PH. D. THESIS

Studies of a random sampling and stochastic processes on the ring of p-adic integers

（p-進整数環上のランダムサンプリングおよび確率過程に関する研究）

(要約)

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Chapter 1

Introduction

1.1 Historical backgrounds and aim of the thesis

Stochastic analysis on fractals was initiated in the 80's and developed mainly by M. Fukushima, K. Hattori, T. Hattori, J. Kigami, S. Kusuoka, T. Kumagai and V. Metz based on its self-similar structure. After the decade, a study on stochastic processes on the field $\mathbb{Q}_p$ of p-adic numbers was started by S. Albeverio and W. Karwowski. Their method relied on its hierarchical structure and established but in an independent fashion of the one in analysis on fractals, whereas Haar measure with a self-similarilty was imported and it played central role in constructing such function spaces as Dirichlet spaces as in the case of fractal analysis.

After Albeverio and Karwowski constructed a Hunt process on $\mathbb{Q}_p$ whose infinitesimal operator covers the Vladimirov operator in [1], many schemes were launched as generalized variants ruling Hunt processes. Exemplary attempts of them are undertaken by Albeverio, Kaneko, Kigami, Yasuda, Karwowski and Vilela-Mendes. Karwowski and Vilela-Mendes focused on the case where the infinitesimal generators are given as modified Vladimirov operator in [18]. The study undertaken Kaneko addressed a class of Hunt processes which admits infinitesimal generators with no apparent symmetry in their description in [29]. Yasuda emphasized that rotational invariance is not essestial in constructing semi-stable processes by taking limit of suitably scaled sums in [44].

Subsequently to those schemes, a research in another direction was undertaken by Albeverio and Karwowski ([2]). They proposed a space consisting of ends of a tree as an even more generalized state space than $\mathbb{Q}_p$. They constructed jump processes on the space consisting of the ends which illustrates a random motion of particle driven by jump to an end from another, where the randomness of the motion is interpreted by the notion of conductance. They revealed that the whole figure of their class is captured within the scheme of the Dirichlet space theory, the infinitesimal generator is written in terms directly of eigenfuctions as in [1]. This approach can be viewed as the intuitive method where the conductance comes first and the families of eigenfunctions and eigenvalues emerges later. In this case, tangible description of the symmetry of the operator is not given similarly to the case in [29].

Recently, Kigami proposed a class of Hunt process on the space Σ^+ consisting of all ends as a wider coverage to overview the existing frameworks on measure symmetric Markov processes on non-Archimedean spaces where the eigenvalues and the eigenfunctions are provided first so as to be the ones associated with the nodewise given Dirichlet forms [22]. Since it turns out that eventually these are the ones of the infinitesimal generator of Hunt process in the scheme, this approach is grasped as the study where the families of eigenfunctions and eigenvalues comes first and the conductance emerges later.

In this thesis, we will present to some results of the the study concerning probability on p-adic integers.

In chapter 2, we will see that a modified p-adic Van der Corput sequence provides us with a reasonable counterpart of Weyl's irrational rotation in the ring. For the ring of p-adic integers, any sequence which plays a similar role to Weyl's irrational rotation has not been proposed yet. We will present a similar random Weyl sampling on the ring to the one proposed by Sugita and Takanobu. In the process of establishing the counterpart, a sampling method based on a function with naturally extended domain to the field of p-adic numbers in terms of the additive characters will be mentioned.

In chapter 3, we will give a method of reconstruction of a random walk in the ring of p-adic integers. Fractals and its self-similarity was investigated well by Hutchinson in [16], Hohlfeld and N. Cohen in [15]. Paying attention to a structural importance in the self similarity, we will perform the construction by means of modified method for constructing canonical stochastic processes on fractals in the Euclidean space. As a result, we will obtain an important subfamily of Albeverio and Karwowski's stochastic processes in [1] with a self-similar randomness.

In chapter 4, we establish an accommodated procedure to show the convergence of Markov processes on the ring of p-adic integers which emerges from a construction of random fractal. So far, the convergence of the stochastic process are discussed by such as Kolesnikov in [23], [24], Kuwae and Shioya in [27]. As seen in other studies on the subject, the notion of generalized Mosco-convergence will be highlighted.

In chapter 5, we will take an advantage of probabilistic counterpart of the Bessel kernels and define Sobolev-Orlicz capacity on ends of a tree. These procedures enable us to derive capacitary estimates from a spectral analytic overview based on recent development of stochastic analytic schemes on the ends of tree. More specifically, we will focus on capacitary estimates for singleton given as an end.

1.2 p-adic numbers and its basic property

For the prime nunber $p \in \mathbb{N}$, we set

$$\mathbb{Q}_p = \left\{ \sum_{i=N}^{\infty} a_i p^i \,\middle|\, a_i \in \{0, 1, \ldots, p-1\},\ N \in \mathbb{Z} \right\}$$

and for any $x \in \mathbb{Q}_p$,

$$\|x\|_p = \begin{cases} p^{-\operatorname{ord}_p x} & (x \neq 0), \\ 0 & (x = 0) \end{cases}$$

where

$$\operatorname{ord}_p x = \begin{cases} \text{the highest power of } p \text{ which divides } x & x \in \mathbb{Z}, \\ \operatorname{ord}_p a - \operatorname{ord}_p b, & x = a/b, a, b \in \mathbb{Z},\ b = 0. \end{cases}$$

$\mathbb{Q}_p$ is cald p-adic field and $\|\cdot\|_p$ is p-adic norm. In paticular, we set the p-adic integers that

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid \|z\|_p \leq 1\}.$$

$\|\cdot\|_p$ is norm on $\mathbb{Q}_p$ and $\mathbb{Z}_p$, futheremore satifying that

$$\|x + y\|_p \leq \max(\|y\|_p, \|y\|_p)\ (\leq \|x\|_p + \|y\|_p).$$

This inequality is called the strong triangle property. Thus, $\mathbb{Q}_p$ becomes ultra metric space with non-Archimedian properties.

Chapter 2

A pairwise independent random sampling method in the ring of p-adic integers

2.1 Hybrid effect of random and algorithmic samplings

For the ring $\mathbb{Z}_p$ of p-adic integers, p being a fixed prime, any sequence which plays a similar role to Weyl's irrational rotation has not been proposed yet. In mainframe of the chapter, we are going to investigate how a sequence of points in the ring can be generated relying on algorithmic procedure, aiming at an approximation to the integral of function with respect to the Haar measure on the ring $\mathbb{Z}_p$ without losing advantages in use of the sequence given by purely random choice of points. To achieve this objective on $\mathbb{Z}_p$ in this chapter, we will introduce a sequence on $\mathbb{Z}_p$ hinted by p-adic Van der Corput sequence, similarly to the sequence with randomness proposed by Sugita and Takanobu on the multidimensional torus.

Numerical approximation methods with an empirical average of function at algorithmically generated points could result unsatisfactory rate of convergence to the integral, if the function takes exceptional values at those sampling points. To avoid such a problem, we can shift our focus onto so called i.i.d.-sampling, which is a core idea supporting the Monte-Carlo method. Sugita and Takanobu mentioned in [39] two facts on the i.i.d.-sampling, one of which says that sampling with large sample size provides us with a secure approximation for square integrable functions and the other says that the i.i.d.-sampling with large sample size responds the quality of the generated pseudorandom numbers, i.e., the statistical bias of them may largely be amplified and diminish the quality of the sampling method. We can find some advantage in a sequence of randomly generated points on the state space. However, it may create a problem arising from statistical bias.

For improving this dichotomous situation, Sugita and Takanobu focused on a sequence with a hybrid effect of random and algorithmic choice of sampling points and proposed a sampling method with the sequence $\{\{x + n\alpha\}\}_{n=0}^{\infty}$ each term of which is given as the fractional part $\{x + n\alpha\}$ of $x + n\alpha$ with a random initial value x and a random common difference α in the k-dimensional torus T^k.

As for the algorithmically generated points on $\mathbb{Z}_p$, it has been revealed in [11] that such non-random sequence of numbers as the sequence of non-negative integers in $\mathbb{Z}_p$ plays a similar role to the p-adic Van der Corput sequence in the unit interval as traditionally studied in [25]. However, non-random sequence of sampling points could again result unsatisfactory rate of convergence, if the integrand has exceptional values at those sampling points.

One might imagine that the sequence $\{x+n\alpha\}_{n=0}^{\infty}$ with randomly taken initial value x and common difference α from $\mathbb{Z}_p$ gives us some hints. However, one fails to achieve this by simply using the sequence. In fact, when x and α is taken from a ball centered at zero with a small radius, the non-archimedean inequality shows that $\|x+n\alpha\|_p \leq \max\{\|x\|_p, \|\alpha\|_p\}$. This fact results that the empirical average can not cover the value of the function at any points outside the ball centered at zero with radius $\max\{\|x\|_p, \|\alpha\|_p\}$.

In this chapter, instead of non-negative integers, we will use the p-adic Van der Corput sequence for approximating the integral of functions on $\mathbb{Z}_p$. Let

$$D_p = \{0, 1, \dots, p-1\} \tag{2.1}$$

and

$$L = \left\{ \frac{a_{-1}}{p} + \cdots + \frac{a_{-M}}{p^M} \;\middle|\; M \in \{1, 2, \dots\},\ a_{-1}, \dots, a_{-M} \in D_p \right\}. \tag{2.2}$$

We define the fractional part $\{x\}_p$ of $x \in \mathbb{Q}_p$ as a unique element $y \in L$ which satisfies $x - y \in \mathbb{Z}_p$. Accordingly, the integer part $[x]_p$ of $x \in \mathbb{Q}_p$ is defined by $[x]_p = x - \{x\}_p$.

In Section 2, we will see that, for any $\alpha \in \mathbb{Z}_p$, the subset $\{[\frac{n\alpha}{p^m}]_p \mid n \in \{0, 1, \dots, p^m - 1\}, m \in \{1, 2, \dots\}\}$ in the ring $\mathbb{Z}_p$ of p-adic integers is dense in the ring if and only if $\alpha \notin \mathbb{Q}$. This suggests that the p-adic Van der Corput sequence provides us with a counterpart of Weyl's irrational rotation in $\mathbb{Z}_p$.

In the one-dimensional case in [39], the method with the Fourier series is employed based on the complete orthogonal system $\{e^{2\pi\sqrt{-1}kt}\}_{k\in\mathbb{Z}}$ in $L^2([0,1))$, which is viewed as a sequence of periodic functions on the real line with period 1. The fundamental system of functions is used for extending domain of functions to the real line without removing the integer part of the variable of functions in the description. In accordance with the procedure, we will take a complete orthogonal system described by the additive characters on $\mathbb{Q}_p$ for extending the original domain $\mathbb{Z}_p$ of functions to the whole space $\mathbb{Q}_p$ without removing the fractional part of variable of functions in their description. For square integrable function f with respect to the Haar measure on $\mathbb{Z}_p$, we will examine the behavior of the sequence $\{\frac{1}{N}\sum_{k=0}^{N-1} f(x + x_k\alpha) - \int_{\mathbb{Z}_p} f(y)\,dy\}$ involving the k-th term x_k of the p-adic Van der Corput sequence and f with the extended domain, under independently random choice of $x, \alpha \in \mathbb{Z}_p$ for achieving a similar result to random sampling method established by Sugita and Takanobu in [39]. We will finally be in a position to regard this approximation for $\int_{\mathbb{Z}_p} f(y)\,dy$ as the one based on a modified random Weyl sampling.

The authors express their gratitude to the reviewer for his or her valuable suggestion. The authors were not able to describe this modified random sampling in a smart fashion without the reviewer's insightful suggestion.

2.2 Fundamental property of p-adic Van der Corput sequence in $\mathbb{Z}_p$

For the random Weyl sampling in the unit interval in the real line based on Weyl's irrational rotation, one takes the sequence each term of which is given as the fractional part of the product of non-negative integer and a fixed irrational real number. As pointed out in Introduction, it is required to find some sequence in $\mathbb{Z}_p$ other than the one involving an irrational number as common difference for creating a similar effect to Weyl's irrational rotation. We will make an attempt of taking the integer part of the terms in the sequence in $\mathbb{Q}_p$ obtained as the product of a fixed number in $\alpha \in \mathbb{Z}_p \setminus \mathbb{Q}$ and the p-adic Van der Corput sequence.

Proposition 2.1 *Let $\alpha \in \mathbb{Z}_p$. The set*

$$U(\alpha) = \left\{ \left[\frac{n\alpha}{p^m} \right]_p \,\middle|\, n \in \{0, 1, \ldots, p^m - 1\}, m \in \{1, 2, \ldots\} \right\}$$

is dense in $\mathbb{Z}_p$, if and only if $\alpha \notin \mathbb{Q}$.

