution of the functional $\int_\Omega u^{-\lambda}$ for some appropriately chosen $\lambda > 0$, we can now derive a time-independent quantitative information ensuring persistence of u in a convenient sense.

Lemma 5.15. *Let $\chi_0 > 0$ and $\mu > 0$, and suppose that r satisfies* (5.5)*. Then there exists $\lambda > 0$ such that for any choice of u_0 satisfying* (5.4)*, one can find $C > 0$ such that*

$$\int_\Omega u^{-\lambda}(x,t)dx \le C \qquad \textit{for all } t > 1. \tag{5.44}$$

Proof. We first claim that thanks to the hypothesis (5.5) we can find $a > 0$ such that

$$a < \chi_0 \tag{5.45}$$

as well as

$$r - a > \frac{(\chi_0 - a)^2}{4}. \tag{5.46}$$

Indeed, if $\chi_0 > 2$ we may take $a := \chi_0 - 2$, so that (5.45) becomes evident and (5.46) results from the observation that then by (5.5),

$$r - a = r + 2 - \chi_0 > 1 = \frac{(\chi_0 - a)^2}{4}.$$

In the case $\chi_0 \le 2$, we note that (5.5) implies that $r > \frac{{\chi_0}^2}{4} \ge \chi_0 - 1$, whence in particular the numbers a_+ and a_- given by

$$a_\pm := \chi_0 - 2 \pm 2\sqrt{r + 1 - \chi_0}$$

are real with $a_- < a_+$, and this definition ensures that (5.46) holds for any $a \in (a_-, a_+)$, because for any such a we have

$$(\chi_0 - a)^2 - 4(r - a) = a^2 - 2(\chi_0 - 2)a + {\chi_0}^2 - 4r = (a - a_+)(a - a_-) < 0.$$

Since moreover (5.5) entails that

$$\left(2\sqrt{r + 1 - \chi_0}\right)^2 = 4r + 4 - 4\chi_0 > {\chi_0}^2 + 4 - 4\chi_0 = (\chi_0 - 2)^2 \ge 0$$

and that hence a_+ is positive, observing that $a_- < \chi_0 - 2 \le 0$ we may fix any $a > 0$ fulfilling $a < \min\{\chi_0, a_+\}$ to achieve that (5.45) and (5.46) are fulfilled simultaneously.

We now let

$$\lambda := \frac{4a}{(\chi_0 - a)^2} \tag{5.47}$$

and multiply the first equation in (5.3) by $u^{-\lambda-1}$ to see that

$$\begin{aligned} &\frac{1}{\lambda}\frac{d}{dt}\int_\Omega u^{-\lambda} + (\lambda+1)\int_\Omega u^{-\lambda-2}|\nabla u|^2 \\ &= (\lambda+1)\chi_0 \int_\Omega \frac{u^{-\lambda-1}}{v}\nabla u \cdot \nabla v - r\int_\Omega u^{-\lambda} + \mu\int_\Omega u^{1-\lambda} \qquad \text{for all } t > 0, \end{aligned} \tag{5.48}$$

where we use the number a as a parameter in the decomposition for all $t > 0$,

$$\begin{aligned} &(\lambda+1)\chi_0 \int_\Omega \frac{u^{-\lambda-1}}{v}\nabla u \cdot \nabla v \\ &= (\lambda+1)a\int_\Omega \frac{u^{-\lambda-1}}{v}\nabla u \cdot \nabla v + (\lambda+1)(\chi_0 - a)\int_\Omega \frac{u^{-\lambda-1}}{v}\nabla u \cdot \nabla v. \end{aligned} \tag{5.49}$$

In the latter summand, we employ Young's inequality to estimate

$$\begin{aligned} &(\lambda+1)(\chi_0 - a)\int_\Omega \frac{u^{-\lambda-1}}{v}\nabla u \cdot \nabla v \\ &\le (\lambda+1)\int_\Omega u^{-\lambda-2}|\nabla u|^2 + \frac{(\lambda+1)(\chi_0-a)^2}{4}\int_\Omega u^{-\lambda}\frac{|\nabla v|^2}{v^2} \end{aligned} \tag{5.50}$$

for all $t > 0$, whereas in the first expression on the right of (5.49) we apply the proof of Lemma 5.14 to obtain, noting that a is nonnegative,

$$\begin{aligned} &(\lambda+1)a\int_\Omega \frac{u^{-\lambda-1}}{v}\nabla u \cdot \nabla v \\ &\le -\frac{(\lambda+1)a}{\lambda}\int_\Omega u^{-\lambda}\frac{|\nabla v|^2}{v^2} + \frac{(\lambda+1)a}{\lambda}\int_\Omega u^{-\lambda} \qquad \text{for all } t > 0. \end{aligned} \tag{5.51}$$

In conclusion, (5.48)-(5.51) yield the inequality

$$\begin{aligned} \frac{1}{\lambda}\frac{d}{dt}\int_\Omega u^{-\lambda} \le &\left\{\frac{(\lambda+1)(\chi_0-a)^2}{4} - \frac{(\lambda+1)a}{\lambda}\right\}\cdot\int_\Omega u^{-\lambda}\frac{|\nabla v|^2}{v^2} \\ &-\left\{r - \frac{(\lambda+1)a}{\lambda}\right\}\cdot\int_\Omega u^{-\lambda} + \mu\int_\Omega u^{1-\lambda} \qquad \text{for all } t > 0, \end{aligned}$$

where

$$\frac{(\lambda+1)(\chi_0-a)^2}{4} - \frac{(\lambda+1)a}{\lambda} = 0$$

due to the definition (5.47) of λ. Observing that (5.47) in conjunction with (5.46) also warrants that

$$c_1 := r - \frac{(\lambda+1)a}{\lambda} = r - a - \frac{a}{\lambda} = r - a - \frac{(\chi_0-a)^2}{4}$$

is positive, we thus obtain that

$$\frac{1}{\lambda}\frac{d}{dt}\int_\Omega u^{-\lambda} + c_1\int_\Omega u^{-\lambda} \le \mu\int_\Omega u^{1-\lambda} \qquad \text{for all } t>0.$$

Now if $\lambda \le 1$, we may use the Hölder inequality along with (5.9) to find that

$$\mu\int_\Omega u^{1-\lambda} \le \mu|\Omega|^\lambda \cdot \Big(\int_\Omega u\Big)^{1-\lambda} \le c_2 := \mu|\Omega|^\lambda m^{1-\lambda} \qquad \text{for all } t>0,$$

whereas if $\lambda > 1$ then an application of Young's inequality yields $c_3 > 0$ such that

$$\mu\int_\Omega u^{1-\lambda} \le \frac{c_1}{2}\int_\Omega u^{-\lambda} + c_3 \qquad \text{for all } t>0.$$

Writing $c_4 := \max\{c_2, c_3\}$, in both of these cases we infer that $y(t) := \int_\Omega u^{-\lambda}(x,t)dx$, satisfies

$$\frac{1}{\lambda}y'(t) + \frac{c_1}{2}y(t) \le c_4 \qquad \text{for all } t>0,$$

and conclude that

$$y(t) \le \max\Big\{\int_\Omega u^{-\lambda}(x,1)dx\,,\,\frac{2c_4}{c_1}\Big\} \qquad \text{for all } t>1.$$

This proves (5.44). □

As a consequence thereof, we obtain the following.

Corollary 5.16. *Let $\chi_0 > 0$ and $\mu > 0$, and assume that r satisfies* (5.5)*. Then whenever* (5.4) *holds, the solution of* (5.3) *satisfies*

(5.52) $$\inf_{t>1}\int_\Omega u(x,t)dx > 0$$

and

(5.53) $$\inf_{(x,t)\in\Omega\times(1,\infty)} v(x,t) > 0.$$

Proof. To verify (5.52), we only need to apply Lemma 5.15 along with the Hölder inequality, which says that for each $\lambda > 0$ we have

$$\int_\Omega u \geq |\Omega|^{\frac{\lambda+1}{\lambda}} \Big(\int_\Omega u^{-\lambda} \Big)^{-\frac{1}{\lambda}} \qquad \text{for all } t > 0.$$

Thereupon, (5.53) can be derived by using Lemma 2.3 and once more recalling the second equation in (5.3). □

Now the claimed boundedness result can be deduced from this and Lemma 5.13 in a straightforward manner.

Proof of Theorem 5.2. Since u is continuous in $\overline{\Omega} \times [0, \infty)$, it is clear that for some $c_1 > 0$ we have

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq c_1 \qquad \text{for all } t \in [0, 1]. \tag{5.54}$$

Moreover, from Corollary 5.16 we know that since (5.5) holds, we can find $\delta > 0$ fulfilling

$$v(x, t) \geq \delta \qquad \text{for all } x \in \Omega \text{ and } t > 1.$$

As a consequence, Lemma 5.13 provides $c_2 > 0$ satisfying

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq c_2 \qquad \text{for all } t > 1.$$

In conjunction with (5.54) this proves (5.6), whereafter (5.7) results upon an application of known elliptic estimates to the solution v of second equation in (5.3). □

Chapter 6

Global existence and boundedness in a two-dimensional parabolic-elliptic Keller–Segel system with general signal-dependent sensitivity

6.1. Problem and result

In this chapter we consider the Neumann initial-boundary value problem for a parabolic-elliptic Keller–Segel system with general signal-dependent sensitivity $\chi(v)$,

$$
\begin{cases}
u_t = \Delta u - \nabla \cdot (u \nabla \chi(v)), & x \in \Omega, \quad t > 0, \\
0 = \Delta v - v + u, & x \in \Omega, \quad t > 0, \\
\dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\
u(x,0) = u_0(x), & x \in \Omega,
\end{cases}
\tag{6.1}
$$

in a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary $\partial\Omega$, where

$$
u_0 \in C^0(\overline{\Omega}), \quad u_0 \geq 0 \quad \text{in } \overline{\Omega}, \quad u_0 \not\equiv 0 \tag{6.2}
$$

and

$$
\chi \in C^{2+\omega}_{\mathrm{loc}}((0,\infty)) \quad \text{with some } \omega \in (0,1), \quad \chi' > 0, \quad \chi'(s) \to 0 \text{ as } s \to \infty. \tag{6.3}
$$

As to the classical case ($\chi(v) \equiv \chi_0 v, \chi_0 > 0$), roughly speaking, the size of initial data determines whether the solution is global and bounded or not. More precisely the solution of (6.1) is global and bounded when $\int_\Omega u_0 < 4\pi/\chi_0$ (see [**46, 66**]; for parabolic-parabolic case, see [**72**]). Whereas, with the large initial data, some blow-up solution is constructed when Ω is a disk by Herrero and Velázquez [**34**], and Nagai [**67**] showed that there are many nonradial blow-up solutions. By using an argument similar to that in [**66**], it is easy to see that there are many radial blow-up solutions, when Ω is a bounded disk and χ satisfies $\inf_{s>0} \chi'(s) > 0$.

On the other hand, we have other point of view in this thesis. We especially focus on a signal-dependent sensitivity function $\chi(v)$. We recall some results in this context. Here let n be spacial dimension. In the radially symmetric setting, we can find a nice picture about the system (6.1) with $\chi(v) = \chi_0 \log v, \chi_0 > 0$ as follows [**71**]:

- If $n = 2$, or $\chi_0 < \frac{2}{n-2}$ and $n \geq 3$ then the *radial* solution is global and bounded.
- If $\chi_0 > \frac{2n}{n-2}$ and $n \geq 3$ then there exists some initial data u_0 such that the *radial* solution blows up in finite time.

Moreover in the case $\chi(v) = \chi_0 v^k (\chi_0 >, k > 0)$ the solution is global and bounded when $\chi_0 > 0, 0 < k < 1$ and $n = 2$; we construct blow-up solution when $\chi_0 > 0, k > 0$ and $n \geq 3$. Without requiring such a symmetry hypothesis, Biler [**5**] showed that in the system (6.1) with $\chi(v) = \chi_0 \log v, \chi_0 > 0$,

- If $\chi_0 \leq 1$ and $n = 2$, or $\chi_0 < \frac{2}{n}$ and $n \geq 3$ then the solution is global.

In Chapter 3 it was established that the above global solution is bounded when $\chi_0 < \frac{2}{n}$ and this method is generalized to the case $\chi(v) = -\chi_0 v^{-k}, k > 0$ with sufficiently small $\chi_0 > 0$. Consequently, in the system (6.1) we can find the gap between *radial* case and *nonradial* case when $\chi(v) = \chi_0 \log v$. Indeed, especially in the two dimensional setting, for all $\chi_0 > 0$ the *radial* solution is global and bounded; on the other hand, the large time behavior of *nonradial* solution of (6.1) with large $\chi_0 > 0$ has been posted as an open problem. In [**5**, Remark 4] Biler and Velázquez gave the following conjecture:

- The optimal range of the coefficient χ_0 guaranteeing the global in time existence is $\chi_0 < \frac{n+2}{n-2}$.

One of the purpose of this chapter is to establish global existence and boundedness of *nonradial* solutions when $\chi(v) = \chi_0 \log v$ for all $\chi_0 > 0$ in the two dimensional setting. The other one is to study the case that sensitivity functions are more general, moreover to establish the *essential* condition on χ for guaranteeing the global solvability. The main result in this chapter reads as follows.

Theorem 6.1. *Suppose that u_0 and χ satisfy* (6.2) *and* (6.3)*, respectively. Then* (6.1) *has a unique global classical positive solution. Moreover the solution is uniformly bounded in time in the sense that*

$$\sup_{t\in[0,\infty)} \|u(t)\|_{L^\infty(\Omega)} < \infty.$$

Remark 6.1. We firstly remark that since $\chi(v) = \chi_0 \log v$ ($\chi_0 > 0$) fulfils the condition (6.3), the above statement gives an answer to the Biler–Velázquez conjecture in the two dimensional setting. Secondly, unlike the previous results [**71, 5, 29, 103, 107, 21, 31**] including Chapters 3, 4 and 5, the method in this chapter does not depend on any particular structure of $\chi(v)$. There are radial blow-up solutions, when Ω is a bounded disk and χ satisfies

$$\inf_{s>0} \chi'(s) > 0.$$

Then, (6.3) is the essential condition for all solutions to exist globally in time.

Before going into details, let us emphasize the main idea underlying the proof of Theorem 6.1. In this chapter we turn our eyes to ε-regularity, which is essentially established in [**81**] and formulated by Sugiyama [**88**]. By testing a cut-off function we focus on concentration of mass around each point of $\overline{\Omega}$ locally. ε-regularity is the property that if the concentration is sufficiently small then the point is not blow-up point. On the other hand, with sufficiently large concentration we invoke the property of Green's function to prove the value of v is sufficiently large. By decaying condition (6.3) the concentrating effect of the cross-diffusive term can be prevented. Finally it is proved that each point of $\overline{\Omega}$ is not blow-up point.

Remark 6.2. The method in this chapter can be applied to the fully parabolic Keller–Segel system under some conditions (see [**27**]).

This chapter is organized as follows. Section 6.2 is devoted to preliminaries, including local existence of solutions. After discussing some basic estimates which play a key role to ensure ε-regularity argument in Section 6.3, we shall establish Theorem 6.1 in Section 6.4. Firstly, we shall deduce uniform in time boundedness of the integral of $u \log u$ over some neighborhood of points of $\overline{\Omega}$ (Proposition 6.7). This will entail some regularity of v (Proposition 6.8, Lemma 6.9) and establish boundedness of u with respect to the norm in $L^p(\Omega), p > 1$ (Proposition 6.10), and finally complete the proof of Theorem 6.1. In Section 6.5 we will give some comments about the application to the fully parabolic cases.

6.2. Preliminaries

We first recall the local existence result established in Proposition 3.1.

Lemma 6.2. *Let* u_0 *and* χ *be as in* (6.2) *and* (6.3)*, respectively. Then there exist* $T_{\max} \le \infty$ (*depending only on* $\|u_0\|_{L^\infty(\Omega)}$) *and exactly one pair* (u, v) *of positive functions*

$$u \in C^{2,1}(\overline{\Omega} \times (0, T_{\max})) \cap C^0([0, T_{\max}); C^0(\overline{\Omega})),$$
$$v \in C^{2,0}(\overline{\Omega} \times (0, T_{\max})) \cap C^0((0, T_{\max}); C^0(\overline{\Omega}))$$

that solves (6.1) *in the classical sense. Also, the solution* (u, v) *satisfies the mass identities*

$$\int_\Omega u(x,t)\,dx = \int_\Omega u_0(x)\,dx \quad \textit{for all } t \in (0, T_{\max})$$

and

$$\int_\Omega v(x,t)\,dx = \int_\Omega u_0(x)\,dx \quad \textit{for all } t \in (0, T_{\max}).$$

Moreover, if $T_{\max} < \infty$*, then*

$$\lim_{t \nearrow T_{\max}} \|u(t)\|_{L^\infty(\Omega)} = \infty.$$

In virtue of the conservation of the total mass $\|u_0\|_{L^1(\Omega)}$, the next lemma is established in [**11**].

Lemma 6.3. *There exists some constant* $M = M(\|u_0\|_{L^1(\Omega)}, p_1, p_2) > 0$ *such that*

$$\sup_{t \in [0, T_{\max})} \|\nabla v\|_{L^{p_1}(\Omega)} + \sup_{t \in [0, T_{\max})} \|v\|_{L^{p_2}(\Omega)} \le M,$$

where $p_1 \in [1, 2)$, $p_2 \in [1, \infty)$.

6.3. Basic estimates

Recalling the Sobolev embedding inequality in the two space dimensions:

$$\|f\|^2_{L^2(\Omega)} \le K\left(\|\nabla f\|^2_{L^1(\Omega)} + \|f\|^2_{L^1(\Omega)}\right) \qquad \text{for all } f \in W^{1,1}(\Omega), \tag{6.4}$$

where $K > 0$ is a constant determined by Ω, we shall show some inequalities in this section.

We firstly introduce a cut-off function ψ. Proceeding similarly as in [**81**, p.26], we can establish the following lemma.

Lemma 6.4. *Let $n \in \mathbb{N}, q \in \overline{\Omega}, B_\delta(q) = B_\delta := \{x \in \mathbb{R}^2 \,|\, |x-q| < \delta\}$. Then there exists a function $\psi = \psi_{q,\delta,n} \in C_0^\infty(\mathbb{R}^2)$ satisfying*

$$\psi(x) = \begin{cases} 1 & (x \in B_{\frac{\delta}{2}}(q)), \\ 0 & (x \in \mathbb{R}^2 \setminus B_\delta(q)), \end{cases}$$

$$0 \le \psi \le 1 \qquad \text{in } \mathbb{R}^2,$$

$$\frac{\partial \psi}{\partial \nu} = 0 \qquad \text{on } \partial\Omega,$$

$$|\nabla \psi| \le A\psi^{1-\frac{1}{n}}, \quad |\Delta \psi| \le B\psi^{1-\frac{2}{n}} \qquad \text{in } \mathbb{R}^2,$$

where $A, B > 0$ are constants determined by δ and n. Moreover for all $\alpha > 0$ it holds that

$$|\nabla \psi^\alpha| \le \alpha A \psi^{\alpha - \frac{1}{n}},$$

$$|\Delta \psi^\alpha| \le \alpha B \psi^{\alpha - \frac{2}{n}} + \alpha(\alpha-1)A^2\psi^{\alpha-\frac{2}{n}} = \alpha((\alpha-1)A^2 + B)\psi^{\alpha - \frac{2}{n}}.$$

We shall prepare some useful estimates.

Lemma 6.5. *Let $\psi = \psi_{q,\delta,n}$ be as in* Lemma 6.4. *Suppose that n is sufficiently large. Then the following inequalities hold*:

(i) *The first component u of the solution satisfies that for all $t \in [0, T_{\max})$,*

$$\int_\Omega u^2\psi \le 2K^2 \int_{B_\delta \cap \Omega} u \int_\Omega u^{-1}|\nabla u|^2\psi + K^2\left(\frac{A^2}{2} + 1\right)\|u_0\|^2_{L^1(\Omega)}.$$

(ii) *Let $p \in (1,2)$. There exists some $C_1 = C_1(p, A, |\Omega|) > 0$ such that for all $s > 1$ and for all $t \in [0, T_{\max})$,*

$$\int_\Omega u^{p+1}\psi \le \frac{2K(p+1)^2}{\log s}\int_{B_\delta\cap\Omega}\left(u\log u + e^{-1}\right)\int_\Omega u^{p-2}|\nabla u|^2\psi$$
$$+ 6s^{p+1}|\Omega| + C_1\|u_0\|^{p+1}_{L^1(\Omega)}.$$

Proof. (i) is proved in [**81**, Lemma 4]. We prove (ii). Putting

$$w := (u^{\frac{p+1}{2}} - s^{\frac{p+1}{2}})_+\psi^{\frac{1}{2}}$$

with $a_+ := \max\{a, 0\}$, we have

$$\|w\|^2_{L^2(\Omega)} = \int_{\{u>s\}} \left(u^{\frac{p+1}{2}} - s^{\frac{p+1}{2}}\right)^2_+ \psi$$
$$\geq \int_{\{u>s\}} \left(\frac{1}{2}u^{p+1} - s^{p+1}\right)\psi$$
$$\geq \frac{1}{2}\int_\Omega u^{p+1}\psi - \frac{3}{2}s^{p+1}|\Omega|,$$

thus

$$\int_\Omega u^{p+1}\psi \leq 2\|w\|^2_{L^2(\Omega)} + 3s^{p+1}|\Omega|. \tag{6.5}$$

On the other hand, we can estimate

$$\left(\int_\Omega |\nabla w|\right)^2 \leq 2\left(\int_{\{u>s\}} |\nabla u^{\frac{p+1}{2}}|\psi^{\frac{1}{2}}\right)^2 + 2\left(\int_{\{u>s\}} (u^{\frac{p+1}{2}} - s^{\frac{p+1}{2}})_+|\nabla\psi^{\frac{1}{2}}|\right)^2 \tag{6.6}$$
$$\leq \frac{(p+1)^2}{2}\left(\int_{\{u>s\}} u^{\frac{p-1}{2}}|\nabla u|\psi^{\frac{1}{2}}\right)^2 + 2\left(\int_{\{u>s\}} u^{\frac{p+1}{2}}|\nabla\psi^{\frac{1}{2}}|\right)^2.$$

Since $s\log s \geq -e^{-1}$ for any $s > 1$ we see that

$$\left(\int_{\{u>s\}} u^{\frac{p-1}{2}}|\nabla u|\psi^{\frac{1}{2}}\right)^2 \leq \int_{B_\delta\cap\{u>s\}} u \cdot \int_\Omega u^{p-2}|\nabla u|^2\psi \tag{6.7}$$
$$\leq \frac{1}{\log s}\int_{B_\delta\cap\Omega} \left(u\log u + e^{-1}\right) \cdot \int_\Omega u^{p-2}|\nabla u|^2\psi.$$

Due to $p \in (1, 2)$, Hölder's inequality and Young's inequality imply that for all $\varepsilon > 0$,

$$\left(\int_{\{u>s\}} u^{\frac{p+1}{2}}|\nabla\psi^{\frac{1}{2}}|\right)^2 \leq \frac{A^2}{4}\left(\int_{\{u>s\}} u^{\frac{p+1}{2}}\psi^{\frac{1}{2}-\frac{1}{n}}\right)^2 \tag{6.8}$$
$$\leq \frac{A^2}{4}\left(\int_{\{u>s\}} u^{p+1}\psi\right)^{\frac{1}{2}}\|u\|^{\frac{p+1}{2}}_{L^1(\Omega)}|\Omega|^{\frac{2-p}{2}}$$
$$\leq \frac{\varepsilon}{2}\int_{\{u>s\}} u^{p+1}\psi + \frac{A^4}{32\varepsilon}\|u\|^{p+1}_{L^1(\Omega)}|\Omega|^{2-p}$$

due to

$$u^{\frac{p+1}{2}}\psi^{\frac{1}{2}-\frac{1}{n}} = u^{\frac{p+1}{4}}\psi^{\frac{1}{4}} \cdot u^{\frac{p+1}{4}}\psi^{\frac{1}{4}-\frac{1}{n}} \cdot 1$$

with sufficiently large $n > 4$. Accordingly, (6.6), (6.7) and (6.8) entail that

$$\left(\int_\Omega |\nabla w|\right)^2 \le \frac{(p+1)^2}{2\log s}\int_{B_\delta\cap\Omega}\left(u\log u + e^{-1}\right)\cdot\int_\Omega u^{p-2}|\nabla u|^2\psi + \varepsilon\int_\Omega u^{p+1}\psi + \frac{A^4}{16\varepsilon}\|u\|_{L^1(\Omega)}^{p+1}|\Omega|^{2-p}. \tag{6.9}$$

Proceeding the same way as the above estimate (6.8) we also have

$$\begin{aligned}\left(\int_\Omega w\right)^2 &= \left(\int_{\{u>s\}}\left(u^{\frac{p+1}{2}} - s^{\frac{p+1}{2}}\right)_+\psi^{\frac12}\right)^2\\ &\le \left(\int_{\{u>s\}} u^{\frac{p+1}{2}}\psi^{\frac12}\right)^2\\ &\le \varepsilon\int_\Omega u^{p+1}\psi + \frac{1}{4\varepsilon}\|u\|_{L^1(\Omega)}^{p+1}|\Omega|^{2-p}.\end{aligned} \tag{6.10}$$

By (6.4), we can combine the estimates (6.5), (6.9) and (6.10) to see

$$\begin{aligned}\int_\Omega u^{p+1}\psi &\le 2\|w\|_{L^2(\Omega)}^2 + 3s^{p+1}|\Omega|\\ &\le 2K\left(\|\nabla w\|_{L^1(\Omega)}^2 + \|w\|_{L^1(\Omega)}^2\right) + 3s^{p+1}|\Omega|\\ &\le \frac{K(p+1)^2}{\log s}\int_{B_\delta\cap\Omega}\left(u\log u + e^{-1}\right)\cdot\int_\Omega u^{p-2}|\nabla u|^2\psi\\ &\quad + 4K\varepsilon\int_\Omega u^{p+1}\psi + \frac{K(A^4+4)}{8\varepsilon}\|u\|_{L^1(\Omega)}^{p+1}|\Omega|^{2-p} + 3s^{p+1}|\Omega|.\end{aligned}$$

Taking $\varepsilon > 0$ as $4K\varepsilon = \frac12$, we can complete the proof of (ii). □

Let us finally provide the next auxiliary lemma which plays a key role in the proof of Lemma 6.9.

Lemma 6.6. *Let* $\psi = \psi_{q,\delta,n}$ *be as in* Lemma 6.4 *with sufficiently large* n*. Then for all* $r \in (2,3)$ *there exist some constants* $C_2 = C_2(r) > 0$ *and* $C_3 = C_3(r, A) > 0$ *fulfilling*

$$\left(\int_\Omega w^{2r}\psi\right)^{\frac1r} \le C_2\left(\int_\Omega |\nabla w|^r\psi\right)^{\frac1r}\left(\int_{\operatorname{supp}\psi\cap\Omega} w^2\right)^{\frac12} + C_3\int_{\operatorname{supp}\psi\cap\Omega} w^2$$

for all $w \in C^1(\overline{\Omega})$.

Proof. We fix $h \in C^1(\overline{\Omega})$. Putting $f = h^2\psi^{\frac{1}{2}}$ we calculate

$$
\begin{aligned}
\|\nabla f\|_{L^1(\Omega)}^2 &= \Big(\int_\Omega |\nabla(h^2\psi^{\frac{1}{2}})|\Big)^2 \\
&= \Big(\int_\Omega |2h\psi^{\frac{1}{2}}\nabla h + h^2\nabla\psi^{\frac{1}{2}}|\Big)^2 \\
&\le 8\Big(\int_\Omega |h\psi^{\frac{1}{2}}\nabla h|\Big)^2 + 2\Big(\int_\Omega h^2|\nabla\psi^{\frac{1}{2}}|\Big)^2 \\
&\le 8\Big(\int_\Omega |\nabla h|^2\psi^\theta\Big)\Big(\int_\Omega h^2\psi^{(1-\theta)}\Big) + C_4\Big(\int_\Omega h^2\psi^{\frac{1}{2}-\frac{1}{n}}\Big)^2
\end{aligned}
\tag{6.11}
$$

with some constant $C_4 = \frac{A^2}{2} > 0$ and where θ is defined as

$$\theta := \frac{r}{2(r-1)} \in \left(\frac{1}{2}, 1\right).$$

We can find a sufficiently large $n > 1$ satisfying

$$\frac{1}{2} - \frac{1}{n} > 1 - \theta = \frac{r-2}{2(r-1)},$$

and thus obtain

$$\int_\Omega h^2\psi^{\frac{1}{2}-\frac{1}{n}} \le \int_\Omega h^2\psi^{1-\theta}. \tag{6.12}$$

Here, thanks to the Sobolev embedding inequality (6.4), (6.11) and (6.12) show that

$$
\begin{aligned}
&\int_\Omega h^4\psi \\
&\le K\Big\{8\Big(\int_\Omega |\nabla h|^2\psi^\theta\Big)\Big(\int_\Omega h^2\psi^{1-\theta}\Big) + C_4\Big(\int_\Omega h^2\psi^{1-\theta}\Big)^2 + \Big(\int_\Omega h^2\psi^{\frac{1}{2}}\Big)^2\Big\} \\
&\le 8K\Big(\int_\Omega |\nabla h|^2\psi^\theta\Big)\Big(\int_\Omega h^2\psi^{1-\theta}\Big) + K(C_4+1)\Big(\int_\Omega h^2\psi^{1-\theta}\Big)^2
\end{aligned}
\tag{6.13}
$$

due to the fact

$$\psi^{\frac{1}{2}} \le \psi^{1-\theta}$$

by $\frac{1}{2} > 1 - \theta$.

Next we fix $w \in C^1(\overline{\Omega})$. Using the substitution $h = w^{\frac{r}{2}}$, Hölder's inequality yields that

$$\int_\Omega h^2 \psi^{1-\theta} = \int_\Omega w^r \psi^{1-\theta} \le \Big(\int_\Omega w^{2r} \psi \Big)^{\frac{r-2}{2(r-1)}} \Big(\int_{\operatorname{supp}\psi \cap \Omega} w^2 \Big)^{\frac{r}{2(r-1)}} \tag{6.14}$$

due to the relation

$$w^r \psi^{1-\theta} = w^{\frac{2r}{\lambda}} \psi^{1-\theta} \cdot w^{\frac{2(\lambda-1)}{\lambda}}$$

with

$$\lambda = \frac{2(r-1)}{r-2} = \frac{1}{1-\theta}.$$

Combining (6.13) and (6.14) we infer that

$$\begin{aligned} \int_\Omega w^{2r} \psi \le 8K \Big(\int_\Omega |\nabla w^{\frac{r}{2}}|^2 \psi^\theta \Big) \Big(\int_\Omega w^r \psi^{1-\theta} \Big) \\ + C_5 \Big(\int_\Omega w^{2r} \psi \Big)^{\frac{r-2}{r-1}} \Big(\int_{\operatorname{supp}\psi \cap \Omega} w^2 \Big)^{\frac{r}{r-1}}, \end{aligned} \tag{6.15}$$

where $C_5 := K(C_4 + 1)$. In the term $\int_\Omega |\nabla w^{\frac{r}{2}}|^2 \psi^\theta$, we may invoke Hölder's inequality to obtain

$$\begin{aligned} \int_\Omega |\nabla w^{\frac{r}{2}}|^2 \psi^\theta &= \Big(\frac{r}{2}\Big)^2 \int_\Omega w^{r-2} |\nabla w|^2 \psi^\theta \\ &\le \Big(\frac{r}{2}\Big)^2 \Big(\int_\Omega |\nabla w|^r \psi \Big)^{\frac{2}{r}} \Big(\int_\Omega w^r \psi^{(1-\theta)} \Big)^{\frac{r-2}{r}}. \end{aligned} \tag{6.16}$$

Collecting (6.15) and (6.16) we have

$$\begin{aligned} \int_\Omega w^{2r} \psi &\le 8K \Big(\frac{r}{2}\Big)^2 \Big(\int_\Omega |\nabla w|^r \psi \Big)^{\frac{2}{r}} \Big(\int_\Omega w^r \psi^{1-\theta} \Big)^{\frac{r-2}{r}} \Big(\int_\Omega w^r \psi^{1-\theta} \Big) \\ &\quad + C_5 \Big(\int_\Omega w^{2r} \psi \Big)^{\frac{r-2}{r-1}} \Big(\int_{\operatorname{supp}\psi \cap \Omega} w^2 \Big)^{\frac{r}{r-1}} \\ &= 8K \Big(\frac{r}{2}\Big)^2 \Big(\int_\Omega |\nabla w|^r \psi \Big)^{\frac{2}{r}} \Big(\int_\Omega w^r \psi^{1-\theta} \Big)^{\frac{2(r-1)}{r}} \\ &\quad + C_5 \Big(\int_\Omega w^{2r} \psi \Big)^{\frac{r-2}{r-1}} \Big(\int_{\operatorname{supp}\psi \cap \Omega} w^2 \Big)^{\frac{r}{r-1}}. \end{aligned}$$

Invoking (6.14) again, we can estimate that

$$(6.17)\quad \int_\Omega w^{2r}\psi$$
$$\le 8K\Big(\frac{r}{2}\Big)^2\Big(\int_\Omega |\nabla w|^r\psi\Big)^{\frac{2}{r}}\Big(\int_\Omega w^{2r}\psi\Big)^{\frac{r-2}{2(r-1)}\cdot\frac{2(r-1)}{r}}\Big(\int_{\operatorname{supp}\psi\cap\Omega} w^2\Big)^{\frac{r}{2(r-1)}\cdot\frac{2(r-1)}{r}}$$
$$+ C_5\Big(\int_\Omega w^{2r}\psi\Big)^{\frac{r-2}{r-1}}\Big(\int_{\operatorname{supp}\psi\cap\Omega} w^2\Big)^{\frac{r}{r-1}}$$
$$= 8K\Big(\frac{r}{2}\Big)^2\Big(\int_\Omega |\nabla w|^r\psi\Big)^{\frac{2}{r}}\Big(\int_\Omega w^{2r}\psi\Big)^{\frac{r-2}{r}}\Big(\int_{\operatorname{supp}\psi\cap\Omega} w^2\Big)$$
$$+ C_5\Big(\int_\Omega w^{2r}\psi\Big)^{\frac{r-2}{r-1}}\Big(\int_{\operatorname{supp}\psi\cap\Omega} w^2\Big)^{\frac{r}{r-1}}.$$

Finally, multiplying (6.17) by $\big(\int_\Omega w^{2r}\psi\big)^{-\frac{r-2}{r}}$ and using Young's inequality, we see that

$$\Big(\int_\Omega w^{2r}\psi\Big)^{\frac{2}{r}} \le 8K\Big(\frac{r}{2}\Big)^2\Big(\int_\Omega |\nabla w|^r\psi\Big)^{\frac{2}{r}}\Big(\int_{\operatorname{supp}\psi\cap\Omega} w^2\Big)$$
$$+ C_5\Big(\int_\Omega w^{2r}\psi\Big)^{\frac{r-2}{r-1}+\frac{2-r}{r}}\Big(\int_{\operatorname{supp}\psi\cap\Omega} w^2\Big)^{\frac{r}{r-1}}$$
$$\le 8K\Big(\frac{r}{2}\Big)^2\Big(\int_\Omega |\nabla w|^r\psi\Big)^{\frac{2}{r}}\Big(\int_{\operatorname{supp}\psi\cap\Omega} w^2\Big)$$
$$+ \frac{1}{2}\Big(\int_\Omega w^{2r}\psi\Big)^{\frac{2}{r}} + C_6\Big(\int_{\operatorname{supp}\psi\cap\Omega} w^2\Big)^2$$

with some constant $C_6 = C_6(r, A) > 0$.

Therefore we can deduce

$$\Big(\int_\Omega w^{2r}\psi\Big)^{\frac{2}{r}}$$
$$\le 16K\Big(\frac{r}{2}\Big)^2\Big(\int_\Omega |\nabla w|^r\psi\Big)^{\frac{2}{r}}\Big(\int_{\operatorname{supp}\psi\cap\Omega} w^2\Big) + 2C_6\Big(\int_{\operatorname{supp}\psi\cap\Omega} w^2\Big)^2.$$

The above inequality completes the proof. □

6.4. Proof of Theorem 6.1

Henceforth we set $q \in \overline{\Omega}$ and $\psi = \psi_{q,\delta,n}$ as in Lemma 6.4. Suppose that n is sufficiently large.

We start by proving boundedness of the functional $\int_\Omega u \log u \cdot \psi$. In the proof of the following estimate, we can see the boundedness mechanism. Indeed, decaying property of sensitivity function $\chi(v)$ enforces boundedness of the solution locally when the solution is likely to blow up.

Proposition 6.7. *If* $|\operatorname{supp}\psi| \le |B_\delta(q)| = \pi\delta^2$ *is sufficiently small, then there exists some constant* $C_7 > 0$ *satisfying*

$$\int_\Omega u(x,t)\log u(x,t)\cdot\psi(x)\,dx \le C_7 \qquad \textit{for all } t \in [0, T_{\max}),$$

where C_7 *depends on* $\delta, A, B, |\Omega|, \|u_0\|_{L^1(\Omega)}$ *and* $\max_{s\in[\gamma,\infty)}\chi'(s)$ *(*γ *is defined in* (2.2)*).*

Proof. Recalling that u is positive, we may multiply the first equation in (6.1) by $\log u \cdot \psi$ and integrate by parts over Ω to see

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u\log u\cdot\psi\,dx - \int_\Omega u_t\psi\,dx \\
&= \int_\Omega u_t \log u\cdot\psi \\
&= \int_\Omega \nabla\cdot(\nabla u - u\nabla\chi(v))\log u\cdot\psi \\
&= -\int_\Omega \nabla u\cdot\nabla(\log u\cdot\psi) + \int_\Omega u\nabla\chi(v)\cdot\nabla(\log u\cdot\psi) \\
&= -\int_\Omega u|\nabla\log u|^2\psi - \int_\Omega \nabla u\cdot\log u\nabla\psi \\
&\quad + \int_\Omega u\nabla\chi(v)\cdot\nabla(\log u)\psi + \int_\Omega u\nabla\chi(v)\cdot\log u\nabla\psi \\
&= -\mathbf{I_1} - \mathbf{I_2} + \mathbf{I_3} + \mathbf{I_4},
\end{aligned} \tag{6.18}$$

where

$$\begin{aligned}
\mathbf{I_1} &:= \int_\Omega u|\nabla\log u|^2\psi, \quad \mathbf{I_2} := \int_\Omega \nabla u\cdot\log u\nabla\psi, \\
\mathbf{I_3} &:= \int_\Omega u\nabla\chi(v)\cdot\nabla(\log u)\psi, \quad \mathbf{I_4} := \int_\Omega u\nabla\chi(v)\cdot\log u\nabla\psi.
\end{aligned}$$

The integral term $\mathbf{I_2}$ can be rewritten as

$$\begin{aligned}\mathbf{I_2} &= -\int_\Omega u\cdot\frac{1}{u}\nabla u\cdot\nabla\psi - \int_\Omega u\log u\cdot\Delta\psi\\ &= -\int_\Omega \nabla u\cdot\nabla\psi - \int_\Omega u\log u\cdot\Delta\psi\\ &= \int_\Omega u\Delta\psi - \int_\Omega u\log u\cdot\Delta\psi\\ &= \int_\Omega u\Delta\psi - \int_\Omega (u\log u + e^{-1})\Delta\psi\end{aligned}$$

due to the equality $\int_\Omega e^{-1}\Delta\psi = \int_{\partial B_\delta} e^{-1}\frac{\partial\psi}{\partial\nu} = 0$. Since $0 \le u\log u + e^{-1}$ we see that

$$\begin{aligned}|\mathbf{I_2}| &\le \int_\Omega u|\Delta\psi| + \int_\Omega (u\log u + e^{-1})|\Delta\psi|\\ &\le B\int_\Omega u\psi^{1-\frac{2}{n}} + B\int_\Omega (u\log u + e^{-1})\psi^{1-\frac{2}{n}}.\end{aligned} \tag{6.19}$$

As to the terms $\mathbf{I_3}$ and $\mathbf{I_4}$, in virtue of Young's inequality we can deduce that

$$|\mathbf{I_3}| \le \frac{1}{2}\int_\Omega u|\nabla\log u|^2\psi + \frac{1}{2}\int_\Omega u|\nabla\chi(v)|^2\psi \tag{6.20}$$

and

$$|\mathbf{I_4}| \le \int_\Omega u|\nabla\chi(v)|^2\psi + \frac{A^2}{4}\int_\Omega u(\log u)^2\psi^{1-\frac{2}{n}}. \tag{6.21}$$

Combining (6.18), (6.19), (6.20) and (6.21) yields that

$$\begin{aligned}\int_\Omega u_t\log u\cdot\psi + \mathbf{I_1} &\le |\mathbf{I_2}| + |\mathbf{I_3}| + |\mathbf{I_4}|\\ &\le \frac{1}{2}\int_\Omega u|\nabla\log u|^2\psi + \frac{3}{2}\int_\Omega u|\nabla\chi(v)|^2\psi + B\int_\Omega u\psi^{1-\frac{2}{n}}\\ &\quad + B\int_\Omega (u\log u + e^{-1})\psi^{1-\frac{2}{n}} + \frac{A^2}{4}\int_\Omega u(\log u)^2\psi^{1-\frac{2}{n}},\end{aligned}$$

so that (6.18) entails the inequality

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u\log u\cdot\psi - \frac{d}{dt}\int_\Omega u\psi + \frac{1}{2}\int_\Omega u|\nabla\log u|^2\psi\\ &\le \frac{3}{2}\int_\Omega u|\nabla\chi(v)|^2\psi + B\int_\Omega u\psi^{1-\frac{2}{n}}\\ &\quad + B\int_\Omega (u\log u + e^{-1})\psi^{1-\frac{2}{n}} + \frac{A^2}{4}\int_\Omega u(\log u)^2\psi^{1-\frac{2}{n}}.\end{aligned} \tag{6.22}$$

Here, we can find some constants $C_8 = C_8(A) > 0$ and $C_9 = C_9(B) > 0$ satisfying

$$\frac{A^2}{4}u(\log u)^2 \le u^{1+\frac{1}{n}} + C_8, \quad Bu\log u \le u^{1+\frac{1}{n}} + C_9,$$

and thus (6.22) implies that

$$\begin{aligned}\frac{d}{dt}\int_\Omega u\log u\cdot\psi - \frac{d}{dt}\int_\Omega u\psi + \frac{1}{2}\int_\Omega u|\nabla\log u|^2\psi \\ \le \frac{3}{2}\int_\Omega u|\nabla\chi(v)|^2\psi + 2\int_\Omega u^{1+\frac{1}{n}}\psi^{1-\frac{2}{n}} \\ + B\int_\Omega u\psi^{1-\frac{2}{n}} + \int_\Omega (Be^{-1}+C_8+C_9)\psi^{1-\frac{2}{n}}.\end{aligned} \tag{6.23}$$

Since $\psi^{1-\frac{2}{n}} \le 1$ there exists some constant $C_{10} = C_{10}(A, B, |\Omega|, \|u_0\|_{L^1(\Omega)}) > 0$ such that

$$B\int_\Omega u\psi^{1-\frac{2}{n}} + \int_\Omega (Be^{-1}+C_8+C_9)\psi^{1-\frac{2}{n}} \le C_{10}, \tag{6.24}$$

whereas we may invoke Hölder's inequality to see

$$\begin{aligned}\int_\Omega u^{1+\frac{1}{n}}\psi^{1-\frac{2}{n}} &= \int_{B_\delta} u^{1+\frac{1}{n}}\psi^{1-\frac{2}{n}} \\ &\le \left(\int_\Omega u^2\psi^{(1-\frac{2}{n})\cdot\frac{2n}{n+1}}\right)^{\frac{n+1}{2n}} \cdot |B_\delta|^{\frac{n-1}{2n}}.\end{aligned}$$

Noting that there exists some $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}\left(1-\frac{2}{n}\right)\cdot\frac{2n}{n+1} &= \frac{n-2}{n}\cdot\frac{2n}{n+1} \\ &= \frac{2(n-2)}{n+1} \ge 1 \qquad \text{for all } n \ge n_0\end{aligned}$$

and $\psi^\alpha \le \psi$ for all $\alpha \ge 1$, we have for sufficiently large $n \ge n_0$,

$$\begin{aligned}\int_\Omega u^{1+\frac{1}{n}}\psi^{1-\frac{2}{n}} &\le |B_\delta|^{\frac{n-1}{2n}}\left(\int_\Omega u^2\psi\right)^{\frac{n+1}{2n}} \\ &\le |B_\delta|^{\frac{n-1}{2n}}\left(\int_\Omega u^2\psi + 1\right)\end{aligned} \tag{6.25}$$

due to the inequality $\frac{n+1}{2n} < 1$. Therefore (6.23), (6.24) and (6.25) yield that

$$\begin{aligned}\frac{d}{dt}\int_\Omega u\log u\cdot\psi - \frac{d}{dt}\int_\Omega u\psi + \frac{1}{2}\int_\Omega u|\nabla\log u|^2\psi \\ \le \frac{3}{2}\int_\Omega u|\nabla\chi(v)|^2\psi + 2|B_\delta|^{\frac{n-1}{2n}}\left(\int_\Omega u^2\psi + 1\right) + C_{10}.\end{aligned} \tag{6.26}$$

Now we focus on the term $\int_\Omega u|\nabla\chi(v)|^2\psi$. A time-independent pointwise lower bound for v was established in Lemma 2.3 (see also (2.2)). Thus there exists $\gamma > 0$ such that

$$\inf_{x\in\Omega} v(x,t) \geq \gamma > 0 \qquad \text{for all } t \in [0, T_{\max}),$$

where γ depends only on $\|u_0\|_{L^1(\Omega)}$ and Ω. Since the function $\chi'(s)$ attains the maximum value $\max_{s\in[\gamma,\infty)} \chi'(s)$ due to the condition (6.3), the above estimate enables us to see

$$\max_{s\in[\gamma,\infty)} \chi'(s) \geq \chi'(v(x,t)) \qquad \text{for all } t \in (0, T_{\max}),\ x \in \Omega.$$

Setting

$$H = H(t,\psi) := \min_{x\in\operatorname{supp}\psi} v(x,t) \geq \gamma > 0,$$

by Hölder's inequality we infer that

$$\begin{aligned}(6.27) \qquad \int_\Omega u|\nabla\chi(v)|^2\psi &\leq \Big(\max_{s\in[H,\infty)} \chi'(s)\Big)^2 \int_\Omega u|\nabla v|^2\psi \\ &\leq \Big(\max_{s\in[H,\infty)} \chi'(s)\Big)^2 \Big(\int_\Omega u^2\psi\Big)^{\frac12} \Big(\int_\Omega |\nabla v|^4\psi\Big)^{\frac12}.\end{aligned}$$

Here, we see

$$\begin{aligned}(6.28) \qquad \Big(\int_\Omega |\nabla v|^4\psi\Big)^{\frac12} &= \Big(\int_\Omega \big|\nabla v\cdot\psi^{\frac14}\big|^4\Big)^{\frac12} \\ &= \Big(\int_\Omega \big|\nabla(v\psi^{\frac14}) - v\nabla\psi^{\frac14}\big|^4\Big)^{\frac12} \\ &\leq 2\|\nabla(v\psi^{\frac14})\|^2_{L^4(\Omega)} + 2\Big(\int_\Omega v^4|\nabla\psi^{\frac14}|^4\Big)^{\frac12} \\ &\leq 2\|\nabla(v\psi^{\frac14})\|^2_{L^4(\Omega)} + C_{11}\Big(\int_\Omega v^4\Big)^{\frac12}\end{aligned}$$

with some constant $C_{11} = C_{11}(A) > 0$. Using the Sobolev embedding theorem and elliptic regularity theory we have

$$\begin{aligned}\|\nabla(v\psi^{\frac14})\|_{L^4(\Omega)} &\leq K_1\|v\psi^{\frac14}\|_{W^{2,\frac43}(\Omega)} \\ &\leq K_1K_2\|-\Delta(v\psi^{\frac14}) + (v\psi^{\frac14})\|_{L^{\frac43}(\Omega)}\end{aligned}$$

with some constants $K_1, K_2 > 0$ and thus Lemma 6.3 yields that

$$
\begin{aligned}
(6.29) \qquad & \|\nabla(v\psi^{\frac{1}{4}})\|_{L^4(\Omega)} \\
& \le K_1K_2\| - \Delta(v\psi^{\frac{1}{4}}) + (v\psi^{\frac{1}{4}})\|_{L^{\frac{4}{3}}(\Omega)} \\
& = K_1K_2\|\psi^{\frac{1}{4}}(-\Delta v + v) - 2\nabla v \cdot \nabla(\psi^{\frac{1}{4}}) - v\Delta\psi^{\frac{1}{4}}\|_{L^{\frac{4}{3}}(\Omega)} \\
& \le K_1K_2\|u\psi^{\frac{1}{4}}\|_{L^{\frac{4}{3}}(\Omega)} + 2K_1K_2\|\nabla v \cdot \nabla(\psi^{\frac{1}{4}})\|_{L^{\frac{4}{3}}(\Omega)} + K_1K_2\|v\Delta\psi^{\frac{1}{4}}\|_{L^{\frac{4}{3}}(\Omega)} \\
& \le K_1K_2\left(\int_\Omega u^{\frac{2}{3}}\psi^{\frac{1}{3}} \cdot u^{\frac{2}{3}}\right)^{\frac{3}{4}} + C_{12}\|\nabla v\|_{L^{\frac{4}{3}}(\Omega)} + C_{13}\|v\|_{L^{\frac{4}{3}}(\Omega)} \\
& \le K_1K_2\left(\int_\Omega u^2\psi\right)^{\frac{1}{4}}\left(\int_\Omega u\right)^{\frac{1}{2}} + C_{14}
\end{aligned}
$$

where $C_{12} = C_{12}(A), C_{13} = C_{13}(A, B)$ and $C_{14} = C_{14}(A, B, \|u_0\|_{L^1(\Omega)})$ are some positive constants. Thus (6.28) and (6.29) imply

$$
\begin{aligned}
(6.30) \qquad \left(\int_\Omega |\nabla v|^4\psi\right)^{\frac{1}{2}} & \le 2\left\{K_1K_2\left(\int_\Omega u^2\psi\right)^{\frac{1}{4}}\left(\int_\Omega u\right)^{\frac{1}{2}} + C_{14}\right\}^2 + C_{11}\left(\int_\Omega v^4\right)^{\frac{1}{2}} \\
& \le 4K_1^2K_2^2\|u_0\|_{L^1(\Omega)}\left(\int_\Omega u^2\psi\right)^{\frac{1}{2}} + C_{15}
\end{aligned}
$$

with some constant $C_{15} = C_{15}(A, B, \|u_0\|_{L^1(\Omega)}) > 0$. Therefore collecting (6.26),(6.27) and (6.30), we obtain that

$$
\begin{aligned}
& \frac{d}{dt}\int_\Omega u\log u \cdot \psi - \frac{d}{dt}\int_\Omega u\psi + \frac{1}{2}\int_\Omega u|\nabla \log u|^2\psi \\
& \le \frac{3}{2}\left(\max_{s\in[H,\infty)} \chi'(s)\right)^2\left(\int_\Omega u^2\psi\right)^{\frac{1}{2}}\left\{4K_1^2K_2^2\|u_0\|_{L^1(\Omega)}\left(\int_\Omega u^2\psi\right)^{\frac{1}{2}} + C_{15}\right\} \\
& \quad + 2|B_\delta|^{\frac{n-1}{2n}}\left(\int_\Omega u^2\psi + 1\right) + C_{10} \\
& \le \left\{12K_1^2K_2^2\|u_0\|_{L^1(\Omega)}\left(\max_{s\in[H,\infty)} \chi'(s)\right)^2 + 2|B_\delta|^{\frac{n-1}{2n}}\right\}\int_\Omega u^2\psi + C_{16}
\end{aligned}
$$

where

$$
C_{16} := \frac{3{C_{15}}^2}{32K_1^2K_2^2\|u_0\|_{L^1(\Omega)}}\left(\max_{s\in[\gamma,\infty)} \chi'(s)\right)^2 + 2|\Omega|^{\frac{n-1}{2n}} + C_{10} > 0.
$$

In light of Lemma 6.5 (i) we see that

$$\frac{d}{dt}\int_\Omega u\log u\cdot\psi - \frac{d}{dt}\int_\Omega u\psi + J(t,\delta)\int_\Omega u|\nabla\log u|^2\psi \le -\frac{1}{8K^2\|u_0\|_{L^1(\Omega)}}\int_\Omega u^2\psi + C_{17}, \tag{6.31}$$

where

$$J(t,\delta) := \frac{1}{2} - \left\{12K_1^2K_2^2\|u_0\|_{L^1(\Omega)}\left(\max_{s\in[H,\infty)}\chi'(s)\right)^2 + 2|B_\delta|^{\frac{n-1}{2n}} + \frac{1}{8K^2\|u_0\|_{L^1(\Omega)}}\right\}\cdot 2K^2\int_{B_\delta\cap\Omega} u,$$

$$C_{17} := C_{16} + \left\{12K_1^2K_2^2\|u_0\|_{L^1(\Omega)}\left(\max_{s\in[\gamma,\infty)}\chi'(s)\right)^2 + 2|B_\delta|^{\frac{n-1}{2n}} + \frac{1}{8K^2\|u_0\|_{L^1(\Omega)}}\right\}\cdot K^2\left(\frac{A^2}{2}+1\right)\|u_0\|_{L^1(\Omega)}^2.$$

Now we let

$$\varepsilon_0 := \frac{\frac{1}{2}}{2K^2\left\{12K_1^2K_2^2\|u_0\|_{L^1(\Omega)}\left(\max_{s\in[\gamma,\infty)}\chi'(s)\right)^2 + 2 + \frac{1}{8K^2\|u_0\|_{L^1(\Omega)}}\right\}}$$

and

$$H_* := \left(\frac{1}{4\pi}\log\frac{1}{2\delta}\right)\varepsilon_0,$$

where we underline that ε_0 is independent of $\delta > 0$. Since the condition (6.3) implies

$$\max_{s\in[H_*,\infty)}\chi'(s) \to 0 \quad \text{as} \quad \delta\to 0,$$

so we can choose sufficiently small $\delta\in(0,\frac{1}{\sqrt{\pi}})$ satisfying that

$$24K^2K_1^2K_2^2\|u_0\|_{L^1(\Omega)}^2\left(\max_{s\in[H_*,\infty)}\chi'(s)\right)^2 \le \frac{1}{8} \quad \text{and} \tag{6.32}$$

$$4K^2|B_\delta|^{\frac{n-1}{2n}}\|u_0\|_{L^1(\Omega)} \le \frac{1}{8}.$$

Moreover we recall Green's function G of $-\Delta+1$ under homogeneous Neumann boundary conditions in Ω:

$$v(x,t) = \int_\Omega G(x,y)u(y,t)\,dy.$$

In the case $q \in \Omega$ we obtain

$$G(x,y) = \frac{1}{2\pi}\log\frac{1}{|x-y|} + K(x,y)$$

with $K \in C^{1,\theta}_{\mathrm{loc}}(\Omega \times \overline{\Omega}), \theta \in (0,1)$, whereas the case $q \in \partial\Omega$ we obtain similar argument (see [**81**, Lemma 6]). Hence by picking up smaller $\delta > 0$ if necessary, we can see that

$$G(x,y) \geq \frac{1}{4\pi}\log\frac{1}{2\delta}, \qquad x, y \in B_\delta(q) \cap \Omega, \tag{6.33}$$

where we remark that $\delta > 0$ is independent of the choice of the point q.

In the rest of this proof we will claim that for fixed $\delta > 0$ as above, the function $J(t,\delta)$ is uniformly nonnegative in time. For each $t \in (0, T_{\max})$, one of the following two cases holds:

$$\textbf{(Case 1):} \quad \int_{B_\delta\cap\Omega} u(x,t)\,dx < \varepsilon_0, \qquad \textbf{(Case 2):} \quad \int_{B_\delta\cap\Omega} u(x,t)\,dx \geq \varepsilon_0.$$

In both cases, we shall show $J(t,\delta) \geq 0$.

(Case 1) We assume that

$$\int_{B_\delta\cap\Omega} u(x,t)\,dx < \varepsilon_0$$

at the time $t \in (0, T_{\max})$.

At the time, since the assumption yields $J(t,\delta) \geq 0$, then (6.31) implies that

$$\frac{d}{dt}\int_\Omega u\log u\cdot\psi - \frac{d}{dt}\int_\Omega u\psi \leq -\frac{1}{8K^2\|u_0\|_{L^1(\Omega)}}\int_\Omega u^2\psi + C_{17}, \tag{6.34}$$

and thus

$$\frac{d}{dt}\left\{\int_\Omega (u\log u + e^{-1})\psi - \int_\Omega u\psi\right\} + \frac{1}{8K^2\|u_0\|_{L^1(\Omega)}}\int_\Omega u^2\psi - \frac{1}{4K^2\|u_0\|_{L^1(\Omega)}}\int_\Omega u\psi \leq C_{17}.$$

Since $u\log u + e^{-1} \le \frac{1}{4}u^2 + C_{18}$ with some constant $C_{18} > 0$, we infer that at the time t,

$$\frac{d}{dt}\left\{ \int_\Omega (u\log u + e^{-1})\psi - \int_\Omega u\psi \right\} + \frac{1}{4K^2\|u_0\|_{L^1(\Omega)}}\left\{ \int_\Omega (u\log u + e^{-1})\psi - \int_\Omega u\psi \right\} + \frac{1}{16K^2\|u_0\|_{L^1(\Omega)}}\int_\Omega u^2\psi \le C_{19}, \tag{6.35}$$

where

$$C_{19} := C_{17} + \frac{C_{18}|\Omega|}{4K^2\|u_0\|_{L^1(\Omega)}}.$$

(Case 2) We assume that

$$\int_{B_\delta\cap\Omega} u(x,t)\,dx \ge \varepsilon_0$$

at the time $t \in (0, T_{\max})$.

By using Green's function G of $-\Delta + 1$, (6.33) yields that for $x \in B_\delta(q)\cap\Omega$,

$$v(x,t) = \int_\Omega G(x,y)u(y,t)\,dy \ge \int_{B_\delta\cap\Omega} G(x,y)u(y,t)\,dy \ge \left(\frac{1}{4\pi}\log\frac{1}{2\delta}\right)\varepsilon_0, \tag{6.36}$$

thus

$$H(t) \ge H_*.$$

Then we may invoke (6.32) to obtain

$$\begin{aligned} J(t_2,\delta) &\ge \frac{1}{2} - \left\{ 12K_1^2K_2^2\|u_0\|_{L^1(\Omega)}\left(\max_{s\in[H_*,\infty)}\chi'(s)\right)^2 + 2|B_\delta|^{\frac{n-1}{2n}} + \frac{1}{8K^2\|u_0\|_{L^1(\Omega)}}\right\}\cdot 2K^2\int_{B_\delta\cap\Omega} u \\ &\ge \frac{1}{2} - \frac{1}{8} - \frac{1}{8} - \frac{1}{4} = 0. \end{aligned}$$

Proceeding similarly as in (Case 1), we get (6.34) and then ensure that (6.35) is valid at the time t.

Consequently, combining (Case 1) and (Case 2) we can confirm (6.35) is valid for all $t \in (0, T_{\max})$ and conclude that there exists some positive constant $C_{20} = C_{20}(\delta, A, B, |\Omega|, \|u_0\|_{L^1(\Omega)}, \max_{s\in[\gamma,\infty)} \chi'(s))$ such that

$$\int_\Omega \left(u(x,t)\log u(x,t) + e^{-1}\right)\psi(x)\,dx - \int_\Omega u(x,t)\psi(x)\,dx \le C_{20} \qquad \text{for all } t \in (0, T_{\max}).$$

Thanks to mass conservation, we complete the proof. □

Next, we proceed to derive a bound for ∇v with respect to the norm in $L^2(\Omega)$.

Proposition 6.8. *If $\delta > 0$ is sufficiently small as in* Proposition 6.7, *then there exists a positive constant C_{21} such that*

$$\int_{B_{\frac{\delta}{2}}\cap\Omega} |\nabla v(x,t)|^2\,dx \le C_{21} \qquad \textit{for all } t \in (0, T_{\max}),$$

where C_{21} depends on δ, A, B, $|\Omega|$, $\|u_0\|_{L^1(\Omega)}$ and $\max_{s\in[\gamma,\infty)} \chi'(s)$.

Proof. Multiplying the second equation of (6.1) by $v\psi^2$ we have

$$\begin{aligned}\int_\Omega uv\psi^2 - \int_\Omega v^2\psi^2 &= -\int_\Omega \Delta v\cdot v\psi^2\\ &= \int_\Omega |\nabla v|^2\psi^2 + \int_\Omega \nabla v\cdot v\nabla\psi^2\\ &= \int_\Omega |\nabla v|^2\psi^2 - \frac{1}{2}\int_\Omega v^2\Delta\psi^2,\end{aligned}$$

and Lemma 6.3 implies

$$\begin{aligned}\int_\Omega |\nabla v|^2\psi^2 &= \int_\Omega uv\psi^2 - \int_\Omega v^2\psi^2 + \frac{1}{2}\int_\Omega v^2\Delta\psi^2\\ &\le \int_\Omega uv\psi^2 + C_{22}\int_\Omega v^2\psi^{2-\frac{2}{n}}\\ &\le \int_\Omega uv\psi^2 + C_{23}\end{aligned} \tag{6.37}$$

with positive constants $C_{22} = C_{22}(A, B)$ and $C_{23} = C_{23}(A, B, \|u_0\|_{L^1(\Omega)})$. We define

$$m := \int_\Omega u\psi.$$

From Jensen's inequality we know that for any $\delta > 0$,

$$
\begin{aligned}
-\log\left(\frac{1}{m}\int_\Omega e^{\delta v\psi}\,dx\right) &= -\log\left(\int_\Omega \frac{e^{\delta v\psi}}{u\psi}\cdot\frac{u\psi}{m}\,dx\right)\\
&\le \int_\Omega -\log\left(\frac{e^{\delta v\psi}}{u\psi}\right)\cdot\frac{u\psi}{m}\,dx\\
&= \frac{1}{m}\int_\Omega u\psi\log(u\psi) - \frac{\delta}{m}\int_\Omega uv\psi^2.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\int_\Omega uv\psi^2 &\le \frac{1}{\delta}\int_\Omega u\psi\log u + \frac{1}{\delta}\int_\Omega u\psi\log\psi + \frac{m}{\delta}\log\left(\int_\Omega e^{\delta v\psi}\right) - \frac{m}{\delta}\log m\\
&\le \frac{1}{\delta}\int_\Omega (u\log u + e^{-1})\psi + \frac{m}{\delta}\log\left(\int_\Omega e^{\delta v\psi}\right) + \frac{e^{-1}}{\delta}
\end{aligned}
\tag{6.38}
$$

due to

$$
x\log x \le 0 \quad \text{for } x\in[0,1] \quad \text{and} \quad -x\log x \le e^{-1} \quad \text{for } x\ge 0.
$$

Now the Trudinger-Moser inequality [**12**] leads to

$$
\begin{aligned}
\log\left(\int_\Omega e^{\delta v\psi}\right) &\le C_{\mathrm{TM}}\delta^2\int_\Omega |\nabla(v\psi)|^2 + C'_{\mathrm{TM}}\delta\|v\|_{L^1(\Omega)}\\
&\le 2C_{\mathrm{TM}}\delta^2\left(\int_\Omega |\nabla v|^2\psi^2 + \int_\Omega v^2|\nabla\psi|^2\right) + C'_{\mathrm{TM}}\delta\|v\|_{L^1(\Omega)}\\
&\le 2C_{\mathrm{TM}}\delta^2\int_\Omega |\nabla v|^2\psi^2 + C_{24}
\end{aligned}
\tag{6.39}
$$

with positive constants C_{TM}, C'_{TM} and $C_{24} = C_{24}(\delta, A, \|u_0\|_{L^1(\Omega)})$ due to Lemma 6.3. Consequently (6.37), (6.38) and (6.39) assert

$$
\begin{aligned}
&\int_\Omega |\nabla v|^2\psi^2\\
&\le \frac{1}{\delta}\int_\Omega (u\log u + e^{-1})\psi + \frac{e^{-1}}{\delta} + 2m\delta C_{\mathrm{TM}}\int_\Omega |\nabla v|^2\psi^2 + \frac{m}{\delta}C_{24} + C_{23}.
\end{aligned}
$$

If $0 < \delta < 1$ is sufficiently small, we see that

$$
\int_\Omega |\nabla v|^2\psi^2 \le C_{25}\int_\Omega (u\log u + e^{-1})\psi + C_{26}
$$

with positive constants C_{25} and C_{26} and hence by Proposition 6.7 we conclude that the desired inequality holds with $C_{21} := C_{25}(C_7 + e^{-1}|\Omega|) + C_{26}$. □

Using Proposition 6.8 we can establish the following regularity argument.

Lemma 6.9. *Let* $r \in (2,3)$ *and* $\delta > 0$ *be as in* Proposition 6.7. *Suppose that* $\psi_1 = \psi_{q,\frac{\delta}{2},n}$ *be as in* Lemma 6.4. *Then there exist positive constants* C_{27} *and* C_{28} *such that*

$$\int_\Omega |\nabla v|^{2r}\psi_1 \le C_{27}\int_\Omega u^r\psi_1 + C_{28},$$

where C_{27} *and* C_{28} *depend on* r, δ, A, B, $|\Omega|$, $\|u_0\|_{L^1(\Omega)}$ *and* $\max_{s\in[\gamma,\infty)}\chi'(s)$.

Proof. Invoking Lemma 6.6 with $w = |\nabla v|$ and $\psi = \psi_1$ we have

$$\begin{aligned}(6.40)\qquad &\Big(\int_\Omega |\nabla v|^{2r}\psi_1\Big)^{\frac{1}{r}}\\ &\le C_2\Big(\int_\Omega |\nabla|\nabla v||^r\psi_1\Big)^{\frac{1}{r}}\Big(\int_{B_{\frac{\delta}{2}}\cap\Omega}|\nabla v|^2\Big)^{\frac{1}{2}} + C_3\int_{B_{\frac{\delta}{2}}\cap\Omega}|\nabla v|^2.\end{aligned}$$

The term $\big(\int_\Omega |\nabla|\nabla v||^r\psi_1\big)^{\frac{1}{r}}$ can be estimated as

$$\begin{aligned}|\nabla|\nabla v||^r\psi_1 &\le \Big|\sum_{i,j=1}^2 \frac{\partial^2 v}{\partial x_i\partial x_j}\Big|^r\psi_1\\ &= \Big|\sum_{i,j=1}^2 \frac{\partial^2 v}{\partial x_i\partial x_j}\cdot\psi_1^{\frac{1}{r}}\Big|^r\\ &= \Big|\sum_{i,j=1}^2 \frac{\partial^2 (v\psi_1^{\frac{1}{r}})}{\partial x_i\partial x_j} - 2\sum_{i,j=1}^2\frac{\partial v}{\partial x_i}\cdot\frac{\partial \psi_1^{\frac{1}{r}}}{\partial x_j} - v\sum_{i,j=1}^2\frac{\partial^2\psi_1^{\frac{1}{r}}}{\partial x_i\partial x_j}\Big|^r\end{aligned}$$

and Lemma 6.3 yields that

$$\begin{aligned}(6.41)\qquad &\Big(\int_\Omega |\nabla|\nabla v||^r\psi_1\Big)^{\frac{1}{r}}\\ &\le \Big(\int_\Omega\Big|\sum_{i,j=1}^2\frac{\partial^2(v\psi_1^{\frac{1}{r}})}{\partial x_i\partial x_j}\Big|^r\Big)^{\frac{1}{r}} + C_{29}\Big(\int_\Omega|\nabla v|^r\psi_1^{1-\frac{r}{n}}\Big)^{\frac{1}{r}} + C_{30}\Big(\int_\Omega v^r\psi_1^{1-\frac{2r}{n}}\Big)^{\frac{1}{r}}\\ &\le \Big(\int_\Omega\Big|\sum_{i,j=1}^2\frac{\partial^2(v\psi_1^{\frac{1}{r}})}{\partial x_i\partial x_j}\Big|^r\Big)^{\frac{1}{r}} + C_{29}\Big(\int_\Omega|\nabla v|^r\psi_1^{1-\frac{r}{n}}\Big)^{\frac{1}{r}} + C_{31},\end{aligned}$$

where $C_{29} = C_{29}(r, A)$, $C_{30} = C_{30}(r, A, B)$ and $C_{31} = C_{31}(r, A, B, \|u_0\|_{L^1(\Omega)})$ are some positive constants. Using elliptic regularity theory and recalling the second equation in (6.1) we infer

$$
\begin{aligned}
(6.42)\qquad & \left(\int_\Omega \left| \sum_{i,j=1}^{2} \frac{\partial^2 (v\psi_1^{\frac{1}{r}})}{\partial x_i \partial x_j} \right|^r \right)^{\frac{1}{r}} \\
& \le \|v\psi_1^{\frac{1}{r}}\|_{W^{2,r}(\Omega)} \\
& \le K_2 \| -\Delta(v\psi_1^{\frac{1}{r}}) + (v\psi_1^{\frac{1}{r}})\|_{L^r(\Omega)} \\
& = K_2 \| -\Delta v \cdot \psi_1^{\frac{1}{r}} - 2\nabla v \cdot \nabla \psi_1^{\frac{1}{r}} - v\Delta\psi_1^{\frac{1}{r}} + (v\psi_1^{\frac{1}{r}})\|_{L^r(\Omega)} \\
& \le K_2 \|u\psi_1^{\frac{1}{r}}\|_{L^r(\Omega)} + C_{32} \left(\int_\Omega |\nabla v|^r \psi_1^{1-\frac{r}{n}} \right)^{\frac{1}{r}} + C_{33} \left(\int_\Omega v^r \psi_1^{1-\frac{2r}{n}} \right)^{\frac{1}{r}} \\
& \le K_2 \|u\psi_1^{\frac{1}{r}}\|_{L^r(\Omega)} + C_{32} \left(\int_\Omega |\nabla v|^r \psi_1^{1-\frac{r}{n}} \right)^{\frac{1}{r}} + C_{34},
\end{aligned}
$$

where $C_{32} = C_{32}(r, A)$, $C_{33} = C_{33}(r, A, B)$ and $C_{34} = C_{34}(r, A, B, \|u_0\|_{L^1(\Omega)})$ are some positive constants.