2.3 A modified Random Weyl sampling on $\mathbb{Z}_p$

In this section, we will present some results for establishing a reasonable modified random Weyl sampling on $\mathbb{Z}_p$ by taking the results of the previous section into account. In what follows, the Haar measure on $\mathbb{Z}_p$ will be denoted by μ and assumed to be normalized as $\mu(\mathbb{Z}_p) = 1$. The integral of complex valued integrable function f on $\mathbb{Z}_p$ with respect to the Haar measure will be denoted by $\int_{\mathbb{Z}_p} f(y)\,dy$.

Definition 2.2 A random element y in $\mathbb{Z}_p$ is said to be uniformly distributed if $P(y \in E) = \mu(E)$ for any topological Borel subset E in $\mathbb{Z}_p$. For any complex-valued square integrable function f on $\mathbb{Z}_p$, the variance $\mathrm{Var}(f)$ of the function is defined by

$$\mathrm{Var}(f) = \int_{\mathbb{Z}_p} \left| f(x) - \int_{\mathbb{Z}_p} f(y)\,dy \right|^2 dx.$$

We introduce the Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{Z}_p} f(x) e^{2\sqrt{-1}\pi\{\xi x\}_p}\,dx, \qquad \xi \in \mathbb{Q}_p,$$

for any complex valued square integrable function f. Then, the original function f is restored as

$$f(x) = \int_{\mathbb{Q}_p} \hat{f}(\xi) e^{-2\sqrt{-1}\pi\{\xi x\}_p}\,d\xi, \qquad x \in \mathbb{Z}_p,$$

by performing the inverse Fourier transform (see Chapter 1, VIII in [42]). Since f can be regarded as a function on $\mathbb{Q}_p$ whose support is contained in $\mathbb{Z}_p$, i.e., the ball $B(0, 1)$ centered at the origin and with radius 1, $\hat{f}$ takes a constant on every ball with radius 1 as seen in Chapter 1, VII in [42]. Accordingly, another representation of the function is given as

$$\begin{aligned} f(x) &= \sum_{\xi \in L} \hat{f}(\xi) \int_{B(0,1)} e^{-2\sqrt{-1}\pi\{(\xi+\eta)x\}_p}\, d\eta \\ &= \sum_{\xi \in L} \hat{f}(\xi) e^{-2\sqrt{-1}\pi\{\xi x\}_p}, \qquad x \in \mathbb{Z}_p, \end{aligned}$$

where L has been defined by (2.2).

As for the additive character $\chi(\xi t) = e^{2\sqrt{-1}\pi\{\xi t\}_p}$, we easily observe that

$$\int_{\mathbb{Z}_p} |\chi(\xi t)|^2\, dt = \int_{\mathbb{Z}_p} \chi(\xi t)\chi(-\xi t)\, dt = \int_{\mathbb{Z}_p} \chi((\xi - \xi)t)\, dt = 1$$

for any $\xi \in L$ and

$$\int_{\mathbb{Z}_p} \chi(\xi t)\overline{\chi}(\eta t)\, dt = \int_{\mathbb{Z}_p} \chi((\xi - \eta)t)\, dt = 0$$

for any pair of distinct $\xi, \eta \in L$. From these identities, we can derive

$$\int_{\mathbb{Z}_p} |f(t)|^2\, dt = \sum_{\xi \in L} |\hat{f}(\xi)|^2.$$

Consequently, we see that the additive characters provide us with the complete orthonormal system $\{\chi(\xi t)\}_{\xi \in L}$ in $L^2(\mathbb{Z}_p, \mu)$.

A natural extension of the domain of f to $\mathbb{Q}_p$ is performed by

$$f(x) = \sum_{\xi \in L} \hat{f}(\xi) e^{-2\sqrt{-1}\pi\{\xi x\}_p} = \sum_{\xi \in L} \hat{f}(\xi)\chi(-\xi x), \qquad x \in \mathbb{Q}_p.$$

For any positive integer M, we take function f_M on $\mathbb{Q}_p$ defined by

$$\begin{aligned} f_M(x) &= \sum_{\xi \in L \cap B(0,p^M)} \hat{f}(\xi) \int_{B(0,1)} e^{-2\sqrt{-1}\pi\{(\xi+\eta)x\}_p} d\eta \\ &= \sum_{\xi \in L \cap B(0,p^M)} \hat{f}(\xi)\chi(-\xi x), \qquad x \in \mathbb{Q}_p, \end{aligned} \tag{2.3}$$

where $B(0, p^M)$ stands for the ball centered at the origin with radius p^M.

We recall that the sequence

$$x_k = \frac{d_0}{p} + \cdots + \frac{d_\ell}{p^{\ell+1}} \quad (k = 0, 1, 2, \dots)$$

determined by the p-adic expansion $k = d_0 + \cdots + d_\ell p^\ell$ of non-negative integer k constitutes the p-adic Van der Corput sequence in [37]. Hereafter, we will take the p-adic Van der Corput sequence $\{x_k\}_{k=0}^{\infty}$. We will focus only on $f(x+x_k\alpha)$ and $f_M(x+x_k\alpha)$ instead of $f(x+[x_k\alpha]_p)$ and $f_M(x+[x_k\alpha]_p)$ respectively without removing fractional part of variables, similarly to the method of extending domain of functions in [39] without removing integer part of the variables.

In the first theorem in this section, we will consider the sequence

$$\left\{\frac{1}{N}\sum_{k=0}^{N-1} f(x+x_k\alpha)\right\}_{N=1}^{\infty}$$

with uniformly distributed independent random variables α and x on $\mathbb{Z}_p$. Our main interest is under what condition the sequence

$$\left\{\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\left(f(x+x_k\alpha)-\int_{\mathbb{Z}_p} f(y)\,dy\right)\right\}_{N=1}^{\infty}$$

with a slower growth rate in the denominator can be expected to converge to zero as $N\to\infty$ for any square integrable function f on $\mathbb{Z}_p$. In this chapter, the method focusing on the sequence for fast approximation to the integral $\int_{\mathbb{Z}_p} f(y)\,dy$ by the empirical average

$$\frac{1}{N}\sum_{k=0}^{N-1} f(x+x_k\alpha)$$

is called modified random Weyl sampling on $\mathbb{Z}_p$.

We see that the modified random Weyl sampling on $\mathbb{Z}_p$ has the robustness in the sense in [40] as the method proposed by Sugita and Takanobu.

Lemma 2.3 *Let f be a complex valued function in $L^2(\mathbb{Z}_p,\mu)$ and $\xi\in L\setminus\{0\}$. Then,*

(i) $f(x+\alpha\xi)\in L^2(\mathbb{Z}_p\times\mathbb{Z}_p,\mu\times\mu)$,

(ii) $\lim\limits_{M\to\infty}\|f(x+\alpha\xi)-f_M(x+\alpha\xi)\|_{L^2(\mathbb{Z}_p\times\mathbb{Z}_p,\mu\times\mu)}=0$,

(iii) $$\iint_{\mathbb{Z}_p\times\mathbb{Z}_p} f(x+\alpha\xi)\,dxd\alpha=\int_{\mathbb{Z}_p} f(y)\,dy,$$

$$\iint_{\mathbb{Z}_p\times\mathbb{Z}_p}\left|f(x+\alpha\xi)-\int_{\mathbb{Z}_p} f(y)\,dy\right|^2 dxd\alpha=\mathrm{Var}(f),$$

(iv) $\xi'\in L\setminus\{0\}$ *and* $\xi'\neq\xi$ *imply*

$$\iint_{\mathbb{Z}_p\times\mathbb{Z}_p}\left(f(x+\alpha\xi)-\int_{\mathbb{Z}_p} f(y)\,dy\right)\overline{\left(g(x+\alpha\xi')-\int_{\mathbb{Z}_p} g(y)\,dy\right)}\,dxd\alpha=0,$$

for any complex valued function g in $L^2(\mathbb{Z}_p,\mu)$.

Theorem 2.4 *For any complex valued function $f \in L^2(\mathbb{Z}_p, \mu)$,*

$$\left\{ f(x+\alpha x_n) - \int_{\mathbb{Z}_p} f(y)\, dy \right\}_{n=0}^{\infty}$$

constitute an orthonormal family in $L^2(\mathbb{Z}_p \times \mathbb{Z}_p, \mu \times \mu)$ satisfying

$$\iint_{\mathbb{Z}_p \times \mathbb{Z}_p} \left| f(x+\alpha x_n) - \int_{\mathbb{Z}_p} f(y)\, dy \right|^2 dx d\alpha = \mathrm{Var}(f).$$

In particular,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(x+\alpha x_n) = \int_{\mathbb{Z}_p} f(y)\, dy \quad \mu \times \mu\text{-a.e. } (x, \alpha),$$

and for any positive integer N,

$$\iint_{\mathbb{Z}_p \times \mathbb{Z}_p} \left| \frac{1}{N} \sum_{n=0}^{N-1} f(x+\alpha x_n) - \int_{\mathbb{Z}_p} f(y)\, dy \right|^2 dx d\alpha = \frac{1}{N} \mathrm{Var}(f).$$

Let us recall $D_p = \{0, 1, 2, \ldots, p-1\}$ (cf. (2.1)). To each integer N with $N-1 \geq p^2$, there corresponds a unique pair of sequence of integers $h_1 > h_2 > \cdots > h_s \geq 0$ and $r_1, r_2, \ldots, r_s \in D_p \setminus \{0\}$ such that

$$N - 1 = r_1 p^{h_1} + r_2 p^{h_2} + \cdots + r_s p^{h_s}, \tag{2.4}$$

where $s = \max\{\ell \in \{1, 2, \ldots\} \mid N-1 \geq p^\ell\}$. We let (ν_1, ν_2) denote the greatest common divisor of integers ν_1 and ν_2.

Lemma 2.5 *For any q with $1 < q < 2$, any $\xi \in L \setminus \{0\}$, and any $N \in \{1, 2, \ldots\}$ with $N-1 \geq p^2$,*

(i) *under the correspondence* (2.4)

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \left| \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \chi(\xi \alpha x_n) \right|^q d\alpha \\
&\leq \frac{1}{\sqrt{N}} + \frac{1}{\sqrt{N}} \sum_{j=1}^{s} \left(\int_{\mathbb{Z}_p} \left| \sum_{a=0}^{r_j-1} \sum_{n=0}^{p^{h_j}-1} e^{2\sqrt{-1}\pi a \left\{ \frac{\xi\alpha}{p^{h_j+1}} \right\}_p} e^{2\sqrt{-1}\pi n \left\{ \frac{\xi\alpha}{p^{h_j}} \right\}_p} \right|^q d\alpha \right)^{1/q},
\end{aligned}$$

where $s = \max\{\ell \mid \ell$ is positive integer satisfying $N-1 \geq p^\ell\}$,

(ii) *for any* $j \in \{1, \cdots, s\}$ *and* $\xi \in L \setminus \{0\}$ *represented as* $\xi = \frac{\nu}{p^M}$ *with* $\nu \in \{0, \ldots, p^M - 1\}$ *and* $(\nu, p) = 1$,

$$\int_{\mathbb{Z}_p} \left| \sum_{a=0}^{r_j - 1} \sum_{n=0}^{p^{h_j}-1} e^{2\sqrt{-1}\pi a \left\{ \frac{\xi\alpha}{p^{h_j+1}} \right\}_p} e^{2\sqrt{-1}\pi n \left\{ \frac{\xi\alpha}{p^{h_j}} \right\}_p} \right|^q d\alpha$$
$$= \frac{1}{p^M} \frac{1}{p^{h_j+1}} \left((r_j p^{h_j})^q + \sum_{c_0 \in D_p \setminus \{0\}} \left| \frac{\sin \pi r_j \frac{c_0}{p}}{\sin \pi \frac{c_0}{p}} p^{h_j} \right|^q \right.$$
$$\left. + \sum_{l=h_j+1}^{M+h_j} \sum_{\substack{c_0 \in D_p \setminus \{0\} \\ c_1, \ldots, c_l \in D_p}} \left| \frac{\sin \pi r_j \frac{c_0 + c_1 p + \cdots + c_l p^l}{p^{l+1}}}{\sin \pi \frac{c_0 + c_1 p + \cdots + c_l p^l}{p^{l+1}}} \frac{\sin \pi p^{h_j} \frac{c_0 + c_1 p + \cdots + c_{l-1} p^{l-1}}{p^l}}{\sin \pi \frac{c_0 + c_1 p + \cdots + c_{l-1} p^{l-1}}{p^l}} \right|^q \right),$$

(iii) *for any positive integer* M, *non-negative integer* h *and* $r \in D_p$,

$$\sum_{l=h+1}^{M+h} \sum_{\substack{c_0 \in D_p \setminus \{0\} \\ c_1, \ldots, c_l \in D_p}} \left| \frac{\sin \pi r \frac{c_0 + c_1 p + \cdots + c_l p^l}{p^{l+1}}}{\sin \pi \frac{c_0 + c_1 p + \cdots + c_l p^l}{p^{l+1}}} \frac{\sin \pi p^h \frac{c_0 + c_1 p + \cdots + c_{l-1} p^{l-1}}{p^l}}{\sin \pi \frac{c_0 + c_1 p + \cdots + c_{l-1} p^{l-1}}{p^l}} \right|^q$$
$$\leq p|r|^q \sum_{l=h+1}^{M+h} \sum_{a=1}^{p^l - 1} \frac{1}{\left| \sin \pi \frac{a}{p^l} \right|^q}$$
$$\leq (p-1)^q \frac{M p^{Mq} \zeta(q)}{2^{q-1}} p^{hq+1},$$

where ζ *stands for the Riemann zeta function, i.e.,* $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$.