Now combining (6.41) and (6.42) implies that

$$
\begin{aligned}
(6.43)\qquad & \left(\int_\Omega |\nabla |\nabla v||^r \psi_1 \right)^{\frac{1}{r}} \\
& \le K_2 \|u\psi_1^{\frac{1}{r}}\|_{L^r(\Omega)} + (C_{29} + C_{32}) \left(\int_\Omega |\nabla v|^r \psi_1^{1-\frac{r}{n}} \right)^{\frac{1}{r}} + (C_{31} + C_{34}).
\end{aligned}
$$

Hence (6.40), (6.43) and Proposition 6.8 yield that

$$
\begin{aligned}
\left(\int_\Omega |\nabla v|^{2r} \psi_1 \right)^{\frac{1}{r}} \le\ & C_2 C_{21}^{\frac{1}{2}} K_2 \|u\psi_1^{\frac{1}{r}}\|_{L^r(\Omega)} + C_2 C_{21}^{\frac{1}{2}} (C_{29} + C_{32}) \left(\int_\Omega |\nabla v|^r \psi_1^{1-\frac{r}{n}} \right)^{\frac{1}{r}} \\
& + C_2 C_{21}^{\frac{1}{2}} (C_{31} + C_{34}) + C_3 C_{21}.
\end{aligned}
$$

Finally, Hölder's inequality and Young's inequality deduce that there exist some constants $C_{35} > 0$ and $C_{36} > 0$ such that

$$
\left(\int_\Omega |\nabla v|^{2r} \psi_1 \right)^{\frac{1}{r}} \le C_{35} \|u\psi_1^{\frac{1}{r}}\|_{L^r(\Omega)} + \frac{1}{2} \left(\int_\Omega |\nabla v|^{2r} \psi_1 \right)^{\frac{1}{r}} + C_{36},
$$

and complete the proof. □

We can now derive L^p-estimate of u around each point of $\overline{\Omega}$ locally.

Proposition 6.10. *Let $\delta > 0$ be as in Proposition 6.7. Then there exist some $p > 1$ and $C_{37} > 0$ such that*

$$\int_{B_{\frac{\delta}{4}} \cap \Omega} u^p(x,t)\,dx \le C_{37} \qquad \text{for all } t \in [0, T_{\max}),$$

where the constant C_{37} depends on $p, \delta, A, B, |\Omega|, \|u_0\|_{L^1(\Omega)}$ and $\max_{s\in[\gamma,\infty)} \chi'(s)$.

Proof. Let $\psi_1 = \psi_{q,\frac{\delta}{2},n}$ be as in Lemma 6.4. Testing $u^{p-1}\psi_1$ to the first equation in (6.1) we see that

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega u^p\psi_1 &= \int_\Omega u^{p-1}\psi_1 u_t \\
&= \int_\Omega u^{p-1}\psi_1 \nabla\cdot(\nabla u - u\nabla\chi(v)) \\
&= -\int_\Omega \nabla(u^{p-1}\psi_1)\cdot\nabla u + \int_\Omega u\nabla(u^{p-1}\psi_1)\cdot\nabla\chi(v) \\
&= -(p-1)\int_\Omega u^{p-2}|\nabla u|^2\psi_1 - \int_\Omega u^{p-1}\nabla\psi_1\cdot\nabla u \\
&\quad + (p-1)\int_\Omega u^{p-1}\psi_1\nabla u\cdot\nabla\chi(v) + \int_\Omega u^p\nabla\psi_1\cdot\nabla\chi(v).
\end{aligned}$$

Hence

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p\psi_1 + (p-1)\int_\Omega u^{p-2}|\nabla u|^2\psi_1 = \mathbf{I_A} + \mathbf{I_B} + \mathbf{I_C} \tag{6.44}$$

where

$$\begin{aligned}
&\mathbf{I_A} := -\int_\Omega u^{p-1}\nabla\psi_1\cdot\nabla u, \quad \mathbf{I_B} := (p-1)\int_\Omega u^{p-1}\psi_1\nabla u\cdot\nabla\chi(v), \\
&\mathbf{I_C} := \int_\Omega u^p\nabla\psi_1\cdot\nabla\chi(v).
\end{aligned}$$

Since the term $\mathbf{I_A}$ can be reduced to

$$\mathbf{I_A} = -\frac{1}{p}\int_\Omega \nabla u^p\cdot\nabla\psi_1 = \frac{1}{p}\int_\Omega u^p\Delta\psi_1,$$

we infer that

$$|\mathbf{I_A}| \le \frac{B}{p}\int_\Omega u^{p+1}\psi_1 + C_{38} \tag{6.45}$$

with some constant $C_{38} = C_{38}(p, B, |\Omega|) > 0$ due to Young's inequality

$$u^p \psi_1^{1-\frac{2}{n}} \le \left(u^p \psi_1^{\frac{p}{p+1}}\right)^{\frac{p+1}{p}} + C\psi_1^{(1-\frac{2}{n}-\frac{p}{p+1})\cdot(p+1)}$$

with some constant $C > 0$. In virtue of the Cauchy–Schwarz inequality we have

$$|\mathbf{I_B}| \le \frac{(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2\psi_1 + \frac{(p-1)}{2}\left(\max_{s\in[\gamma,\infty)}\chi'(s)\right)^2\int_\Omega u^p|\nabla v|^2\psi_1. \tag{6.46}$$

As to the term $\mathbf{I_C}$ Hölder's inequality and Young's inequality yield that

$$\begin{aligned}|\mathbf{I_C}| &\le A\max_{s\in[\gamma,\infty)}\chi'(s)\int_\Omega u^p\psi_1^{1-\frac{1}{n}}|\nabla v|\\ &\le A\max_{s\in[\gamma,\infty)}\chi'(s)\left(\int_\Omega u^{p+1}\psi_1\right)^{\frac{p}{p+1}}\left(\int_\Omega|\nabla v|^{p+1}\psi_1^{(1-\frac{1}{n}-\frac{p}{p+1})\cdot(p+1)}\right)^{\frac{1}{p+1}}\\ &\le \int_\Omega u^{p+1}\psi_1 + C_{39}\int_\Omega|\nabla v|^{p+1}\psi_1^{(1-\frac{1}{n}-\frac{p}{p+1})\cdot(p+1)}\end{aligned}$$

with a positive constant $C_{39} = C_{39}(p, A, \max_{s\in[\gamma,\infty)}\chi'(s))$. Moreover the Cauchy–Schwarz inequality deduces

$$\begin{aligned}\int_\Omega|\nabla v|^{p+1}\psi_1^{(1-\frac{1}{n}-\frac{p}{p+1})\cdot(p+1)} &\le \left(\int_\Omega|\nabla v|^{2(p+1)}\psi_1\right)^{\frac{1}{2}}\left(\int_\Omega\psi_1^{1-\frac{2(p+1)}{n}}\right)^{\frac{1}{2}}\\ &\le \frac{1}{2}\int_\Omega|\nabla v|^{2(p+1)}\psi_1 + \frac{1}{2}|\Omega|\end{aligned}$$

with sufficiently large n and then

$$|\mathbf{I_C}| \le \int_\Omega u^{p+1}\psi_1 + \frac{C_{39}}{2}\int_\Omega|\nabla v|^{2(p+1)}\psi_1 + \frac{C_{39}|\Omega|}{2}. \tag{6.47}$$

Consequently, (6.44), (6.45),(6.46) and (6.47) imply that

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p\psi_1 + \frac{(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2\psi_1\\ &\le C_{40}\int_\Omega u^p|\nabla v|^2\psi_1 + C_{41}\int_\Omega u^{p+1}\psi_1 + C_{42}\int_\Omega|\nabla v|^{2(p+1)}\psi_1 + C_{43},\end{aligned} \tag{6.48}$$

where

$$\begin{aligned}C_{40} &:= \frac{(p-1)}{2}\left(\max_{s\in[\gamma,\infty)}\chi'(s)\right)^2, \quad C_{41} := \frac{B}{p} + 1,\\ C_{42} &:= \frac{C_{39}}{2}, \quad C_{43} := C_{38} + \frac{C_{39}|\Omega|}{2}.\end{aligned}$$

We focus on the term $\int_\Omega u^p|\nabla v|^2\psi_1$. Using Hölder's inequality and Young's inequality we see that

$$\begin{aligned}\int_\Omega u^p|\nabla v|^2\psi_1 &\le \Big(\int_\Omega u^{p+1}\psi_1\Big)^{\frac{p}{p+1}}\Big(\int_\Omega |\nabla v|^{2(p+1)}\psi_1\Big)^{\frac{1}{p+1}}\\ &\le \frac{p}{p+1}\int_\Omega u^{p+1}\psi_1 + \frac{1}{p+1}\int_\Omega |\nabla v|^{2(p+1)}\psi_1.\end{aligned}$$

By (6.48) and recalling Proposition 6.9 with $r = p+1$, we see that

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p\psi_1 + \frac{p-1}{2}\int_\Omega u^{p-2}\psi_1|\nabla u|^2 \le C_{44}\int_\Omega u^{p+1}\psi_1 + C_{45} \tag{6.49}$$

where

$$C_{44} := \frac{pC_{40}}{p+1} + C_{41} + C_{27}\bigg(\frac{C_{40}}{p+1} + C_{42}\bigg) > 0,$$

$$C_{45} := C_{28}\bigg(\frac{C_{40}}{p+1} + C_{42}\bigg) + C_{43} > 0.$$

In light of Lemma 6.5 (ii) we ensure that

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p\psi_1 + C_{44}\int_\Omega u^{p+1}\psi_1\\ &\qquad + \left\{\frac{p-1}{2} - \frac{4C_{44}K(p+1)^2}{\log s}\int_{B_{\frac{\delta}{2}}\cap\Omega}(u\log u + e^{-1})\right\}\int_\Omega u^{p-2}|\nabla u|^2\psi_1\\ &\le 2C_{44}\Big(6s^{p+1}|\Omega| + C_1\|u_0\|_{L^1(\Omega)}^{p+1}\Big) + C_{45}.\end{aligned}$$

Due to Proposition 6.7 we can pick up sufficiently large s satisfying that there exists some constant $C_{46} > 0$ such that

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p\psi_1 + C_{44}\int_\Omega u^{p+1}\psi_1 \le C_{46}.$$

Here Hölder's inequality implies

$$\begin{aligned}\int_\Omega u^p\psi_1 &= \int_\Omega u^p{\psi_1}^{\frac{p}{p+1}}\cdot{\psi_1}^{\frac{1}{p+1}}\\ &\le \Big(\int_\Omega u^{p+1}\psi_1\Big)^{\frac{p}{p+1}}\Big(\int_\Omega \psi_1\Big)^{\frac{1}{p+1}}\\ &\le |\Omega|^{\frac{1}{p+1}}\Big(\int_\Omega u^{p+1}\psi_1\Big)^{\frac{p}{p+1}},\end{aligned}$$

and so

$$|\Omega|^{-\frac{1}{p}}\left(\int_\Omega u^p\psi_1\right)^{\frac{p+1}{p}} \le \int_\Omega u^{p+1}\psi_1.$$

Therefore we have

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p\psi_1 + C_{44}|\Omega|^{-\frac{1}{p}}\left(\int_\Omega u^p\psi_1\right)^{\frac{p+1}{p}} \le C_{46}$$

and complete the proof. □

We are now in a position to prove Theorem 6.1.

Proof of Theorem 6.1. Since the domain Ω is bounded, we can find the family of balls $\{B_\delta(q_i)\}_i$ ($q_i \in \overline{\Omega}$, $i = 1, ..., \ell$) satisfying

$$\overline{\Omega} \subset \bigcup_{i=1}^{\ell} B_{\frac{\delta}{4}}(q_i)$$

where $\delta > 0$ is defined in Proposition 6.7. For each $B_{\frac{\delta}{4}}(q_i)$ we apply Proposition 6.10 and have

$$\|u(t)\|_{L^p(\Omega)} \le \sum_{i=1}^{\ell}\|u(t)\|_{L^p(B_{\frac{\delta}{4}}(q_i))} \le \ell C_{37}^{\frac{1}{p}}.$$

Now proceeding similarly as in Proposition 3.6 we complete the proof. □

6.5. Further application to the parabolic-parabolic case

The purpose of this section is to introduce an overview of Fujie–Senba [**27**], in which the method in this chapter is applied to the fully parabolic Keller–Segel system. We note that due to an essential difference between the parabolic-elliptic system and the parabolic-parabolic system, we need new ideas which will be explained below. In this section we consider global existence and boundedness of solutions to the Neumann initial-boundary value problem for the parabolic-parabolic Keller–Segel system with signal-dependent sensitivity,

$$\begin{cases} u_t = \Delta u - \nabla\cdot(u\nabla\chi(v)) & \text{in } \Omega\times(0,\infty), \\ \tau v_t = \Delta v - v + u & \text{in } \Omega\times(0,\infty), \\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = 0 & \text{on } \partial\Omega\times(0,\infty), \\ u(\cdot,0) = u_0, \quad v(\cdot,0) = v_0 & \text{in } \Omega, \end{cases} \tag{6.50}$$

in a ball $\Omega = B_R(0) := \{x \in \mathbb{R}^2 \,|\, |x| < R\} \subset \mathbb{R}^2$ ($R > 0$) with parameter $\tau \in (0, 1]$, where $(u_0, v_0) \in C^0(\overline{\Omega}) \times C^2(\overline{\Omega})$ satisfies

$$(6.51) \qquad \begin{cases} u_0 \geq 0 \quad \text{in } \overline{\Omega}, \quad u_0 \not\equiv 0 \quad \text{and} \quad \text{radially symmetric}, \\ v_0 > 0 \quad \text{in } \overline{\Omega}, \quad \dfrac{\partial v_0}{\partial \nu} = 0 \quad \text{on } \partial\Omega \quad \text{and} \quad \text{radially symmetric}, \end{cases}$$

and the sensitivity function χ is assumed to satisfy

$$(6.52) \qquad \chi \in C^{2+\theta}_{\mathrm{loc}}((0,\infty)) \quad \text{with some } \theta \in (0,1), \quad \chi' > 0, \quad \chi'(s) \to 0 \quad \text{as } s \to \infty.$$

Main result. Our main result reads as follows.

Theorem 6.11 ([**27**])**.** *Suppose that (u_0, v_0) and χ satisfy (6.51) and (6.52), respectively. Then there exists $\tau_0 > 0$ such that for all $\tau \in (0, \tau_0)$, the problem (6.50) has a unique global classical positive radially symmetric solution (u, v) such that*

$$u \in C^{2,1}(\overline{\Omega} \times (0,\infty)) \cap C^0([0,\infty); C^0(\overline{\Omega})),$$
$$v \in C^{2,1}(\overline{\Omega} \times (0,\infty)) \cap C^0([0,\infty); C^0(\overline{\Omega})) \cap L^\infty_{\mathrm{loc}}([0,\infty); W^{1,\infty}(\Omega)).$$

Moreover the solution is uniformly bounded in time in the sense that

$$\sup_{t \in [0,\infty)} \Big(\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{W^{1,\infty}(\Omega)} \Big) < \infty.$$

Remark 6.3. This theorem establishes global existence and boundedness of (6.50) with $\chi(v) = \chi_0 \log v$ for all $\chi_0 > 0$ in the two dimensional setting. Unfortunately, removing the smallness condition on $\tau > 0$ and the assumption of radial symmetry have been left as an open problem.

Strategy and main difficulty. We first recall the method in the parabolic-elliptic system in this chapter. The cornerstone is the *local-in-space* lower bound for v (6.36): for each $t > 0$ and $\varepsilon_0 > 0$,

$$\text{if} \int_{B_\delta(q) \cap \Omega} u(x,t)\,dx, \geq \varepsilon_0 \quad \text{then} \quad \inf_{x \in B_\delta(q) \cap \Omega} v(x,t) \geq \frac{\varepsilon_0}{4\pi} \log \frac{1}{2\delta},$$

where $B_\delta(q) = \{x \in \mathbb{R}^2 \,|\, |x - q| < \delta\}$ ($\delta > 0, q \in \overline{\Omega}$). Here we consider the fully parabolic system (6.50). The essential difference lies on *local-in-space* lower bound for v. Actually, in the fully parabolic system it follows (Lemma 6.12): for $T > 0$ and $\varepsilon_1 > 0$,

$$\text{if} \min_{t \in [T-\tau, T]} \left(\int_{B_\delta(0) \cap \Omega} u(x,t)\,dx \right) \geq \varepsilon_1 \quad \text{then} \quad \inf_{x \in B_\delta(0) \cap \Omega} v(x,T) \geq \frac{\varepsilon_1 e^{-2}}{4\pi} \log \frac{1}{\delta^2}.$$

As compared with the parabolic-elliptic system, we should control the *local-in-space* mass $\int_{B_\delta(0)} u(t)$ on some interval $[T-\tau, T]$. To overcome this difficulty, we will establish Hölder continuity of a *local-in-space* mass with respect to time (Proposition 6.13, Lemma 6.14):

$$t \mapsto \int_{B_\delta(0)\cap\Omega} u(t).$$

We remark that such a continuity was established by the symmetry of Green's function in the parabolic-elliptic case ([**81**]). However, this technique cannnot be applied to the parabolic-parabolic case.

Sketch of proof. In the rest of this section sketch of the proof of Theorem 6.11 will be presented. Henceforth, we assume (6.51) and (6.52), moreover denote by (u, v) the solution of (6.50) in $\Omega \times (0, T_{\max})$. The cornerstone of the poof is the *local-in-space* lower bound for v as stated above. By calculating a fundamental solution the following lemma can be established.

Lemma 6.12 ([**27**])**.** *Let $\tau > 0$, $T \in (\tau, T_{\max})$ and $0 < \delta < \min\{1, R\}$. If there exists some $\varepsilon_1 > 0$ such that*

$$\int_{B_\delta(0)} u(x,t)\,dx \geq \varepsilon_1 \qquad \textit{for all } t \in [T-\tau, T],$$

then

$$\inf_{x\in B_\delta(0)} v(x,T) \geq \frac{\varepsilon_1 e^{-2}}{4\pi} \log\frac{1}{\delta^2}.$$

Next we will deduce a continuity of a local-in-space mass with respect to time. Using the assumption of radial symmetry, we can derive some stability on the neighbourhood of the origin. We point out that the radial symmetry assumption is required in the proof of the next lemma.

Proposition 6.13 ([**27**])**.** *Let $0 < t_1 < t_2 < T_{\max}$ and $\psi = \psi_{0,\delta,n}$ be as in* Lemma 6.4. *Assume that $\delta > 0$ is sufficiently small. Then there exists a positive constant C which is independent of t_1, t_2 and $\tau \in (0, 1]$ satisfying*

$$\left| e^{t_2}\int_\Omega u(x,t_2)\psi(x)\,dx - e^{t_1}\int_\Omega u(x,t_1)\psi(x)\,dx \right| \leq Ce^{t_2}\left(|t_2-t_1| + |t_2-t_1|^{\frac{1}{4}}\right).$$

Thus, invoking the continuity of a local-in-space mass with respect to time, we can guarantee a lower bound for v on some interval.

Lemma 6.14 ([**27**]). *Let* $\varepsilon_1 > 0$ *and* $\psi = \psi_{0,\delta,n}$ *be as in* Lemma 6.4 *with sufficiently large* n. *Assume that* $\delta > 0$ *is sufficiently small. Then there exists some* $\tau_0 \in (0, 1]$ *satisfying that if there exists some* $t_0 \in (0, T_{\max})$ *such that*

$$\int_\Omega u(x, t_0)\psi(x)\, dx \geq \varepsilon_1$$

then for all $\tau \in (0, \tau_0)$,

$$\int_\Omega u(x, t)\psi(x)\, dx \geq \frac{\varepsilon_1}{2} \qquad (\max\{0, t_0 - \tau\} \leq t \leq t_0).$$

In light of Lemma 6.14, we can establish a *local-in-space* energy estimate based on the same spirit in Proposition 6.7. Indeed, introducing the following modified Lyapunov functional

$$W(t) := M \int_\Omega u \log u\psi - \int_\Omega uv\psi + \frac{1}{2}\int_\Omega |\nabla v|^2 \psi + \frac{1}{2}\int_\Omega v^2\psi,$$

with a sufficiently large $M > 0$, we can establish uniform-in-time boundedness of $W(t)$ as follows.

Proposition 6.15 ([**27**]). *Let* $\psi = \psi_{0,\delta,n}$ *be as in* Lemma 6.4 *with sufficiently large* n. *Assume that* $\tau > 0$ *is sufficiently small. If* $M > 0$ *is sufficiently large and* $\delta > 0$ *is sufficiently small then there exists a positive constant* C *satisfying*

$$W(t) \leq C \qquad \textit{for all } t \in [0, T_{\max}).$$

In view of the above lemma, we can prove Theorem 6.11.

PART II:

PARABOLIC-PARABOLIC KELLER–SEGEL SYSTEM

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla \chi(v)), \\ v_t = \Delta v - v + u. \end{cases}$$

Chapter 7

Global existence and boundedness in a parabolic-parabolic Keller–Segel system with singular sensitivity

7.1. Problem and result

In this chapter we consider the Neumann initial-boundary value problem for a fully parabolic chemotaxis system with singular sensitivity $\chi(v) = \chi_0 \log v$, that is,

$$
\begin{cases}
u_t = \Delta u - \chi_0 \nabla \cdot (\frac{u}{v} \nabla v), & x \in \Omega, \quad t > 0, \\
v_t = \Delta v - v + u, & x \in \Omega, \quad t > 0, \\
\dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\
u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega,
\end{cases}
\tag{7.1}
$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$ with smooth boundary, where $\chi_0 > 0$ and

$$
\begin{cases}
u_0 \in C^0(\overline{\Omega}), \quad u_0 \geq 0 \quad \text{in } \overline{\Omega}, \quad u_0 \not\equiv 0, \\
v_0 \in W^{1,\infty}(\Omega), \quad v_0 > 0 \quad \text{in } \overline{\Omega}.
\end{cases}
\tag{7.2}
$$

Winkler [**107**] proved that if $\chi_0 < \sqrt{\frac{2}{n}}$, then (7.1) possesses a global classical solution. As pointed out in [**107**], the result did not rule out the possibility that the solution may become unbounded as $t \to \infty$. The question of boundedness of the solution to

(7.1) has been posted as an open problem. Global existence of weak solutions was also established when $\chi_0 < \sqrt{\frac{n+2}{3n-4}}$ ([**107**]). In the radially symmetric setting, Stinner and Winkler [**85**] constructed certain weak solutions under the condition $\chi_0 < \sqrt{\frac{n}{n-2}}$. Moreover, in virtue of additional dampening kinetic terms, Manásevich, Phan and Souplet [**61**] proved global existence and boundedness in a related system for all $\chi_0 > 0$.

In this chapter we improve the approach in [**107**] and establish uniform-in-time boundedness of solutions to (7.1). The main result reads as follows.

Theorem 7.1. *Let $n \geq 2$. Assume that χ_0 satisfies*

$$0 < \chi_0 < \sqrt{\frac{2}{n}},$$

and suppose that u_0 and v_0 satisfy (7.2)*. Then the global solution of* (7.1) *is bounded in the sense that there exists $C > 0$ such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq C \qquad \text{for all } t > 0.$$

The above theorem states *uniform-in-time* boundedness of solutions under the same condition on $\chi_0 > 0$ as in [**107**]. There are two difficulties in deriving boundedness. The first difficulty stems from the singularity of $\frac{1}{v}$. By establishing a *time-independent* pointwise lower bound for v (Lemma 2.6), we overcome this difficulty. Note that the strong maximum principle easily implies

$$v(\cdot,t) \geq \eta(t) := \min_{x\in\overline{\Omega}} v_0(x) \cdot e^{-t} \qquad \text{for all } t > 0.$$

However, this is useless in proving uniform-in-time boundedness of solutions, since $\eta(t) \to 0$ as $t \to \infty$. The second difficulty lies in deducing *time-independent* L^p-boundedness of solutions. Although the L^p-estimate in [**107**] depends on time, we shall reconstruct the method in [**107**] and remove the dependence. Invoking the above two *time-independent* estimates, we establish boundedness.

This chapter is organized as follows. Section 7.2 will be concerned with preliminaries. In Section 7.3 we deduce time-independent L^p-boundedness of solutions and complete the proof of Theorem 7.1.

7.2. Preliminaries

We first recall the global existence result established in [**107**].

Lemma 7.2. *Assume that*

$$0 < \chi_0 < \sqrt{\frac{2}{n}}.$$

If the initial data (u_0, v_0) *satisfies* (7.2), *then* (7.1) *has a global classical positive solution*

$$u \in C^{2,1}(\overline{\Omega} \times (0,\infty)) \cap C^0([0,\infty); C^0(\overline{\Omega})),$$
$$v \in C^{2,1}(\overline{\Omega} \times (0,\infty)) \cap C^0([0,\infty); C^0(\overline{\Omega})).$$

Moreover, the first component of the solution satisfies the mass identity

$$\int_\Omega u(x,t)\,dx = \int_\Omega u_0(x)\,dx \qquad \text{for all } t > 0. \tag{7.3}$$

To achieve boundedness of the norm of $u(\cdot,t)$ in $L^p(\Omega)$ we shall use the following lemmas.

Lemma 7.3. *Let* $p \in \mathbb{R}$ *and* $q \in \mathbb{R}$. *Then the following identity holds for all* $t > 0$:

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u^p v^q + q\int_\Omega u^p v^q - q\int_\Omega u^{p+1} v^{q-1}\\
&= -\,p(p-1)\int_\Omega u^{p-2} v^q |\nabla u|^2 + \int_\Omega u^p v^{q-2}\cdot\big[-q(q-1) + pq\chi_0\big]\cdot|\nabla v|^2\\
&\quad + \int_\Omega u^{p-1}v^{q-1}\cdot\big[-2pq + p(p-1)\chi_0\big]\nabla u\cdot\nabla v.
\end{aligned}$$

Proof. Proceeding analogously to [**107**, Lemma 2.3], we can prove the desired identity. □

Lemma 7.4. *Let* $1 \le \theta, \mu \le \infty$.

(i) *If* $\frac{n}{2}(\frac{1}{\theta} - \frac{1}{\mu}) < 1$, *then there exists* $C > 0$ *such that*

$$\|v(\cdot,t)\|_{L^\mu(\Omega)} \le C\Big(1 + \sup_{s\in(0,\infty)} \|u(\cdot,s)\|_{L^\theta(\Omega)}\Big) \qquad \text{for all } t > 0.$$

(ii) *If* $\frac{1}{2} + \frac{n}{2}(\frac{1}{\theta} - \frac{1}{\mu}) < 1$, *then there exists* $C > 0$ *such that*

$$\|\nabla v(\cdot,t)\|_{L^\mu(\Omega)} \le C\Big(1 + \sup_{s\in(0,\infty)} \|u(\cdot,s)\|_{L^\theta(\Omega)}\Big) \qquad \text{for all } t > 0.$$

Proof. We can argue similarly as in [**107**, Lemma 2.4] due to the estimate for $e^{t(\Delta-1)}$:

$$\|e^{t(\Delta-1)}\varphi\|_{L^\mu(\Omega)} \le c\,t^{-\frac{n}{2}(\frac{1}{\theta}-\frac{1}{\mu})}e^{-\delta t}\|\varphi\|_{L^\theta(\Omega)} \qquad \text{for all } t>0,\ \varphi \in L^\theta(\Omega),$$

with some constants $c, \delta > 0$. □

7.3. Proof of Theorem 7.1

We follow the same way as in [**107**]. The difference is that our estimates are independent of time.

Lemma 7.5. *Let* $n \geq 2$ *and* $0 < \chi_0 < \sqrt{\frac{2}{n}}$. *Assume that* $p \in (1, \frac{1}{{\chi_0}^2})$ *and* $r \in (r_-(p), r_+(p))$*, where*

$$r_\pm(p) := \frac{p-1}{2}(1 \pm \sqrt{1 - p{\chi_0}^2}).$$

If there exists a constant $c > 0$ *such that*

$$\|v(\cdot, t)\|_{L^{p-r}(\Omega)} \leq c \qquad \text{for all } t > 0, \tag{7.4}$$

then there exists $C > 0$ *such that*

$$\int_\Omega u^p(x,t)v^{-r}(x,t)\,dx \leq C \qquad \text{for all } t > 0.$$

Proof. Choosing $q := -r$ in Lemma 7.3, we obtain

$$\begin{aligned} \mathbf{I} := & \frac{d}{dt}\int_\Omega u^p v^{-r} - r\int_\Omega u^p v^{-r} + r\int_\Omega u^{p+1}v^{-r-1} \\ = & -p(p-1)\int_\Omega u^{p-2}v^{-r}|\nabla u|^2 - \int_\Omega u^p v^{-r-2}\big[r(r+1) + pr\chi_0\big] \cdot |\nabla v|^2 \\ & + \int_\Omega u^{p-1}v^{-r-1}\big[2pr + p(p-1)\chi_0\big]\nabla u \cdot \nabla v \end{aligned} \tag{7.5}$$

for $t > 0$. Applying Young's inequality to the last term, we have

$$\begin{aligned} & \Big| \int_\Omega u^{p-1}v^{-r-1}\big[2pr + p(p-1)\chi_0\big]\nabla u \cdot \nabla v \Big| \\ & \leq p(p-1)\int_\Omega u^{p-2}v^{-r}|\nabla u|^2 + \frac{1}{4p(p-1)}\int_\Omega u^p v^{-r-2}\big[2pr + p(p-1)\chi_0\big]^2 \cdot |\nabla v|^2. \end{aligned}$$

Therefore (7.5) yields

$$\mathbf{I} \leq -\int_\Omega u^p v^{-r-2} h(p, r, \chi_0)|\nabla v|^2, \tag{7.6}$$

where

$$h(p, r, \chi_0) := r(r+1) + pr\chi_0 - \frac{\big[2pr + p(p-1)\chi_0\big]^2}{4p(p-1)}. \tag{7.7}$$

As $p \in (1, \frac{1}{{\chi_0}^2})$ and $r \in (r_-(p), r_+(p))$, we thus obtain

$$\begin{aligned} 4(p-1)h(p,r,\chi_0) &= -4r^2 + 4(p-1)r - p(p-1)^2{\chi_0}^2 \\ &= 4(r_+(p)-r)(r-r_-(p)) > 0. \end{aligned}$$

In view of the positivity $h > 0$, (7.5) and (7.6) imply

$$\frac{d}{dt}\int_\Omega u^p v^{-r} + r\int_\Omega u^{p+1}v^{-r-1} \le r\int_\Omega u^p v^{-r} \qquad \text{for all } t > 0. \tag{7.8}$$

Now unlike the proof of [**107**, Lemma 4.2] we pay attention to the term $r\int_\Omega u^{p+1}v^{-r-1}$. Hölder's inequality implies that

$$\begin{aligned} \int_\Omega u^p v^{-r} &= \int_\Omega (u^{p+1}v^{-r-1})^{\frac{p}{p+1}} \cdot v^{-r-\frac{p(-r-1)}{p+1}} \\ &\le \Big(\int_\Omega u^{p+1}v^{-r-1}\Big)^{\frac{p}{p+1}} \Big(\int_\Omega v^{p-r}\Big)^{\frac{1}{p+1}}. \end{aligned}$$

In virtue of the assumption (7.4), we see that

$$\int_\Omega u^p v^{-r} \le c^{\frac{p-r}{p+1}} \Big(\int_\Omega u^{p+1}v^{-r-1}\Big)^{\frac{p}{p+1}}.$$

Hence we have that

$$c^{-\frac{p-r}{p}} \Big(\int_\Omega u^p v^{-r}\Big)^{\frac{p+1}{p}} \le \int_\Omega u^{p+1}v^{-r-1}. \tag{7.9}$$

Combining (7.9) with (7.8), we establish the following inequality:

$$\frac{d}{dt}\int_\Omega u^p v^{-r} \le -rc^{-\frac{p-r}{p}} \Big(\int_\Omega u^p v^{-r}\Big)^{\frac{p+1}{p}} + r\int_\Omega u^p v^{-r}.$$

Since we find $\frac{p+1}{p} > 1$, thus the standard ODE technique completes the proof. □

We are now in a position to prove Theorem 7.1.

Proof of Theorem 7.1. The proof is divided into two steps.

(Step 1) In this step we shall gain L^p-boundedness of solutions. We will prove that there exist some $p > \frac{n}{2}$ and $C_p > 0$ such that

$$\|u(\cdot,t)\|_{L^p(\Omega)} \le C_p \qquad \text{for all } t > 0. \tag{7.10}$$

We consider an iterative argument. First we pick a pair (p_0, r_0) such that

$$\begin{cases} p_0 \in \Big(1,\, \min\Big\{\dfrac{1}{{\chi_0}^2},\, n+1,\, \dfrac{n+2}{n-2}\Big\}\Big), \\ r_0 := \dfrac{p_0-1}{2}. \end{cases} \tag{7.11}$$

Then we can confirm that

$$p_0 > r_0, \quad r_0 < \frac{n}{2}, \quad r_0 \in (r_-(p_0), r_+(p_0)) \quad \text{and} \quad p_0 - r_0 = \frac{p_0+1}{2} < \frac{n}{n-2}.$$

Since $\frac{n}{2}(1 - \frac{1}{p_0-r_0}) < 1$ due to the inequality $p_0 - r_0 < \frac{n}{n-2}$, Lemma 7.4 (i) together with the mass identity (7.3) allows us to find a constant $c_0 > 0$ fulfilling

$$\|v(\cdot,t)\|_{L^{p_0-r_0}(\Omega)} \le C\Big(1 + \sup_{s\in(0,\infty)} \|u(\cdot,s)\|_{L^1(\Omega)}\Big) \le c_0 \qquad \text{for all } t > 0.$$

Therefore Lemma 7.5 yields that there exists a constant $c_0' > 0$ such that

$$\int_\Omega u^{p_0} v^{-r_0} \le c_0' \qquad \text{for all } t > 0.$$

Now we claim that for all $q_0 \in (1, \min\{p_0, \frac{n(p_0-r_0)}{n-2r_0}\})$ there exists a constant $c_0'' > 0$ such that

$$\int_\Omega u^{q_0} \le c_0'' \qquad \text{for all } t > 0. \tag{7.12}$$

Indeed, applying Hölder's inequality, we obtain

$$\begin{aligned} \int_\Omega u^{q_0} &= \int_\Omega (u^{p_0} v^{-r_0})^{\frac{q_0}{p_0}} \cdot v^{\frac{r_0 q_0}{p_0}} \\ &\le \Big(\int_\Omega u^{p_0} v^{-r_0}\Big)^{\frac{q_0}{p_0}} \cdot \Big(\int_\Omega v^{\frac{q_0 r_0}{p_0-q_0}}\Big)^{\frac{p_0-q_0}{p_0}} \\ &\le {c_0'}^{\frac{q_0}{p_0}} \cdot \Big(\int_\Omega v^{\frac{q_0 r_0}{p_0-q_0}}\Big)^{\frac{p_0-q_0}{p_0}}. \end{aligned} \tag{7.13}$$

Since $\frac{n}{2}(\frac{1}{q_0} - \frac{p_0-q_0}{q_0 r_0}) < 1$ due to $q_0 < \frac{n(p_0-r_0)}{n-2r_0}$, it follows from Lemma 7.4 (i) that

$$\sup_{t>0} \|v(\cdot,t)\|_{L^{\frac{q_0 r_0}{p_0-q_0}}(\Omega)} \le K_0\Big(1 + \sup_{t>0} \|u(\cdot,t)\|_{L^{q_0}(\Omega)}\Big)$$

with $K_0 > 0$. Applying this estimate to (7.13), we have

$$\sup_{t>0} \|u(\cdot,t)\|_{L^{q_0}(\Omega)} \le K_0'\Big(1 + (\sup_{t>0} \|u(\cdot,t)\|_{L^{q_0}(\Omega)})^{\frac{r_0}{p_0}}\Big)$$

with $K_0' > 0$. Since $\frac{r_0}{p_0} < 1$, we can verify (7.12).