Proposition 2.6 *For any* $\xi \in L \setminus \{0\}$,

$$\lim_{N \to \infty} \left(\int_{\mathbb{Z}_p} \left| \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \chi(\xi \alpha x_n) \right|^q d\alpha \right)^{\frac{1}{q}} = 0.$$

Theorem 2.7 *For any* q *satisfying* $1 < q < 2$ *and complex-valued function* $f \in L^2(\mathbb{Z}_p, \mu)$,

$$\lim_{N \to \infty} \iint_{\mathbb{Z}_p \times \mathbb{Z}_p} \left| \sqrt{N} \left(\frac{1}{N} \sum_{n=0}^{N-1} f(x + \alpha x_n) - \int_{\mathbb{Z}_p} f(y)\, dy \right) \right|^q dx d\alpha = 0.$$

Chapter 3

A Dirichlet space associated with consistent networks on the ring of p-adic integers

3.1 Various compatible sequences of networks

Stochastic analysis on fractals on was initiated from the 80's to 90's and developed mainly by M. Fukushima, K. Hattori, T. Hattori, J. Kigami, S. Kusuoka, T. Kumagai and V. Metz based on its self-similar structure. In the latter the decade, a study on stochastic processes on the field of p-adic numbers was started by S. Albeverio and W. Karwowski. Their method relied on its hierarchical structure and was established but in an independent fashion of the one in analysis on fractals, whereas Haar measure with a self-similarilty was imported and it played central role in constructing such function spaces as Dirichlet spaces as in the case of fractal analysis.

A remarkable development of analysis on fractals has been made by taking an increasing family $\{V_m\}_{m=0}^{\infty}$ of finite sets and introducing a family $\{H_m\}_{m=0}^{\infty}$ of Laplacians on $\{V_m\}_{m=0}^{\infty}$ so that each Laplacian H_m in the family is given as a linear map $H_m : \ell(V_m) \to \ell(V_m)$, where $\ell(V_m)$ stands for the set of maps from V_m to $\mathbb{R}$. Then, the sequence $\{\mathcal{E}_{H_m}\}_{m=0}^{\infty}$ of Dirichlet forms is associated with H_m through the identity $\mathcal{E}_{H_m}(u,v) = -(u, H_m v)$ $(u, v \in \ell(V_m))$ for each non-negative integer m, as described in the monograph written by J. Kigami [20]. When the maps $F_0, \ldots, F_{N-1}$ are contractions characterizing the self-similar structure of a fractal K as $K = F_0(K) \cup \cdots \cup F_{N-1}(K)$ in the Eucldean space and $(H_0, \mathbf{r})$ is a regular harmonic structure with some sequence $\mathbf{r} = (r_0, \ldots, r_{N-1})$ of N positive real parameters s in the sense in [20], a canonical diffusion process on K is obtained by the Dirichlet space given as a limit of the Dirichlet forms recursively determined by

$$\begin{cases} V_{m+1} = F_0(V_m) \cup \cdots \cup F_{N-1}(V_m), \\ \mathcal{E}_{H_{m+1}}(u,v) = \displaystyle\sum_{i=0}^{N-1} \frac{1}{r_i} \mathcal{E}_{H_m}(u \circ F_i, v \circ F_i) \quad \text{for } u, v \in \ell(V_{m+1}) \end{cases}$$

with an initial set V_0 of geometrically configured vertices in K.

The harmonic structure, which is de facto standard structural grip on fractals, is represented by a pair $(H_0, \mathbf{r})$ of the initial Laplacian H_0 on V_0 and the sequence $\mathbf{r}$, when

$$\mathcal{E}_{H_0}(u,u) = \mathcal{E}_{T_{V_0} - {}^t J_{V_0} X_{V_0}^{-1} J_{V_0}}(u,u) = \min_{v \in \ell(V_1), v|_{V_0} = u} \mathcal{E}_{H_1}(v,v)$$

is satisfied for the submatrices T_{V_0}, J_{V_0} and X_{V_0} of H_0 each of which is given as the linear map $T_{V_0} : \ell(V_0) \to \ell(V_0)$, $J_{V_0} : \ell(V_0) \to \ell(V_1 \backslash V_0)$ and $X_{V_0} : \ell(V_1 \backslash V_0) \to \ell(V_1 \backslash V_0)$ respectively. Then, it turns out that $\{(V_m, H_m)\}_{m=0}^{\infty}$ is a consistent sequence of $\mathbf{r}$-networks in the sense in the monograph [20]. If $0 < r_i < 1$ for all $i = 0, 1, \ldots, N-1$, a harmonic structure $(H_0, \mathbf{r})$ is said to be regular and a pilot scheme for analysis on fractals was built based on the regular harmonic structure. In fact, if a harmonic structure $(H_0, \mathbf{r})$ is given, for any $x, y \in V_* = \bigcup_m V_m$, the effective resistance

$$R(x,y) = (\min\{\mathcal{E}_{H_m}(u,u) \mid u \in \ell(V_m), u(x) = 1, u(y) = 0\})^{-1},$$

determined independently by the choice of m with $x, y \in V_m$, defines a metric on V_* and the regularity gives a sufficient condition for the completion Ω of V_* with respect to R to be embedded into the fractal K. The metric R plays a crucial role in the main scheme in [20].

In this chapter, we will take V_* as the set of all non-negative integers contained in the ring $\mathbb{Z}_p$ of p-adic integers so that a similar object to the compatible sequence $\{(V_m, H_m)\}_{m=0}^{\infty}$ of $\mathbf{r}$-networks is constructed. We will see that the completion of V_* with respect to the resistance metric is homeomorphic to $\mathbb{Z}_p$ with the p-adic distance. As a result, we will see the networks obtained by our procedure generates a Dirichet form corresponding to the Markov process in the subfamily of random walk constructed by Albeverio and Karwowski in [1].

The existing theories of Markov processes on fractals are widely appreciated as a versatile probabilistic coverage for such typical fractals as Sierpinski gasket, pentaflake and snowflake with continuous random motion guided by contractions charcterizing their geometric self-similarities. By contrast, p-adic case, jumps communicating any couple of disjoint balls are required to be feasible at each hierarchical level in the iterative method, as seen in the random walk constructed by Albeverio and Karwowski in [1]. Accrodingly some maps inducing random jumps can not be contractive in this case and it will be required to cover all rings $\mathbb{Z}_p$ determined by an arbitrarily given prime number p. Similarly to the existing approach through the compatible sequence $\{(V_m, H_m)\}_{m=0}^{\infty}$ of $\mathbf{r}$-network associated with some family of maps $F_0, \ldots, F_{N-1} : V_* \to V_*$ with $V_* = \bigcup_{m=0}^{\infty} V_m$, we apply the network method to cover potential theoretic features in the ring of the p-adic integers and construct a canonical Hunt process on the ring, however, we will modify the method for our use in non-Archimedean environment.

In Section 2, we will introduce a family of maps $F_0, \ldots, F_{p+p^p-1}$ from $V_{0,\mathbb{Z}_p} = \{0, \ldots, p-1\}$ to $V_{1,\mathbb{Z}_p} = \{0, \ldots, p^2-1\}$ guiding the self-similar random jumps of particle's motion with Markov property. In Section 3, we will slightly modify the framework established in [20] so that jumps guided by map with no contraction is allowed in the modified method. In Section 4, we will see a Dirichlet space naturally built on the ring $\mathbb{Z}_p$ corresponding to a random walk constructed by S. Albeverio and W. Karwowski.

3.2 Sequence of r-networks on $\mathbb{Z}_p$

We will define a family of maps $\{F_0, \dots, F_{p-1}\}$ which characterizes a self-similarity of $\mathbb{Z}_p$ and works cohesively with the existing probabilistic fractal theory.

In the ring $\mathbb{Z}_p$, viewed as the ball centered at the origin with the radius 1 in the field $\mathbb{Q}_p$ of p-adic integers, for any $x \in \mathbb{Z}_p$ and non-negative integer m, we denote the ball $\{y \in \mathbb{Z}_p \mid \|y-x\|_p \leq p^{-m}\}$ by $B_m(x)$ and we introduce contraction $F_i : \mathbb{Z}_p \to B_1(i)$ given by $F_i(x) = px + i$ for each $i = 0, 1, 2, \dots, p-1$. Then, we see that the direct sum $\mathbb{Z}_p = F_0(\mathbb{Z}_p) \cup \dots \cup F_{p-1}(\mathbb{Z}_p)$ shows the simplest self-similar structure for $\mathbb{Z}_p$. By taking $V_{0,\mathbb{Z}_p} = \{0, 1, 2, \dots, p-1\}$, as the union of the images $F_0(V_{0,\mathbb{Z}_p}), \dots, F_{p-1}(V_{0,\mathbb{Z}_p})$ we have $V_{1,\mathbb{Z}_p} = \{0, 1, 2, \dots, p^2-1\}$ and inductively we obtain $V_{m,\mathbb{Z}_p} = \{0, 1, 2, \dots, p^{m+1}-1\}$ as the union of images $F_0(V_{m-1,\mathbb{Z}_p}), \dots, F_{p-1}(V_{m-1,\mathbb{Z}_p})$ for any positive integer m.

Let us establish a framework which presides over random motions of the particle by means of the Dirichlet form and Laplacian as given by J. Kigami based on Definition 2.1.1 and Definition 2.1.2 in the monograph [20]. For that purpose, we introduce the Laplacian $H_{0,\mathbb{Z}_p}$ on $V_{0,\mathbb{Z}_p} = \{0, \dots, p-1\}$ by

$$H_{0,\mathbb{Z}_p} = \begin{pmatrix} -(p-1) & 1 & \dots & 1 & 1 \\ 1 & -(p-1) & 1 & \dots & 1 \\ \vdots & & \ddots & & \vdots \\ 1 & \dots & \dots & 1 & -(p-1) \end{pmatrix}$$

and the linear space $\ell(V_{m,\mathbb{Z}_p})$ given as the set of real-valued functions defined on $V_{m,\mathbb{Z}_p}$ for any non-negative integer m. For the initial setup, the Dirichlet form $\mathcal{E}_{H_{0,\mathbb{Z}_p}}$ with the domain $\ell(V_{0,\mathbb{Z}_p})$ associated with the Laplacian $H_{0,\mathbb{Z}_p}$ is given by $\mathcal{E}_{H_{0,\mathbb{Z}_p}}(u,v) = -(u, H_{0,\mathbb{Z}_p} v)$ for $u, v \in \ell(V_{0,\mathbb{Z}_p})$.

Since the probability of transition from $B_2(k)$ to $B_2(\ell)$ with $\|k-\ell\|_p = \frac{1}{p}$ is positive and larger than the one from $B_2(k')$ to $B_2(\ell')$ with $\|k'-\ell'\|_p = 1$ for typical Hunt processes constructed by Albeverio and Karwowski [1], which means the matrix representation of the Laplacian $H_{1,\mathbb{Z}_p}$ on $V_{1,\mathbb{Z}_p}$ is required to be obtained with distinct off-diagonal components, another family $\{F_p, \dots, F_{p+p^p-1}\}$ of maps from $V_{0,\mathbb{Z}_p}$ to $V_{1,\mathbb{Z}_p}$ is required to induce jumps from $B_2(k')$ to $B_2(\ell')$ with $\|k'-\ell'\|_p = 1$.

Accordingly, we introduce the family $\{F_p, \dots, F_{p+p^p-1}\}$ of maps by

$$F_j(x) = \begin{cases} px + m_0 p & \text{on } B_1(0), \\ px + m_1 p - (p-1) & \text{on } B_1(1), \\ px + m_2 p - 2(p-1) & \text{on } B_1(2), \\ \dots & \\ px + m_{p-1} p - (p-1)^2 & \text{on } B_1(p-1), \end{cases}$$

where $j = p + m_0 + m_1 p + \dots + m_{p-1} p^{p-1}$ with $m_0, \dots, m_{p-1} \in \{0, \dots, p-1\}$.

By taking $\mathbf{r} = (r_0, r_1, \ldots, r_{p+p^p-1}) = \Big(\underbrace{\frac{p^2}{c}, \frac{p^2}{c}, \ldots, \frac{p^2}{c}}_{p}, \underbrace{p^p, p^p, \ldots, p^p}_{p^p}\Big)$ with some positive real constant c and the inductive procedure already introduced in fractal analysis, the sequence $\{\mathcal{E}_{H_{m,\mathbb{Z}_p}}\}_{m=0}^{\infty}$ of Dirichlet forms is determined by the fornula:

$$\mathcal{E}_{H_{m+1,\mathbb{Z}_p}}(u, v) = \sum_{i=0}^{p+p^p-1} \frac{1}{r_i} \mathcal{E}_{H_{m,\mathbb{Z}_p}}(u \circ F_i, v \circ F_i),$$

each domain of which is given as $\ell(V_{m+1,\mathbb{Z}_p})$. Through these procedure, we have obtained a sequence $\{(V_{m,\mathbb{Z}_p}, H_{m+1,\mathbb{Z}_p})\}$ of $\mathbf{r}$-networks on $\mathbb{Z}_p$.

Example 3.1 (Consistency of Dirichlet forms on $\mathbb{Z}_2$) In the case $p = 2$, the Laplacian $H_{0,\mathbb{Z}_2}$ on $V_{0,\mathbb{Z}_2} = \{0, 1\}$ is given by

$$H_{0,\mathbb{Z}_2} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

and the sequence of real parameters $\mathbf{r}$ is given by $\mathbf{r} = (2^2/c, 2^2/c, 2^2, 2^2, 2^2, 2^2)$ and the following maps $F_0, \ldots, F_5$ from $V_{0,\mathbb{Z}_2} = \{0, 1\}$ to $V_{1,\mathbb{Z}_2} = \{0, 2, 1, 3\}$ are required for the sequence of Laplacians:

$$F_0(x) = \quad 2x \quad \text{on } \mathbb{Z}_2, \qquad F_1(x) = \quad 2x+1 \quad \text{on } \mathbb{Z}_2,$$

$$F_2(x) = \begin{cases} 2x & \text{on } B_1(0), \\ 2x-1 & \text{on } B_1(1), \end{cases} \qquad F_3(x) = \begin{cases} 2x+2 & \text{on } B_1(0), \\ 2x-1 & \text{on } B_1(1), \end{cases}$$

$$F_4(x) = \begin{cases} 2x & \text{on } B_1(0), \\ 2x+1 & \text{on } B_1(1), \end{cases} \qquad F_5(x) = \begin{cases} 2x+2 & \text{on } B_1(0), \\ 2x+1 & \text{on } B_1(1), \end{cases}$$

for the recursive definition of the Laplacians. In fact, these maps associate the matrices

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad M_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad M_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad M_5 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

respectively, and we obtain

$$\begin{aligned} H_{1,\mathbb{Z}_2} &= \frac{1}{2^2}\{c({}^tM_0 H_{0,\mathbb{Z}_2} M_0 + {}^tM_1 H_{0,\mathbb{Z}_2} M_1) \\ &\quad + {}^tM_2 H_{0,\mathbb{Z}_2} M_2 + {}^tM_3 H_{0,\mathbb{Z}_2} M_3 + {}^tM_4 H_{0,\mathbb{Z}_2} M_4 + {}^tM_5 H_{0,\mathbb{Z}_2} M_5\} \\ &= \begin{pmatrix} -\alpha(-2) & \frac{c}{2^2} & \frac{1}{2^2} & \frac{1}{2^2} \\ \frac{c}{2^2} & -\alpha(-2) & \frac{1}{2^2} & \frac{1}{2^2} \\ \frac{1}{2^2} & \frac{1}{2^2} & -\alpha(-2) & \frac{c}{2^2} \\ \frac{1}{2^2} & \frac{1}{2^2} & \frac{c}{2^2} & -\alpha(-2) \end{pmatrix} \end{aligned}$$

where $\alpha(-2) = \frac{c+2}{2^2}$. By repeating this procedure, we have

$$
\begin{aligned}
H_{2,\mathbb{Z}_2} = \frac{1}{2^2}\{c({}^tM_0H_{1,\mathbb{Z}_2}M_0 + {}^tM_1H_{1,\mathbb{Z}_2}M_1) \\
+ {}^tM_2H_{1,\mathbb{Z}_2}M_2 + {}^tM_3H_{1,\mathbb{Z}_2}M_3 + {}^tM_4H_{1,\mathbb{Z}_2}M_4 + {}^tM_5H_{1,\mathbb{Z}_2}M_5\}
\end{aligned}
$$

$$
= \begin{pmatrix}
-\alpha(-3) & \frac{c(c+2)}{2^4} & \frac{c}{2^4} & \frac{c}{2^4} & \frac{1}{2^4} & \frac{1}{2^4} & \frac{1}{2^4} & \frac{1}{2^4} \\
\frac{c(c+2)}{2^4} & -\alpha(-3) & \frac{c}{2^4} & \frac{c}{2^4} & \frac{1}{2^4} & \frac{1}{2^4} & \frac{1}{2^4} & \frac{1}{2^4} \\
\frac{c}{2^4} & \frac{c}{2^4} & -\alpha(-3) & \frac{c(c+2)}{2^4} & \frac{1}{2^4} & \frac{1}{2^4} & \frac{1}{2^4} & \frac{1}{2^4} \\
\frac{c}{2^4} & \frac{c}{2^4} & \frac{c(c+2)}{2^4} & -\alpha(-3) & \frac{1}{2^4} & \frac{1}{2^4} & \frac{1}{2^4} & \frac{1}{2^4} \\
\frac{1}{2^4} & \frac{1}{2^4} & \frac{1}{2^4} & \frac{1}{2^4} & -\alpha(-3) & \frac{c(c+2)}{2^4} & \frac{c}{2^4} & \frac{c}{2^4} \\
\frac{1}{2^4} & \frac{1}{2^4} & \frac{1}{2^4} & \frac{1}{2^4} & \frac{c(c+2)}{2^4} & -\alpha(-3) & \frac{c}{2^4} & \frac{c}{2^4} \\
\frac{1}{2^4} & \frac{1}{2^4} & \frac{1}{2^4} & \frac{1}{2^4} & \frac{c}{2^4} & \frac{c}{2^4} & -\alpha(-3) & \frac{c(c+2)}{2^4} \\
\frac{1}{2^4} & \frac{1}{2^4} & \frac{1}{2^4} & \frac{1}{2^4} & \frac{c}{2^4} & \frac{c}{2^4} & \frac{c(c+2)}{2^4} & -\alpha(-3)
\end{pmatrix}
$$

where $\alpha(-3) = \left(\frac{c+2}{2^2}\right)^2$ and the adjustment in size of the matrices $M_0, \ldots, M_5$ is required as

$$
M_0 = \begin{pmatrix} E & O & O & O \\ O & E & O & O \end{pmatrix}, \qquad M_1 = \begin{pmatrix} O & O & E & O \\ O & O & O & E \end{pmatrix},
$$
$$
M_2 = \begin{pmatrix} E & O & O & O \\ O & O & E & O \end{pmatrix}, \qquad M_3 = \begin{pmatrix} O & E & O & O \\ O & O & E & O \end{pmatrix},
$$
$$
M_4 = \begin{pmatrix} E & O & O & O \\ O & O & O & E \end{pmatrix}, \qquad M_5 = \begin{pmatrix} O & E & O & O \\ O & O & O & E \end{pmatrix}
$$

with the submatrices

$$
E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
$$

In the rest of part of the example, we will denote $V_{m,\mathbb{Z}_2}$ simply by V_m for any non-negative integer m. We observe that $B_1(0) = \{x \in \mathbb{Z}_2 \mid \|x\|_2 \leq 1/2\}$ is represented by the element 0 in V_0 and $B_1(0)$ is represented by the element 0 and by 2 in V_1 as well. This observation allows us to denote $B_1(0)$ by the representative subset $F_0(V_0) \cap V_0$ of V_0 and by the representative subset $F_0(V_0)$ of V_1. Similarly, $B_2(0)$ can be represented by the subset $F_0(F_0(V_0) \cap V_0)$ in V_1 and by the subset $F_0(F_0(V_0))$ in V_2 and so on.

The consistency of the bilinear forms $\mathcal{E}_{H_{1,\mathbb{Z}_2}}$ and $\mathcal{E}_{H_{0,\mathbb{Z}_2}}$ in the sense that

$$
\mathcal{E}_{H_{1,\mathbb{Z}_2}}(1_{F_0(V_0)}, 1_{F_1(V_0)}) = \mathcal{E}_{H_{0,\mathbb{Z}_2}}(1_{F_0(V_0)\cap V_0}, 1_{F_1(V_0)\cap V_0}),
$$

is verified by the following procedure with $\alpha(-2) = \frac{c+2}{2^2}$:

$$\begin{aligned}
\mathcal{E}_{H_{1,\mathbb{Z}_2}}(1_{F_0(V_0)}, 1_{F_1(V_0)}) &= -\begin{pmatrix}1 & 1 & 0 & 0\end{pmatrix} H_1 \begin{pmatrix}0\\0\\1\\1\end{pmatrix} \\
&= -\begin{pmatrix}1 & 1 & 0 & 0\end{pmatrix} \begin{pmatrix} -\alpha(-2) & \frac{c}{2^2} & \frac{1}{2^2} & \frac{1}{2^2} \\ \frac{c}{2^2} & -\alpha(-2) & \frac{1}{2^2} & \frac{1}{2^2} \\ \frac{1}{2^2} & \frac{1}{2^2} & -\alpha(-2) & \frac{c}{2^2} \\ \frac{1}{2^2} & \frac{1}{2^2} & \frac{c}{2^2} & -\alpha(-2) \end{pmatrix} \begin{pmatrix}0\\0\\1\\1\end{pmatrix} \\
&= -\begin{pmatrix}1 & 0\end{pmatrix} \begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix} \begin{pmatrix}0\\1\end{pmatrix} \\
&= -\begin{pmatrix}1 & 0\end{pmatrix} H_0 \begin{pmatrix}0\\1\end{pmatrix} \\
&= \mathcal{E}_{H_{0,\mathbb{Z}_2}}(1_{F_0(V_0)\cap V_0}, 1_{F_1(V_0)\cap V_0}).
\end{aligned}$$

This can be viewed as the consistency of $\mathcal{E}_{H_{1,\mathbb{Z}_2}}$ and $\mathcal{E}_{H_{0,\mathbb{Z}_2}}$ under the identifications of the indicators on both sides.

Subsequently, we can verify the consistency between $\mathcal{E}_{H_{2,\mathbb{Z}_2}}$ and $\mathcal{E}_{H_{1,\mathbb{Z}_2}}$ through the formula $\mathcal{E}_{H_{2,\mathbb{Z}_2}}(u,v) = \sum_{i=0}^{5} \frac{1}{r_i} \mathcal{E}_{H_{1,\mathbb{Z}_2}}(u \circ F_i, v \circ F_i)$. In fact,

$$\begin{aligned}
\mathcal{E}_{H_{2,\mathbb{Z}_2}}(1_{F_0(F_0(V_0))}, 1_{F_1(F_1(V_0))}) &= \frac{1}{2^2} \mathcal{E}_{H_{1,\mathbb{Z}_2}}(1_{F_0(F_0(V_0))} \circ F_4, 1_{F_1(F_1(V_0))} \circ F_4) \\
&= \frac{1}{2^2} \mathcal{E}_{H_{1,\mathbb{Z}_2}}(1_{F_0(V_0)}, 1_{F_1(V_0)}), \\
\mathcal{E}_{H_{1,\mathbb{Z}_2}}(1_{F_0(F_0(V_0)\cap V_0)}, 1_{F_1(F_1(V_0)\cap V_0)}) &= \frac{1}{2^2} \mathcal{E}_{H_{0,\mathbb{Z}_2}}(1_{F_0(F_0(V_0)\cap V_0)} \circ F_4, 1_{F_1(F_1(V_0)\cap V_0)} \circ F_4) \\
&= \frac{1}{2^2} \mathcal{E}_{H_{0,\mathbb{Z}_2}}(1_{F_0(V_0)\cap V_0}, 1_{F_1(V_0)\cap V_0})
\end{aligned}$$

and

$$\mathcal{E}_{H_{1,\mathbb{Z}_2}}(1_{F_0(V_0)}, 1_{F_1(V_0)}) = \mathcal{E}_{H_{0,\mathbb{Z}_2}}(1_{F_0(V_0)\cap V_0}, 1_{F_1(V_0)\cap V_0})$$

imply

$$\mathcal{E}_{H_{2,\mathbb{Z}_2}}(1_{F_0(F_0(V_0))}, 1_{F_1(F_1(V_0))}) = \mathcal{E}_{H_{1,\mathbb{Z}_2}}(1_{F_0(F_0(V_0)\cap V_0)}, 1_{F_1(F_1(V_0)\cap V_0)}).$$