In the above argument, if $p_0 > \frac{n}{2}$, then we can pick $q_0 > \frac{n}{2}$ and we establish (7.10). On the other hand, if $p_0 \le \frac{n}{2}$, then we consequently deduce that for all $q_0 \in (1, \frac{n(p_0+1)}{2(n-p_0+1)})$ there exists $c_0'' > 0$ satisfying

$$\int_\Omega u^{q_0} \le c_0'' \qquad \text{for all } t > 0 \tag{7.14}$$

due to $p_0 \ge \frac{n(p_0-r_0)}{n-2r_0} = \frac{n(p_0+1)}{2(n-p_0+1)}$ when $p_0 \le \frac{n}{2}$.

We proceed the second iteration. We fix a pair (p_1, r_1) such that

$$\begin{cases} p_1 \in \Big(p_0,\, \min\Big\{\dfrac{1}{{\chi_0}^2},\, n+1,\, \dfrac{p_0(n+2)}{n-2p_0}\Big\}\Big), \\ r_1 := \dfrac{p_1 - 1}{2}. \end{cases} \tag{7.15}$$

Then we see that

$$p_1 > r_1, \quad r_1 < \frac{n}{2} \quad \text{and} \quad r_1 \in (r_-(p_1), r_+(p_1)).$$

Moreover, we can calculate that

$$\begin{aligned} p_1 - r_1 = \frac{p_1+1}{2} &< \frac{\frac{p_0(n+2)}{n-2p_0} + 1}{2} \\ &= \frac{n(p_0+1)}{2(n-2p_0)} = \frac{n(p_0+1)}{2\{(n-p_0+1)-(p_0+1)\}} = \frac{n \cdot \frac{n(p_0+1)}{2(n-p_0+1)}}{n - 2 \cdot \frac{n(p_0+1)}{2(n-p_0+1)}}. \end{aligned}$$

Hence, we can find some $q_0 \in (1, \frac{n(p_0+1)}{2(n-p_0+1)})$ satisfying

$$p_1 - r_1 < \frac{nq_0}{n-2q_0}.$$

Noting that $\frac{n}{2}(\frac{1}{q_0} - \frac{1}{p_1-r_1}) < 1$, we deduce from Lemma 7.4 (i) and (7.14) that there exists a constant $c_1 > 0$ such that

$$\|v(\cdot,t)\|_{L^{p_1-r_1}(\Omega)} \le C\Big(1 + \sup_{s\in(0,\infty)} \|u(\cdot,s)\|_{L^{q_0}(\Omega)}\Big) \le c_1 \qquad \text{for all } t > 0$$

and Lemma 7.5 yields that there exists a constant $c_1' > 0$ fulfilling

$$\int_\Omega u^{p_1} v^{-r_1} \le c_1' \qquad \text{for all } t > 0.$$

Using a similar estimate as the first iteration, we have for all $q_1 \in (1, \min\{p_1, \frac{n(p_1-r_1)}{n-2r_1}\})$ there exists a constant $c_1'' > 0$ such that

$$\int_\Omega u^{q_1} \le c_1'' \qquad \text{for all } t > 0.$$

If we can choose $p_1 > \frac{n}{2}$, then we can pick $q_1 > \frac{n}{2}$ and establish (7.10). Moreover if $p_1 \le \frac{n}{2}$, then we have that for all $q_1 \in (1, \frac{n(p_1+1)}{2(n-p_1+1)})$ there exists a constant $c_1'' > 0$ satisfying

$$\int_\Omega u^{q_1} \le c_1'' \qquad \text{for all } t > 0.$$

Consequently, we can define a pair (p_k, r_k) $(k \in \mathbb{N})$:

$$\begin{cases} p_k \in \Big(p_{k-1}, \min\Big\{\dfrac{1}{{\chi_0}^2}, n+1, \dfrac{p_{k-1}(n+2)}{n-2p_{k-1}}\Big\}\Big), \\ r_k := \dfrac{p_k - 1}{2}, \end{cases} \tag{7.16}$$

and if $p_k \le \frac{n}{2}$, then we deduce that for all $q_k \in (1, \frac{n(p_k+1)}{2(n-p_k+1)})$

$$\int_\Omega u^{q_k} \le c_k'' \qquad \text{for all } t > 0$$

with constant $c_k'' > 0$. Because $\frac{2}{n} < \min\{\frac{1}{{\chi_0}^2}, n+1\}$ due to the condition $\chi_0 < \sqrt{\frac{2}{n}}$ and the increasing function

$$f(x) := \frac{x(n+2)}{n-2x}$$

satisfies $f(x) > 1\,(x > 1)$ and

$$f(x) \to \infty \qquad \text{as } x \to \frac{n}{2},$$

we can obtain some k_0 large enough such that $p_{k_0} > \frac{n}{2}$ and hence $q_{k_0} > \frac{n}{2}$. Therefore we prove (7.10).

(Step 2) By L^p-boundedness of solutions (Step 1), we show L^∞-boundedness in this step. Building on Lemma 7.4 (ii), we invoke the standard semigroup technique (e.g. [**107**, Lemma 3.4]) to imply that there exists $C > 0$ such that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \le C \qquad \text{for all } t > 0.$$

Thus we can complete the proof. □

Remark 7.1. The method in this chapter can be applied to the general case:

$$\begin{cases} u_t = \Delta u - \chi_0 \nabla \cdot (\frac{u}{v^k} \nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases} \tag{7.17}$$

with $k > 1$. Indeed, instead of $h(p, r, \chi_0)$ in (7.7), set

$$\begin{aligned} h(p, r, \chi_0, v) :&= r(r+1) + pr\chi_0 \cdot \frac{1}{v^{k-1}} - \frac{\left[2pr + p(p-1)\chi_0 \cdot \frac{1}{v^{k-1}}\right]^2}{4p(p-1)} \\ &\geq r(r+1) + pr\chi_0 \cdot \frac{1}{\eta^{k-1}} - \frac{\left[2pr + p(p-1)\chi_0 \cdot \frac{1}{\eta^{k-1}}\right]^2}{4p(p-1)}. \end{aligned}$$

Replacing χ_0 with $\overline{\chi_0} := \frac{\chi_0}{\eta^{k-1}}$, we can argue similarly as our proofs. Hence, if $\chi_0 < \sqrt{\frac{2}{n}} \cdot \eta^{k-1}$ we can establish boundedness of solutions to (7.17) with $k > 1$.

Chapter 8

Global existence and boundedness in a parabolic-parabolic Keller–Segel system with strongly singular sensitivity

8.1. Problem and result

In this chapter we consider the following Neumann initial-boundary value problem for the fully parabolic chemotaxis system with the *strongly* singular sensitivity $\chi(v)$,

$$
\begin{cases}
u_t = \Delta u - \nabla \cdot (u \nabla \chi(v)), & x \in \Omega, \quad t > 0, \\
v_t = \Delta v - v + u, & x \in \Omega, \quad t > 0, \\
\dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\
u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega,
\end{cases}
\tag{8.1}
$$

where we assume that $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded domain with smooth boundary, and χ satisfies

$$
\chi \in C^{2+\omega}_{\mathrm{loc}}((0,\infty)) \quad \text{with some } \omega \in (0,1)
\tag{8.2}
$$

and

$$
0 < \chi'(v) \leq \frac{\chi_0}{v^k} \quad \text{for some } \chi_0 > 0 \text{ and } k > 1;
\tag{8.3}
$$

moreover, the initial data (u_0, v_0) fulfils

$$\begin{cases} u_0 \in C^0(\overline{\Omega}), \quad u_0 \geq 0 \quad \text{in } \overline{\Omega}, \quad u_0 \not\equiv 0, \\ v_0 \in W^{1,\infty}(\Omega), \quad v_0 > 0 \quad \text{in } \overline{\Omega}. \end{cases} \tag{8.4}$$

From a mathematical point of view, in this context, the questions of global existence and boundedness appear to be quite challenging. As to the *regular* case that

$$0 < \chi'(v) \leq \frac{\chi_0}{(1+\alpha v)^k} \quad (\alpha > 0,\ \chi_0 > 0,\ k > 1),$$

it was established that (8.1) possesses a unique globally bounded classical solution ([**103**]). On the other hand, our interest is in the *singular* case. When $\chi'(v) = \frac{\chi_0}{v}$, global existence of classical (resp. weak) solutions of (8.1) was proved under the condition

$$\chi_0 < \sqrt{\frac{2}{n}} \quad \left(\text{resp. } \chi_0 < \sqrt{\frac{n+2}{3n-4}}\right)$$

by Winkler [**107**]. Boundedness is also established in Chapter 7.

In this chapter we consider global existence and uniform-in-time boundedness of solutions in the "strongly" singular case such as $\chi'(v) = \frac{\chi_0}{v^k}$ with "$k > 1$". We cannot directly apply Winkler's method as in [**103**] to this case due to the singularity of $\chi(v)$. In this chapter we turn our eyes to a *uniform-in-time* lower estimate for v (Lemma 2.6) and this estimate enables us to develop Winkler's strategy to the singular case easily. Note that the strong maximum principle easily implies $v(\cdot,t) \geq \eta(t) := \min v_0 \cdot e^{-t}$ for fixed $t > 0$; however, this is useless in proving uniform-in-time boundedness of solutions because $\eta(t) \to 0$ as $t \to \infty$. Our main result reads as follows.

Theorem 8.1. *Suppose that χ satisfy* (8.2) *and* (8.3)*, and assume that (u_0, v_0) fulfils* (8.4)*. Then the problem* (8.1) *has a global classical solution (u, v) and moreover the solution is bounded in the sense that there exists $C > 0$ such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq C \qquad \textit{for all } t > 0.$$

Remark 8.1. In Remark 7.1, it has been shown that (8.1) with $\chi'(v) = \frac{\chi_0}{v^k}$ $(k \geq 1)$ has a globally bounded solution, provided that "$\chi_0 > 0$ is sufficiently small". By virtue of Theorem 8.1 we can remove this smallness condition on χ_0 in the strongly singular case $k > 1$.

8.2. Proof of Theorem 8.1

We consider the following regularization of (8.1):

$$
\begin{cases}
u_{\varepsilon t} = \Delta u_\varepsilon - \nabla \cdot (u_\varepsilon \nabla \chi_\varepsilon(v_\varepsilon)), & x \in \Omega, \quad t > 0, \\
v_{\varepsilon t} = \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon, & x \in \Omega, \quad t > 0, \\
\dfrac{\partial u_\varepsilon}{\partial \nu} = \dfrac{\partial v_\varepsilon}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\
u_\varepsilon(x,0) = u_0(x), \quad v_\varepsilon(x,0) = v_0(x), & x \in \Omega,
\end{cases}
\tag{8.5}
$$

where $\varepsilon \in (0,1)$ and $\chi_\varepsilon \in C^{2+\omega}_{\mathrm{loc}}([0,\infty))$ with some $\omega > 0$ satisfies

$$
\chi_\varepsilon(s) := \chi(s+\varepsilon), \quad s \geq 0.
$$

Then we have

$$
\begin{aligned}
0 < \chi_\varepsilon'(s) &= \chi'(s+\varepsilon) \\
&\leq \frac{\chi_0}{(s+\varepsilon)^k} = \frac{\varepsilon^{-k}\chi_0}{(1+\frac{1}{\varepsilon}s)^k}.
\end{aligned}
$$

Therefore we can invoke the method in [**103**] to obtain global classical solutions of (8.5). Moreover, we can easily find that u_ε fulfils the mass conservation property $\int_\Omega u_\varepsilon(x,t)\,dx \equiv \int_\Omega u_0$.

We apply Winkler's method [**103**] to the approximate problem (8.5) and accomplish the passage to the limit of approximate solutions due to *uniform-in-time* lower bound for v (Lemma 2.6).

Proof of Theorem 8.1. The proof is divided into three steps.

(Step 1) In this step we prove an independent-in-ε bound on the L^p norm for the approximate solutions u_ε. Based the similar spirit as in [**103**, Lemma 3.1], we will see that there exists a constant $C_1 > 0$ such that

$$
\sup_{t>0} \|u_\varepsilon(t)\|_{L^p(\Omega)} \leq C_1 \qquad \text{for all } \varepsilon \in (0,1), \ p > 1,
\tag{8.6}
$$

where the constant C_1 is independent of ε. Indeed, from Lemma 2.6 (see (2.12)) we confirm the following upper estimate for χ_ε' on $[\eta,\infty)$:

$$
\begin{aligned}
\chi_\varepsilon'(s) &\leq \frac{\chi_0}{(s+\varepsilon)^k} \\
&\leq \frac{\chi_0}{s^k} = \frac{2^k\chi_0}{(s+s)^k} \leq \frac{2^k\chi_0}{(\eta+s)^k} \qquad \text{for all } s \geq \eta.
\end{aligned}
$$

Here we will give a rigorous proof of (8.6). In this proof we denote (u, v) and χ instead of $(u_\varepsilon, v_\varepsilon)$ and χ_ε, respectively. Given $p > 1$, we choose $\kappa > 0$ sufficiently small such as

$$\kappa \le \frac{p-1}{16p} \qquad \text{and} \qquad \kappa + 2 < 2k \tag{8.7}$$

and also set $\beta > 0$ sufficiently small satisfying

$$\frac{2^{2k+1}p(p-1){\chi_0}^2}{\kappa(\kappa+1)\beta^2} \cdot \frac{(1+\beta\eta)^{\kappa+2}}{(2\eta)^{2k}} \le 1 \qquad \text{and} \qquad (\kappa+2)\beta\eta - k \le 0. \tag{8.8}$$

We define the *suitable* test function

$$\varphi(s) := e^{(1+\beta s)^{-\kappa}} (\ge 0) \qquad \text{for } s \ge 0.$$

This test function $\varphi(s)$ has the following properties:

$$\varphi'(s) = -\kappa\beta(1+\beta s)^{-\kappa-1}\varphi(s) \le 0 \qquad \text{for } s \ge 0,$$
$$\varphi''(s) = \kappa(\kappa+1)\beta^2(1+\beta s)^{-\kappa-2}\varphi(s) + \kappa^2\beta^2(1+\beta s)^{-2\kappa-2}\varphi(s) \qquad \text{for } s \ge 0,$$

moreover there exists some $C > 0$ such that

$$-s\varphi'(s) = \kappa\beta s(1+\beta s)^{-\kappa-1}\varphi(s) \le C\varphi(s) \qquad \text{for } s \ge 0. \tag{8.9}$$

Using both PDEs in (8.5), we have that

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega u^p\varphi(v) &= \int_\Omega u^{p-1}\varphi(v)u_t + \frac{1}{p}\int_\Omega u^p\varphi'(v)v_t \\
&= \int_\Omega u^{p-1}\varphi(v)\Delta u - \int_\Omega u^{p-1}\varphi(v)\nabla\cdot(u\nabla\chi(v)) \\
&\quad + \frac{1}{p}\int_\Omega u^p\varphi'(v)\Delta v - \frac{1}{p}\int_\Omega u^p v\varphi'(v) + \frac{1}{p}\int_\Omega u^{p+1}\varphi'(v) \\
&= -(p-1)\int_\Omega u^{p-2}\varphi(v)|\nabla u|^2 - \int_\Omega u^{p-1}\varphi'(v)\nabla v\cdot\nabla u \\
&\quad + (p-1)\int_\Omega u^{p-1}\varphi(v)\nabla u\cdot\nabla\chi(v) + \int_\Omega u^p\varphi'(v)\chi'(v)|\nabla v|^2 \\
&\quad - \int_\Omega u^{p-1}\varphi'(v)\nabla u\cdot\nabla v - \frac{1}{p}\int_\Omega u^p\varphi''(v)|\nabla v|^2 \\
&\quad - \frac{1}{p}\int_\Omega u^p v\varphi'(v) + \frac{1}{p}\int_\Omega u^{p+1}\varphi'(v).
\end{aligned}$$

Since $\chi'(v) > 0$ and $\varphi'(v) \le 0$, we have that

$$\begin{aligned}(8.10)\qquad & \frac{1}{p}\frac{d}{dt}\int_\Omega u^p\varphi(v) + (p-1)\int_\Omega u^{p-2}\varphi(v)|\nabla u|^2 \\ & \le -2\int_\Omega u^{p-1}\varphi'(v)\nabla u\cdot\nabla v + (p-1)\int_\Omega u^{p-1}\varphi(v)\chi'(v)\nabla u\cdot\nabla v \\ & \quad -\frac{1}{p}\int_\Omega u^p\varphi''(v)|\nabla v|^2 - \frac{1}{p}\int_\Omega u^p v\varphi'(v).\end{aligned}$$

First the property (8.9) implies

$$(8.11)\qquad -\frac{1}{p}\int_\Omega u^p v\varphi'(v) \le \frac{C}{p}\int_\Omega u^p\varphi(v).$$

Moreover Young's inequality yields that

$$\begin{aligned}(8.12)\qquad & -2\int_\Omega u^{p-1}\varphi'(v)\nabla u\cdot\nabla v \\ & \le \frac{p-1}{8}\int_\Omega u^{p-2}\varphi(v)|\nabla u|^2 + \frac{8}{p-1}\int_\Omega u^p\frac{(\varphi')^2(v)}{\varphi(v)}|\nabla v|^2,\end{aligned}$$

and

$$\begin{aligned}(8.13)\qquad & (p-1)\int_\Omega u^{p-1}\varphi(v)\chi'(v)\nabla u\cdot\nabla v \\ & \le \frac{p-1}{8}\int_\Omega u^{p-2}\varphi(v)|\nabla u|^2 + 2(p-1)\int_\Omega u^p\varphi(v)(\chi')^2(v)|\nabla v|^2.\end{aligned}$$

Therefore plugging (8.11), (8.12) and (8.13) into (8.10) implies that

$$\begin{aligned}(8.14)\qquad & \frac{1}{p}\frac{d}{dt}\int_\Omega u^p\varphi(v) + \frac{p-1}{2}\int_\Omega u^{p-2}\varphi(v)|\nabla u|^2 \\ & \le \int_\Omega \mathbf{H}(v)u^p|\nabla v|^2 + \frac{C}{p}\int_\Omega u^p\varphi(v),\end{aligned}$$

where

$$\mathbf{H}(v) := \frac{8}{p-1}\frac{(\varphi')^2(v)}{\varphi(v)} + 2(p-1)\varphi(v)(\chi')^2(v) - \frac{1}{p}\varphi''(v).$$

We will prove $\mathbf{H}(v) \le 0$. To this purpose, we compute that

$$\begin{aligned}\mathbf{I_1} &:= \frac{8}{p-1}\frac{(\varphi')^2(v)}{\varphi(v)} = \frac{8}{p-1}\beta^2\kappa^2(1+\beta s)^{-2\kappa-2}\varphi(v), \\ \mathbf{I_2} &:= 2(p-1)\varphi(v)(\chi')^2(v) \le 2(p-1)\varphi(v)\frac{2^{2k}{\chi_0}^2}{(\eta+v)^{2k}}, \\ \mathbf{I_3} &:= \frac{1}{p}\varphi''(v) \ge \frac{1}{p}\kappa(\kappa+1)\beta^2(1+\beta v)^{-\kappa-2}\varphi(v)\end{aligned}$$

and then by (8.7) we have

$$\begin{aligned}\frac{\mathbf{I_1}}{\frac{1}{2}\mathbf{I_3}} &\le \frac{16p\kappa}{(p-1)(\kappa+1)} \cdot \frac{1}{(1+\beta s)^{\kappa}}\\ &\le \frac{16p\kappa}{p-1}\\ &\le 1.\end{aligned}$$

On the other hand, we also deduce that

$$\frac{\mathbf{I_2}}{\frac{1}{2}\mathbf{I_3}} \le \frac{2^{2k+1}p(p-1){\chi_0}^2}{\kappa(\kappa+1)\beta^2} \cdot \frac{(1+\beta v)^{\kappa+2}}{(\eta+v)^{2k}}.$$

Defining the function $\psi(s)$ for $s > 0$ by

$$\psi(s) := \frac{(1+\beta s)^{\kappa+2}}{(\eta+s)^{2k}},$$

then by (8.7) and (8.8) we confirm that the function ψ satisfies

$$\begin{aligned}\psi'(s) &= (\eta+s)^{-k-1}(1+\beta s)^{\kappa+1} \cdot \{(\kappa+2-2k)\beta s + (\kappa+2)\beta\eta - k\}\\ &\le 0\end{aligned}$$

so that $\psi(s) \le \psi(\eta)$ for all $s \ge \eta$. In light of $v(t) \ge \eta$, we obtain that

$$\begin{aligned}\frac{\mathbf{I_2}}{\frac{1}{2}\mathbf{I_3}} &\le \frac{2^{2k+1}p(p-1){\chi_0}^2}{\kappa(\kappa+1)\beta^2} \cdot \frac{(1+\beta\eta)^{\kappa+2}}{(2\eta)^{2k}}\\ &\le 1\end{aligned}$$

due to the condition (8.8). Consequently, we deduce that

$$\mathbf{H}(v) = \mathbf{I_1} + \mathbf{I_2} - \mathbf{I_3} \le 0,$$

and then have

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p\varphi(v) + \frac{p-1}{2}\int_\Omega u^{p-2}\varphi(v)|\nabla u|^2 \le C\int_\Omega u^p\varphi(v). \tag{8.15}$$

By invoking the Gagliardo–Nirenberg inequality it follows that

$$\begin{aligned}\int_\Omega u^p\varphi(v) &\le e\int_\Omega u^p\\ &= e\|u^{\frac{p}{2}}\|^2_{L^2(\Omega)}\\ &\le e \cdot C_{\mathrm{GN}}\left(\|\nabla(u^{\frac{p}{2}})\|_{L^2(\Omega)} + \|u^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}\right)^{2a} \cdot \|u^{\frac{p}{2}}\|^{2(1-a)}_{L^{\frac{2}{p}}(\Omega)}\end{aligned}$$

with some constant $C_{\mathrm{GN}} > 0$ and $a = \frac{p-1}{p} \in (0,1)$. Since we see that

$$\|u^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)} = \int_\Omega u = \|u_0\|_{L^1(\Omega)},$$

there exists some constant $C' > 0$ such that

$$\int_\Omega u^p \varphi(v) \le C' \left(\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + 1 \right)^a.$$

Hence it follows that

$$\begin{aligned} \frac{p-1}{2} \int_\Omega u^{p-2} \varphi(v) |\nabla u|^2 &\ge \frac{p-1}{2} \int_\Omega u^{p-2} |\nabla u|^2 \\ &= \frac{2(p-1)}{p^2} \int_\Omega |\nabla u^{\frac{p}{2}}|^2 \\ &\ge \frac{2(p-1)}{(C')^{\frac{1}{a}} p^2} \left(\int_\Omega u^p \varphi(v) \right)^{\frac{1}{a}} - \frac{2(p-1)}{p^2}. \end{aligned} \tag{8.16}$$

Collecting (8.15) and (8.16), we have

$$\frac{1}{p} \frac{d}{dt} \int_\Omega u^p \varphi(v) \le - \frac{2(p-1)}{(C')^{\frac{1}{a}} p^2} \left(\int_\Omega u^p \varphi(v) \right)^{\frac{1}{a}} + C \int_\Omega u^p \varphi(v) \psi + \frac{2(p-1)}{p^2}.$$

By standard ODE technique implies the boundedness (8.6).

We remark that the constants C and C' are independent of ε, so that the constant C_1 is independent of ε.

(Step 2) Using Lemma 2.6, we can proceed as in the proof of [**103**, Theorem 3.2] to deduce an independent-in-ε bound on the L^∞ norm for u_ε: there exists a constant $C_2 > 0$ such that

$$\sup_{t>0} \|u_\varepsilon(t)\|_{L^\infty(\Omega)} \le C_2 \quad \text{for all } \varepsilon \in (0,1).$$

(Step 3) Finally we construct a solution of (8.1) as the limit of a net of solutions to (8.5). This method is due to the proof of [**107**, Theorem 3.5]. For convenience we recall the proof. Since $(u_\varepsilon)_{\varepsilon \in (0,1)}$ is bounded in $L^\infty(\overline{\Omega} \times [0,\infty))$, parabolic Schauder estimate (Lemma 2.7) entails that both nets $(u_\varepsilon)_{\varepsilon \in (0,1)}$ and $(v_\varepsilon)_{\varepsilon \in (0,1)}$ are bounded in $C^{2+\theta, 1+\frac{\theta}{2}}_{\mathrm{loc}}(\overline{\Omega} \times (0,\infty))$ for some $\theta > 0$. We apply the Arzelà-Ascoli theorem and then infer that there exist a suitable sequence of numbers $\varepsilon_k \searrow 0$ and a pair (u, v) such that $u_{\varepsilon_k} \to u$ and $v_{\varepsilon_k} \to v$ in $C^{2,1}_{\mathrm{loc}}(\overline{\Omega} \times (0,\infty))$. This pair (u, v) solves the PDEs and the Neumann conditions in (8.1). The initial condition is also checked by parabolic regularity theory and semigroup techniques. Consequently, we have a global classical solution (u, v) of (8.1) such that u belongs to $L^\infty(\overline{\Omega} \times [0, \infty))$ in light of boundedness of $(u_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^\infty(\overline{\Omega} \times [0, \infty))$; note that this boundedness property is uniform with respect to ε. $\square$

Chapter 9

Global existence and boundedness in a quasilinear parabolic-parabolic Keller–Segel system with sensitivities $S(u)$ and $\log v$

9.1. Problem and result

In this chapter we consider the following quasilinear fully parabolic Keller–Segel system with sensitivities $S(u)$ and $\chi(v) = \log v$, that is,

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = \nabla \cdot (D(u)\nabla u) - \nabla \cdot \left(\dfrac{S(u)}{v}\nabla v\right), & x \in \Omega, \quad t > 0, \\
\dfrac{\partial v}{\partial t} = \Delta v - v + u, & x \in \Omega, \quad t > 0, \\
\dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\
u(x,0) = u_0(x), \ v(x,0) = v_0(x), & x \in \Omega,
\end{cases}
\tag{9.1}
$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, $n \in \mathbb{N}$ and $\frac{\partial}{\partial \nu}$ denotes differentiation with respect to the outward normal of $\partial\Omega$. The initial data (u_0, v_0) is assumed to be a pair of functions fulfilling

$$
\begin{cases}
u_0 \in C^2(\overline{\Omega}), \quad u_0 \geq 0 \quad \text{in } \overline{\Omega}, \quad u_0 \not\equiv 0, \\
v_0 \in C^1(\overline{\Omega}), \quad v_0 > 0 \quad \text{in } \overline{\Omega}.
\end{cases}
\tag{9.2}
$$

Moreover we suppose that D and S satisfy the following conditions:

$$D, S \in C^2([0,\infty)) \quad \text{with } S(0) = 0 \quad \text{and } S \geq 0, \tag{9.3}$$

$$D(u) \geq K_0(u+1)^{m-1} \quad \text{with } m \in \mathbb{R} \quad \text{and } K_0 > 0 \quad \text{for all } u \geq 0, \tag{9.4}$$

$$D(u) \leq K_1(u+1)^{M-1} \quad \text{with } M \in \mathbb{R} \quad \text{and } K_1 > 0 \quad \text{for all } u \geq 0, \tag{9.5}$$

$$\frac{S(u)}{D(u)} \leq K(u+1)^{\alpha} \quad \text{with } \alpha < \frac{2}{n} \quad \text{and } K > 0 \quad \text{for all } u \geq 0. \tag{9.6}$$

From a mathematical point of view it is important to study whether solutions remain bounded or blow up. As to the problem without $\frac{1}{v}$, i.e., in the case that the chemotaxis term in the first equation in (9.1) is replaced with $-\nabla\cdot(S(u)\nabla v)$, Tao and Winkler [**96**] proved boundedness of solutions, provided that D and S satisfy (9.3), (9.4), (9.5) and (9.6) and Ω is convex. Recently this convexity condition of Ω was removed in [**39**]. As to blow-up of solutions to the problem (9.1) without $\frac{1}{v}$, Winkler [**108, 106**] and Ciéslak and Stinner [**16**] established that the solutions blow up in finite time under the conditions that

$$\frac{S(u)}{D(u)} \geq Ku^{\frac{2}{n}+\eta} \quad \text{for } u > 1 \quad \text{with } K > 0,\ \eta > 0$$

and that

$$S(u) \geq cu \quad \text{for some } c > 0.$$

Therefore the optimal exponent is known as $\frac{2}{n}$.

In the last decade, a growing literature has been concerned with signal-dependent sensitivity. However, to the best of our knowledge, no results are available for the system with both nonlinear diffusion and signal-dependent sensitivity. As opposed to the case without $\frac{1}{v}$, we find that all solutions of (9.1) are global and bounded in the case $D(u) \equiv 1$ and $S(u) \equiv \chi_0 u$ with sufficiently small $\chi_0 > 0$ in [**107**] and Chapter 7. This means that the case $\alpha = 1$ and sufficiently small $K > 0$ admits global existence and boundedness. As $1 > \frac{2}{n}$ for $n \geq 3$, this fact indicates that the constant $\frac{2}{n}$ is not optimal in the condition (9.6). The question of optimality of (9.6) remains an open problem.

The purpose of this chapter is to establish global existence and boundedness of solutions of the Keller–Segel system with not only the nonlinear diffusion $\nabla\cdot(D(u)\nabla u)$ but also the singular sensitivity function $\frac{S(u)}{v}$. The main result in this chapter reads as follows.

Theorem 9.1. *Assume that* (u_0, v_0) *fulfils* (9.2). *Let* D *and* S *satisfy* (9.3), (9.4), (9.5) *and* (9.6) *with some* $m \in \mathbb{R}$, $M \in \mathbb{R}$, $\alpha < \frac{2}{n}$, $K_0 > 0$, $K_1 > 0$ *and* $K > 0$. *Then there exists a couple* (u, v) *of nonnegative functions such that*

$$u \in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)),$$
$$v \in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty))$$

which solves (9.1) *classically and moreover there exists* $C > 0$ *such that*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \le C \quad \textit{for all } t > 0.$$

The difficulty in the proof of Theorem 9.1 lies in the singularity of $\frac{1}{v}$. A *uniform-in-time* lower bound for v (Lemma 2.6) builds a "bridge" between the regular case ([**96, 39**]) and the singular case. We will consider approximate problems in Section 9.2 and prepare some estimates. Section 9.3 is devoted to discussing convergence of approximate solutions and completing the proof of Theorem 9.1.

9.2. Approximate problem

We consider the following regularization of (9.1):

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} = \nabla \cdot (D(u_\varepsilon) \nabla u_\varepsilon) - \nabla \cdot \left(\dfrac{S(u_\varepsilon)}{v_\varepsilon + \varepsilon} \nabla v_\varepsilon \right), & x \in \Omega, \quad t > 0, \\ \dfrac{\partial v_\varepsilon}{\partial t} = \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon, & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = \dfrac{\partial v_\varepsilon}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\ u_\varepsilon(x, 0) = u_0(x), \ v_\varepsilon(x, 0) = v_0(x), & x \in \Omega, \end{cases} \tag{9.7}$$

where $\varepsilon > 0$. For all (u_0, v_0) satisfying (9.2) we may invoke [**96**, Lemmas 1.1 and 1.2] to establish local existence of solutions to (9.7) as the following lemma.

Lemma 9.2. *Let* $\varepsilon > 0$. *Suppose that* (u_0, v_0) *fulfils* (9.2). *Assume that* D *and* S *satisfy* (9.3), (9.4) *and* (9.5). *Then there exist* $T_{\max} \in (0, \infty]$ *and a pair* $(u_\varepsilon, v_\varepsilon)$ *of nonnegative functions from* $C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max}))$ *solving* (9.7) *classically in* $\Omega \times (0, T_{\max})$. *Moreover,*

$$\textit{either} \quad T_{\max} = \infty \quad \textit{or} \quad \limsup_{t \nearrow T_{\max}} (\|u_\varepsilon(t)\|_{L^\infty(\Omega)} + \|v_\varepsilon(t)\|_{L^\infty(\Omega)}) = \infty;$$

furthermore, u_ε *has the following mass conservation*:

$$\|u_\varepsilon(t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} \quad \textit{for all } t \in (0, T_{\max}).$$

The following lemma is a cornerstone of this work, which was essentially established in Lemma 2.6. Mass conservation property plays a key role in the proof of the lemma. In view of the lemma we can ensure a *uniform-in-time* estimate for v_ε.

Lemma 9.3. *Let $\varepsilon > 0$ and $T > 0$. Suppose that (u_0, v_0) fulfils (9.2). Assume that D and S satisfy (9.3), (9.4) and (9.5). Let $(u_\varepsilon, v_\varepsilon)$ be a solution of (9.7) on $[0, T)$. Then there exists $\delta > 0$ such that*

$$\inf_{x\in\Omega} v_\varepsilon(x,t) \geq \delta > 0 \quad \textit{for all } t \in (0,T),\ \varepsilon > 0,$$

where δ does not depend on ε and T.

As a preparation for the passage to the limit, we present three lemmas.

Lemma 9.4. *Let $\varepsilon > 0$ and $T > 0$. Suppose that (u_0, v_0) fulfils (9.2). Assume that D and S satisfy (9.3), (9.4), (9.5) and (9.6). Let $(u_\varepsilon, v_\varepsilon)$ be a solution of (9.7) on $[0, T)$. Then for all $p \in [1,\infty)$ and each $q \in [1,\infty)$ there exist $C_p > 0$ and $C'_{2q} > 0$ such that*

$$\begin{aligned} \|u_\varepsilon(t)\|_{L^p(\Omega)} &\leq C_p && \textit{for all } t \in (0,T), \\ \|\nabla v_\varepsilon(t)\|_{L^{2q}(\Omega)} &\leq C'_{2q} && \textit{for all } t \in (0,T), \end{aligned}$$

where C_p and C'_{2q} do not depend on ε and T.

Proof. Proceeding similarly as in [**96**, Lemma 3.3] and [**39**, Proposition 3.2], we define ϕ as

$$\phi(r) := \int_0^r \int_0^\rho \frac{(\sigma+1)^{m+p-3}}{D(\sigma)}\, d\sigma d\rho.$$

Thus we can calculate

$$\begin{aligned} \frac{d}{dt}\int_\Omega \phi(u_\varepsilon) &= \int_\Omega \phi'(u_\varepsilon)\nabla\cdot(D(u_\varepsilon)\nabla u_\varepsilon) - \int_\Omega \phi'(u_\varepsilon)\nabla\cdot\left(\frac{S(u_\varepsilon)}{v_\varepsilon+\varepsilon}\nabla v_\varepsilon\right) \\ &= -\int_\Omega \phi''(u_\varepsilon)D(u_\varepsilon)|\nabla u_\varepsilon|^2 + \int_\Omega \phi''(u_\varepsilon)\frac{S(u_\varepsilon)}{v_\varepsilon+\varepsilon}\nabla u_\varepsilon\cdot\nabla v_\varepsilon \\ &= -\int_\Omega (u_\varepsilon+1)^{m+p-3}|\nabla u_\varepsilon|^2 \\ &\quad + \int_\Omega (u_\varepsilon+1)^{m+p-3}\frac{S(u_\varepsilon)}{D(u_\varepsilon)}\frac{1}{v_\varepsilon+\varepsilon}\nabla u_\varepsilon\cdot\nabla v_\varepsilon. \end{aligned}$$

Now in virtue of Lemma 9.3 we have the following independent-in-ε bound:

$$\frac{1}{v_\varepsilon+\varepsilon} \leq \frac{1}{\delta}, \tag{9.8}$$

and we are in the same position as [**96**, (3.10)]. The rest of this proof is the same procedure as in the proofs of [**96**, Lemma 3.3] and [**39**, Proposition 3.2]. □

Lemma 9.5. *Let $\varepsilon > 0$ and $T > 0$. Suppose that (u_0, v_0) fulfils* (9.2)*. Assume that D and S satisfy* (9.3)*,* (9.4)*,* (9.5) *and* (9.6)*. Let $(u_\varepsilon, v_\varepsilon)$ be a solution of* (9.7) *on $[0, T)$. Then there exist $C_\infty > 0$ and $C'_\infty > 0$ such that*

$$\|u_\varepsilon(t)\|_{L^\infty(\Omega)} \le C_\infty \quad \text{for all } t \in (0, T), \tag{9.9}$$

$$\|\nabla v_\varepsilon(t)\|_{L^\infty(\Omega)} \le C'_\infty \quad \text{for all } t \in (0, T), \tag{9.10}$$

where C_∞ and C'_∞ do not depend on ε and T.