Similarly we can derive from

$$\begin{aligned}
\mathcal{E}_{H_{2,\mathbb{Z}_2}}(1_{F_0(F_0(V_0))}, 1_{F_0(F_1(V_0))}) &= \frac{c}{2^2} \mathcal{E}_{H_{1,\mathbb{Z}_2}}(1_{F_0(F_0(V_0))} \circ F_0, 1_{F_0(F_1(V_0))} \circ F_0) \\
&= \frac{c}{2^2} \mathcal{E}_{H_{1,\mathbb{Z}_2}}(1_{F_0(V_0)}, 1_{F_1(V_0)}), \\
\mathcal{E}_{H_{1,\mathbb{Z}_2}}(1_{F_0(F_0(V_0)\cap V_0)}, 1_{F_0(F_1(V_0)\cap V_0)}) &= \frac{c}{2^2} \mathcal{E}_{H_{0,\mathbb{Z}_2}}(1_{F_0(F_0(V_0)\cap V_0)} \circ F_0, 1_{F_0(F_1(V_0)\cap V_0)} \circ F_0) \\
&= \frac{c}{2^2} \mathcal{E}_{H_{0,\mathbb{Z}_2}}(1_{F_0(V_0)\cap V_0}, 1_{F_1(V_0)\cap V_0})
\end{aligned}$$

and

$$\mathcal{E}_{H_{1,\mathbb{Z}_2}}(1_{F_0(V_0)}, 1_{F_1(V_0)}) = \mathcal{E}_{H_{0,\mathbb{Z}_2}}(1_{F_0(V_0)\cap V_0}, 1_{F_1(V_0)\cap V_0})$$

that

$$\mathcal{E}_{H_{2,\mathbb{Z}_2}}(1_{F_0(F_0(V_0))}, 1_{F_0(F_1(V_0))}) = \mathcal{E}_{H_{1,\mathbb{Z}_2}}(1_{F_0(F_0(V_0)\cap V_0)}, 1_{F_0(F_1(V_0)\cap V_0)}).$$

We need to notice we come across a little bit more complicated identity in the inductive procedure. In fact, we see

$$\begin{aligned}
&\mathcal{E}_{H_{3,\mathbb{Z}_2}}(1_{F_0(F_0(F_0(V_0)))}, 1_{F_0(F_0(F_1(V_0)))}) \\
&= \frac{c}{2^2}\mathcal{E}_{H_{2,\mathbb{Z}_2}}(1_{F_0(F_0(F_0(V_0)))} \circ F_0, 1_{F_0(F_0(F_1(V_0)))} \circ F_0) \\
&\qquad + \frac{1}{2^2}\mathcal{E}_{H_{2,\mathbb{Z}_2}}(1_{F_0(F_0(F_0(V_0)))} \circ F_2, 1_{F_0(F_0(F_1(V_0)))} \circ F_2) \\
&\qquad + \frac{1}{2^2}\mathcal{E}_{H_{2,\mathbb{Z}_2}}(1_{F_0(F_0(F_0(V_0)))} \circ F_4, 1_{F_0(F_0(F_1(V_0)))} \circ F_4) \\
&= \frac{1}{2^2}(c+2)\mathcal{E}_{H_{2,\mathbb{Z}_2}}(1_{(F_0(F_0(V_0))}, 1_{F_0(F_1(V_0))}),
\end{aligned}$$

$$\begin{aligned}
&\mathcal{E}_{H_{2,\mathbb{Z}_2}}(1_{F_0(F_0(F_0(V_0)\cap V_0))}, 1_{F_0(F_0(F_1(V_0)\cap V_0))}) \\
&= \frac{c}{2^2}\mathcal{E}_{H_{1,\mathbb{Z}_2}}(1_{F_0(F_0(F_0(V_0)\cap V_0))} \circ F_0, 1_{F_0(F_0(F_1(V_0)\cap V_0))} \circ F_0) \\
&\qquad + \frac{1}{2^2}\mathcal{E}_{H_{1,\mathbb{Z}_2}}(1_{F_0(F_0(F_0(V_0)\cap V_0))} \circ F_2, 1_{F_0(F_0(F_1(V_0)\cap V_0))} \circ F_2) \\
&\qquad + \frac{1}{2^2}\mathcal{E}_{H_{1,\mathbb{Z}_2}}(1_{F_0(F_0(F_0(V_0)\cap V_0))} \circ F_4, 1_{F_0(F_0(F_1(V_0)\cap V_0))} \circ F_4) \\
&= \frac{1}{2^2}(c+2)\mathcal{E}_{H_{1,\mathbb{Z}_2}}(1_{F_0(F_0(V_0)\cap V_0)}, 1_{F_0(F_1(V_0)\cap V_0)})
\end{aligned}$$

and

$$\mathcal{E}_{H_{2,\mathbb{Z}_2}}(1_{(F_0(F_0(V_0))}, 1_{F_0(F_1(V_0))}) = \mathcal{E}_{H_{1,\mathbb{Z}_2}}(1_{F_0(F_0(V_0)\cap V_0)}, 1_{F_0(F_1(V_0)\cap V_0)})$$

imply

$$\mathcal{E}_{H_{3,\mathbb{Z}_2}}(1_{F_0(F_0(F_0(V_0)))}, 1_{F_0(F_0(F_1(V_0)))}) = \mathcal{E}_{H_{2,\mathbb{Z}_2}}(1_{F_0(F_0(F_0(V_0)\cap V_0))}, 1_{F_0(F_0(F_1(V_0)\cap V_0))}).$$

3.3 A resistance metric on set of vertices

In this section, we will establish a general framework for a consistent family of Dirichlet forms on the set of vertices. For the objective, we get back at the general situation where an increasing sequence $\{V_m\}_{m=0}^{\infty}$ of finite sets, a family $\{F_0, \dots, F_{N-1}\}$ of injective maps from $V_* = \bigcup_{m=0}^{\infty} V_m$ to itself satisfying $F_j : V_m \to V_{m+1}$ for any $m = 0, 1, \dots$ and $j \in \{0, \dots, N-1\}$ and the $N \times N$ square matrix

$$H_0 = \begin{pmatrix} -(N-1) & 1 & \dots & 1 & 1 \\ 1 & -(N-1) & 1 & \dots & 1 \\ \vdots & & \ddots & & \vdots \\ 1 & \dots & \dots & 1 & -(N-1) \end{pmatrix}$$

are given.

In the same fashion as shown in [20], we start with the Dirichlet form $\mathcal{E}_{H_0}(u,v)$ determined by $\mathcal{E}_{H_0}(u,v) = -(u, H_0 v)$ for $u, v \in \ell(V_0)$ and take the sequence $\{\mathcal{E}_{H_m}\}_{m=1}^{\infty}$ of Dirichlet forms given by the formula

$$\mathcal{E}_{H_m}(u,v) = \sum_{i=0}^{N-1} \frac{1}{r_i} \mathcal{E}_{H_{m-1}}(u \circ F_i, v \circ F_i), \qquad u, v \in \ell(V_m) \quad m = 1, 2, \ldots,$$

where $\ell(V_m)$ stands for the set of real valued function on V_m and $\mathbf{r} = (r_0, \ldots, r_{N-1})$ for a sequence of N positive numbers. Here, we see that each $\mathcal{E}_{H_m}$ meets the notion of Dirichlet form given by J. Kigami in Definition 2.1.1 in [20], which means that each H_m gives a Laplacian in the sense in Definition 2.1.2 in [20]. We have just obtained the sequence $\{(V_m, H_m)\}_{m=0}^{\infty}$ of $\mathbf{r}$-networks. The next theorem shows a sufficient condition for the sequence to be associated with consistent Dirichlet forms.

Theorem 3.2 *Suppose that there exists $I \subset \{0, \ldots, N-1\}$ such that*

$$\bigcup_{i \in I} F_i(V_m) = V_{m+1}, \qquad \bigcup_{j \notin I} F_j(F_i(V_*)) \subset F_i(V_*)$$

for any $i \in I$, $F_i(V_*) \cap F_{i'}(V_*) = \emptyset$ and

$$\mathcal{E}_{H_1}(1_{F_i(V_0)}, 1_{F_{i'}(V_0)}) = \mathcal{E}_{H_0}(1_{F_i(V_0) \cap V_0}, 1_{F_{i'}(V_0) \cap V_0})$$

for any $i, i' \in I$ with $i \neq i'$. If $F_j(F_i(V_*)) \cap F_{j'}(F_i(V_*)) \neq \emptyset$ implies $F_j = F_{j'}$ on $F_i(V_*)$ for any $i \in I$ and $j, j' \in \{0, \ldots, N-1\}$, then

$$\mathcal{E}_{H_{m+1}}(1_{F_{j_1} \circ \cdots \circ F_{j_m} \circ F_i(V_0)}, 1_{F_{j_1} \circ \cdots \circ F_{j_m} \circ F_{i'}(V_0)}) = \mathcal{E}_{H_m}(1_{F_{j_1} \circ \cdots \circ F_{j_m}(F_i(V_0) \cap V_0)}, 1_{F_{j_1} \circ \cdots \circ F_{j_m}(F_{i'}(V_0) \cap V_0)})$$

for any $i, i' \in I$ and $j_1, \ldots, j_m \in \{0, \ldots, N-1\}$, where the conventional notations for representative subsets in the previous example are employed.

In particular, the family $\{F_0, \ldots, F_{p+p^p-1}\}$ of maps from $\mathbb{Z}_p$ to itself as given in the previous section provides us with the sequence $\{\mathcal{E}_{H_{m,\mathbb{Z}_p}}\}_{m=0}^{\infty}$ of Dirichlet forms determined by the formula

$$\mathcal{E}_{H_{m,\mathbb{Z}_p}}(u,v) = \sum_{i=0}^{p+p^p-1} \frac{1}{r_i} \mathcal{E}_{H_{m-1,\mathbb{Z}_p}}(u \circ F_i, v \circ F_i)$$

for $u, v \in \ell(V_{m,\mathbb{Z}_p})$ by starting with the Dirichlet form $\mathcal{E}_{H_{0,\mathbb{Z}_p}}$ with the domain $\ell(V_{0,\mathbb{Z}_p})$ and $\mathbf{r} = (\underbrace{\frac{p^2}{c}, \frac{p^2}{c}, \ldots, \frac{p^2}{c}}_{p}, \underbrace{p^p, p^p, \ldots, p^p}_{p^p})$ with some positive real number c.

The family of functions taking constant on every ball with the radius $p^{-(m+1)}$ in $\mathbb{Z}_p$ will be denoted by $\mathcal{C}_m$. Then, a natural identification between $\mathcal{C}_m$ and $\ell(V_{m,\mathbb{Z}_p})$ is induced by the restriction of domain of function in $\mathcal{C}_m$ to $V_{m,\mathbb{Z}_p}$, where the reverse correspondence is given by the extension of function u in $\ell(V_{m,\mathbb{Z}_p})$ to the one on $\mathbb{Z}_p$ taking the same value on every ball with the radius $p^{-(m+1)}$ as the value of u at the point of V_m contained in the ball. Hereafter, we will take this identification.

We can verify the consistency in the sequence $\{\mathcal{E}_{H_{m,\mathbb{Z}_p}}\}_{m=0}^{\infty}$ of the Dirichlet forms in the following corollary. Here and in the sequel, the standard Haar measure on $\mathbb{Z}_p$ will be denoted by μ.

Corollary 3.3 (i) *For the sequence $\{\mathcal{E}_{H_{m,\mathbb{Z}_p}}\}_{m=0}^{\infty}$ of Dirichlet forms,*

$$\mathcal{E}_{H_{m,\mathbb{Z}_p}}(u,v) = \mathcal{E}_{H_{m-1,\mathbb{Z}_p}}(u|_{V_{m-1,\mathbb{Z}_p}}, v|_{V_{m-1,\mathbb{Z}_p}})$$

for any functions u, v in $\mathcal{C}_m$ on $\mathbb{Z}_p$.

(ii) *There exists a positive measurable function $K(x,y)$ on $\mathbb{Z}_p \times \mathbb{Z}_p \setminus \Delta = \{(x,y) \mid x, y \in \mathbb{Z}_p, x \neq y\}$ such that*

$$\mathcal{E}_{H_{m,\mathbb{Z}_p}}(u,v) = \frac{1}{2}\iint_{\mathbb{Z}_p\times\mathbb{Z}_p\setminus\Delta}(u(x)-u(y))(v(x)-v(y))K(x,y)\,d\mu(x)d\mu(y)$$

for any $u, v \in \mathcal{C}_m$.

Apart from the specific situation in $\mathbb{Z}_p$, we shift our focus back to a general theory after Theorem 3.2. In what follows, we deal with the case that the sequence $\{(V_m, H_m)\}_{m=0}^{\infty}$ of $\mathbf{r}$-networks associated with consistent Dirichlet forms satisfies all assumptions in Theorem 3.2 and conditions in the theorem with some I with $\#I = \#V_0$. We assume that $r_i = r_{i'}$ for any $i, i' \in I$, $r_j = r_{j'}$ for any $j, j' \in J \setminus I$ and

$$\sum_{F_j(V_*)\cap F_i(V_*)\neq\emptyset, F_j(V_*)\cap F_{i'}(V_*)\neq\emptyset} \frac{1}{r_j} < \min\left\{\frac{1}{r_i} \;\middle|\; i \in I\right\}$$

for any distinct elements i, i' in I described in Theorem 3.2. In particular, the condition $\#I = \#V_0$ admits to assign each element in V_0 by the notation z_i according to the choice of element $i \in I$.

Proposition 3.4 *Suppose that*

$$\#\big\{j \in \{0,\ldots,N-1\} \;\big|\; F_j(V_*) \cap F_{i_1}(F_{i_2}(V_*)) \neq \emptyset \text{ and } F_j(V_*) \cap F_{i_1'}(F_{i_2'}(V_*)) \neq \emptyset\big\}$$

depends only on $\min\{k \mid i_k = i_k'\}$ for any $i_1, i_2, i_1', i_2' \in I$ and that $F_j(V_*) \cap F_{i_1}(F_{i_2}(V_*)) \neq \emptyset$ implies $F_j \circ F_{i_2(j)} = F_{i_1} \circ F_{i_2}$ with some $i_2(j) \in I$ for any $i_1, i_2 \in I$ and $j \in \{0,\ldots,N-1\}$. Then

$$\mathcal{E}_{H_m}(1_{F_{i_1}\circ F_{i_2}\circ\cdots\circ F_{i_m}(\{z_{i_{m+1}}\})}, 1_{F_{i_1'}\circ F_{i_2'}\circ\cdots\circ F_{i_m'}(\{z_{i_{m+1}'}\})})$$

depends only on $\min\{k \mid i_k \neq i_k'\}$ and takes negative value decreasing with respect to $\min\{k \mid i_k \neq i_k'\}$, where $i_1, i_1', i_2, i_2', \ldots, i_m, i_m', i_{m+1}, i_{m+1}' \in I$.

By applying the proposition to the specific sequence $\{\mathcal{E}_{H_{m,\mathbb{Z}_p}}\}$ of Dirichlet forms introduced in the previous section, we can obtain the following corollary:

Corollary 3.5 *The value of the measurable function $K(x,y)$ obtained in the corollary of Theorem 3.2 depends only on $\|x-y\|_p$ and decreases with respect to $\|x-y\|_p$ on $\mathbb{Z}_p \times \mathbb{Z}_p \setminus \Delta$.*

We will restrict our attention to the case that all conditions in Theorem 3.2 and Proposition 3.4 are satisfied. Let us define effective resistance by

$$R_{H_m}(x,y) = (\min\{\mathcal{E}_{H_m}(u,u) \mid u \in \ell(V_m) \text{ with } u(x) = 1 \text{ and } u(y) = 0\})^{-1}$$

for any pair x, y in V_m. Theorem 3.2 shows that for any $u \in \ell(V_n)$ and integer m with $m > n$, there exists a function $\overline{u}$ in $\ell(V_m)$ satisfying $\mathcal{E}_{H_m}(\overline{u},\overline{u}) = \mathcal{E}_{H_n}(u,u)$. The consistency between $\mathcal{E}_{H_m}$ and $\mathcal{E}_{H_{m+1}}$ derived from this observation ensures that $R_{H_m}(x,y) \leq R_{H_{m+1}}(x,y)$ for x, y in V_m, which allows us to define $R(x,y) = \sup_m R_{H_m}(x,y)$ for x, y in $V_* = \bigcup_m V_m$. Afterward we will see that $R(x,y)$ is a metric on V_* unless it diverges.

Definition 3.6 Any pair of distinct points x, y in V_* admits representations

$$x = F_{i_1} \circ \cdots \circ F_{i_{k-1}} \circ F_{i_k} \circ F_{i_{k+1}} \circ \cdots \circ F_{i_m}(z_{i_{m+1}})$$

and

$$y = F_{i_1} \circ \cdots \circ F_{i_{k-1}} \circ F_{i'_k} \circ F_{i'_{k+1}} \circ \cdots \circ F_{i'_m}(z_{i'_{m+1}})$$

with $i_k \neq i'_k$ and $z_{i_{m+1}}, z_{i'_{m+1}} \in V_0$ by taking sufficiently large m. The distance between x and y is defined by $d(x,y) = \frac{1}{\# I^k}$.

Proposition 3.7 *For $x, y \in V_*$, $R(x,y)$ depends only on $d(x,y)$. In particular, $R(x,y)$ is a non-Archimedean metric.*

Lemma 3.8 *For any point $o, x \in V_m$, $\min\{\mathcal{E}_{H_m}(u,u) \mid u \in \ell(V_m)$ with $u(o) = 1$ and $u(z) = \frac{1}{2}$ for any z with $d(z,o) \geq d(z,x)\}$ is attained by $\mathcal{E}_{H_m}(u_0,u_0)$ with a unique function u_0 decreasing with respect to the distance $d(o,x)$.*

Proposition 3.9 (i) *$R(x,y)$ is positive for distinct points $x, y \in V_*$ and enjoys the triangle inequality.*

(ii) *$\sum_{m=1}^{\infty} 1/(-H_m)_{z_0,z_0} < \infty$ for some point $z_0 \in V_0$ implies that $R(x,y)$ is finite for any points $x, y \in V_*$.*

Remark 3.10 For any $u \in \ell(V_n)$, we can find a unique function $w \in \ell(V_n)$ attaining $\min_{w \in \ell(V_n), w|_{V_m} = u} \mathcal{E}_{H_n}(w,w)$ for every $u \in \ell(V_m)$ with $m < n$ by applying Lemma 2.1.5 in [20]. Theorem 3.2 shows that w satisfies $\mathcal{E}_{H_n}(w,w) \leq \mathcal{E}_{H_n}(\overline{u},\overline{u}) \leq \mathcal{E}_{H_m}(u,u)$.

3.4 Completion with respect to resistance metric

If a compatible sequence of $\mathbf{r}$-networks determined by a family of Laplacians $\{H_m\}_{m=0}^{\infty}$ based on a finite sequence $\mathbf{r}$ of real parameters is obtained, the pair $(H_0, \mathbf{r})$ is said to be a harmonic structure in [20]. The harmonic structure $(H_0, \mathbf{r})$ associated with the sequence of real parameters $\mathbf{r} = (r_0, r_1, \ldots, r_{N-1})$ satisfying $0 < r_i < 1$ for all $i \in \{0, 1, \ldots, N-1\}$ is said to be regular in [20], in contrast this condition is not always satisfied as seen in the previous sections in our case. A key method in [20] is focusing on contractions in the family of maps characterizing self-similarity of fractals. We are taking similar procedure but not exactly the same one in respect that all maps guiding the random motion are not always contractions. However, under the assumption that $r_i < 1$ for any i in the subset I of $\{0, \ldots, N-1\}$ in Theorem 3.2, we will see that we can take a compactification with respect to the resistance metric R to construct a Hunt process characterized by the sequence $\{(V_m, H_m)\}_{m=0}^{\infty}$ of $\mathbf{r}$-networks associated with consistent Dirichlet forms. In what follows, if this assumption is satisfied, the sequence $\{(V_m, H_m)\}_{m=0}^{\infty}$ of $\mathbf{r}$-networks associated with consistent Dirichlet forms is said to be pseudo-regular.

Now we take the completion of V_* with respect to the metric R and denote it by S. Then R is clearly extended so as to be a non-Archimedean metric and the following assertion on the metric is obtained:

Theorem 3.11 *If the sequence $\{(V_m, H_m)\}_{m=0}^{\infty}$ of $\mathbf{r}$-networks associated with consistent Dirichlet forms is pseudo-regular, then*

(i) *F_i is extended to a contraction on S for each $i \in I$,*

(ii) *R is a bounded metric on S,*

(iii) *(S, R) is a compact metric space,*

(iv) *for the family $\{F_i\}_{i \in I}$ of contractions with the extended domain S in (i), $S = \bigcup_{i \in I} F_i(S)$,*

We consider the case that pseudo-regular sequence $\{(V_m, H_m)\}_{m=0}^{\infty}$ of $\mathbf{r}$-networks associated with consistent Dirichlet forms is given. Since the space $W_*^{\#I}$ in [20], which is viewed also as the shift space $\Sigma^{\#I}$, admits an injective continuous map with the compact image S, we can take the induced measure $\mu_{\Sigma^{\#I}}$ on S by the uniform measure on $\Sigma^{\#I}$ normalized by $\mu_{\Sigma^{\#I}}(S) = 1$. Any function $u \in \ell(V_n)$ admits its extension to S taking the same value on every image $F_{i_1} \circ F_{i_2} \circ \cdots \circ F_{i_{n+1}}(S)$ with $i_1, i_2, \ldots, i_{n+1} \in I$ as the value of u at the point of V_n contained in the image. The function with the extended domain will be denoted by the same notation u. Since R is non-Archimedean, the function defined on S is continuous with respect to the metric R.

In this situation, any sequence $\{u_n\}$ satisfying $u_k \in \ell(V_{m(k)})$ with some $m(k) \geq k$ for any $k = 0, 1, 2 \ldots$ and $\lim_{k,\ell \to \infty} \mathcal{E}_{H_{\max\{m(k), m(\ell)\}}}(u_k - u_\ell, u_k - u_\ell) = 0$ provides us with a sequence $\{\widetilde{u}_k\}$ determined by $\widetilde{u}_k(x) = u_k(x) - \int_S u_k(y)\, d\mu_{\Sigma^{\#I}}(y)$, which is a Cauchy sequence in the sense that

$$\lim_{k,\ell \to \infty} \mathcal{E}_{H_{\max\{m(k), m(\ell)\}}}(\widetilde{u}_k - \widetilde{u}_\ell, \widetilde{u}_k - \widetilde{u}_\ell) = 0.$$

Here, we note the validity of the inequality $|u(x)-u(y)| \leq R_m(x,y)\mathcal{E}_{H_m}(u,u)$ for any $x, y \in V_m$ and $u \in \ell(V_m)$ which is derived from the definition of R_m. Since the metric R is bounded, the uniform convergence of the sequence follows from

$$\begin{aligned}
|\widetilde{u}_k(x) - \widetilde{u}_\ell(x)| &\leq \left| u_k(x) - u_\ell(x) - \int_S (u_k(y) - u_\ell(y))\, d\mu_{\Sigma^{\# I}}(y) \right| \\
&\leq \sqrt{\sup_{x,y\in V_*} R_{\max\{m(k),m(\ell)\}}(x,y)} \sqrt{\mathcal{E}_{H_{\max\{m(k),m(l)\}}}(\widetilde{u}_k - \widetilde{u}_\ell, \widetilde{u}_k - \widetilde{u}_\ell)} \\
&\leq \sqrt{\sup_{x,y\in V_*} R(x,y)} \sqrt{\mathcal{E}_{H_{\max\{m(k),m(\ell)\}}}(\widetilde{u}_k - \widetilde{u}_\ell, \widetilde{u}_k - \widetilde{u}_\ell)}.
\end{aligned}$$

Let us introduce the function space

$$\widetilde{\mathcal{F}} = \left\{ \widetilde{u} \in \mathcal{C}(V_*) \;\middle|\; \begin{array}{l} \widetilde{u}(x) = \lim_{k\to\infty} \widetilde{u}_k(x) \text{ for each } x \in V_* \\ \text{for some sequence } \{u_k\}_{k=0}^\infty \text{with } u_k \in \ell(V_{m(k)}) \text{ satisfying} \\ \lim\limits_{k,\ell\to\infty} \mathcal{E}_{H_{\max\{m(k),m(\ell)\}}}(u_k - u_\ell, u_k - u_\ell) = 0 \end{array} \right\}$$

and the symmetric bilinear form

$$\mathcal{E}(\widetilde{u}, \widetilde{v}) = \lim_{n\to\infty} \mathcal{E}_{H_{m(k)}}(u_k, v_k) \quad \text{for} \quad \widetilde{u}, \widetilde{v} \in \widetilde{\mathcal{F}}$$

with approximating sequences $\{u_k\}, \{v_k\}$ respectively for $\widetilde{u}, \widetilde{v}$ as in the definition of $\widetilde{\mathcal{F}}$.