Proof. In light of (9.8), we can proceed as in [**96**, Lemma A.1] and so Lemma 9.4 implies (9.9). As to (9.10), using the representation formula for v_ε and standard smoothing estimates, we see that

$$\begin{aligned}
\|\nabla v_\varepsilon(t)\|_{L^\infty(\Omega)} &\le \|\nabla e^{t(\Delta-1)} v_0\|_{L^\infty(\Omega)} + \int_0^t \|\nabla e^{(t-s)(\Delta-1)} u_\varepsilon(s)\|_{L^\infty(\Omega)}\, ds \\
&\le c\left(\|v_0\|_{L^\infty(\Omega)} + \int_0^t (t-s)^{-\frac{1}{2}-\frac{n}{2}\cdot\frac{1}{\theta}} e^{-\eta(t-s)} \|u_\varepsilon(s)\|_{L^\theta(\Omega)}\, ds \right)
\end{aligned}$$

with constants $c > 0$, $\eta > 0$ and $\theta > 1$. Now we can choose $\theta > 1$ large enough satisfying

$$\frac{1}{2} + \frac{n}{2} \cdot \frac{1}{\theta} < 1$$

and (9.9) ensures boundedness of the right-hand side of the above inequality which leads to the conclusion. □

Lemma 9.6. *Let $\varepsilon > 0$ and $T > 0$. Suppose that (u_0, v_0) fulfils* (9.2)*. Assume that D and S satisfy* (9.3)*,* (9.4)*,* (9.5) *and* (9.6)*. Let $(u_\varepsilon, v_\varepsilon)$ be a solution of* (9.7) *on $[0, T)$. Then there exists $C''_\infty > 0$ such that*

$$\|\nabla u_\varepsilon(t)\|_{L^\infty(\Omega)} \le C''_\infty \quad \text{for all } t \in (0, T), \tag{9.11}$$

where C''_∞ does not depend on ε and T.

Proof. We can calculate the first equation in (9.7) as

$$\begin{aligned}
\frac{\partial u_\varepsilon}{\partial t} &= \nabla \cdot (D(u_\varepsilon)\nabla u_\varepsilon) - \nabla\left(\frac{S(u_\varepsilon)}{v_\varepsilon + \varepsilon}\right) \cdot \nabla v_\varepsilon - \frac{S(u_\varepsilon)}{v_\varepsilon + \varepsilon}\Delta v_\varepsilon \\
&= \nabla \cdot (D(u_\varepsilon)\nabla u_\varepsilon) + \frac{S(u_\varepsilon)}{(v_\varepsilon + \varepsilon)^2}|\nabla v_\varepsilon|^2 - \frac{S'(u_\varepsilon)}{v_\varepsilon + \varepsilon}\nabla u_\varepsilon \cdot \nabla v_\varepsilon - \frac{S(u_\varepsilon)}{v_\varepsilon + \varepsilon}\Delta v_\varepsilon.
\end{aligned}$$

From (9.8) we have the following upper estimates:

$$\left|\frac{S(u_\varepsilon)}{(v_\varepsilon+\varepsilon)^2}|\nabla v_\varepsilon|^2\right| \le \frac{S(u_\varepsilon)}{\delta^2}|\nabla v_\varepsilon|^2,$$

$$\left|\frac{S'(u_\varepsilon)}{v_\varepsilon+\varepsilon}\nabla u_\varepsilon\cdot\nabla v_\varepsilon\right| \le \frac{|S'(u_\varepsilon)|}{\delta}|\nabla u_\varepsilon||\nabla v_\varepsilon|$$

and

$$\left|\frac{S(u_\varepsilon)}{v_\varepsilon+\varepsilon}\Delta v_\varepsilon\right| \le \frac{S(u_\varepsilon)}{\delta}|\Delta v_\varepsilon|.$$

By noting that $u_0 \in C^2(\overline{\Omega})$, these estimates allow us to apply standard parabolic theory (Lemma 2.9) and to complete the proof. □

9.3. Proof of Theorem 9.1

We start by showing that $\{u_\varepsilon\}$ and $\{v_\varepsilon\}$ satisfy the Cauchy condition.

Lemma 9.7. *Let $\varepsilon > 0$ and $T > 0$. Suppose that (u_0, v_0) fulfils (9.2). Assume that D and S satisfy (9.3), (9.4), (9.5) and (9.6). Let $(u_\varepsilon, v_\varepsilon)$ be a solution of (9.7) on $[0, T)$. Then there exist $c_1 > 0$, $c_2 > 0$, $c_3 > 0$, $c_4 > 0$ and $c_5 > 0$ such that for all $\mu > 0$, $\nu > 0$ and $t \in [0, T]$,*

$$\begin{aligned}(9.12)\quad & \|u_\mu(t)-u_\nu(t)\|^2_{L^2(\Omega)} + c_1\|v_\mu(t)-v_\nu(t)\|^2_{L^2(\Omega)} \\ &+ c_2\int_0^t \|\nabla(u_\mu(s)-u_\nu(s))\|^2_{L^2(\Omega)}\,ds + c_3\int_0^t \|\nabla(v_\mu(s)-v_\nu(s))\|^2_{L^2(\Omega)}\,ds \\ &\le c_4|\mu-\nu|^2 e^{c_5 T}.\end{aligned}$$

Proof. Let $\mu > 0$ and $\nu > 0$. Multiplying the difference of the first equations in (9.7) by $(u_\mu - u_\nu)$, we see that

$$\begin{aligned}(9.13)\quad \frac{1}{2}\frac{d}{dt}\|u_\mu-u_\nu\|^2_{L^2(\Omega)} &= \int_\Omega \nabla\cdot(D(u_\mu)\nabla u_\mu - D(u_\nu)\nabla u_\nu)(u_\mu-u_\nu) \\ &\quad - \int_\Omega \nabla\cdot\left(\frac{S(u_\mu)}{v_\mu+\mu}\nabla v_\mu - \frac{S(u_\nu)}{v_\nu+\nu}\nabla v_\nu\right)(u_\mu-u_\nu) \\ &= -\int_\Omega (D(u_\mu)\nabla u_\mu - D(u_\nu)\nabla u_\nu)\cdot\nabla(u_\mu-u_\nu) \\ &\quad + \int_\Omega \left(\frac{S(u_\mu)}{v_\mu+\mu}\nabla v_\mu - \frac{S(u_\nu)}{v_\nu+\nu}\nabla v_\nu\right)\cdot\nabla(u_\mu-u_\nu) \\ &=: I_1 + I_2.\end{aligned}$$

As to the first term I_1, it follows from (9.3), (9.4), (9.9) and (9.11) that

$$\begin{aligned}
I_1 &= -\int_\Omega (D(u_\mu)\nabla u_\mu - D(u_\nu)\nabla u_\nu)\cdot\nabla(u_\mu - u_\nu)\\
&= -\int_\Omega D(u_\mu)|\nabla(u_\mu - u_\nu)|^2 - \int_\Omega (D(u_\mu) - D(u_\nu))\nabla u_\nu\cdot\nabla(u_\mu - u_\nu)\\
&\le -\tilde{K}_0\int_\Omega |\nabla(u_\mu - u_\nu)|^2 + C_{\max}\int_\Omega |\nabla u_\nu||u_\mu - u_\nu||\nabla(u_\mu - u_\nu)|\\
&\le -\tilde{K}_0\|\nabla(u_\mu - u_\nu)\|^2_{L^2(\Omega)} + C_{\max}C''_\infty\int_\Omega |u_\mu - u_\nu||\nabla(u_\mu - u_\nu)|,
\end{aligned}$$

where $\tilde{K}_0 := K_0\min\{1, (C_\infty + 1)^{m-1}\}$ and $C_{\max} := \max_{\sigma\in[0,C_\infty]} D'(\sigma)$. In light of Young's inequality we deduce that

$$\begin{aligned}
I_1 &\le -\tilde{K}_0\|\nabla(u_\mu - u_\nu)\|^2_{L^2(\Omega)}\\
&\quad + \frac{5C^2_{\max}C''^2_\infty}{2\tilde{K}_0}\|u_\mu - u_\nu\|^2_{L^2(\Omega)} + \frac{\tilde{K}_0}{10}\|\nabla(u_\mu - u_\nu)\|^2_{L^2(\Omega)}.
\end{aligned}\tag{9.14}$$

As to the second term I_2 in (9.13), we write it as follows:

$$\begin{aligned}
I_2 &= \int_\Omega \left(\frac{S(u_\mu)}{v_\mu + \mu}\nabla v_\mu - \frac{S(u_\nu)}{v_\nu + \nu}\nabla v_\nu\right)\cdot\nabla(u_\mu - u_\nu)\\
&= \int_\Omega \frac{S(u_\mu) - S(u_\nu)}{v_\mu + \mu}\nabla v_\mu\cdot\nabla(u_\mu - u_\nu)\\
&\quad + \int_\Omega S(u_\nu)\left(\frac{1}{v_\mu + \mu}\nabla v_\mu - \frac{1}{v_\nu + \nu}\nabla v_\nu\right)\cdot\nabla(u_\mu - u_\nu).
\end{aligned}$$

Then (9.3) and (9.9) entail that

$$\begin{aligned}
I_2 &\le \tilde{C}_{\max}\int_\Omega |u_\mu - u_\nu|\frac{1}{v_\mu + \mu}|\nabla v_\mu||\nabla(u_\mu - u_\nu)|\\
&\quad + \hat{C}_{\max}\int_\Omega |H(v_\mu, v_\nu, \mu, \nu)||\nabla(u_\mu - u_\nu)|,
\end{aligned}$$

where $\tilde{C}_{\max} := \max_{\sigma\in[0,C_\infty]} S'(\sigma)$, $\hat{C}_{\max} := \max_{\sigma\in[0,C_\infty]} S(\sigma)$ and

$$H(v_\mu, v_\nu, \mu, \nu) := \left(\frac{1}{v_\mu + \mu} - \frac{1}{v_\nu + \nu}\right)\nabla v_\mu + \frac{1}{v_\nu + \nu}\nabla(v_\mu - v_\nu).$$

From Lemma 9.3 we find that

$$\begin{aligned}
|H(v_\mu, v_\nu, \mu, \nu)| &\le \frac{1}{(v_\mu + \mu)(v_\nu + \nu)}|v_\nu - v_\mu||\nabla v_\mu|\\
&\quad + \frac{1}{(v_\mu + \mu)(v_\nu + \nu)}|\nu - \mu||\nabla v_\mu| + \frac{1}{v_\nu + \nu}|\nabla(v_\mu - v_\nu)|\\
&\le \frac{1}{\delta^2}|v_\nu - v_\mu||\nabla v_\mu| + \frac{1}{\delta^2}|\nu - \mu||\nabla v_\mu| + \frac{1}{\delta}|\nabla(v_\mu - v_\nu)|.
\end{aligned}$$

Thus applying (9.10) and Lemma 9.3, we infer

$$
\begin{aligned}
I_2 \leq & \frac{\tilde{C}_{\max} C'_\infty}{\delta} \int_\Omega |u_\mu - u_\nu| |\nabla(u_\mu - u_\nu)| \\
& + \frac{\hat{C}_{\max} C'_\infty}{\delta^2} \int_\Omega |v_\nu - v_\mu| |\nabla(u_\nu - u_\mu)| \\
& + \frac{\hat{C}_{\max} C'_\infty}{\delta^2} \int_\Omega |\nu - \mu| |\nabla(u_\nu - u_\mu)| \\
& + \frac{\hat{C}_{\max}}{\delta} \int_\Omega |\nabla(v_\mu - v_\nu)| |\nabla(u_\mu - u_\nu)|.
\end{aligned}
$$

Hence Young's inequality says that

$$
\begin{aligned}
I_2 \leq & \frac{5\tilde{C}^2_{\max} C'^2_\infty}{2\delta^2 \tilde{K}_0} \|u_\mu - u_\nu\|^2_{L^2(\Omega)} + \frac{\tilde{K}_0}{10} \|\nabla(u_\mu - u_\nu)\|^2_{L^2(\Omega)} \\
& + \frac{5\hat{C}^2_{\max} C'^2_\infty}{2\delta^4 \tilde{K}_0} \|v_\mu - v_\nu\|^2_{L^2(\Omega)} + \frac{\tilde{K}_0}{10} \|\nabla(u_\mu - u_\nu)\|^2_{L^2(\Omega)} \\
& + |\Omega| \frac{5\hat{C}^2_{\max} C'^2_\infty}{2\delta^4 \tilde{K}_0} |\nu - \mu|^2 + \frac{\tilde{K}_0}{10} \|\nabla(u_\mu - u_\nu)\|^2_{L^2(\Omega)} \\
& + \frac{5\hat{C}^2_{\max}}{2\delta^2 \tilde{K}_0} \|\nabla(v_\mu - v_\nu)\|^2_{L^2(\Omega)} + \frac{\tilde{K}_0}{10} \|\nabla(u_\mu - u_\nu)\|^2_{L^2(\Omega)}.
\end{aligned} \tag{9.15}
$$

Consequently, combining (9.13) with (9.14) and (9.15), we see that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt} \|u_\mu - u_\nu\|^2_{L^2(\Omega)} + & \frac{\tilde{K}_0}{2} \|\nabla(u_\mu - u_\nu)\|^2_{L^2(\Omega)} \\
& \leq C_1 \|u_\mu - u_\nu\|^2_{L^2(\Omega)} + C_2 \|v_\nu - v_\mu\|^2_{L^2(\Omega)} \\
& \quad + C_3 |\nu - \mu|^2 + C_4 \|\nabla(v_\mu - v_\nu)\|^2_{L^2(\Omega)},
\end{aligned} \tag{9.16}
$$

where C_1, C_2, C_3 and C_4 are given by

$$
\begin{aligned}
C_1 &:= \frac{5C^2_{\max} C''^2_\infty}{2\tilde{K}_0} + \frac{5\tilde{C}^2_{\max} C'^2_\infty}{2\delta^2 \tilde{K}_0}, \quad & C_2 &:= \frac{5\hat{C}^2_{\max} C'^2_\infty}{2\delta^4 \tilde{K}_0}, \\
C_3 &:= |\Omega| \frac{5\hat{C}^2_{\max} C'^2_\infty}{2\delta^4 \tilde{K}_0}, & C_4 &:= \frac{5\hat{C}^2_{\max}}{2\delta^2 \tilde{K}_0}.
\end{aligned}
$$

Similarly, Young's inequality yields

$$
\frac{1}{2}\frac{d}{dt} \|v_\mu - v_\nu\|^2_{L^2(\Omega)} \leq -\|\nabla(v_\mu - v_\nu)\|^2_{L^2(\Omega)} + \frac{1}{4} \|u_\mu - u_\nu\|^2_{L^2(\Omega)}. \tag{9.17}
$$

Multiplying (9.17) by $2C_4$ and adding (9.16), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|u_\mu - u_\nu\|_{L^2(\Omega)}^2 + 2C_4\|v_\mu - v_\nu\|_{L^2(\Omega)}^2) \\
&\qquad + \frac{\tilde{K}_0}{2}\|\nabla(u_\mu - u_\nu)\|_{L^2(\Omega)}^2 + C_4\|\nabla(v_\mu - v_\nu)\|_{L^2(\Omega)}^2 \\
&\le \left(C_1 + \frac{C_4}{2}\right)\|u_\mu - u_\nu\|_{L^2(\Omega)}^2 + C_2\|v_\mu - v_\nu\|_{L^2(\Omega)}^2 + C_3|\mu - \nu|^2 \\
&\le C_5(\|u_\mu - u_\nu\|_{L^2(\Omega)}^2 + 2C_4\|v_\mu - v_\nu\|_{L^2(\Omega)}^2) + C_3|\mu - \nu|^2,
\end{aligned}$$

where $C_5 := \max\{C_1 + \frac{C_4}{2}, \frac{C_\infty'^2}{2\delta^2}\}$, and thus Gronwall's lemma yields

$$\begin{aligned}
&\|u_\mu(t) - u_\nu(t)\|_{L^2(\Omega)}^2 + 2C_4\|v_\mu(t) - v_\nu(t)\|_{L^2(\Omega)}^2 \\
&\quad + \int_0^t e^{2C_5(t-s)}\Big(\tilde{K}_0\|\nabla(u_\mu(s) - u_\nu(s))\|_{L^2(\Omega)}^2 + 2C_4\|\nabla(v_\mu(s) - v_\nu(s))\|_{L^2(\Omega)}^2\Big)\,ds \\
&\le \frac{C_3}{C_5}|\mu - \nu|^2 e^{2C_5T}
\end{aligned}$$

for all $t \in [0,T]$. Since $e^{2C_5(t-s)} \ge 1\,(s \in [0,t])$, we obtain the desired inequality. □

We are now in a position to prove Theorem 9.1.

Proof of Theorem 9.1. We have $T_{\max} = \infty$ from Lemma 9.5. For all $T > 0$, in view of Lemma 9.7 we find u and v from $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ such that

$$\begin{aligned}
u_\varepsilon &\to u && \text{in } L^\infty(0,T;L^2(\Omega)) && \text{as } \varepsilon \to 0, \\
v_\varepsilon &\to v && \text{in } L^\infty(0,T;L^2(\Omega)) && \text{as } \varepsilon \to 0, \\
\nabla u_\varepsilon &\to \nabla u && \text{in } L^2(0,T;L^2(\Omega)) && \text{as } \varepsilon \to 0, \qquad (9.18)\\
\nabla v_\varepsilon &\to \nabla v && \text{in } L^2(0,T;L^2(\Omega)) && \text{as } \varepsilon \to 0. \qquad (9.19)
\end{aligned}$$

We will prove that (u,v) is a classical solution of (9.1) and bounded. The proof is divided into two steps.

(Step 1) In this step we prove that (u,v) is a weak solution of (9.1). Let $\varphi \in C_c^\infty(\Omega \times [0,\infty))$. We can fix $T > 0$ such that $\operatorname{supp}\varphi \subset \Omega \times [0,T)$. Multiplying the first equation in (9.7) by φ and integrating it over $\Omega \times (0,T)$, we can see

$$\begin{aligned}
-\int_0^T\int_\Omega u_\varepsilon\frac{d\varphi}{dt} &= -\int_0^T\int_\Omega D(u_\varepsilon)\nabla u_\varepsilon \cdot \nabla\varphi \\
&\quad + \int_0^T\int_\Omega \frac{S(u_\varepsilon)}{v_\varepsilon + \varepsilon}\nabla v_\varepsilon \cdot \nabla\varphi + \int_\Omega u_0\varphi(\cdot,0).
\end{aligned} \qquad (9.20)$$

To accomplish the passage to the limit of approximate solutions we will confirm convergence of each term. Firstly from the convergence $u_\varepsilon \to u$ in $L^2(0,T;L^2(\Omega))$ as $\varepsilon \to 0$ due to (9.12), we easily check

$$-\int_0^T \int_\Omega u_\varepsilon \frac{d\varphi}{dt} \to -\int_0^T \int_\Omega u \frac{d\varphi}{dt} \quad \text{as } \varepsilon \to 0.$$

Next we consider convergence of the first term on the right-hand side of (9.20). We observe that

$$|D(u_\varepsilon)\nabla\varphi| \le \max_{\sigma\in[0,C_\infty]} D(\sigma)\cdot|\nabla\varphi| \in L^2(0,T;L^2(\Omega))$$

due to (9.3) and $D(u_\varepsilon) \to D(u)$ pointwisely as $\varepsilon \to 0$. Thus it follows that

$$D(u_\varepsilon)\nabla\varphi \to D(u)\nabla\varphi \quad \text{in } L^2(0,T;L^2(\Omega)) \quad \text{as } \varepsilon \to 0. \tag{9.21}$$

Therefore invoking (9.18) and (9.21), we can show the following convergence:

$$-\int_0^T \int_\Omega D(u_\varepsilon)\nabla u_\varepsilon \cdot \nabla\varphi \to -\int_0^T \int_\Omega D(u)\nabla u \cdot \nabla\varphi \quad \text{as } \varepsilon \to 0.$$

As to the second term, (9.3) and Lemma 9.3 yield

$$\left|\frac{S(u_\varepsilon)}{v_\varepsilon+\varepsilon}\nabla\varphi\right| \le \frac{\hat{C}_{\max}}{\delta}|\nabla\varphi| \in L^2(0,T;L^2(\Omega))$$

and $\frac{S(u_\varepsilon)}{v_\varepsilon+\varepsilon} \to \frac{S(u)}{v}$ pointwisely as $\varepsilon \to 0$, and hence we can establish

$$\frac{S(u_\varepsilon)}{v_\varepsilon+\varepsilon}\nabla\varphi \to \frac{S(u)}{v}\nabla\varphi \quad \text{in } L^2(0,T;L^2(\Omega)) \quad \text{as } \varepsilon \to 0.$$

In the same fashion as before (9.19) implies

$$\int_0^T \int_\Omega \frac{S(u_\varepsilon)}{v_\varepsilon+\varepsilon}\nabla v_\varepsilon \cdot \nabla\varphi \to \int_0^T \int_\Omega \frac{S(u)}{v}\nabla v \cdot \nabla\varphi \quad \text{as } \varepsilon \to 0.$$

Therefore we can accomplish the passage of the limit and hence

$$-\int_0^T \int_\Omega u\frac{d\varphi}{dt} = -\int_0^T \int_\Omega D(u)\nabla u \cdot \nabla\varphi + \int_0^T \int_\Omega \frac{S(u)}{v}\nabla v \cdot \nabla\varphi + \int_\Omega u_0\varphi(\cdot,0).$$

As to the second equation in (9.1), we can similarly deduce the following identity:

$$-\int_0^T \int_\Omega v\frac{d\varphi}{dt} = -\int_0^T \int_\Omega \nabla v \cdot \nabla\varphi - \int_0^T \int_\Omega v\varphi + \int_\Omega u\varphi + \int_\Omega v_0\varphi(\cdot,0).$$

Thus we conclude that (u, v) is a weak solution of (9.1).

(Step 2) Invoking standard semigroup techniques and parabolic Schauder estimates (Lemmas 2.8 and 2.10), we deduce from straightforward regularity arguments that (u, v) is a global classical solution of (9.1). Consequently, we have a globally bounded classical solution (u, v) of (9.1) such that u belongs to $L^\infty(\overline{\Omega} \times [0, \infty))$ in light of boundedness of $\{u_\varepsilon\}_{\varepsilon>0}$ in $L^\infty(\overline{\Omega} \times [0, \infty))$ (Lemma 9.5). $\square$

PART III:

CHEMOTAXIS SYSTEM FOR TUMOR INVASION

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), \\ v_t = \Delta v + wz, \\ w_t = -wz, \\ z_t = \Delta z - z + u. \end{cases}$$

Chapter 10

Local existence and uniqueness in a chemotaxis model for tumor invasion

10.1. Purposes

The main purposes of this chapter are to propose a new modified tumor invasion model of Chaplain–Anderson type in [**13**] and to establish a new result on existence and uniqueness of local-in-time classical solutions to the simplified model. In particular, we propose a tumor invasion model in which the role of an active extracellular matrix is taken into consideration. Actually, in the new tumor invasion model the active extracellular matrices are produced by the biochemical reaction between the extracellular matrices and the matrix degrading enzymes and play a role as an attractant of tumor cells. Moreover, as pointed out in [**43**] and [**92**] we also take the effects of some functional proteins on the motility and haptotaxis of tumor cells into consideration. Mathematical results on existence of solutions to some related models have been obtained by [**49, 50, 51, 52**], where the equation is modified by adding the subdifferential of the indicator function, that is, a constraint condition is added. However, they did not succeed in showing uniqueness of solutions, which was left as an open problem. The main result of this chapter says that existence and uniqueness of solutions hold true when we add one equation modeling the effect of an active extracellular matrix.

The plan of this chapter is as follows. In Section 10.2 we give a mathematical control method of tumor invasion phenomenon and propose a modified tumor invasion model. Section 10.3 is devoted to showing existence and uniqueness of local classical solutions to a simplified model.

10.2. Mathematical control method

In this section we give one of ideas to control the tumor invasion phenomenon from a mathematical point of view. In Subsections 10.2.1 and 10.2.2, we give one of the ideas to control HSPs by using the temperature. Subsections 10.2.3 and 10.2.4 are devoted to proposing a new model of the tumor invasion phenomenon. Subsection 10.2.5 is the goal of this section.

10.2.1. Temperature as the control parameter

As pointed out in [**92**], a certain HSP (heat shock protein) has an influence on the random motility and the haptotaxis of tumor cells. Such HSPs are synthesized a lot in order to overcome the stress brought about the changes of the external environment, for example, temperature, pressure, ultraviolet rays and metal ions, as soon as possible.

Hence we can consider the temperature as one of the control parameters which have an influence on the behaviors of HSPs. As the typical example which describes the kinetics of the temperature θ, we give the following system (θ):

$$(\theta) \qquad \begin{cases} \theta_t = D_\theta \Delta\theta + h_1 & \text{a.e. in } Q_T = \Omega \times (0,T), \\ \dfrac{\partial\theta}{\partial\nu} + n_0\theta = h_2 & \text{a.e. on } \Sigma_T = \partial\Omega \times (0,T), \\ \theta(0) = \theta_0 & \text{a.e. in } \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^3$ containing tumor with a smooth boundary Γ; h_1 and h_2 are the prescribed internal and boundary heat sources, respectively; $D_\theta > 0$ and $n_0 > 0$ are constants. In this model we can control the temperature θ by the heat sources h_1 and h_2.

10.2.2. Kinetics of HSPs under heat stress

Since the temperature is one of the external environment surrounding tumor, the heat stress has an influence on the behaviors of HSPs. Here the heat stress means the sudden change of the temperature.

For example, in [**44**] Ito et al. considered a signal cascade, shown in Figure 1 (see the next page), which describes the HSP synthesis process. This synthesis process starts as soon as the heat stress is given.

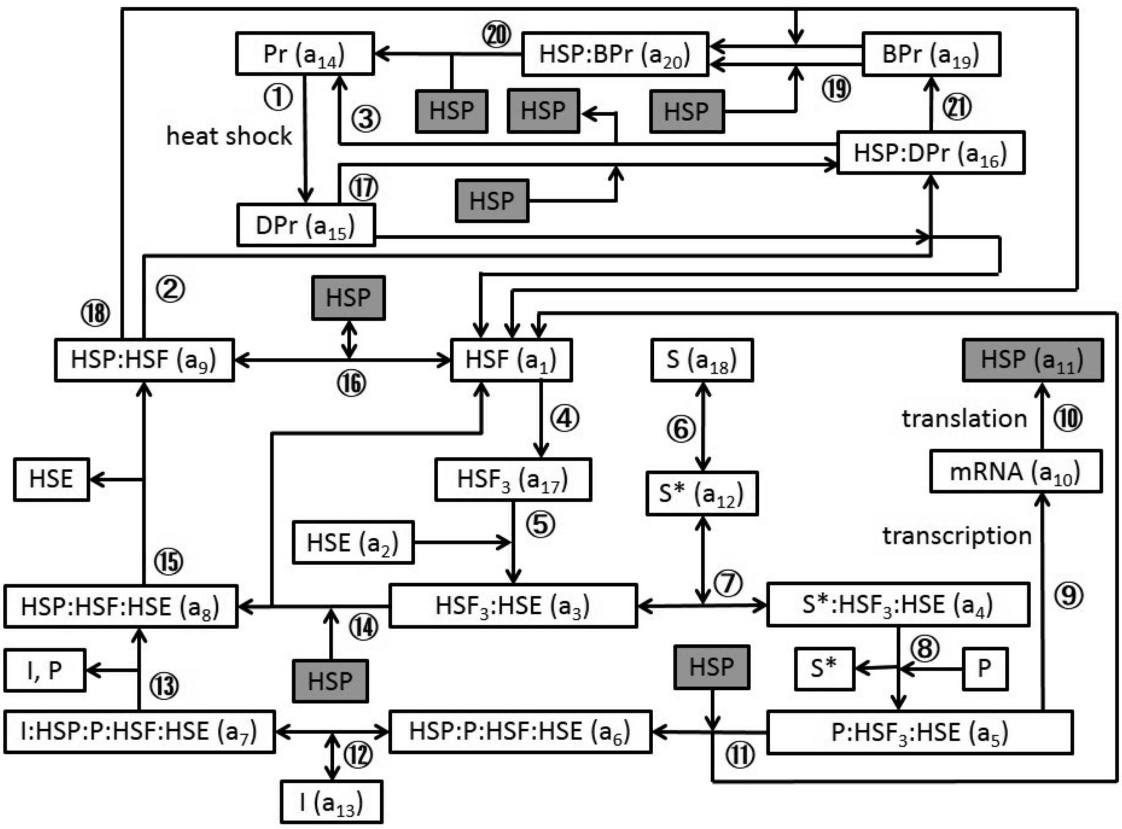

Figure 10.1: Signal cascade of the HSP synthesis process under heat shock

Using the approach of systems biology, they proposed the following mathematical model (HSP):={(10.1)-(10.20)}:

(10.1) $$a_1' = (k_2 a_{15} + k_{16b} + k_{18} a_{19}) a_9 + 2(k_{11} a_5 + k_{14} a_3) a_{11} - 3k_4 a_1^3 - k_{16} a_1 a_{11},$$

(10.2) $$a_2' = k_{15} a_8 - k_5 a_2 a_{17},$$

(10.3) $$a_3' = k_5 a_2 a_{17} + k_{7b} a_4 - (k_7 a_{12} + k_{14} a_{11}) a_3,$$

(10.4) $$a_4' = k_7 a_3 a_{12} - (k_{7b} + k_8) a_4,$$

(10.5) $$a_5' = k_8 a_4 - k_{11} a_5 a_{11},$$

(10.6) $$a_6' = k_{11} a_5 a_{11} + k_{12b} a_7 - k_{12} a_6 a_{13},$$

(10.7) $$a_7' = k_{12} a_6 a_{13} - (k_{12b} + k_{13}) a_7,$$

(10.8) $$a_8' = k_{13} a_7 + k_{14} a_3 a_{11} - k_{15} a_8,$$

(10.9) $$a_9' = k_{15} a_8 + k_{16} a_1 a_{11} - (k_2 a_{15} + k_{16b} + k_{18} a_{19}) a_9,$$

$$a'_{10} = k_9 a_5 - \frac{k_{mRNA}}{k_{mRNA} + a_{12}} a_{10}, \tag{10.10}$$

$$a'_{11} = k_3 a_{16} + k_{10} a_{10} + k_{16b} a_9 + k_{20} a_{20} - (k_{11} a_5 + k_{14} a_3 + k_{16} a_1 + k_{17} a_{15} + k_{19} a_{19} + k_{HSP}) a_{11}, \tag{10.11}$$

$$a'_{12} = \frac{k_{6MAX} a_{18}}{k_6 + a_{18}} + (k_{7b} + k_8) a_4 - \frac{k_{6bMAX} a_{12}}{k_{6b} + a_{12}} - k_7 a_3 a_{12}, \tag{10.12}$$

$$a'_{13} = (k_{12b} + k_{13}) a_7 - k_{12} a_6 a_{13}, \tag{10.13}$$

$$a'_{14} = k_3 a_{16} + k_{20} a_{20} - k_1(\theta) a_{14}, \tag{10.14}$$

$$a'_{15} = k_1(\theta) a_{14} - (k_2 a_9 + k_{17} a_{11}) a_{15}, \tag{10.15}$$

$$a'_{16} = (k_2 a_9 + k_{17} a_{11}) a_{15} - (k_3 + k_{21}) a_{16}, \tag{10.16}$$

$$a'_{17} = k_4 a_1^3 - k_5 a_2 a_{17}, \tag{10.17}$$

$$a'_{18} = \frac{k_{6bMAX} a_{12}}{k_{6b} + a_{12}} - \frac{k_{6MAX} a_{18}}{k_6 + a_{18}}, \tag{10.18}$$

$$a'_{19} = k_{21} a_{16} - (k_{18} a_9 + k_{19} a_{11}) a_{19}, \tag{10.19}$$

$$a'_{20} = (k_{18} a_9 + k_{19} a_{11}) a_{19} - k_{20} a_{20}. \tag{10.20}$$

As you see from (10.14) and (10.15) in (HSP), HSPs are synthesized a lot in order to refold denatured proteins and overcome the damage given by the heat stress as soon as possible. As a result, from (θ) and (HSP), the quantities of HSPs must be controlled by the heat stress.

Recently, in [**56**] Komatsu et al. showed existence and uniqueness of nonnegative global-in-time solutions to the initial value problem for (HSP) whenever all initial values $a_i(0)$, $1 \le i \le 20$, are nonnegative.

10.2.3. Biochemical reactions between ECM and MDE

In this subsection we explain the biochemical reaction between ECM (extracellular matrix) and MDE (matrix degrading enzyme) in detail by using the following reactions:

$$\mathrm{ECM} + \mathrm{MDE} \ \rightleftharpoons \ \mathrm{ECM{:}MDE} \ \rightharpoonup \ \mathrm{C}_1 + \mathrm{C}_2 + \cdots + \mathrm{C}_k + \mathrm{MDE}. \tag{10.21}$$

In (10.21) we assume that the following properties are satisfied:

(H1) ECM combines with MDE and the enzyme-substrate complex ECM:MDE is formed.

(H2) ECM:MDE generates the substances C_j, $j = 1, 2, \ldots, k$, and releases MDE. Otherwise, it is resolved into ECM and MDE again.

Now we denote by c and c_j the densities of ECM:MDE and C_j, respectively. By employing the approach of systems biology, we derive the following system (EM):

$$\text{(EM)} \qquad \begin{cases} f_t = -v_1^+ mf + v_1^- c, \\ c_t = v_1^+ mf - v_1^- c - v_2^+ c, \\ (c_j)_t = v_2^+ c, \quad j = 1, 2, \ldots, k, \\ m_t = -v_1^+ mf + v_1^- c + v_2^+ c, \end{cases}$$

where $v_1^\pm$ and v_2^+ are velocities of the following biochemical reactions:

$$\begin{aligned} v_1^+ &: \mathrm{ECM} + \mathrm{MDE} \quad \rightharpoonup \quad \mathrm{ECM{:}MDE}, \\ v_1^- &: \mathrm{ECM} + \mathrm{MDE} \quad \leftharpoondown \quad \mathrm{ECM{:}MDE}, \\ v_2^+ &: \mathrm{ECM{:}MDE} \quad \rightharpoonup \quad \mathrm{C}_1 + \mathrm{C}_2 + \cdots + \mathrm{C}_k + \mathrm{MDE}. \end{aligned}$$

By using the Michaelis–Menten kinetics, i.e., the density of complex ECM:MDE always stays at the dynamic equilibrium state, we have $c_t(t) = 0$ for all $t > 0$. Then we can rewrite (EM) as the following system {(10.22), (10.23)}, which is denoted by the same notation (EM):

$$f_t = -\alpha mf, \tag{10.22}$$

$$(c_j)_t = \alpha mf, \quad j = 1, 2, \ldots, k, \tag{10.23}$$

where $\alpha > 0$ is a constant given by

$$\alpha = \frac{v_1^+}{v_1^- + v_2^+}.$$

Moreover we assume that all substances C_j, $j = 1, 2, \ldots, k$, are so small that they can diffuse uniformly in the space. Then we can derive the kinetic equations below instead of (10.23):

$$(c_j)_t = D_j \Delta c_j + \alpha mf, \quad j = 1, 2, \ldots, k,$$

where each $D_j > 0$ is a diffusion constant of the substance C_j.

At last we derive the following new system again denoted by (EM), which comes from the biochemical reaction between ECM and MDE with (1.6):

$$\text{(EM)} \qquad \begin{cases} f_t = -\alpha mf, \\ (c_j)_t = D_j \Delta c_j + \alpha mf, \quad j = 1, 2, \ldots, k, \\ m_t = D_m \Delta m + P(n, m) - G(n, f, m). \end{cases}$$

It has to be noted that we can derive another system, which may be more complicated, when the Michaelis–Menten kinetics is not used.

10.2.4. Kinetics of tumor cells

In this subsection we propose the modified kinetic equation of tumor cells. For this purpose we assume that the following hypotheses are satisfied:

(H3) ECM does not have any influences on the haptotaxis of tumor cells.

(H4) One substance among C_j, $j = 1, 2, \ldots, k$, has an influence on the haptotaxis of tumor cells, which is denoted by ECM* and whose density is f^* throughout this chapter. We call ECM* an active ECM. Roughly speaking, the active ECM is an attractant for tumor cells.

(H5) A certain family of HSPs has an influence on the coefficients of the random motility and haptotaxis of tumor cells. We denote the densities of such HSPs by a vector $\boldsymbol{p} = (p_1, p_2, \ldots, p_\ell)$ for some ℓ.

Under the above hypotheses we derive the following kinetic equation for tumor cells:

$$n_t = \nabla \cdot (D_n(n, f, \boldsymbol{p})\nabla n) - \nabla \cdot (\lambda(\boldsymbol{p}) n \nabla f^*),$$

which is the modified first equation of (1.6) taking (H3)–(H5) into account.

On the other hand, in [**45**] Ito et al. obtained the experimental data of the proliferation curves of HepG2, which is one of the human hepatocellular carcinoma. Their experimental data imply that the heat stress as well as the temperature have a huge influence on the proliferation and apoptosis of HepG2. Moreover, they proposed the following modified Verhulst model and estimated the nonlinear functions α, $n_{\max}$ and β by using the extended Kalman filter:

$$n' = \alpha(\theta) n \left(n - \frac{n}{n_{\max}(\theta)} \right) - \beta(\theta) n.$$

So we assume that the following hypothesis is satisfied as well as (H3)–(H5):

(H6) The proliferation and apoptosis of tumor cells depend on the temperature. We denote by F_p and F_a the proliferation and the apoptosis rates, respectively, which are nonnegative functions.

Finally, we derive the following kinetic equation for tumor cells:

$$n_t = \nabla \cdot (D_n(n, f, \boldsymbol{p})\nabla n) - \nabla \cdot (\lambda(\boldsymbol{p}) n \nabla f^*) + F_p(\theta, n, f) - F_a(\theta, n, f).$$

In this model we can dominate the behavior of tumor cells by controlling the quantities of HSPs in order to make $D_n(n, f, \boldsymbol{p})$ and $\lambda(\boldsymbol{p})$ smaller and smaller, respectively. Furthermore the quantities of HSPs are also controlled by the temperature stated in Subsection 10.2.2.

10.2.5. New model

We assume that the temperature θ is a prescribed function on $\overline{\Omega} \times [0, T]$, for example, which is decided by the system (θ) in Subsection 10.2.1. Then by using the system (HSP), we can derive the vector field $\boldsymbol{p}$ of HSPs on $\overline{\Omega} \times [0, T]$. Thus we propose the following tumor invasion model (TIM):

$$\text{(TIM)} \quad \begin{cases} n_t = \nabla \cdot (D_n(n, f, \boldsymbol{p})\nabla n) - \nabla \cdot (\lambda(\boldsymbol{p}) n \nabla f^*) + F_p(\theta, n, f) - F_a(\theta, n, f), \\ f_t = -\alpha m f, \\ f_t^* = D_* \Delta f^* + \alpha m f, \\ m_t = D_m \Delta m + P(n, f) - G(n, f, m). \end{cases}$$

This model gives one mathematical scenario to control the behavior of tumor cells by using the temperature effect. Moreover it is also one idea to control the haptotaxis of tumor cells by introducing an inhibitor of ECM*.

10.3. Mathematical result and proof

In this section we will establish existence and uniqueness of solutions to the new model proposed in Section 10.2.5.

Let Ω be a bounded domain in $\mathbb{R}^N$ ($N \in \mathbb{N}$) with smooth boundary $\partial\Omega$. We assume that the coefficients D_n, λ are constants for simplicity, moreover we do not consider the proliferation term F_p, and the apoptosis term F_a. We choose the production term P and the decay term G as typical functions n and m, respectively. We treat the constant parameters α, D_*, D_m as 1. Namely, we consider the case where

$$\begin{cases} D_n(n,f,\boldsymbol{p}) \equiv 1, \\ \lambda(\boldsymbol{p}) \equiv 1, \\ F_p(\theta,n,f) \equiv 0, \\ F_a(\theta,n,f) \equiv 0, \\ P(n,f) = n, \\ G(n,f,m) = m, \\ \alpha = D_* = D_m = 1 \end{cases}$$

in (TIM) proposed in Section 10.2.5.

Consequently, we deal with the problem (P):={(10.24)-(10.29)}:

$$n_t = \Delta n - \nabla \cdot (n \nabla f^*) \quad \text{in } \Omega \times (0,T), \tag{10.24}$$

$$f_t = -mf \quad \text{in } \Omega \times (0,T), \tag{10.25}$$

$$f^*_t = \Delta f^* + mf \quad \text{in } \Omega \times (0,T), \tag{10.26}$$

$$m_t = \Delta m + n - m \quad \text{in } \Omega \times (0,T), \tag{10.27}$$

$$\frac{\partial n}{\partial \nu} = \frac{\partial f^*}{\partial \nu} = \frac{\partial m}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{10.28}$$

$$n(\cdot,0) = n_0,\ f(\cdot,0) = f_0,\ f^*(\cdot,0) = f^*_0,\ m(\cdot,0) = m_0 \quad \text{in } \Omega. \tag{10.29}$$

We assume that the initial data n_0, f_0, f^*_0 and m_0 satisfy the following conditions:

(A1) $\quad n_0 \in C^0(\overline{\Omega}), \qquad n_0 \geq 0,$

(A2) $\quad f_0 \in C^1(\overline{\Omega}), \qquad f_0 \geq 0,$

(A3) $\quad f^*_0 \in W^{1,\infty}(\Omega), \qquad f^*_0 \geq 0,$

(A4) $\quad m_0 \in C^0(\overline{\Omega}), \qquad m_0 \geq 0.$

Now we are in a position to give the main theorem in this chapter.

Theorem 10.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \in \mathbb{N})$ with smooth boundary $\partial\Omega$. Assume that* (A1)-(A4) *are satisfied. Then there exists $T_{\max} \in (0, \infty]$ (depending only on $\|n_0\|_{L^\infty(\Omega)}$, $\|f_0^*\|_{W^{1,\infty}(\Omega)}$ and $\|m_0\|_{L^\infty(\Omega)}$) such that* (P) *possesses a unique classical solution (n, f, f^*, m) with the following properties:*

(P1) $\quad n \in C^{2,1}(\overline{\Omega} \times (0, T_{\max})) \cap C^0([0, T_{\max}); C^0(\overline{\Omega})), \qquad n \ge 0,$

(P2) $\quad f(t) = f_0 \exp\Big(- \int_0^t m(s)\, ds \Big), \qquad f \ge 0,$

(P3) $\quad f^* \in C^{2,1}(\overline{\Omega} \times (0, T_{\max})) \cap C^0([0, T_{\max}); C^0(\overline{\Omega})) \cap L^\infty_{\mathrm{loc}}([0, T_{\max}); W^{1,\infty}(\Omega)),$
$f^* \ge 0,$

(P4) $\quad m \in C^{2,1}(\overline{\Omega} \times (0, T_{\max})) \cap C^0([0, T_{\max}); C^0(\overline{\Omega})), \qquad m \ge 0.$

Moreover, if $T_{\max} < \infty$, then

$$\lim_{t \nearrow T_{\max}} \Big(\|n(t)\|_{L^\infty(\Omega)} + \|f^*(t)\|_{W^{1,\infty}(\Omega)} + \|m(t)\|_{L^\infty(\Omega)} \Big) = \infty.$$

Remark 10.1. Using the method in the proof of Theorem 10.1 with minor changes, we can obtain the assertion also for (1.6) if $D_n(n, f)$ is constant. Thus there is not a big difference between (1.6) and (TIM) in the local solvability.

We prove Theorem 10.1. In the first half we give a proof of existence of classical solutions by the Banach fixed point theorem, while in the second half uniqueness of solutions is proved. The argument used here are based on [**38**].

Proof of existence. The existence proof follows a standard contraction argument. Assume that n_0, f_0, f_0^* and m_0 satisfy (A1)-(A4).

(Step 1) Suitable mapping and space.

With $R > 0$ and $T \in (0, 1)$ to be fixed below, let X be the Banach space defined as

$$X := C^0([0, T]; C^0(\overline{\Omega})) \times L^\infty(0, T; W^{1,\infty}(\Omega)) \times C^0([0, T]; C^0(\overline{\Omega}))$$

with norm

$$\|(n, f^*, m)\|_X := \|n\|_{C^0([0,T]; C^0(\overline{\Omega}))} + \|f^*\|_{L^\infty(0,T; W^{1,\infty}(\Omega))} + \|m\|_{C^0([0,T]; C^0(\overline{\Omega}))}.$$

We claim that if T is sufficiently small, then on the closed set

$$S := \Big\{ (n, f^*, m) \in X \Big| \ \|(n, f^*, m)\|_X \le R \Big\},$$

the mapping

$$\Psi((n,f^*,m))(t) := \begin{pmatrix} \Psi_1((n,f^*,m))(t) \\ \Psi_2((n,f^*,m))(t) \\ \Psi_3((n,f^*,m))(t) \end{pmatrix} \tag{10.30}$$

$$:= \begin{pmatrix} e^{t\Delta} n_0 - \displaystyle\int_0^t e^{(t-s)\Delta} \nabla \cdot (n(s)\nabla f^*(s))\, ds \\ e^{t\Delta} f_0^* + \displaystyle\int_0^t e^{(t-s)\Delta} m(s) f(s)\, ds \\ e^{t(\Delta-1)} m_0 + \displaystyle\int_0^t e^{(t-s)(\Delta-1)} n(s)\, ds \end{pmatrix},$$

where

$$f(s) := f_0 \exp\Big(- \int_0^s m(r)\, dr \Big), \tag{10.31}$$

acts as a contraction from S into itself.

(Step 2) $\Psi(S) \subset S$**.**

We let $(n, f^*, m) \in S$ and let $f(s)$ be given by (10.31). First, from elementary properties of the heat semigroup we have $\Psi((n, f^*, m)) \in X$.

Next, to prove that $\Psi((n, f^*, m)) \in S$, it remains to show that $\|\Psi(n, f^*, m)\|_X \le R$ if T is appropriately small. We fix $q_1 > N$ and choose θ such that $\theta \in (\frac{N}{2q_1}, \frac{1}{2})$. Applying (2.1) with $(m, p) = (0, \infty)$ and using Lemma 2.2 **(i)** and **(iii)**, we see that for all $t \in [0, T]$,

$$\begin{aligned}
&\|\Psi_1((n,f^*,m))(t)\|_{L^\infty(\Omega)} \\
&\le \|e^{t\Delta} n_0\|_{L^\infty(\Omega)} + c_{0,\infty} \int_0^t \|(-\Delta+1)^\theta e^{(t-s)\Delta} \nabla \cdot (n(s)\nabla f^*(s))\|_{L^{q_1}(\Omega)}\, ds \\
&\le \|n_0\|_{L^\infty(\Omega)} + c_{0,\infty} C_1 \int_0^t \left(\frac{t-s}{2}\right)^{-\theta} \|e^{\frac{t-s}{2}\Delta} \nabla \cdot (n(s)\nabla f^*(s))\|_{L^{q_1}(\Omega)}\, ds \\
&\le \|n_0\|_{L^\infty(\Omega)} + c_{0,\infty} C_1 C_4 \int_0^t \left(\frac{t-s}{2}\right)^{-\theta-\frac{1}{2}} e^{-\frac{\nu_2(t-s)}{2}} \|n(s)\nabla f^*(s)\|_{L^{q_1}(\Omega)}\, ds,
\end{aligned}$$

where $T < 1$ is used. Using the fact $(n, f^*, m) \in S$, we note that for a.a. $s \in [0, T]$,

$$\|n(s)\nabla f^*(s)\|_{L^{q_1}(\Omega)} \le |\Omega|^{\frac{1}{q_1}} \|n(s)\|_{L^\infty(\Omega)} \|\nabla f^*(s)\|_{L^\infty(\Omega)} \le |\Omega|^{\frac{1}{q_1}} R^2.$$

Therefore it follows that

$$\|\Psi_1((n,f^*,m))(t)\|_{L^\infty(\Omega)} \le \|n_0\|_{L^\infty(\Omega)} + c_{0,\infty}C_1C_4|\Omega|^{\frac{1}{q_1}}R^2\int_0^t \left(\frac{t-s}{2}\right)^{-\theta-\frac{1}{2}} ds$$
$$\le \|n_0\|_{L^\infty(\Omega)} + \Gamma_1 R^2 T^{\frac{1}{2}-\theta} \quad \text{for all } t\in[0,T],$$

where $\Gamma_1 := \frac{2^{\theta+\frac{1}{2}}}{\frac{1}{2}-\theta}c_{0,\infty}C_1C_4|\Omega|^{\frac{1}{q_1}} > 0$. Picking $q_2 \in (1,\infty)$ and $\gamma \in (\frac{1}{2},1)$ satisfying $1 < 2\gamma - \frac{N}{q_2}$, applying (2.1) with $(m,p)=(1,\infty)$ and using Lemma 2.2 **(i)**, we have

$$\|\Psi_2((n,f^*,m))(t)\|_{W^{1,\infty}(\Omega)}$$
$$\le \|e^{t\Delta}f_0^*\|_{W^{1,\infty}(\Omega)} + c_{1,\infty}\int_0^t \|(-\Delta+1)^\gamma e^{(t-s)\Delta}m(s)f(s)\|_{L^{q_2}(\Omega)}\,ds$$
$$\le K\|f_0^*\|_{W^{1,\infty}(\Omega)} + c_{1,\infty}C_1\int_0^t (t-s)^{-\gamma}\|m(s)f(s)\|_{L^{q_2}(\Omega)}\,ds$$

for some constant $K>1$. Since $\|m\|_{C^0([0,T];C^0(\overline{\Omega}))} \le R$, we have that for all $s\in[0,T]$,

$$\|m(s)f(s)\|_{L^{q_2}(\Omega)} \le \|m(s)\|_{L^\infty(\Omega)}\bigg(\int_\Omega \bigg(f_0\exp\bigg(-\int_0^s m(r)\,dr\bigg)\bigg)^{q_2} dx\bigg)^{\frac{1}{q_2}}$$
$$\le R\|f_0\|_{L^\infty(\Omega)}e^{RT}|\Omega|^{\frac{1}{q_2}}.$$

Therefore we see that for all $t\in[0,T]$,

$$\|\Psi_2((n,f^*,m))(t)\|_{W^{1,\infty}(\Omega)} \le K\|f_0^*\|_{W^{1,\infty}(\Omega)} + \Gamma_2 Re^{RT}T^{1-\gamma},$$

where $\Gamma_2 := \frac{1}{1-\gamma}c_{1,\infty}C_1\|f_0\|_{L^\infty(\Omega)}|\Omega|^{\frac{1}{q_2}} \ge 0$. Taking $q_3 > N$, we can choose β such that $\beta \in (\frac{N}{2q_3},1)$. Applying (2.1) with $(m,p)=(0,\infty)$ and using Lemma 2.2 **(ii)**, we obtain

$$\|\Psi_3((n,f^*,m))(t)\|_{L^\infty(\Omega)}$$
$$\le \|e^{t(\Delta-1)}m_0\|_{L^\infty(\Omega)} + c_{0,\infty}\int_0^t \|(-\Delta+1)^\beta e^{(t-s)(\Delta-1)}n(s)\|_{L^{q_3}(\Omega)}\,ds$$
$$\le \|m_0\|_{L^\infty(\Omega)} + c_{0,\infty}C_2\int_0^t (t-s)^{-\beta}\|n(s)\|_{L^{q_3}(\Omega)}\,ds$$
$$\le \|m_0\|_{L^\infty(\Omega)} + \Gamma_3 RT^{1-\beta} \quad \text{for all } t\in[0,T],$$

where $\Gamma_3 := \frac{1}{1-\beta} c_{0,\infty} C_2 |\Omega|^{\frac{1}{q_3}} > 0$. Therefore it follows that

$$\|(n, f^*, m)\|_X \le \|n_0\|_{L^\infty(\Omega)} + K\|f_0^*\|_{W^{1,\infty}(\Omega)} + \|m_0\|_{L^\infty(\Omega)} + \Gamma_1 R^2 T^{\frac{1}{2}-\theta} + \Gamma_2 R e^{RT} T^{1-\gamma} + \Gamma_3 R T^{1-\beta}.$$

If we fix

$$R := \|n_0\|_{L^\infty(\Omega)} + K\|f_0^*\|_{W^{1,\infty}(\Omega)} + \|m_0\|_{L^\infty(\Omega)} + 1,$$

then we can choose T as

$$\Gamma_1 R^2 T^{\frac{1}{2}-\theta} + \Gamma_2 R e^{RT} T^{1-\gamma} + \Gamma_3 R T^{1-\beta} < 1.$$

This yields that $\|\Psi(n, f^*, m)\|_X \le R$ and hence $\Psi(S) \subset S$.

(Step 3) Contractivity of Ψ.

We proceed to check that on further diminishing T if necessary we obtain that Ψ is a contraction mapping. Let $(n, f^*, m), (\overline{n}, \overline{f^*}, \overline{m}) \in S$ and let f be given by (10.31) and $\overline{f}$ be defined by

$$\overline{f}(s) := f_0 \exp\Big(- \int_0^s \overline{m}(r)\, dr \Big).$$

As in the proof of **(Step 2)**, it follows that for all $t \in [0, T]$,

$$\|\Psi_1((n, f^*, m))(t) - \Psi_1((\overline{n}, \overline{f^*}, \overline{m}))(t)\|_{L^\infty(\Omega)} \le c_{0,\infty} C_1 C_4 \int_0^t \left(\frac{t-s}{2} \right)^{-\theta-\frac{1}{2}} \|n(s)\nabla f^*(s) - \overline{n}(s)\nabla \overline{f^*}(s)\|_{L^{q_1}(\Omega)}\, ds.$$

Since $(n, f^*, m), (\overline{n}, \overline{f^*}, \overline{m}) \in S$, we have that for a.a. $s \in [0, T]$,

$$\begin{aligned} \|n(s)\nabla f^*(s) - \overline{n}(s)\nabla \overline{f^*}(s)\|_{L^{q_1}(\Omega)} &\le \|n(s)\|_{L^\infty(\Omega)} \|\nabla (f^*(s) - \overline{f^*}(s))\|_{L^{q_1}(\Omega)} \\ &\quad + \|n(s) - \overline{n}(s)\|_{L^\infty(\Omega)} \|\nabla \overline{f^*}(s)\|_{L^{q_1}(\Omega)} \\ &\le 2|\Omega|^{\frac{1}{q_1}} R \|(n, f^*, m) - (\overline{n}, \overline{f^*}, \overline{m})\|_X. \end{aligned} \tag{10.32}$$

Therefore we obtain

$$\|\Psi_1((n, f^*, m)) - \Psi_1((\overline{n}, \overline{f^*}, \overline{m}))\|_X \le 2\Gamma_1 R T^{\frac{1}{2}-\theta} \|(n, f^*, m) - (\overline{n}, \overline{f^*}, \overline{m})\|_X.$$

Next we consider the estimate for Ψ_2. We see that for all $t \in [0,T]$,

$$
\begin{aligned}
&\|\Psi_2((n,f^*,m))(t) - \Psi_2((\overline{n},\overline{f^*},\overline{m}))(t)\|_{W^{1,\infty}(\Omega)} \\
&\le c_{1,\infty} C_1 \int_0^t (t-s)^{-\gamma} \|m(s)f(s) - \overline{m}(s)\overline{f}(s)\|_{L^{q_2}(\Omega)}\, ds \\
&\le c_{1,\infty} C_1 \int_0^t (t-s)^{-\gamma} \|m(s)(f(s) - \overline{f}(s))\|_{L^{q_2}(\Omega)}\, ds \\
&\quad + c_{1,\infty} C_1 \int_0^t (t-s)^{-\gamma} \|(m(s) - \overline{m}(s))\overline{f}(s)\|_{L^{q_2}(\Omega)}\, ds \\
&=: \boldsymbol{I}_1 + \boldsymbol{I}_2.
\end{aligned}
$$

First, we estimate the term $\boldsymbol{I}_1$. Noting that for all $s \in [0,T]$,

$$
\begin{aligned}
|f(s) - \overline{f}(s)| &= \left| f_0 \exp\left(- \int_0^s m(r)\, dr \right) - f_0 \exp\left(- \int_0^s \overline{m}(r)\, dr \right) \right| \\
&\le \|f_0\|_{L^\infty(\Omega)} e^{RT} \left| - \int_0^s m(r)\, dr + \int_0^s \overline{m}(r)\, dr \right| \\
&\le \|f_0\|_{L^\infty(\Omega)} e^{RT} \|m - \overline{m}\|_{C^0([0,T];C^0(\overline{\Omega}))} T,
\end{aligned}
$$

we obtain that

$$
\begin{aligned}
\boldsymbol{I}_1 &\le c_{1,\infty} C_1 R |\Omega|^{\frac{1}{q_2}} \int_0^t (t-s)^{-\gamma} \|f(s) - \overline{f}(s)\|_{L^\infty(\Omega)}\, ds \\
&\le c_{1,\infty} C_1 R |\Omega|^{\frac{1}{q_2}} \|f_0\|_{L^\infty(\Omega)} e^{RT} \|m - \overline{m}\|_{C^0([0,T];C^0(\overline{\Omega}))} T \int_0^t (t-s)^{-\gamma}\, ds \\
&\le \Gamma_2 e^{RT} T^{1-\gamma} RT \|(n,f^*,m) - (\overline{n},\overline{f^*},\overline{m})\|_X.
\end{aligned}
$$

On the other hand, it turns out that

$$
\begin{aligned}
\boldsymbol{I}_2 &\le c_{1,\infty} C_1 \|f_0\|_{L^\infty(\Omega)} e^{RT} |\Omega|^{\frac{1}{q_2}} \|m(s) - \overline{m}(s)\|_{C^0([0,T];C^0(\overline{\Omega}))} \int_0^t (t-s)^{-\gamma}\, ds \\
&\le \Gamma_2 e^{RT} T^{1-\gamma} \|(n,f^*,m) - (\overline{n},\overline{f^*},\overline{m})\|_X.
\end{aligned}
$$

Consequently, it follows that

$$
\|\Psi_2((n,f^*,m)) - \Psi_2((\overline{n},\overline{f^*},\overline{m}))\|_X \le \Gamma_2 e^{RT} T^{1-\gamma} (RT+1) \|(n,f^*,m) - (\overline{n},\overline{f^*},\overline{m})\|_X.
$$

Proceeding similarly as in **Step 2**, we have

$$\|\Psi_3((n,f^*,m)) - \Psi_3((\overline{n},\overline{f^*},\overline{m}))\|_X \le \Gamma_3 T^{1-\beta}\|(n,f^*,m)-(\overline{n},\overline{f^*},\overline{m})\|_X$$

and conclude that Ψ is a contraction mapping if T is sufficiently small. From the Banach fixed point theorem we thus obtain existence of $(n,f^*,m)\in X$ such that

$$(n,f^*,m) = \Psi((n,f^*,m)).$$

(Step 4) Regularity and nonnegativity of solutions.

By standard parabolic regularity argument (Lemma 2.8) and semigroup techniques, we can observe that (n,f,f^*,m) solves (P) in the classical sense. It is clear by definition that f is nonnegative. To show that n is nonnegative, we multiply (10.24) by $n_- := -\min\{n,0\}$ and integrate it over Ω. Then we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega n_-^2 &= -\int_\Omega |\nabla n_-|^2 + \int_\Omega n_-\nabla f^*\cdot\nabla n_- \\ &\le \frac{1}{4}\|\nabla f^*\|^2_{L^\infty(\Omega\times(0,T))}\int_\Omega n_-^2.\end{aligned}$$

Integrating this inequality yields

$$\int_\Omega n_-(t)^2 \le e^{\frac{1}{2}t\|\nabla f^*\|^2_{L^\infty(\Omega\times(0,T))}}\int_\Omega n_-(0)^2 = 0.$$

This implies that n is nonnegative. Moreover, using the positivity preserving properties of $\{e^{t\Delta}\}_{t\ge0}$ and $\{e^{t(\Delta-1)}\}_{t\ge0}$ (or using the maximal principle) implies that f^* and m are nonnegative.

Therefore 4 steps yield existence of solutions to (P) with (P1)-(P4). □

Finally we prove uniqueness of solutions to (P).

Proof of uniqueness. To prove uniqueness of solutions in the indicated class, let us assume that (n,f,f^*,m) and $(\overline{n},\overline{f},\overline{f^*},\overline{m})$ are the solutions on some interval $[0,T]$. Put

$$\begin{aligned}\overline{N} &:= n-\overline{n},\\ \overline{F} &:= f-\overline{f},\\ \overline{F^*} &:= f^*-\overline{f^*},\\ \overline{M} &:= m-\overline{m}.\end{aligned}$$

By subtracting the equations (10.24), (10.25), (10.26) and (10.27), multiplying by $\overline{N}$, $\overline{F}$, $\overline{F^*}$ and $\overline{M}$, respectively, and integrating them in space, we deduce the energy inequalities which play a key role to prove uniqueness of solutions. First, we see that for all $t \in (0,T)$,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \overline{N}^2 = -\int_\Omega |\nabla \overline{N}|^2 + \int_\Omega \Big(n\nabla f^* - \overline{n}\nabla \overline{f^*}\Big)\cdot \nabla \overline{N}$$
$$\leq \frac{1}{4}\int_\Omega |n\nabla f^* - \overline{n}\nabla\overline{f^*}|^2.$$

Noting that

$$n\nabla f^* - \overline{n}\nabla \overline{f^*} = \overline{N}\nabla f^* + \overline{n}\nabla \overline{F^*},$$

we have the following energy inequality for all $t \in (0,T)$,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega \overline{N}^2 &\leq \frac{1}{4}\int_\Omega |\overline{N}\nabla f^* + \overline{n}\nabla\overline{F^*}|^2 \\
&\leq \frac{1}{2}\int_\Omega |\overline{N}\nabla f^*|^2 + \frac{1}{2}\int_\Omega |\overline{n}\nabla \overline{F^*}|^2 \\
&\leq \frac{1}{2}\sup_{t\in[0,T]}\|\nabla f^*(t)\|^2_{L^\infty(\Omega)}\int_\Omega \overline{N}^2 + \frac{1}{2}\sup_{t\in[0,T]}\|\nabla \overline{n}(t)\|^2_{L^\infty(\Omega)}\int_\Omega |\nabla\overline{F^*}|^2.
\end{aligned} \tag{10.33}$$

Moreover proceeding as in the above, we obtain that for all $t \in (0,T)$,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega \overline{F}^2 &= -\int_\Omega m\overline{F}^2 - \int_\Omega \overline{f}\,\overline{M}\,\overline{F} \\
&\leq \sup_{t\in[0,T]}\|m(t)\|_{L^\infty(\Omega)}\int_\Omega \overline{F}^2 \\
&\quad + \sup_{t\in[0,T]}\|\overline{f}(t)\|_{L^\infty(\Omega)}\Big(\frac{1}{2}\int_\Omega \overline{M}^2 + \frac{1}{2}\int_\Omega \overline{F}^2\Big),
\end{aligned} \tag{10.34}$$

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega (\overline{F^*})^2 &= -\int_\Omega |\nabla \overline{F^*}|^2 + \int_\Omega (mf - \overline{m}\overline{f})(f^* - \overline{f^*}) \\
&\leq -\int_\Omega |\nabla\overline{F^*}|^2 + \frac{1}{2}\int_\Omega (\overline{F^*})^2 + \frac{1}{2}\int_\Omega (\overline{M}f + \overline{m}\overline{F})^2 \\
&\leq -\int_\Omega |\nabla\overline{F^*}|^2 + \frac{1}{2}\int_\Omega (\overline{F^*})^2 + \sup_{t\in[0,T]}\|f(t)\|^2_{L^\infty(\Omega)}\int_\Omega \overline{M}^2 \\
&\quad + \sup_{t\in[0,T]}\|\overline{m}(t)\|^2_{L^\infty(\Omega)}\int_\Omega \overline{F}^2,
\end{aligned} \tag{10.35}$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \overline{M}^2 = -\int_\Omega |\nabla \overline{M}|^2 + \int_\Omega \overline{M}\,\overline{N} - \int_\Omega \overline{M}^2 \leq \frac{1}{4}\int_\Omega \overline{N}^2. \tag{10.36}$$

Finally (10.33)-(10.36) imply

$$\frac{d}{dt}\Big(\int_\Omega \overline{N}^2 + \int_\Omega \overline{F}^2 + C\int_\Omega (\overline{F^*})^2 + \int_\Omega \overline{M}^2\Big)$$
$$\leq C'\Big(\int_\Omega \overline{N}^2 + \int_\Omega \overline{F}^2 + C\int_\Omega (\overline{F^*})^2 + \int_\Omega \overline{M}^2\Big),$$

where $C, C' > 0$ are some constants. Applying the Gronwall lemma, we have

$$\int_\Omega \overline{N}^2(t) + \int_\Omega \overline{F}^2(t) + C\int_\Omega (\overline{F^*})^2(t) + \int_\Omega \overline{M}^2(t)$$
$$\leq e^{C't}\Big(\int_\Omega \overline{N}^2(0) + \int_\Omega \overline{F}^2(0) + C\int_\Omega (\overline{F^*})^2(0) + \int_\Omega \overline{M}^2(0)\Big).$$

This completes the proof. □

Chapter 11

Stabilization in a chemotaxis model for tumor invasion

11.1. Problem and results

This chapter is concerned with the chemotaxis system

$$
\begin{cases}
u_t = \Delta u - \nabla \cdot (u \nabla v), \\
v_t = \Delta v + wz, \\
w_t = -wz, \\
z_t = \Delta z - z + u,
\end{cases} \tag{11.1}
$$

which has been proposed in Chapter 10 as a modification of the tumor invasion model originally introduced by Chaplain and Anderson in [**13**]. A particular focus of the model (11.1) consists in accounting for a chemotactic attraction induced by a so-called *active extracellular matrix*, ECM*, which is produced by a biological reaction between the extracellular matrix, ECM, and a matrix-degrading enzyme, MDE. Accordingly, besides the densities u, w and z of tumor cells, ECM and MDE, a fourth relevant quantity becomes the concentration of ECM*, which is represented by the function v in (11.1).

As compared to previous works studying tumor invasion phenomena, cross-diffusion in (11.1) is of *chemotaxis* type in that it is directed toward the diffusible ECM*, the latter being produced by the static ECM in conjunction with the chemical MDE. From a mathematical point of view, one might expect this additional influence of diffusion to entail certain improved regularity properties of solutions. On the other hand, the literature shows that also such chemotactic cross-diffusion may have a strong destabilizing effect: For instance, in the Keller–Segel system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla z), \\ z_t = \Delta z - z + u, \end{cases} \tag{11.2}$$

widely considered as a prototypical model for chemoattractive processes, it is known that solutions are global and remain bounded if either $n = 1$ ([**76**]), or $n \geq 2$ and the initial data are suitably small ([**72**], [**104**]), but that some large-data solutions become unbounded even within finite time in the cases $n = 2$ ([**34**]) and $n \geq 3$ ([**108**]), where n denotes the space dimension.

As opposed to (11.1), in the Keller–Segel system (11.2) the substance secreted by the cells is immediately directing chemoattraction, whereas in (11.1) this chemical only has an indirect taxis effect by stimulating the signal production. It is the purpose of this chapter to clarify how far this indirect chemotactic feedback may enhance the regularity and boundedness properties of solutions. Indeed, we shall see that any type of blow-up is thereby entirely suppressed in the physically relevant case $n \leq 3$, and that furthermore basically all solutions approach a spatially homogeneous equilibrium in the large time limit.

In order to precisely formulate the results in this direction, let us specify the full problem setting by considering the initial-boundary value problem

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & x \in \Omega, \quad t > 0, \\ v_t = \Delta v + wz, & x \in \Omega, \quad t > 0, \\ w_t = -wz, & x \in \Omega, \quad t > 0, \\ z_t = \Delta z - z + u, & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = \dfrac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \\ w(x,0) = w_0(x), \quad z(x,0) = z_0(x), & x \in \Omega, \end{cases} \tag{11.3}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, where throughout this chapter we shall assume that

$$
(11.4) \qquad \begin{cases} u_0 \in C^0(\overline{\Omega}) & u_0 \geq 0, \\ v_0 \in W^{1,\infty}(\Omega) & v_0 \geq 0, \\ w_0 \in C^1(\overline{\Omega}) & w_0 \geq 0 \qquad \text{and} \\ z_0 \in C^0(\overline{\Omega}) & z_0 \geq 0. \end{cases}
$$

The first of the main results asserts that under this condition, (11.3) admits for global existence of a bounded classical solution when $n \leq 3$. We underline that the following statement on this does not require any smallness condition on the initial data, such as necessary for global boundedness in the Keller–Segel system.

Theorem 11.1. *Let $n \leq 3$, and suppose that (11.4) holds. Then there exists a uniquely determined quadruple (u, v, w, z) of nonnegative functions*

$$
\begin{aligned}
u &\in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \\
v &\in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)) \cap L^\infty_{loc}([0,\infty); W^{1,\infty}(\Omega)), \\
w &\in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{0,1}(\overline{\Omega} \times (0,\infty)) \qquad \textit{and} \\
z &\in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)),
\end{aligned}
$$

which solve (11.3) classically in $\Omega \times (0,\infty)$. Moreover the solution (u, v, w, z) of (11.3) is bounded in the sense that there exists $C > 0$ such that

$$
\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} + \|z(\cdot,t)\|_{L^\infty(\Omega)} \leq C
$$

for all $t > 0$.