Proposition 3.12 *Any function $\widetilde{u} \in \widetilde{\mathcal{F}}$ is extended to a continuous function on S satisfying $|\widetilde{u}(x) - \widetilde{u}(y)|^2 \leq \mathcal{E}(\widetilde{u}, \widetilde{u})R(x,y)$ for any $x, y \in S$.*

We introduce

$$\mathcal{F} = \left\{ u \in \mathcal{C}(V_*) \;\middle|\; \begin{array}{l} u(x) = \lim\limits_{k\to\infty} u_k(x) \text{ for each } x \in V_* \\ \text{for some sequence } \{u_k\}_{k=0}^\infty \text{ with } u_k \in \ell(V_{m(k)}) \text{ satisfying} \\ \lim\limits_{k,\ell\to\infty} \mathcal{E}_{H_{\max\{m(k),m(\ell)\}}}(u_k - u_\ell, u_k - u_\ell) = 0 \end{array} \right\}.$$

Then it turns out that the function $u(x) - \int_S u(y)\, d\mu_{\Sigma^{\# I}}(dy)$ with $u \in \mathcal{F}$ can be written by $\widetilde{u}$, consistently with the notation introduced in the definition of $\widetilde{\mathcal{F}}$. We define $\mathcal{E}(u,v) = \lim_{k\to\infty} \mathcal{E}_{H_{m(k)}}(u_k, v_k)$ by taking representative sequences $\{u_k\}_{k=0}^\infty$, $\{v_k\}_{k=0}^\infty$ respectively for u and v and then we see $\mathcal{E}(\widetilde{u}, \widetilde{v}) = \mathcal{E}(u,v)$ for any $u, v \in \mathcal{F}$. In what follows, we will deal with functions in $\mathcal{F}$ and in $\widetilde{\mathcal{F}}$ as continuous functions on S.

Lemma 3.13 *If a sequence $\{\widetilde{u}_n\}$ in $\widetilde{\mathcal{F}}$ satisfies $\lim_{n\to\infty} \mathcal{E}(\widetilde{u}_n, \widetilde{u}_n) = 0$, $\{\widetilde{u}_n\}$ converges to 0 everywhere on S.*

Lemma 3.14 *If a sequence $\{\widetilde{u}_n\}$ in $\widetilde{\mathcal{F}}$ satisfies $\lim_{n,m\to\infty} \mathcal{E}(\widetilde{u}_n - \widetilde{u}_m, \widetilde{u}_n - \widetilde{u}_m) = 0$, $\{\widetilde{u}_n\}$ converges uniformly to a continous function everywhere on S.*

Corollary 3.15 *For the sequence $\{\mathcal{E}_{H_{m,\mathbb{Z}_p}}\}_{m=0}^\infty$ of Dirichlet forms introduced in Section 2, $\mathcal{E}(\widetilde{u}, \widetilde{v}) = \frac{1}{2} \iint_{\mathbb{Z}_p\times\mathbb{Z}_p\setminus\Delta} (\widetilde{u}(x) - \widetilde{u}(y))(\widetilde{v}(x) - \widetilde{v}(y))K(x,y)\, d\mu(x)d\mu(y)$ for any $\widetilde{u}, \widetilde{v} \in \widetilde{\mathcal{F}}$.*

3.5 Dirichlet space for Hunt process on $\mathbb{Z}_p$

In this section, we will pave the way to the Dirichlet space on $\mathbb{Z}_p$ so that the family of Hunt processes associated with our Dirichlet forms corresponding to a subclass of Albeverio and Karwowski's stochastic process on $\mathbb{Z}_p$ is obtained. We will denote $\bigcup_m \mathcal{C}_m$ by $\mathcal{C}_*$.

In the case that the contractions F_i's are relevant to geometric figure of fractals, a maximal principle established in [20] gives an approximation for function $u \in \ell(V_*)$ satisfying $\lim_{m\to\infty} \mathcal{E}_{H_m}(u|_{V_m}, u|_{V_m}) < \infty$ by functions with some harmonicity on V_m. Pivotal parts of the research were undertaken essentially under the assumption $\#(F_i(V_0)\cap V_0) \leq 1$ for all F_i's and a Dirichlet space theory on fractals was established. However, we need to pay attention to the fact that this assumption is not satisfied on some maps in $\{F_p, \ldots, F_{p^p+p-1}\}$ in our case.

Hereafter, we will focus on the specific Drirchlet space determined by the sequence $\{(V_{m,\mathbb{Z}_p}, H_{m,\mathbb{Z}_p})\}$ of **r**-networks associated with consistent Dirichlet forms in Section 2 and assume that $c > p^2$ so that all conditions imposed so far are satisfied.

We note the fact that $\widetilde{\mathcal{F}}$ is a Hilbert space with the inner product $\mathcal{E}(\widetilde{u}, \widetilde{v})$. This is because, by an elementary theory of Cauchy sequence, for any sequence $\{\widetilde{u}_k\}$ in $\widetilde{\mathcal{F}}$ satisfying $\lim_{k\to\infty} \mathcal{E}(\widetilde{u}_k - \widetilde{u}, \widetilde{u}_k - \widetilde{u}) = 0$, we can take a sequence $\{w_k\}$ in $\mathcal{C}_*$ satisfying $w_k \in \mathcal{C}_{m(k)}$ with $m(k) \geq k$ and so small value $\mathcal{E}(\widetilde{u}_k - \widetilde{w}_k, \widetilde{u}_k - \widetilde{w}_k)$ for sufficiently large k that $\lim_{k\to\infty} \mathcal{E}_{H_{\max\{m(k),m(\ell)\},\mathbb{Z}_p}}(\widetilde{w}_k - \widetilde{w}, \widetilde{w}_k - \widetilde{w}) = 0$. The descriptions before Proposition 3.12 show that $\lim_{k\to\infty} \mathcal{E}(\widetilde{w}_k - u, \widetilde{w}_k - u) = 0$ for some element u in $\widetilde{\mathcal{F}}$ with the property

$$\lim_{k\to\infty} \sup_{x\in S} |u(x) - \widetilde{w}_k(x)| = 0.$$

Since $\lim_{k\to\infty} \mathcal{E}(\widetilde{u}_k - u, \widetilde{u}_k - u) = 0$, $\lim_{k\to\infty} \sup_{x\in S} |u(x) - \widetilde{u}_k(x)| = 0$ can be justified by applying Lemma 3.13 for the sequence $\{u - \widetilde{u}_k\}$.

Proposition 3.16 *S is homeomorphic to $\mathbb{Z}_p$ and the family of function $\mathcal{F}$ is a dense subfamily of $\mathcal{C}(\mathbb{Z}_p)$.*

We note that this condition is fulfilled in the particular case on $\mathbb{Z}_p$ what we have detailed so far. In fact, it is not difficult to see that S is homeomorphic to $\mathbb{Z}_p$ and it is well known fact that the family of continuous functions on $\mathbb{Z}_p$ has a dense subfamily $\mathcal{C}_*$, which is contained in $\mathcal{F}$ and there exists a Radon measure J on $\{(x,y) \in \mathbb{Z}_p \times \mathbb{Z}_p \mid x \neq y\}$ to describe the Dirichlet form $\mathcal{E}(u,v)$ as

$$\frac{1}{2}\int_{\{(x,y)\in\mathbb{Z}_p\times\mathbb{Z}_p|x\neq y\}} (u(x) - u(y))(v(x) - v(y))\, dJ(x,y)$$

for locally constant function u, v on $\mathbb{Z}_p$.

Theorem 3.17 *$(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(\mathbb{Z}_p; \mu)$.*

In [1], S. Albeverio and W. Karwowski initiated a theory of random walks on $\mathbb{Q}_p$ by starting with a sequence $\{a(m)\}_{m=-\infty}^{\infty}$ satisfying

(i) $a(m) \geq a(m+1)$,

(ii) $\lim_{m\to\infty} a(m) = 0$ and $\lim_{m\to-\infty} a(m) > 0$ or $= \infty$.

By comparing the description of $\mathcal{E}_{H_{m-1},\mathbb{Z}_p}(1_{B_m(0)}, 1_{B_m(0)})$ in terms of the sequence with the one involving the diagonal component of H_{m-1}, we have the following assertion.

Proposition 3.18 *The regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathbb{Z}_p; \mu)$ generates the random walk characterized by the sequence $\{a(m)\}_{m=-\infty}^{\infty}$ given by $a(-m) = \frac{\{(p-1)(c+p)\}^{m-1}}{p^m}$ for positive integer m and by $a(m) = 0$ for non-negative integer m.*

Chapter 4

A convergence of Hunt processes on the ring of p-adic integers and its application to a random fractal

4.1 Discussing convergence of Hunnt process based on Dirichlet space theory

The convergence of stochastic processes is one of subjects founded on the importance of the numerical analysis and physical models with stability. Such practical importance inspires us with vast range of interests as to on which space the convergence can be addressed and which sort of accommodated method is required to demonstrate the convergence.

In a progress in the Dirichlet space theory, Mosco introduced the notion of Γ-convergence to encompass the convergence in terms of Dirichlet forms. The insight showed that the convergence as bilinear forms is not always sufficient to deduce the convergence of stochastic processes and clarified which sort of extra sufficient conditions must be imposed for the rigorous justification of the phenomenon.

In [27], Kuwae and Shioya implemented the framework by complying those sufficient conditions to the case where the state space is deformed and the Dirichlet form alters. After those progresses, various cases with convergent sequence of Dirichlet forms are investigated. For instance, on typical fractals emerged as the limit of sequence of closed sets with fundamental shapes, the convergence of stochastic processes on the closed sets was figured out due to the framework. In fact, Kumagai and Sturm demonstrated in [26] a vast framework on the convergence of Hunt processes which naturally encompasses the celebrated approximation method invented by Kigami in [20], where a sequence of Dirichlet forms are introduced on such closed sets aiming at the standard Hunt process on typical fractals embedded in the Euclidean space. Recently, in [41], Suzuki studied a convergence of Markov processes arising from tree structure as in [22] and [34].

In this chapter, we will propose an accommodated method to deal with convergence of Hunt processes with the same spirit as Hinz in [14] so that it covers the convergence arising from construction of random fractals on the ring of p-adic integers where the shapes of the closed sets are deformed randomly. Since we employ Dirichlet space theory in describing a sequence of Hunt processes on the randomly deformed closed sets, an analysis on the sequence of random Dirichlet forms will be required.

In the second section, we will point out examples where convergence of Hunt processes on $\mathbb{Z}_p$ is expected to be verified and propose a framework which covers these examples. As a main object in our investigation, we will consider Markov processes which admit the characterization by Kolmogorov's equations

$$\begin{cases} \dfrac{d}{dt}P_{K_f,K_i}(t) = -\widetilde{a}(K_f)P_{K_f,K_i}(t) + \displaystyle\sum_{j\neq f}^{\infty}\widetilde{u}(K_f,K_j)P_{K_j,K_i}(t), \\ \dfrac{d}{dt}P_{K_f,K_i}(t) = -\widetilde{a}(K_i)P_{K_f,K_i}(t) + \displaystyle\sum_{j\neq i}^{\infty}P_{K_f,K_j}(t)\widetilde{u}(K_j,K_j), \end{cases}$$

on $P_{K_f,K_i}(t) = P(X(t) \in K_f \mid X(0) \in K_i)$ with reasonably given coefficients $\widetilde{u}(K_f,K_j)$ and $\widetilde{a}(K_f)$ as originally proposed by Karwowski and Vilela-Mendes in [18], by introducing disjoint balls $\{K_f\}$ with identical radius satisfying $\bigcup_f K_f = \mathbb{Z}_p$. We will discuss convergence of Hunt processes associated with a construction of a random fractal. To be more precise, in accordance with the removal of randomly taken ball with the radius in the cut-out procedure in [33], we remove the ball from the support of the measure involved in the coefficients $\widetilde{a}(K_f)$ and $\widetilde{u}(K_f,K_j)$ and renormalize the measure in the coefficients so that the Markov process associated with the equation with the coefficients involving the cut-out effect is obtained. In accordance with the cut-out effect with the framework of the generalized Mosco-convergence, we will take an equivalent procedure to the cut-out within the framework of Dirichlet space and study the sequence of Hunt processes constructed as step-by-step cut-outs are processed.