Moreover, whenever u_0 is nontrivial, the above solution approaches a certain spatially homogeneous steady state:

Theorem 11.2. *Let $n \leq 3$. Assume that u_0, v_0, w_0 and z_0 comply with (11.4), and that $u_0 \not\equiv 0$. Then the solution (u, v, w, z) of (11.3) satisfies*

$$
\begin{aligned}
&\|u(\cdot,t) - \overline{u_0}\|_{L^\infty(\Omega)} \to 0, \\
&\|v(\cdot,t) - (\overline{v_0} + \overline{w_0})\|_{L^\infty(\Omega)} \to 0, \\
&\|w(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \qquad \textit{and} \\
&\|z(\cdot,t) - \overline{u_0}\|_{L^\infty(\Omega)} \to 0
\end{aligned}
$$

as $t \to \infty$, where the constants $\overline{u_0}$, $\overline{v_0}$ and $\overline{w_0}$ are given by

$$
(11.5) \qquad \overline{u_0} := \frac{1}{|\Omega|}\int_\Omega u_0, \qquad \overline{v_0} := \frac{1}{|\Omega|}\int_\Omega v_0, \qquad \textit{and} \qquad \overline{w_0} := \frac{1}{|\Omega|}\int_\Omega w_0.
$$

In consequence, the indirect mechanism of signal production in (11.3) is apparently insufficient to generate any significant instability of homogeneous distributions: In fact, the results from Theorems 11.1 and 11.2 indicate that at least when $n \le 3$, the cross-diffusive term in the first equation in (11.3) is substantially overbalanced by diffusion, and that hence the overall behavior of the model, with respect to both global solvability and asymptotic behavior, is essentially the same as that of the correspondingly modified system obtained on fully disregarding this taxis mechanism.

This chapter is organized as follows. After collecting some preliminary facts including local existence in Section 11.2, we shall establish Theorem 11.1 in Section 11.3 by deriving suitable a priori estimates through a two-step bootstrap argument which eventually yields a bound for the crucial component u with respect to the norm in $L^\infty(\Omega)$ (Lemma 11.10). The large time behavior will be addressed in Section 11.4, as a starting point using the integrability property

$$\int_0^\infty \int_\Omega w(x,t)z(x,t)dxdt < \infty$$

(Lemma 11.12). Thanks to global regularity estimates implied by the boundedness of solutions (Lemma 11.11), this will entail convergence of v to some nonnegative constant L in $W^{1,\infty}(\Omega)$ (Lemma 11.14). This in turn warrants stabilization of u (Lemma 11.15) and then of z (Lemma 11.16) in the sense claimed by Theorem 11.2, where the latter property along with the assumption $\overline{u_0} > 0$ enforces decay of w (Lemma 11.17) and thereupon allows for determining L (Lemma 11.18), thus completing the proof of Theorem 11.2. Finally further results on the cases of nonlinear diffusion will be presented in Section 11.5.

11.2. Local existence and basic estimates

The following statement on local existence and uniqueness is contained in Chapter 10.

Lemma 11.3. *Let $n \ge 1$, and assume that u_0, v_0, w_0 and z_0 satisfy* (11.4)*. Then there exist $T_{\max} \in (0,\infty]$ and a unique classical solution (u,v,w,z) of* (11.3) *in $\Omega \times (0,T_{\max})$ which is such that*

$$\begin{aligned}
&0 \le u \in C^0(\overline{\Omega} \times [0,T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0,T_{\max})),\\
&0 \le v \in C^0(\overline{\Omega} \times [0,T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0,T_{\max})) \cap L^\infty_{loc}([0,T_{\max}); W^{1,\infty}(\Omega)),\\
&0 \le w \in C^0(\overline{\Omega} \times [0,T_{\max})) \cap C^{0,1}(\overline{\Omega} \times (0,T_{\max})) \qquad \textit{and}\\
&0 \le z \in C^0(\overline{\Omega} \times [0,T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0,T_{\max})),
\end{aligned}$$

and such that

$$\text{(11.6)} \qquad \textit{if } T_{\max} < \infty \quad \textit{then} \quad \lim_{t \nearrow T_{\max}} \Big(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|z(\cdot,t)\|_{L^\infty(\Omega)} \Big) = \infty.$$

Throughout the sequel, we suppose that (u_0, v_0, w_0, z_0) is given such that (11.4) holds, and let (u, v, w, z) and $T_{\max} \in (0, \infty]$ denote the corresponding solution of (11.3) and its maximal existence time as specified in Lemma 11.3.

The following statement on conservation of the total mass $\int_\Omega u$ of cells is obvious but essential to the analysis in this chapter.

Lemma 11.4. *The first solution component u satisfies*

$$\text{(11.7)} \qquad \int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx \qquad \textit{for all } t \in (0, T_{\max}).$$

Proof. This can immediately be seen upon integrating the first equation in (11.3) over $\Omega \times (0,t)$ for $t \in (0, T_{\max})$. $\square$

Likewise, it is evident from (11.3) that w is nonincreasing with time. We shall frequently use the following implication thereof.

Lemma 11.5. *The third solution component w fulfills*

$$\|w(\cdot,t)\|_{L^\infty(\Omega)} \le \|w_0\|_{L^\infty(\Omega)} \qquad \textit{for all } t \in (0, T_{\max}).$$

Proof. Since both w and z are nonnegative, this estimate is obvious from the third equation in (11.3). $\square$

The particular structure of the nonlinearities in the second and third equations in (11.3) moreover enables us to derive boundedness of v with respect to the norm in $L^1(\Omega)$.

Lemma 11.6. *The second solution component has the property that*

$$\text{(11.8)} \qquad \int_\Omega v(x,t)dx \le \int_\Omega v_0(x)dx + \int_\Omega w_0(x)dx \qquad \textit{for all } t \in (0, T_{\max}).$$

Proof. We add the third to the second equation in (11.3) and integrate with respect to $x \in \Omega$ to obtain

$$\frac{d}{dt} \int_\Omega (v + w) = \int_\Omega \Delta v = 0 \qquad \text{for all } t \in (0, T_{\max}),$$

because $\frac{\partial v}{\partial \nu} = 0$ on $\partial\Omega$. Thus,

$$\text{(11.9)} \qquad \int_\Omega v(x,t)dx + \int_\Omega w(x,t)dx = \int_\Omega v_0(x)dx + \int_\Omega w_0(x)dx$$

for all $t \in (0, T_{\max})$, from which (11.8) follows by nonnegativity of w. $\square$

11.3. Boundedness. Proof of Theorem 11.1

Throughout the subsequent analysis in this chapter, we shall frequently make use of well-known smoothing properties of the Neumann heat semigroup $(e^{t\Delta})_{t\ge0}$ in Ω.

Let us first use these regularization properties to derive the following estimate for the solution component z under an appropriate boundedness assumption on u.

Lemma 11.7. *Let $p \ge 1$ and*

$$\begin{cases} q \in [1, \frac{np}{n-2p}) & \text{if } p \le \frac{n}{2}, \\ q \in [1, \infty] & \text{if } p > \frac{n}{2}. \end{cases} \tag{11.10}$$

Then for all $M > 0$ there exists $C_z(p, q, M) > 0$ such that if for some $T \in (0, T_{\max})$ we have

$$\|u(\cdot, t)\|_{L^p(\Omega)} \le M \qquad \text{for all } t \in (0, T), \tag{11.11}$$

then

$$\|z(\cdot, t)\|_{L^q(\Omega)} \le C_z(p, q, M) \qquad \text{for all } t \in (0, T). \tag{11.12}$$

Proof. In view of the Hölder inequality, we may clearly assume that $q > p$. Then according to standard L^p-L^q estimates for $(e^{t\Delta})_{t\ge0}$, we can find $c_1 > 0$ such that

$$\|e^{\tau\Delta}\varphi\|_{L^q(\Omega)} \le c_1\Big(1 + \tau^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\Big) \cdot \|\varphi\|_{L^p(\Omega)} \qquad \text{for all } \tau > 0 \text{ and any } \varphi \in L^p(\Omega),$$

and using the maximum principle for the heat equation, we easily obtain $c_2 > 0$ fulfilling

$$\|e^{\tau\Delta}\varphi\|_{L^q(\Omega)} \le c_2\|\varphi\|_{L^\infty(\Omega)} \qquad \text{for all } \tau > 0 \text{ and arbitrary } \varphi \in L^\infty(\Omega).$$

Therefore, from the variation-of-constants representation of z,

$$z(\cdot, t) = e^{t(\Delta-1)}z_0 + \int_0^t e^{(t-s)(\Delta-1)}u(\cdot, s)ds \qquad \text{for all } t \in (0, T),$$

we infer that the assumption (11.11) entails the inequality

$$\begin{aligned} \|z(\cdot, t)\|_{L^q(\Omega)} &\le e^{-t}\|e^{t\Delta}z_0\|_{L^q(\Omega)} + \int_0^t e^{-(t-s)}\|e^{(t-s)\Delta}u(\cdot, s)\|_{L^q(\Omega)}ds \\ &\le c_2e^{-t} \cdot \|z_0\|_{L^\infty(\Omega)} + c_1M\int_0^t e^{-(t-s)} \cdot \Big(1 + (t-s)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\Big)ds \end{aligned}$$

for all $t \in (0, T)$. Since (11.10) ensures that $c_3 := \int_0^\infty (1 + \sigma^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}) \cdot e^{-\sigma}d\sigma$ is finite, this implies that

$$\|z(\cdot, t)\|_{L^q(\Omega)} \le c_2\|z_0\|_{L^\infty(\Omega)} + c_1c_3M \qquad \text{for all } t \in (0, T)$$

and thereby proves (11.12). □

Next, a boundedness property of z of the above form entails a certain regularity property for ∇v.

Lemma 11.8. *Let $q \geq 1$ and*

$$\begin{cases} r \in [1, \frac{nq}{n-q}) & \text{if } q \leq n, \\ r \in [1, \infty] & \text{if } q > n. \end{cases} \tag{11.13}$$

Then for all $M > 0$ there exists $C_v(q,r,M) > 0$ with the property that if $T \in (0, T_{\max})$ is such that

$$\|z(\cdot,t)\|_{L^q(\Omega)} \leq M \qquad \text{for all } t \in (0,T), \tag{11.14}$$

then

$$\|\nabla v(\cdot,t)\|_{L^r(\Omega)} \leq C_v(q,r,M) \qquad \text{for all } t \in (0,T).$$

Proof. Again in view of the Hölder inequality, we need to consider the case $r \geq q$ only, in which according to known regularization properties of $(e^{t\Delta})_{t\geq 0}$, as contained in [**104**, Lemma 1.3], for all $s \in [1,q]$ we can find $c_1(s) > 0$ such that

$$\|\nabla e^{\tau\Delta}\varphi\|_{L^r(\Omega)} \leq c_1(s)\Big(1 + \tau^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{s}-\frac{1}{r})}\Big)\|\varphi\|_{L^s(\Omega)} \tag{11.15}$$

for all $\tau > 0$ and each $\varphi \in L^s(\Omega)$, and moreover there exists $c_2 > 0$ satisfying

$$\|\nabla e^{\tau\Delta}\varphi\|_{L^r(\Omega)} \leq c_2\|\varphi\|_{W^{1,\infty}(\Omega)} \qquad \text{for all } \tau > 0 \text{ and any } \varphi \in W^{1,\infty}(\Omega). \tag{11.16}$$

We now fix a nonnegative integer k and represent $v(\cdot,t)$ according to

$$v(\cdot,t) = e^{(t-k)\Delta}v(\cdot,k) + \int_k^t e^{(t-s)\Delta}w(\cdot,s)z(\cdot,s)ds \qquad \text{for all } t \in (k,\infty)\cap(0,T). \tag{11.17}$$

Here if $k \geq 1$, we may apply (11.15) to $s := 1$ and use Lemma 11.6 to estimate

$$\begin{aligned} \|\nabla e^{(t-k)\Delta}v(\cdot,k)\|_{L^r(\Omega)} &\leq c_1(1)\Big(1 + (t-k)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})}\Big)\|v(\cdot,k)\|_{L^1(\Omega)} \\ &\leq c_1(1)c_3\Big(1 + (t-k)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})}\Big) \\ &\leq 2c_1(1)c_3 \qquad \text{for all } t \in [k+1,\infty)\cap(0,T) \end{aligned} \tag{11.18}$$

with $c_3 := \int_\Omega v_0 + \int_\Omega w_0$. In the case $k = 0$, we instead employ (11.16) to obtain

$$\begin{aligned} \|\nabla e^{(t-k)\Delta}v(\cdot,k)\|_{L^r(\Omega)} &= \|\nabla e^{t\Delta}v_0\|_{L^r(\Omega)} \\ &\leq c_2\|v_0\|_{W^{1,\infty}(\Omega)} \qquad \text{for all } t > 0. \end{aligned} \tag{11.19}$$

In the second summand on the right-hand side of (11.17) we use (11.15) with $s := q$ to see that

$$
\begin{aligned}
(11.20) \qquad & \left\| \nabla \int_k^t e^{(t-s)\Delta} w(\cdot,s) z(\cdot,s) ds \right\|_{L^r(\Omega)} \\
& \le c_1(q) \int_k^t \Big(1 + (t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}\Big) \|w(\cdot,s)z(\cdot,s)\|_{L^q(\Omega)} ds \\
& \qquad\qquad \text{for all } t \in (k,\infty) \cap (0,T),
\end{aligned}
$$

where thanks to the hypothesis (11.14) and Lemma 11.5 we know that

$$
\begin{aligned}
\|w(\cdot,s)z(\cdot,s)\|_{L^q(\Omega)} &\le \|w(\cdot,s)\|_{L^\infty(\Omega)} \|z(\cdot,s)\|_{L^q(\Omega)} \\
&\le c_4 M \qquad \text{for all } s \in (0,T)
\end{aligned}
$$

with $c_4 := \|w_0\|_{L^\infty(\Omega)}$. Therefore, (11.20) entails that

$$
\begin{aligned}
(11.21) \qquad & \left\| \nabla \int_k^t e^{(t-s)\Delta} w(\cdot,s) z(\cdot,s) ds \right\|_{L^r(\Omega)} \\
& \le c_1(q) c_4 M \int_k^t \Big(1 + (t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}\Big) ds \\
& \le c_1(q) c_4 M \cdot c_5 \qquad \text{for all } t \in (k,k+2) \cap (0,T),
\end{aligned}
$$

where the assumption (11.13) on r warrants that

$$
c_5 := \int_0^2 \Big(1 + \sigma^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}\Big) d\sigma
$$

is finite. Hence, in the case $t \in (0,2) \cap (0,T)$ we infer from (11.17), (11.19) and (11.21) that

$$
\|\nabla v(\cdot,t)\|_{L^r(\Omega)} \le c_2 \|v_0\|_{W^{1,\infty}(\Omega)} + c_1(q) c_4 c_5 M,
$$

whereas whenever $t \in (0,T)$ is such that $t \ge 2$, we can pick an integer $k \ge 1$ such that $t \in [k+1,k+2)$ and thereupon obtain from (11.17), (11.18) and (11.21) that

$$
\|\nabla v(\cdot,t)\|_{L^r(\Omega)} \le c_1(1) c_3 + c_1(q) c_4 c_5 M.
$$

The proof is thus complete. □

We can now prepare a closure of the regularity reasoning by deriving an estimate for u from a supposedly present appropriate boundedness property of ∇v.

Lemma 11.9. *Suppose that $r > n$. Then for all $M > 0$ there exists $C_u(r, M) > 0$ such that if*

$$\|\nabla v(\cdot, t)\|_{L^r(\Omega)} \le M \qquad \textit{for all } t \in (0, T) \tag{11.22}$$

with some $T \in (0, T_{\max})$, then

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \le C_u(r, M) \qquad \textit{for all } t \in (0, T).$$

Proof. Since $r > n$, we can fix a number θ such that

$$\theta > \frac{r}{r+1} \tag{11.23}$$

and

$$n < \theta < r. \tag{11.24}$$

Then according to Lemma 2.2 **(iv)** there exists $c_1 > 0$ fulfilling

$$\|e^{\tau\Delta}\nabla \cdot \varphi\|_{L^\infty(\Omega)} \le c_1 \tau^{-\frac{1}{2} - \frac{n}{2\theta}} \|\varphi\|_{L^\theta(\Omega)} \quad \text{for all } \varphi \in C^1(\overline{\Omega}; \mathbb{R}^n) \text{ such that } \varphi \cdot \nu = 0 \text{ on } \partial\Omega. \tag{11.25}$$

Moreover, standard L^p-L^q estimates yield $c_2 > 0$ satisfying

$$\|e^{\tau\Delta}\varphi\|_{L^\infty(\Omega)} \le c_2 \tau^{-\frac{n}{2}} \|\varphi\|_{L^1(\Omega)} \quad \text{for all } \tau > 0 \text{ and each } \varphi \in L^1(\Omega) \text{ such that } \int_\Omega \varphi = 0. \tag{11.26}$$

Now proceeding in a way similar to that in the proof of Lemma 11.8, for a given integer $k \ge 0$ we use a variation-of-constants representation of u to estimate

$$\begin{aligned} \|u(\cdot, t)\|_{L^\infty(\Omega)} &= \left\| e^{(t-k)\Delta} u(\cdot, k) - \int_k^t e^{(t-s)\Delta} \nabla \cdot \Big(u(\cdot, s) \nabla v(\cdot, s) \Big) ds \right\|_{L^\infty(\Omega)} \\ &\le \|e^{(t-k)\Delta} u(\cdot, k)\|_{L^\infty(\Omega)} + \int_k^t \left\| e^{(t-s)\Delta} \nabla \cdot (u(\cdot, s) \nabla v(\cdot, s) \right\|_{L^\infty(\Omega)} ds \end{aligned} \tag{11.27}$$

for all $t > k$. Here when $k = 0$, by the maximum principle we obtain

$$\begin{aligned} \|e^{(t-k)\Delta} u(\cdot, k)\|_{L^\infty(\Omega)} &= \|e^{t\Delta} u_0\|_{L^\infty(\Omega)} \\ &\le \|u_0\|_{L^\infty(\Omega)} \qquad \text{for all } t > 0, \end{aligned} \tag{11.28}$$

while in the case $k \geq 1$ we use (11.26) and recall (11.7) to see that

$$\begin{aligned}
\|e^{(t-k)\Delta}u(\cdot,k)\|_{L^\infty(\Omega)} &\leq \|e^{(t-k)\Delta}(u(\cdot,k)-\overline{u_0})\|_{L^\infty(\Omega)} + \overline{u_0} \\
&\leq c_2(t-k)^{-\frac{n}{2}}\|u(\cdot,k)-\overline{u_0}\|_{L^1(\Omega)} + \overline{u_0} \\
&\leq 2c_2(t-k)^{-\frac{n}{2}}\|u_0\|_{L^1(\Omega)} + \overline{u_0} \\
&\leq 2c_2\|u_0\|_{L^1(\Omega)} + \overline{u_0} \qquad \text{for all } t \geq k+1,
\end{aligned} \tag{11.29}$$

due to the relation $e^{t\Delta}\overline{u_0} \equiv \overline{u_0}$ for all $t > 0$. In the rightmost integral in (11.27), we invoke (11.25) to find that

$$\begin{aligned}
&\int_k^t \Big\|e^{(t-s)\Delta}\nabla\cdot\Big(u(\cdot,s)\nabla v(\cdot,s)\Big)\Big\|_{L^\infty(\Omega)} ds \\
&\leq c_1 \int_k^t (t-s)^{-\frac{1}{2}-\frac{n}{2\theta}}\|u(\cdot,s)\nabla v(\cdot,s)\|_{L^\theta(\Omega)} ds \\
&\qquad \text{for all } t \in (k,\infty)\cap(0,T),
\end{aligned} \tag{11.30}$$

where an application of the Hölder inequality combined with the hypothesis (11.22) shows that

$$\begin{aligned}
\|u(\cdot,s)\nabla v(\cdot,s)\|_{L^\theta(\Omega)} &\leq \|\nabla v(\cdot,s)\|_{L^r(\Omega)} \cdot \|u(\cdot,s)\|_{L^{\frac{r\theta}{r-\theta}}(\Omega)} \\
&\leq M\cdot\|u(\cdot,s)\|_{L^{\frac{r\theta}{r-\theta}}(\Omega)} \qquad \text{for all } s \in (0,T).
\end{aligned} \tag{11.31}$$

Since the property (11.23) ensures that $\frac{r\theta}{r-\theta} > 1$ and that hence $\kappa := \frac{r-\theta}{r\theta} \in (0,1)$, we may once again use the Hölder inequality and (11.7) to estimate

$$\begin{aligned}
\|u(\cdot,s)\|_{L^{\frac{r\theta}{r-\theta}}(\Omega)} &\leq \|u(\cdot,s)\|_{L^1(\Omega)}^\kappa \cdot \|u(\cdot,s)\|_{L^\infty(\Omega)}^{1-\kappa} \\
&= \|u_0\|_{L^1(\Omega)}^\kappa \cdot \|u(\cdot,s)\|_{L^\infty(\Omega)}^{1-\kappa} \qquad \text{for all } s \in (0,T),
\end{aligned}$$

so that (11.30) and (11.31) imply that

$$\begin{aligned}
&\int_k^t \Big\|e^{(t-s)\Delta}\nabla\cdot\Big(u(\cdot,s)\nabla v(\cdot,s)\Big)\Big\|_{L^\infty(\Omega)} ds \\
&\leq c_1 M\cdot\|u_0\|_{L^1(\Omega)}^\kappa \cdot \int_k^t (t-s)^{-\frac{1}{2}-\frac{n}{2\theta}}\|u(\cdot,s)\|_{L^\infty(\Omega)}^{1-\kappa} ds
\end{aligned} \tag{11.32}$$

for all $t \in (k,\infty)\cap(0,T)$. Thus, writing

$$K \equiv K(T) := \sup_{t\in(0,T)} \|u(\cdot,t)\|_{L^\infty(\Omega)},$$

from (11.27), (11.28) and (11.32) we obtain that if $t \in (0,2) \cap (0,T)$ then

$$\begin{aligned} \|u(\cdot,t)\|_{L^\infty(\Omega)} &\le \|u_0\|_{L^\infty(\Omega)} + c_1 M \|u_0\|_{L^1(\Omega)}^\kappa \cdot K^{1-\kappa} \cdot \int_0^t (t-s)^{-\frac{1}{2}-\frac{n}{2\theta}} ds \\ &\le \|u_0\|_{L^\infty(\Omega)} + c_1 c_3 M \|u_0\|_{L^1(\Omega)}^\kappa \cdot K^{1-\kappa} \end{aligned} \tag{11.33}$$

holds with $c_3 := \int_0^2 \sigma^{-\frac{1}{2}-\frac{n}{2\theta}} d\sigma$ being finite due to the left inequality in (11.24). On the other hand, if $t \in (0,T)$ is such that $t \ge 2$ then for some integer $k \ge 1$ we have $t \in [k+1, k+2)$ and hence infer from (11.27), (11.29) and (11.32) that

$$\begin{aligned} &\|u(\cdot,t)\|_{L^\infty(\Omega)} \\ &\le 2c_2 \|u_0\|_{L^1(\Omega)} + \overline{u_0} + c_1 M \|u_0\|_{L^1(\Omega)}^\kappa \cdot K^{1-\kappa} \cdot \int_k^t (t-s)^{-\frac{1}{2}-\frac{n}{2\theta}} ds \\ &\le 2c_2 \|u_0\|_{L^1(\Omega)} + \overline{u_0} + c_1 c_3 M \|u_0\|_{L^1(\Omega)}^\kappa \cdot K^{1-\kappa}. \end{aligned} \tag{11.34}$$

Combining (11.33) with (11.34) thus shows that

$$K \le c_4 + c_5 M K^{1-\kappa},$$

where $c_4 := \max\{\|u_0\|_{L^\infty(\Omega)}, 2c_2\|u_0\|_{L^1(\Omega)} + \overline{u_0}\}$ and $c_5 := c_1 c_3 \|u_0\|_{L^1(\Omega)}^\kappa$, from which upon an elementary argument we conclude that

$$K \le \max\left\{ (2c_5 M)^{\frac{1}{\kappa}}, \left(\frac{c_4}{c_5 M}\right)^{\frac{1}{1-\kappa}} \right\},$$

as desired. □

Combining Lemmas 11.7, 11.8 and 11.9 and using the mass conservation property (11.7) as a starting point, we can now prove that u in fact must be bounded when $n \le 3$.

Lemma 11.10. *Suppose that $n \le 3$. Then there exists $C > 0$ such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \text{for all } t \in (0, T_{\max}). \tag{11.35}$$

Proof. Since $n \le 3$, we have $\frac{n}{2} < \frac{n}{(n-2)_+}$, so that it is possible to find $q \in [1,n]$ satisfying

$$\frac{n}{2} < q < \frac{n}{(n-2)_+}. \tag{11.36}$$

Here the left inequality warrants that $\frac{nq}{n-q} > n$, whence we can pick a number r fulfilling

$$n < r < \frac{nq}{n-q}. \tag{11.37}$$

We now write $M_1 := \|u_0\|_{L^1(\Omega)}$, let

$$M_2 := C_z(1, q, M_1)$$

be as provided by Lemma 11.7 and

$$M_3 := C_v(q, r, M_2)$$

be as given by Lemma 11.8, and claim that then for any choice of $T \in (0, T_{\max})$ we have

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \le C_u(r, M_3) \qquad \text{for all } t \in (0, T) \tag{11.38}$$

with $C_u(r, M_3)$ taken from Lemma 11.9. Indeed, for any such T, thanks to the right inequality in (11.36) we may apply Lemma 11.7 which in view of (11.7) and the definitions of M_1 and M_2 shows that

$$\|z(\cdot, t)\|_{L^q(\Omega)} \le M_2 \qquad \text{for all } t \in (0, T).$$

Due to the right inequality in (11.37), we thus obtain from Lemma 11.8 that

$$\|\nabla v(\cdot, t)\|_{L^r(\Omega)} \le M_3 \qquad \text{for all } t \in (0, T),$$

whereupon Lemma 11.9 implies (11.38), because $r > n$ by (11.37). Since $T \in (0, T_{\max})$ was arbitrary, this directly yields (11.35). □

In light of the extensibility statement in Lemma 11.3, the above readily shows that the local solution actually exists globally in time and has some further boundedness properties.

Lemma 11.11. *Let $n \le 3$. Then the solution (u, v, w, z) of (11.3) is global in time; that is, $T_{\max} = \infty$. Moreover, there exist $\alpha \in (0, 1)$ and $C > 0$ such that*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)} + \|z(\cdot, t)\|_{L^\infty(\Omega)} \le C \tag{11.39}$$

for all $t > 0$, as well as

$$\|u\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{\Omega} \times [t, t+1])} + \|v\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{\Omega} \times [t, t+1])} + \|z\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{\Omega} \times [t, t+1])} \le C \tag{11.40}$$

for all $t \ge 1$.

Proof. As shown in Lemma 11.10, we know that $\sup_{t\in(0,T_{\max})}\|u(\cdot,t)\|_{L^\infty(\Omega)}$ is finite, whence applying Lemma 11.7 to some conveniently large $p \geq 1$ and then Lemma 11.8 to suitably large $q \geq 1$ we can observe that also $\sup_{t\in(0,T_{\max})}\|z(\cdot,t)\|_{L^\infty(\Omega)}$ and $\sup_{t\in(0,T_{\max})}\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)}$ are finite. In conjunction with Lemma 11.5 and the extensibility criterion (11.6) in Lemma 11.3, this shows that $T_{\max} = \infty$ and, by independence of the obtained estimate with respect to $t \in (0, T_{\max}) = (0,\infty)$, establishes (11.39). Thereupon, straightforward bootstrap arguments involving standard interior parabolic regularity theory (Lemma 2.7) readily yield (11.40). □

Now the proof of the main result on global well-posedness and boundedness is obvious.

Proof of Theorem 11.1. We only need to combine Lemma 11.3 with Lemma 11.11. □

11.4. Large time behavior. Proof of Theorem 11.2

The core of the proof of the stabilization result in Theorem 11.2 consists in the following observation.

Lemma 11.12. *The solution of* (11.3) *has the property that*

$$\int_0^\infty \int_\Omega w(x,t)z(x,t)dxdt < \infty. \tag{11.41}$$

Proof. For arbitrary $t > 0$, integrating the third equation in (11.3) over $\Omega \times (0,t)$ we obtain

$$\int_0^t \int_\Omega w(x,s)z(x,s)dxds = \int_\Omega w_0(x)dx - \int_\Omega w(x,t)dx.$$

Since w is nonnegative, this implies (11.41). □

When combined with appropriate compactness properties such as e.g. implied by Lemma 11.11, the above integrability statement can step by step be turned into the convergence results from Theorem 11.2. We first derive a weak version of the claimed stabilization property of v.

Lemma 11.13. *There exists a constant* $L \geq 0$ *such that*

$$\|v(\cdot,t) - L\|_{L^1(\Omega)} \to 0 \qquad \textit{as } t \to \infty. \tag{11.42}$$

Proof. According to Lemma 11.11 and e.g. the Arzelà-Ascoli theorem, we can find $(t_k)_{k\in\mathbb{N}} \subset (1,\infty)$ and a nonnegative function $v_\infty \in C^0(\overline{\Omega})$ such that $t_k \to \infty$ and

$$v(\cdot,t_k) \to v_\infty \qquad \text{in } L^1(\Omega) \tag{11.43}$$

as $k \to \infty$. To show that we actually have

$$v(\cdot,t) \to \overline{v_\infty} := \frac{1}{|\Omega|}\int_\Omega v_\infty \quad \text{in } L^1(\Omega) \qquad \text{as } k \to \infty, \tag{11.44}$$

we let $\varepsilon > 0$ be given. Then in view of (11.43) and Lemma 11.12 we can fix $k \in \mathbb{N}$ large enough such that

$$\|v(\cdot,t_k) - v_\infty\|_{L^1(\Omega)} < \frac{\varepsilon}{3} \tag{11.45}$$

and

$$\int_{t_k}^\infty \int_\Omega w(x,t)z(x,t)\,dxdt < \frac{\varepsilon}{3}. \tag{11.46}$$

Moreover, using the well-known fact that for any $\varphi \in L^1(\Omega)$ we have $e^{\tau\Delta}\varphi \to \frac{1}{|\Omega|}\int_\Omega \varphi$ in $L^1(\Omega)$ as $\tau \to \infty$, we can choose some suitably large $\tau_0 > 0$ fulfilling

$$\|e^{\tau\Delta}v_\infty - \overline{v_\infty}\|_{L^1(\Omega)} < \frac{\varepsilon}{3} \qquad \text{for all } \tau > \tau_0. \tag{11.47}$$

Then by means of the variation-of-constants representation of v we see that

$$\begin{aligned} v(\cdot,t) - \overline{v_\infty} = {} & e^{(t-t_k)\Delta}\Big(v(\cdot,t_k) - v_\infty\Big) + \Big(e^{(t-t_k)\Delta}v_\infty - \overline{v_\infty}\Big) \\ & + \int_{t_k}^t e^{(t-s)\Delta}w(\cdot,s)z(\cdot,s)ds \qquad \text{for all } t > t_k, \end{aligned} \tag{11.48}$$

where from (11.47) we obtain

$$\|e^{(t-t_k)\Delta}v_\infty - \overline{v_\infty}\|_{L^1(\Omega)} < \frac{\varepsilon}{3} \qquad \text{for all } t > t_k + \tau_0. \tag{11.49}$$

Next, since $e^{\tau\Delta}$ acts as a contraction on $L^1(\Omega)$, we can use (11.45) to estimate

$$\begin{aligned} \Big\| e^{(t-t_k)\Delta}\Big(v(\cdot,t_k) - v_\infty\Big)\Big\|_{L^1(\Omega)} & \le \|v(\cdot,t_k) - v_\infty\|_{L^1(\Omega)} \\ & < \frac{\varepsilon}{3} \qquad \text{for all } t > t_k, \end{aligned} \tag{11.50}$$

and invoke (11.46) to infer that

$$
\begin{aligned}
\left\| \int_{t_k}^t e^{(t-s)\Delta} w(\cdot,s)z(\cdot,s)ds \right\|_{L^1(\Omega)} &\le \int_{t_k}^t \|w(\cdot,s)z(\cdot,s)\|_{L^1(\Omega)} ds \\
&\le \int_{t_k}^\infty \int_\Omega w(x,s)z(x,s)\,dxds \\
&< \frac{\varepsilon}{3}.
\end{aligned}
\tag{11.51}
$$

Collecting (11.48)-(11.51) shows that

$$
\|v(\cdot,t) - \overline{v_\infty}\|_{L^1(\Omega)} < \varepsilon \qquad \text{for all } t > t_k + \tau_0,
$$

which establishes (11.44) and thereby proves (11.42) with $L := \overline{v_\infty} \ge 0$. □

According to Lemma 11.11 and the Arzelà-Ascoli theorem, the above convergence actually takes place in the space $W^{1,\infty}(\Omega)$.

Lemma 11.14. *With $L \ge 0$ as in* Lemma 11.13*, we have*

$$
\|v(\cdot,t) - L\|_{W^{1,\infty}(\Omega)} \to 0 \qquad \text{as } t \to \infty. \tag{11.52}
$$

In particular,

$$
\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \qquad \text{as } t \to \infty. \tag{11.53}
$$

Proof. Since Lemma 11.11 asserts that $(v(\cdot,t))_{t\ge1}$ is bounded in $C^2(\overline{\Omega})$ and hence relatively compact in $C^1(\overline{\Omega})$ thanks to the Arzelà-Ascoli theorem, (11.52) and thus also (11.53) immediately result from Lemma 11.13. □

Having asserted appropriate decay of the gradient responsible for cross-diffusion in (11.3), we can proceed to make sure that u approaches its spatial mean in the large time limit.

Lemma 11.15. *The first component of the solution of* (11.3) *satisfies*

$$
\|u(\cdot,t) - \overline{u_0}\|_{L^\infty(\Omega)} \to 0 \qquad \text{as } t \to \infty.
$$

where $\overline{u_0}$ is given by (11.5)*.*

Proof. In view of Lemma 11.11 and the Arzelà-Ascoli theorem, it is sufficient to show that

$$
\|u(\cdot,t) - \overline{u_0}\|_{L^2(\Omega)} \to 0 \qquad \text{as } t \to \infty. \tag{11.54}
$$

To accomplish this, we first recall that if $\lambda_1 > 0$ denotes the first nonzero eigenvalue of the Neumann Laplacian in Ω, then

$$\|e^{\tau\Delta}\varphi\|_{L^2(\Omega)} \le e^{-\lambda_1\tau}\|\varphi\|_{L^2(\Omega)} \qquad \text{for all } \tau > 0 \text{ and all } \varphi \in L^2(\Omega) \text{ fulfilling } \int_\Omega \varphi = 0, \tag{11.55}$$

because for any such φ, by the variational characterization of λ_1 a standard testing procedure shows that

$$\begin{aligned} \frac{d}{d\tau}\int_\Omega |e^{\tau\Delta}\varphi|^2 &= -2\int_\Omega |\nabla e^{\tau\Delta}\varphi|^2 \\ &\le -2\lambda_1 \int_\Omega |e^{\tau\Delta}\varphi|^2 \qquad \text{for all } \tau > 0. \end{aligned}$$

Moreover, with some $c_1 > 0$ we have

$$\|e^{\tau\Delta}\nabla\cdot\varphi\|_{L^2(\Omega)} \le c_1(1+\tau^{-\frac{1}{2}})\cdot e^{-\lambda_1\tau}\cdot\|\varphi\|_{L^2(\Omega)} \qquad \text{for all } \tau > 0 \text{ and any } \varphi \in C^1(\overline{\Omega};\mathbb{R}^n) \text{ such that } \varphi\cdot\nu = 0 \text{ on } \partial\Omega \tag{11.56}$$

(cf. e.g. [**104**, Lemma 1.3]). We next let

$$h(x,t) := u(x,t)\nabla v(x,t) \qquad \text{for } x \in \overline{\Omega} \text{ and } t > 0,$$

and note that according to Lemma 11.11 we can find $c_2 > 0$ such that

$$\|h(\cdot,t)\|_{L^2(\Omega)} \le c_2 \qquad \text{for all } t > 0, \tag{11.57}$$

whereas Lemma 11.11 combined with Lemma 11.14 entails that

$$\|h(\cdot,t)\|_{L^2(\Omega)} \to 0 \qquad \text{as } t \to \infty. \tag{11.58}$$

Now in order to prove (11.54) we let $\varepsilon > 0$ be given and can thereupon choose $t_0 > 0$ large enough such that

$$e^{-\lambda_1 t}\|u_0 - \overline{u_0}\|_{L^2(\Omega)} < \frac{\varepsilon}{3} \qquad \text{for all } t > t_0 \tag{11.59}$$

as well as

$$c_1 c_2 \cdot \int_{\frac{t}{2}}^\infty (1+\sigma^{-\frac{1}{2}})\cdot e^{-\lambda_1\sigma} d\sigma < \frac{\varepsilon}{3} \qquad \text{for all } t > t_0, \tag{11.60}$$

and such that furthermore

$$c_1 \|h(\cdot,t)\|_{L^2(\Omega)} \cdot \int_0^\infty (1+\sigma^{-\frac{1}{2}}) \cdot e^{-\lambda_1 \sigma} d\sigma < \frac{\varepsilon}{3} \qquad \text{for all } t > \frac{t_0}{2}, \tag{11.61}$$

where in achieving the latter we make use of (11.58). Then since constants are invariant under the action of $e^{t\Delta}$, we have $e^{t\Delta}\overline{u_0} \equiv \overline{u_0}$ for all $t > 0$ and thus can represent u according to

$$u(\cdot,t) - \overline{u_0} = e^{t\Delta}(u_0 - \overline{u_0}) - \int_0^t e^{(t-s)\Delta} \nabla \cdot h(\cdot,s) ds, \qquad \text{for all } t > 0.$$

Here we apply (11.56) to estimate

$$\begin{aligned} &\|u(\cdot,t) - \overline{u_0}\|_{L^2(\Omega)} \\ &\quad \le \|e^{t\Delta}(u_0 - \overline{u_0})\|_{L^2(\Omega)} + c_1 \int_0^t \Big(1 + (t-s)^{-\frac{1}{2}}\Big) \cdot e^{-\lambda_1 (t-s)} \cdot \|h(\cdot,s)\|_{L^2(\Omega)} ds \end{aligned} \tag{11.62}$$

for all $t > 0$, where due to (11.55) and (11.59) we have

$$\|e^{t\Delta}(u_0 - \overline{u_0})\|_{L^2(\Omega)} \le e^{-\lambda_1 t} \|u_0 - \overline{u_0}\|_{L^2(\Omega)} < \frac{\varepsilon}{3} \qquad \text{for all } t > t_0. \tag{11.63}$$

Moreover, (11.57) and (11.60) ensure that

$$\begin{aligned} & c_1 \int_0^{\frac{t}{2}} \Big(1 + (t-s)^{-\frac{1}{2}}\Big) \cdot e^{-\lambda_1 (t-s)} \cdot \|h(\cdot,s)\|_{L^2(\Omega)} ds \\ & \le c_1 c_2 \int_0^{\frac{t}{2}} \Big(1 + (t-s)^{-\frac{1}{2}}\Big) \cdot e^{-\lambda_1 (t-s)} ds \\ & = c_1 c_2 \int_{\frac{t}{2}}^t \big(1 + \sigma^{-\frac{1}{2}}\big) \cdot e^{-\lambda_1 \sigma} d\sigma \\ & < \frac{\varepsilon}{3} \qquad \text{for all } t > t_0, \end{aligned} \tag{11.64}$$

while from (11.58) and (11.61) we infer that

$$\begin{aligned} & c_1 \int_{\frac{t}{2}}^t \Big(1 + (t-s)^{-\frac{1}{2}}\Big) \cdot e^{-\lambda_1 (t-s)} \cdot \|h(\cdot,s)\|_{L^2(\Omega)} ds \\ & \le c_1 \cdot \sup_{s > \frac{t}{2}} \|h(\cdot,s)\|_{L^2(\Omega)} \cdot \int_{\frac{t}{2}}^t \Big(1 + (t-s)^{-\frac{1}{2}}\Big) \cdot e^{-\lambda_1 (t-s)} ds \\ & = c_1 \cdot \sup_{s > \frac{t}{2}} \|h(\cdot,s)\|_{L^2(\Omega)} \cdot \int_0^{\frac{t}{2}} \big(1 + \sigma^{-\frac{1}{2}}\big) \cdot e^{-\lambda_1 \sigma} d\sigma \\ & \le c_1 \cdot \sup_{s > \frac{t}{2}} \|h(\cdot,s)\|_{L^2(\Omega)} \cdot \int_0^\infty \big(1 + \sigma^{-\frac{1}{2}}\big) \cdot e^{-\lambda_1 \sigma} d\sigma \\ & \le \frac{\varepsilon}{3} \qquad \text{for all } t > t_0. \end{aligned}$$

Along with (11.63), (11.64) and (11.62), this shows (11.54) and thus completes the proof. □

Now the above convergence property has a straightforward consequence for z.

Lemma 11.16. *The fourth component of the solution of* (11.3) *satisfies*

$$\|z(\cdot,t) - \overline{u_0}\|_{L^\infty(\Omega)} \to 0 \qquad \textit{as } t \to \infty \tag{11.65}$$

with $\overline{u_0}$ *determined by* (11.5).

Proof. As a consequence of Lemma 11.11, we can find $c_1 > 0$ such that

$$\|u(\cdot,t) - \overline{u_0}\|_{L^\infty(\Omega)} \le c_1 \qquad \text{for all } t > 0. \tag{11.66}$$

Now Lemma 11.15 says that given $\varepsilon > 0$ we can fix some sufficiently large $t_0 > 0$ such that

$$\|u(\cdot,t) - \overline{u_0}\|_{L^\infty(\Omega)} < \frac{\varepsilon}{4} \qquad \text{for all } t > \frac{t_0}{2}, \tag{11.67}$$

where enlarging t_0 if necessary we can also achieve that

$$\|z_0\|_{L^\infty(\Omega)} \cdot e^{-t} < \frac{\varepsilon}{4} \qquad \text{for all } t > t_0 \tag{11.68}$$

and

$$\overline{u_0} \cdot e^{-t} < \frac{\varepsilon}{4} \qquad \text{for all } t > t_0 \tag{11.69}$$

as well as

$$c_1 \cdot e^{-\frac{t}{2}} < \frac{\varepsilon}{4} \qquad \text{for all } t > t_0. \tag{11.70}$$

By the variation-of-constants representation of z, we can write

$$\begin{aligned} z(\cdot,t) - \overline{u_0} = e^{-t}e^{t\Delta}z_0 &+ \int_0^t e^{-(t-s)}e^{(t-s)\Delta}\Big(u(\cdot,s) - \overline{u_0}\Big)ds \\ &+ \int_0^t e^{-(t-s)}e^{(t-s)\Delta}\overline{u_0}ds \ - \overline{u_0} \qquad \text{for all } t > 0, \end{aligned} \tag{11.71}$$

and use the maximum principle and (11.68) in estimating

$$\begin{aligned} \|e^{-t}e^{t\Delta}z_0\|_{L^\infty(\Omega)} &\le e^{-t}\|z_0\|_{L^\infty(\Omega)} \\ &< \frac{\varepsilon}{4} \qquad \text{for all } t > t_0. \end{aligned} \tag{11.72}$$

As $e^{(t-s)\Delta}\overline{u_0} \equiv \overline{u_0}$, by (11.69) we moreover have

$$\left\| \int_0^t e^{-(t-s)} e^{(t-s)\Delta} \overline{u_0} ds \; - \overline{u_0} \right\|_{L^\infty(\Omega)} = \left| \int_0^t e^{-(t-s)} ds - 1 \right| \cdot \overline{u_0} \tag{11.73}$$

$$= e^{-t}\overline{u_0}$$

$$< \frac{\varepsilon}{4} \qquad \text{for all } t > t_0.$$

Finally, again by means of the maximum principle we obtain

$$\left\| \int_0^t e^{-(t-s)} e^{(t-s)\Delta} \Big(u(\cdot, s) - \overline{u_0} \Big) ds \right\|_{L^\infty(\Omega)} \tag{11.74}$$

$$\leq \int_0^t e^{-(t-s)} \|u(\cdot, s) - \overline{u_0}\|_{L^\infty(\Omega)} ds$$

for all $t > 0$, where from (11.66) and (11.70) we know that

$$\int_0^{\frac{t}{2}} e^{-(t-s)} \|u(\cdot, s) - \overline{u_0}\|_{L^\infty(\Omega)} ds \leq c_1 \int_0^{\frac{t}{2}} e^{-(t-s)} ds \tag{11.75}$$

$$= c_1 (e^{-\frac{t}{2}} - e^{-t})$$

$$< \frac{\varepsilon}{4} \qquad \text{for all } t > t_0,$$

and where (11.67) guarantees that

$$\int_{\frac{t}{2}}^t e^{-(t-s)} \|u(\cdot, s) - \overline{u_0}\|_{L^\infty(\Omega)} ds \leq \frac{\varepsilon}{4} \cdot \int_{\frac{t}{2}}^t e^{-(t-s)} ds \tag{11.76}$$

$$= \frac{\varepsilon}{4} \cdot (1 - e^{-\frac{t}{2}})$$

$$< \frac{\varepsilon}{4} \qquad \text{for all } t > t_0.$$

Inserting (11.72)-(11.76) into (11.71) yields (11.65). □

Whenever the limit in Lemma 11.16 is nontrivial, we can finally show that the monotone limit of $w(\cdot, t)$ as $t \to \infty$ actually must be zero.

Lemma 11.17. *Suppose that $u_0 \not\equiv 0$. Then*

$$\|w(\cdot, t)\|_{L^\infty(\Omega)} \to 0 \qquad \text{as } t \to \infty. \tag{11.77}$$

Proof. Since $\int_\Omega u_0 > 0$, the uniform stabilization of z, as asserted by Lemma 11.16, enables us to find $c_1 > 0$ and $t_0 > 0$ such that

$$z(x, t) \geq c_1 \qquad \text{for all } x \in \Omega \text{ and } t > t_0.$$

Integrating the third equation in (11.3) with respect to the time variable, in view of Lemma 11.5 we thus infer that

$$\begin{aligned} w(x,t) &= w(x,t_0) \cdot \exp\Big(- \int_{t_0}^t z(x,s)ds \Big) \\ &\le \|w_0\|_{L^\infty(\Omega)} \cdot e^{-c_1(t-t_0)} \qquad \text{for all } x \in \Omega \text{ and } t > t_0, \end{aligned}$$

which immediately implies (11.77). □

For completing the knowledge on the asymptotic of solutions, it remains to determine the value of the above number L. If $u_0 \not\equiv 0$, this can easily be achieved by using Lemma 11.17 in conjunction with (11.9) and Lemma 11.14.

Lemma 11.18. *Suppose that $u_0 \not\equiv 0$. Then the number L provided by* Lemma 11.13 *satisfies*

$$L = \overline{v_0} + \overline{w_0}, \tag{11.78}$$

where $\overline{v_0}$ and $\overline{w_0}$ are given by (11.5)*.*

Proof. According to (11.9) and Lemma 11.17, we obtain that

$$\int_\Omega v(x,t)dx \to \int_\Omega v_0(x)dx + \int_\Omega w_0(x)dx \qquad \text{as } t \to \infty.$$

On the other hand, Lemma 11.14 shows that

$$\int_\Omega v(x,t)dx \to |\Omega| L \qquad \text{as } t \to \infty.$$

Combining these relations immediately yields (11.78). □

Now the main result on stabilization is evident.

Proof of Theorem 11.2. We only need to collect Lemmas 11.15, 11.14, 11.18, 11.17 and 11.16. □

Remark 11.1. An interesting question left open in this chapter concerns the respective rates of convergence in Theorem 11.2, which is basically due to the fact that the approach in this chapter is based on a compactness method. The only evident implication of the results concerns the solution component w, for which it is clear that according to the uniform convergence property of z, given any $\varepsilon > 0$ one can find $C_\varepsilon > 0$ such that

$$w(x,t) \le C_\varepsilon \cdot e^{-(\overline{u_0}-\varepsilon)t} \qquad \text{for all } t > t_0.$$

Remark 11.2. By straightforward adaptation, for the corresponding variant of (11.3) given by

$$\begin{cases} u_t = d_u \Delta u - \lambda \nabla \cdot (u \nabla v), & x \in \Omega, \quad t > 0, \\ v_t = d_v \Delta v + awz, & x \in \Omega, \quad t > 0, \\ w_t = -awz, & x \in \Omega, \quad t > 0, \\ z_t = d_z \Delta z - bz + cu, & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = \dfrac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \\ w(x,0) = w_0(x), \quad z(x,0) = z_0(x), & x \in \Omega, \end{cases}$$

with positive parameters d_u, d_v, d_z, λ, a, b, $c > 0$, one can derive similar statements on global existence and asymptotic stabilization. In this general setting, the convergence results then read

$$u(x,t) \to \overline{u_0}, \quad v(x,t) \to \overline{v_0} + \overline{w_0}, \quad w(x,t) \to 0 \quad \text{and} \quad z(x,t) \to \frac{c}{b}\overline{u_0},$$

uniformly with respect to $x \in \Omega$, whenever $u_0 \not\equiv 0$.

11.5. Further results on nonlinear diffusion cases

In this chapter the linear diffusion case was studied via the Duhamel formula using the heat semigroup, whereas this method cannot be applied to the case of nonlinear diffusion as follows:

$$\begin{cases} u_t = \nabla \cdot (D(u,w)\nabla u) - \lambda \nabla \cdot (u \nabla v), & x \in \Omega, \quad t > 0, \\ v_t = \Delta v + wz, & x \in \Omega, \quad t > 0, \\ w_t = -wz, & x \in \Omega, \quad t > 0, \\ z_t = \Delta z - z + u, & x \in \Omega, \quad t > 0, \end{cases}$$

where the diffusion coefficient function D is a smooth function. This section is devoted to introducing results of Fujie–Ishida–Ito–Yokota [**22**], whose subject is to develop an approach to the system with some variants of nonlinear diffusion depending on both u and w in the two cases *nondegenerate* and *degenerate*.

In this section we consider the following initial-boundary value problem:

$$
\begin{cases}
u_t = \nabla \cdot (D(u,w)\nabla u) - \lambda \nabla \cdot (u \nabla v), & x \in \Omega,\ t > 0, \\
v_t = d_v \Delta v + awz, & x \in \Omega,\ t > 0, \\
w_t = -awz, & x \in \Omega,\ t > 0, \\
z_t = d_z \Delta z - bz + cu, & x \in \Omega,\ t > 0, \\
D(u,w)\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\
u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \\
w(x,0) = w_0(x), \quad z(x,0) = z_0(x), & x \in \Omega,
\end{cases}
\tag{11.79}
$$

where $\Omega \subset \mathbb{R}^n$ $(1 \le n \le 3)$ is a bounded domain with smooth boundary $\partial\Omega$; ν is an outer unit normal vector on $\partial\Omega$; D is a nonnegative function on $\mathbb{R} \times \mathbb{R}$; $\lambda, d_v, d_z, a, b, c$ are prescribed positive constants; u_0, v_0, w_0, z_0 are prescribed nonnegative initial data.

11.5.1. The case of nondegenerate diffusion

In this subsection we consider the case that there exists some constant $c_0 > 0$ fulfilling

$$
D(u,w) \ge c_0 > 0 \quad \forall\, u, w \ge 0. \tag{11.80}
$$

Assuming the regularity of initial data

$$
\begin{cases}
(u_0, v_0, w_0, z_0) \in C^0(\overline{\Omega}) \times W^{1,\infty}(\Omega) \times C^1(\overline{\Omega}) \times C^0(\overline{\Omega}), \\
u_0, v_0, w_0, z_0 \ge 0,
\end{cases}
\tag{11.81}
$$

we will deduce stabilization in (11.79). The first main result reads as follows.

Theorem 11.19 ([**22**]). *Suppose that* (11.80) *and* (11.81) *hold. Then there exists a uniquely determined quadruple* (u,v,w,z) *of nonnegative functions*

$$
\begin{aligned}
u, v, z &\in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)) \qquad \textit{and} \\
w &\in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{0,1}(\overline{\Omega} \times (0,\infty)),
\end{aligned}
$$

which solves (11.79) *in the classical sense in* $\Omega \times (0,\infty)$. *Moreover there exists* $C > 0$ *such that*

$$
\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} + \|z(\cdot,t)\|_{L^\infty(\Omega)} \le C \quad \forall\, t > 0.
$$

We next determine the large time behavior of these solutions whenever u_0 is nontrivial.

Theorem 11.20 ([**22**]). *Suppose that* (11.80) *holds. Assume that* u_0, v_0, w_0 *and* z_0 *comply with* (11.81)*, and that* $u_0 \not\equiv 0$*. Then the solution* (u, v, w, z) *of* (11.79) *satisfies*

$$u(\cdot,t) \to \overline{u_0} \quad \text{in } L^p(\Omega) \quad (t \to \infty) \quad \forall\, p \in [1,\infty),$$
$$v(\cdot,t) \to \overline{v_0} + \overline{w_0}, \quad w(\cdot,t) \to 0 \quad \text{and} \quad z(\cdot,t) \to \frac{c}{b}\overline{u_0} \quad \text{in } L^\infty(\Omega) \quad (t \to \infty),$$

where

$$\overline{u_0} := \frac{1}{|\Omega|}\int_\Omega u_0, \quad \overline{v_0} := \frac{1}{|\Omega|}\int_\Omega v_0 \quad \text{and} \quad \overline{w_0} := \frac{1}{|\Omega|}\int_\Omega w_0.$$

Remark 11.3. Unfortunately, as compared with the convergence result in Section 11.4, the above theorem does not assure convergence of u in $L^\infty(\Omega)$. Nevertheless, convergences of v, w, z take place in $L^\infty(\Omega)$.

Sketch of Proof of large time behavior. The case of linear diffusion is analyzed by a straightforward estimate for $u(t) - \overline{u_0}$ via the Duhamel formula. In the case of nonlinear diffusion we need other methods. If the diffusion is of *nondegenerate*, we invoke enough smoothness of u to establish the decaying property $\|\nabla f(u(t))\|_{L^2(\Omega)} \to 0$ as $t \to \infty$, where f is an increasing function determined by $D(u, w)$. Then we have the convergence $u(t) \to f^{-1}(L)$ as $t \to \infty$ with some constant $L \geq 0$. Finally noting the mass conservation law in the problem (11.79), we can precisely determine the limit function such that $f^{-1}(L) = \overline{u_0}$.

11.5.2. The case of degenerate diffusion

This subsection is devoted to considering the case of degenerate diffusion such that $D \in C^1([0,\infty) \times [0,\infty))$ requires the conditions that $D(0, r_2) = 0$ or all $r_2 \geq 0$ and there exist $c_1, \gamma > 0$ and $m > 1$ fulfilling

$$D(r_1, r_2) \geq \widetilde{D}(r_1) := \begin{cases} c_1 {r_1}^{m-1} & (r_1 \leq \gamma), \\ c_1 \gamma^{m-1} & (r_1 > \gamma). \end{cases} \tag{11.82}$$

Moreover we assume that there exist $p_0 > \max\{1, m-1\}$ and $\{\overline{D}(r_2)\,;\, r_2 \geq 0\}$ such that

$$G(r_1, r_2) := \lim_{(\overline{r}_1, \overline{r}_2) \to (r_1, r_2)} \frac{D(\overline{r}_1, \overline{r}_2)}{{\overline{r}_1}^{p_0-2}} = \begin{cases} \dfrac{D(r_1, r_2)}{{r_1}^{p_0-2}} & (r_1 > 0 \text{ and } r_2 \geq 0), \\ \overline{D}(r_2) & (r_1 = 0 \text{ and } r_2 \geq 0). \end{cases} \tag{11.83}$$

Under the regularity of initial data

$$
\begin{cases} (u_0, v_0, w_0, z_0) \in L^\infty(\Omega) \times W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega) \times L^\infty(\Omega), \\ u_0, v_0, w_0, z_0 \geq 0, \end{cases} \tag{11.84}
$$

we will study large time behavior in (11.79). Before stating the main results, we give the definition of weak solutions of (11.79). We adopt a natural concept of weak solutions by testing procedures.

A quadruple (u, v, w, z) of nonnegative functions defined on $\Omega \times (0, \infty)$ is called a weak solution of (11.79) on $[0, \infty)$ if for all $T > 0$,

(i) $u \in L^\infty(0, T; L^\infty(\Omega))$ with $\displaystyle\int_0^u D(\sigma, w)\, d\sigma \in L^2(0, T; H^1(\Omega))$,

(ii) $v \in L^\infty(0, T; W^{1,\infty}(\Omega))$,

(iii) $w \in L^\infty(0, T; L^\infty(\Omega))$,

(iv) $z \in L^\infty(0, T; L^\infty(\Omega)) \cap L^2(0, T; H^1(\Omega))$,

(v) (u, v, w, z) satisfies (11.79) in the following sense: for all $\varphi \in C_0^\infty(\Omega \times [0, \infty))$,

$$
\begin{aligned}
&\int_0^\infty \int_\Omega \left(\left[\nabla \left(\int_0^u D(\sigma, w)\, d\sigma \right) - \int_0^u \frac{\partial D}{\partial r_2}(\sigma, w) \nabla w \right] \cdot \nabla \varphi - \lambda u \nabla v \cdot \nabla \varphi - u \varphi_t \right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \int_\Omega u_0 \varphi(0), \\
&\int_0^\infty \int_\Omega (d_v \nabla v \cdot \nabla \varphi + a w z \varphi - v \varphi_t) = \int_\Omega v_0 \varphi(0), \\
&w(x, t) = w_0(x) \exp\left(-a \int_0^t z(x, s)\, ds \right) \quad \text{a.e. } (x, t) \in \Omega \times (0, \infty), \\
&\int_0^\infty \int_\Omega (d_z \nabla z \cdot \nabla \varphi - b z \varphi + c u \varphi - z \varphi_t) = \int_\Omega z_0 \varphi(0).
\end{aligned}
$$

The first main result reads as follows.

Theorem 11.21 ([**22**]). *Suppose that* (11.82), (11.83), (11.84) *hold. Then there exists a uniquely determined quadruple* (u, v, w, z) *of nonnegative functions which is a weak solution of* (11.79) *on* $[0, \infty)$. *Moreover the weak solution* (u, v, w, z) *of* (11.79) *is bounded in the sense that there exists* $C > 0$ *such that*

$$
\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{W^{1,\infty}(\Omega)} + \|w(t)\|_{L^\infty(\Omega)} + \|z(t)\|_{L^\infty(\Omega)} \leq C \qquad \text{a.e. } t > 0.
$$

Furthermore, the solution has the following properties:

$$\|u(t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} \quad a.e.\ t > 0,$$
$$\|v(t)\|_{L^1(\Omega)} + \|w(t)\|_{L^1(\Omega)} = \|v_0\|_{L^1(\Omega)} + \|w_0\|_{L^1(\Omega)} \quad \forall t > 0. \tag{11.85}$$

Remark 11.4. Since the nonlinear terms in the second, third and fourth equations in the problem belong to $L^\infty(0,T;L^\infty(\Omega))$, we can confirm that the solutions v, w and z are in $C^0([0,\infty);L^\infty(\Omega))$. From this reason, the conservation law (11.85) is valid for all $t > 0$.

We shall next discuss the large time behavior of these solutions whenever u_0 is nontrivial.

Theorem 11.22 ([**22**])**.** *Assume* (11.82), (11.83), (11.84) *and* $u_0 \not\equiv 0$. *Then for each* $p \in [1,\infty)$ *the weak solution* (u,v,w,z) *of* (11.79) *satisfies either* (1) *or* (2) *stated below holds:*

(1) *For any* $\delta > 0$ *and* $\xi > 0$ *there exists* $T_{p,\delta,\xi} > 0$ *such that* $|\mathcal{L}_{p,\delta} \cap (T_{p,\delta,\xi},\infty)| \le \xi$, *where* $\mathcal{L}_{p,\delta}$ *is a measurable set given by* $\mathcal{L}_{p,\delta} := \{t \in (0,\infty) \mid \|u(t)-\overline{u_0}\|_{L^p(\Omega)} \ge \delta\}$.

(2) *There exist a constant* $\delta_p > 0$ *and a sequence* $\{s_k\}_{k\in\mathbb{N}}$ *such that for all* $k \in \mathbb{N}$,

$$|\mathcal{L}_{p,\delta_p} \cap (s_k, s_{k+1})| = 1 \quad \text{and} \quad \lim_{k\to\infty}(s_{k+1}-s_k) = \infty.$$

Moreover, if the above (1) *is satisfied, then*

$$v(t) \to \overline{v_0} + \overline{w_0}, \quad w(t) \to 0, \quad z(t) \to \frac{c}{b}\overline{u_0} \quad \text{in } L^\infty(\Omega) \quad (t \to \infty).$$

Difficulty and key lemma. In the case of nondegenerate diffusion, we invoke some integrability in time and space and a uniform continuity argument to discuss convergence of u. Unfortunately, due to the lack of regularity of the solution, this method cannot be applied to the case of degenerate diffusion. Indeed, for a just integrable function $f : (0,\infty) \to [0,\infty)$ with

$$\int_0^\infty f(t)\,dt < \infty,$$

we only have that there exists some sequence $\{t_k\}_k$ such that $f(t_k) \to 0$ as $k \to \infty$. Instead of the uniform continuity argument, we generalized the usual subsequence technique (c.f. [**53**, Proof of Lemma 2.5 (a)]).

Lemma 11.23 ([**22**]). *Let $(X, \|\cdot\|_X)$ be a Banach space. Let x_∞ be any element of X and $x : [0,\infty) \to X$ a measurable function satisfying that there exists a null set $\mathcal{N}$ so that $\mathcal{N} \supset \{t \in (0,\infty) \,|\, \|x(t)\|_X = \infty\}$. For any $\delta > 0$ define a measurable set $\mathcal{L}_\delta \subset (0,\infty) \setminus \mathcal{N}$ by $\mathcal{L}_\delta := \{t \in (0,\infty) \setminus \mathcal{N} \,|\, \|x(t) - x_\infty\|_X \geq \delta\}$. Assume that for any sequence $\{t_k\}_{k\in\mathbb{N}} \subset (1,\infty) \setminus \mathcal{N}$ satisfying $t_k \to \infty$ $(k \to \infty)$ there exists a null set $\mathcal{N}(\{t_k\}_{k\in\mathbb{N}}) \subset (0,1)$ such that the following property $(\star)$ holds:*

$$(\star) \begin{pmatrix} \text{for any } \tau \in (0,1) \setminus \mathcal{N}(\{t_k\}_{k\in\mathbb{N}}) \text{ there exists a subsequence } \{t_{k_\ell(\tau)}\}_{\ell\in\mathbb{N}} \text{ of } \{t_k\}_{k\in\mathbb{N}} \\ \text{such that} \quad x(t_{k_\ell(\tau)} + \tau) \to x_\infty \quad \text{in } X \quad (\ell \to \infty). \end{pmatrix}$$

Then either (1) or (2) stated below holds:

(1) *For any $\delta > 0$ and $\xi > 0$ there exists $T_{\delta,\xi} > 0$ such that $|\mathcal{L}_\delta \cap (T_{\delta,\xi}, \infty)| \leq \xi$.*

(2) *There exist a constant $\delta_0 > 0$ and a sequence $\{s_k\}_{k\in\mathbb{N}}$ such that the following properties are satisfied:*

$$|\mathcal{L}_{\delta_0} \cap (s_k, s_{k+1})| = 1 \quad \forall\, k \in \mathbb{N}, \quad \lim_{k\to\infty} (s_{k+1} - s_k) = \infty.$$

Notation

General notations

$\mathbb{N}$	$= \{1, 2, 3, \cdots\}$ = natural number.
$\mathbb{R}^n$	n-dimensional real Euclidean space $(n \in \mathbb{N})$, $\mathbb{R} = \mathbb{R}^1$.
U^c	$\{x \in \mathbb{R}^n \mid x \notin U\}$, where U is a subset of $\mathbb{R}^n$.
$U \setminus V$	$= U \cap V^c$, where U, V are subsets of $\mathbb{R}^n$.
$\lvert U \rvert$	measure of U.
∂U	boundary of U.
$\overline{U}$	$= U \cup \partial U$ = closure of U.
$U \subset\subset V$	$\overline{U}$ is a compact subset of V.
$B_r(x)$	= open ball with center x, radius $r > 0$.
Ω	denotes domain, i.e. a nonempty, connected, open subset of $\mathbb{R}^n$.
D^α	$= \dfrac{\partial^{\lvert\alpha\rvert}}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n}$, $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\lvert\alpha\rvert = \sum_{i=1}^{n} \alpha_i$.
∇u	$= (u_{x_1}, \cdots, u_{x_n})$ = gradient of u.
Δu	$= \sum_{i=1}^{n} \dfrac{\partial^2 u}{\partial x_i^2}$ = Laplacian of u.
$\nabla \cdot \mathbf{u}$	$= \sum_{i=1}^{n} \dfrac{\partial u_i}{\partial x_i} = \operatorname{div} \mathbf{u}$, where $\mathbf{u} = (u_1, \cdots, u_n)$.
$\dfrac{\partial u}{\partial \nu}$	$= \nabla u \cdot \nu$ = outward normal derivative.
0^0	$= 1$.

Function spaces

$$C^0(\Omega) = \{u : \Omega \to \mathbb{R} \mid u \text{ is continuous on } \Omega\}.$$

$$C^0_c(\Omega) = \left\{u \in C^0(\Omega) \mid u \text{ has a compact support in } \Omega\right\}.$$

$$C^m(\Omega) = \{u : \Omega \to \mathbb{R} \mid u \text{ is } m \text{ times continuously differentiable on } \Omega\}.$$

$$C^\infty(\Omega) = \bigcap_{m\geq 1} C^m(\Omega).$$

$$C^m(\overline{\Omega}) = \big\{u \in C^m(\Omega) \mid \text{for every multi-index } \alpha \text{ with } |\alpha| \leq m, \text{ the function } x \mapsto D^\alpha u(x) \text{ admits a continuous extension to } \partial\Omega\big\}.$$

$$\|u\|_{C^m(\overline{\Omega})} = \sum_{|\alpha|\leq m} \sup_{x\in\Omega} |D^\alpha u(x)|.$$

$$C^{m+\theta}(\overline{\Omega}) = \left\{u \in C^m(\overline{\Omega}) \;\middle|\; \sup_{x,y\in\Omega,\ x\neq y} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\theta} < \infty \quad \text{for all } |\alpha| \leq m\right\}$$

$$(m \in \mathbb{N} \cup \{0\},\ \theta \in (0,1)).$$

$$\|u\|_{C^{m+\theta}(\overline{\Omega})} = \|u\|_{C^m(\overline{\Omega})} + \sum_{|\alpha|\leq m} \sup_{x,y\in\Omega,\ x\neq y} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\theta}.$$

$$C^{m+\theta}_{\text{loc}}(\Omega) = \bigcap_{\omega\subset\subset\Omega} C^{m+\theta}(\omega).$$

$$L^p(\Omega) = \left\{u : \Omega \to \mathbb{R} \mid u \text{ is measurable on } \Omega \text{ such that } \int_\Omega |u|^p < \infty\right\}$$

$$(1 \leq p < \infty).$$

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p\right)^{\frac{1}{p}} \ (1 \leq p < \infty).$$

$$L^\infty(\Omega) = \big\{u : \Omega \to \mathbb{R} \mid u \text{ is measurable on } \Omega \text{ such that } \operatorname{esssup}_{x\in\Omega}|u(x)| < \infty\big\}.$$

$$\|u\|_{L^\infty(\Omega)} = \operatorname{esssup}_{x\in\Omega}|u(x)|.$$

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) \mid D^\alpha u \in L^p(\Omega) \quad \text{for all } |\alpha| \leq m\}.$$

$$\|u\|_{W^{m,p}(\Omega)} = \sum_{|\alpha|\leq m} \|D^\alpha u\|_{L^p(\Omega)}.$$

$$H^m(\Omega) = W^{m,2}(\Omega).$$

$$\|u\|_{H^m(\Omega)} = \|u\|_{W^{m,2}(\Omega)}.$$

Let $U \subset \mathbb{R}^n$, $J \subset \mathbb{R}$ be open bounded subsets, $m \in \mathbb{N} \cup \{0\}$, $\theta \in (0,1)$ and X be a Banach space.

$$C^{2m,m}(\overline{U} \times \overline{J}) = \left\{u : U \times J \to \mathbb{R} \mid D_x^\alpha \partial_t^j u \in C^0(\overline{U \times J}) \text{ for } |\alpha| + 2j \le 2m\right\}.$$

$$\|u\|_{C^{2m,m}(\overline{U}\times\overline{J})} = \sum_{|\alpha|+2j\le 2m} \|D_x^\alpha \partial_t^j u\|_{C^0(\overline{U\times J})}.$$

$$C^{2m+\theta,m+\frac{\theta}{2}}(\overline{U} \times \overline{J}) = \left\{u \in C^{2m,m}(\overline{U} \times \overline{J}) \;\middle|\; \|u\|_{C^{2m+\theta,m+\frac{\theta}{2}}(\overline{U}\times\overline{J})} < \infty\right\},$$

where

$$\begin{aligned}\|u\|_{C^{2m+\theta,m+\frac{\theta}{2}}(\overline{U}\times\overline{J})} &= \|u\|_{C^{2m,m}(\overline{U}\times\overline{J})} \\ &+ \sum_{|\alpha|+2j\le 2m} \sup_{x,y\in U,\, x\neq y,\, t,s\in J} \frac{|D_x^\alpha \partial_t^j u(x,t) - D_x^\alpha \partial_t^j u(y,s)|}{|x-y|^\theta} \\ &+ \sum_{0<2m+\theta-|\alpha|-2j<2} \sup_{x,y\in U,\, t,s\in J,\, t\neq s,} \frac{|D_x^\alpha \partial_t^j u(x,t) - D_x^\alpha \partial_t^j u(y,s)|}{|t-s|^{\frac{2m+\theta-|\alpha|-2j}{2}}}.\end{aligned}$$

$$C^{2m,m}(U \times I) = \bigcup_{\text{open bounded subsets } U'\subset\subset U,\, J'\subset\subset J} C^{2m,m}(\overline{U} \times \overline{J}).$$

$$C^m(J;X) = \left\{u : J \to X \;\middle|\; \frac{d^j}{dt^j}u \text{ exists and is uniformly continuous on } J \text{ for } j \le m\right\}.$$

$$L^p(J;X) = \left\{u : J \to X \;\middle|\; u \text{ is strongly measurable and } \big\| \|u\|_X \big\|_{L^p(J)} < \infty\right\}.$$

Let $k \in \mathbb{N}$. We shall say that Ω is class C^k (see [**10**]), if for every $x \in \partial U$ there exist a neighborhood U of x in $\mathbb{R}^n$ and a bijective mapping $H : Q \to U$ such that

$$H \in C^m(\overline{Q}), \quad H^{-1} \in C^m(\overline{U}), \quad H(Q_+) = U \cap \Omega, \quad H(Q_0) = U \cap \partial\Omega,$$

where

$$\begin{aligned}Q &= \{x = (x', x_n) \mid |x'| < 1 \quad \text{and} \quad |x_n| < 1\},\\ Q_+ &= \{x = (x', x_n) \in Q \mid x_n > 0\},\\ Q_0 &= \{x = (x', x_n) \mid |x'| < 1 \quad \text{and} \quad x_n = 0\}.\end{aligned}$$

In this thesis we always assume that Ω is of class C^2.

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