In the third section, we will prepare some lemmas to prove the convergence relying the notion of KS-convergence established by Kuwae and Shioya. In the final section, sufficient conditions for the convergence of Hunt processes including condition for the tightness will be verified with the terminology in the Dirichlet space theory.

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4.2 Convergence of Dirichlet forms

As Kumagai and Strum studied, a natural interest on the convergence of stochastic processes can arise from improving precision level in various approximations. Since the phenomenon in the ring $\mathbb{Z}_p$ of p-adic integers is illustrated by a randomly moving ball with the radius determined by the precision level, the sort of convergence is expected to be observed when the radius of the ball goes to zero as the ball turns to a particle. The convergence is describable by taking sequence of Dirichlet forms k-th term of which admits the family of functions $\mathcal{C}_k$ taking constant on every ball with the radius $p^{-(k+1)}$ in $\mathbb{Z}_p$ as its domain. As a fundamental observation on the convergence in such primitive phenomenon, the sequence $\{\mathcal{E}_{H_{k,\mathbb{Z}_p}}\}$ of Dirichlet forms arising from the construction of a Dirichlet space on $\mathbb{Z}_p$ in [1] is expected to be covered. In fact, each of the Dirichlet forms admits the family of functions $\mathcal{C}_k$ taking constant on every ball with the radius $p^{-(k+1)}$ in $\mathbb{Z}_p$ as its domain. From the consitency of the Dirichlet forms

$$\mathcal{E}_{H_{k+1,\mathbb{Z}_p}}(u,v) = \mathcal{E}_{H_{k,\mathbb{Z}_p}}(u,v)$$

for any pair of functions $u, v \in \mathcal{C}_k$ in the sense of [1], we can derive

$$\lim_{k\to\infty} \mathcal{E}_{H_{k,\mathbb{Z}_p}}(u,v) = \mathcal{E}_K(u,v)$$

for any locally constant functions u, v on $\mathbb{Z}_p$, where $\mathcal{E}_K(u,v)$ stands for the bilinear form in the article.

For a coverage to deal with this sort of convergence, we may focus on the sequence $\{(\mathcal{E}_k, \mathcal{C}_k)\}$ of pairs of symmetric bilinear forms and their domains each of which admits the representation

$$\mathcal{E}_k(u,v) = \frac{1}{2}\iint_{(F_k\times F_k)\cap\{x\neq y\}} (u(x)-u(y))(v(x)-v(y))\rho_k(x,y)\, d\mu_k(x)d\mu_k(y), \tag{4.1}$$

for $u, v \in \mathcal{C}_k$, where $\{\mu_k\}$ stands for a sequence of Radon measures with a vague limit μ_F. $\{F_k\}$ for a sequence of closed sets in $\mathbb{Z}_p$ and $\rho_k(x,y)$ for a non-negative locally constant function on $\mathbb{Z}_p \times \mathbb{Z}_p \setminus \{(x,x) \mid x \in \mathbb{Z}_p\}$. In fact, particularly by choosing μ_k as an identical Radon measure, $\rho_k(x,y)$ as $K(x,y)$ and F_k as the whole space $\mathbb{Z}_p$ independent of k, we can cover the case which was dealt with in [1].

A main reason to focus on such sequence of Dirichlet forms can be found in a construction of a random fractal. In fact, we take a positive real number α satisfying $\alpha < 1 - t_2$ with the real parameter t_2 and the sequence $\{\mu_k\}$ of random Radon measures in the article [33] and introduce a sequence of random symmetric bilinear forms $\{\mathcal{E}_{F_k}\}$ defined by

$$\mathcal{E}_{F_k}(u,v) = \frac{1}{2}\iint_{(F_k\times F_k)\cap\{x\neq y\}} (u(x)-u(y))(v(x)-v(y))\, \|x-y\|_p^{-\alpha}\ d\mu_k(x)d\mu_k(y),$$

for locally constant functions u, v on $\mathbb{Z}_p$, where $\|x-y\|_p$ stands for the p-adic norm of $x - y \in \mathbb{Z}_p$. Hereafter, the family of locally constant function on $\mathbb{Z}_p$ will be denoted by $\mathcal{C}$ and the Haar measure on $\mathbb{Z}_p$ by μ.

In the construction of the random fractal F in [33], F_k is obtained right after removal of a ball with radius $p^{-\ell_k}$ in [33], F_k is described as a union of balls with radius $p^{-\ell_k}$. By recalling the fact that $\ell_k < k$ for sufficiently large k, we will regard F_k as a union of balls with radius $p^{-(k+1)}$ to subdue the use of suffix in notations. We already saw in [33] that, for any ball A in $\mathbb{Z}_p$, $\{\mu_k(A)\}$ is an $\{\mathcal{F}_k\}$-martingale with the deterministic initial value given by the Haar measure $\mu(A)$ of A. In particular, we focused on a sequence $\{\mu_k\}_{k=1}^{\infty}$ of random Radon measures on $\mathbb{Z}_p$ defined by $\mu_k(A) = \left(\Pi_{j=1}^{k}(1-p^{-\ell_j})^{-1}\right)\mu(A\cap F_k)$ and satisfying $E(\mu_k(A)) = \mu(A)$ for any ball A in $\mathbb{Z}_p$. We obtained the random Radon measure μ_F on $\mathbb{Z}_p$ which is uniquely characterized by $\mu_F(A) = \lim_{k\to\infty} \mu_k(A)$ for any ball $A \subset \mathbb{Z}_p$ with probability one. In our compact space $\mathbb{Z}_p$, Radon measure has finite total mass and the finiteness of μ_F with probability one follows from $E(\mu_F(\mathbb{Z}_p)) = \mu(\mathbb{Z}_p) = 1$ obtained in [33].

If the sequence $\{A_n\}$ of the balls each of which is taken for the n-th cut out procedure satisfies $\liminf_{k\to\infty} k\mu(A_k) = t_1$ and $\limsup_{k\to\infty} k\mu(A_k) = t_2$ for some t_1 and t_2 with $0 < t_1 < t_2 < 1$, then it turned out in [33] that the Hausdorff dimension $\dim_H F$ of the random fractal F given by $F = \bigcap_{k=1}^{\infty} A_k^c$ satisfies $\dim_H F \leq 1 - t_1$ with probability one and $\dim_H F \geq 1 - t_2$ with a positive probability. To be more precise, μ_F concentrates only on F and

$$\int_F \int_F \frac{1}{\|x-y\|_p^{\alpha}}\, d\mu_F(x) d\mu_F(y) < \infty$$

holds valid with probability one for any α satisfying $0 < \alpha < 1 - t_2$ and $\mu_F(F) > 0$ with a positive probability. Therefore, we can introduce the bilinear form

$$\mathcal{E}_F(u,v) = \frac{1}{2} \iint_{(F\times F)\cap\{x\neq y\}} (u(x)-u(y))(v(x)-v(y))\, \|x-y\|_p^{-\alpha}\, d\mu_F(x) d\mu_F(y),$$

for any pair of functions u, v in $\mathcal{C}$ with probability one.

Proposition 4.1 *The sequence $\{\mathcal{E}_{F_k}\}$ of symmetric bilinear forms converges to $\mathcal{E}_F$, in the sense that*

$$\lim_{k\to\infty} \mathcal{E}_{F_k}(u,v) = \mathcal{E}_F(u,v),$$

for any pair of functions u, v in $\mathcal{C}$ with probability one.

Thanks to these observations, we may concentrate our interest to the case that a non-random Radon measure μ_F satisfying

$$\lim_{k\to\infty} \mathcal{E}_{F_k}(u,v) = \mathcal{E}_F(u,v)$$

for any pair of functions u, v in $\mathcal{C}$ is given as the vague limit of non-random Radon measures $\{\mu_k\}$.

Since $\{\mathcal{E}_{F_k}\}$ is represented as $\{\mathcal{E}_k\}$ by taking the function $1/\|x-y\|_p^{\alpha}$ as $\rho_k(x,y)$ independently chosen in k, we may concentrate our attention only on the sequence $\{\mathcal{E}_k\}$ represented by (4.1). By denoting either the Haar measure used in [1] or μ_F introduced in [33] by μ_0 in either case discussed so far, the bilinear forms obtained as the limit are written by a single representation

$$\mathcal{E}_0(u,v) = \frac{1}{2} \iint_{(\mathbb{Z}_p\times\mathbb{Z}_p)\cap\{x\neq y\}} (u(x)-u(y))(v(x)-v(y))\rho(x,y)\, d\mu_0(x) d\mu_0(y) \tag{4.2}$$

for any pair of functions u, v in $\mathcal{C}$.

Similarly to [29], for any sequence $\{c(m)\}_{m\in\mathbb{Z}_-\cup\{0\}}$ with $\mathbb{Z}_- = \{m \in \mathbb{Z} \mid m < 0\}$ satisfying $c(0) \leq c(-1) \leq \cdots \leq c(m) \leq \cdots$ and Radon measure ν on $\mathbb{Z}_p$, we can define $\overline{b}_{x,m}^{(\nu)} = \nu(B(x,p^m))$ and $b_{x,\ell}^{(\nu)} = \nu(B(x,p^{\ell}) \setminus B(x,p^{\ell-1}))$ non-positive integers m, ℓ and take a symmetric bilinear form

$$\mathcal{E}^{(\{c\},\nu)}(u,v) = \frac{1}{2} \iint_{(\mathbb{Z}_p\times\mathbb{Z}_p)\cap\{x\neq y\}} (u(x)-u(y))(v(x)-v(y))\rho^{(\{c\},\nu)}(x,y)\, d\nu(x) d\nu(y)$$

where $\rho^{(\{c\},\nu)}(x,y) = \sum_{\ell=m+1}^{0} b_{x,\ell}^{(\nu)} c(\ell)(2c(m) - c(\ell)) - \overline{b}_{x,m}^{(\nu)} c(m)^2$ with the integer m satisfying $p^m = \|x-y\|_p$.

For the sequence $\{\mu_k\}$ of the Radon measure arising from the random fractal and its vague limit μ_0, we obtain the sequence $\{\mathcal{E}^{(\{c\},\mu_k)}\}$ of the symmetric bilinear forms and $\{\mathcal{E}^{(\{c\},\mu_0)}\}$. As a corollary of Proposition 1.1, we have the following assertion:

Corollary 4.2 *The sequence $\{\mathcal{E}^{(\{c\},\mu_k)}\}$ converges to $\mathcal{E}^{(\{c\},\mu_0)}$, in the sense that*

$$\lim_{k\to\infty} \mathcal{E}^{(\{c\},\mu_k)}(u,v) = \mathcal{E}^{(\{c\},\mu_0)}(u,v),$$

for any pair of functions u, v in $\mathcal{C}$ with probability one.

This corollary allows us to choose $\rho^{(\{c\},\mu_k)}$ as ρ_k in (4.1) for regarding $\mathcal{E}^{(\{c\},\mu_k)}$ as $\mathcal{E}_k$ and choose $\rho^{(\{c\},\mu_0)}$ as ρ in (4.2) for regarding $\mathcal{E}^{(\{c\},\mu_0)}$ as $\mathcal{E}_0$. In what follows, we will establish a method to address the convergence of Hunt processes relevant to the sequences $\{\mathcal{E}_{H_{k,\mathbb{Z}_p}}\}$, $\{\mathcal{E}_{F_k}\}$ and $\{\mathcal{E}^{(\{c\},\mu_k)}\}$ of Dirichlet forms each of which converges to $\mathcal{E}_K, \mathcal{E}_F$ and $\mathcal{E}^{(\{c\},\mu_0)}$ respectively as symmetric bilinear forms. From now on, whichever sequence of Dirichlet forms is addressed, we will denote the sequence of Dirichlet forms by $\{\mathcal{E}_k\}$ and its limit by $\mathcal{E}_0$. In building our scheme, we may concentrate our focus on the case that ρ and any ρ_k in the sequence take constant on every Cartesian product $B \times B'$ of distinct balls B and B' with the same radius in $\mathbb{Z}_p$ and that $\lim_{k\to\infty} \rho_k(x,y) = \rho(x,y)$ for any pair x, y of distinct points in $\mathbb{Z}_p$.

Lemma 4.3 *The symmetric bilinear form $(\mathcal{E}_0, \mathcal{C})$ is closable on $L^2(\mathbb{Z}_p;\mu_0)$.*

This lemma shows that $\mathcal{E}_0$ admits the symmetric bilinear form $\mathcal{E}$ with the domain $\mathcal{F}$ obtained as the smallest closed extension of $\mathcal{E}_0$ and as its domain.

4.3 Lemmas

For any $\delta > 0$, we define the symmetric bilinear form

$$\mathcal{E}_k^{(\delta)}(u,v) = \frac{1}{2}\iint_{(\mathbb{Z}_p\times\mathbb{Z}_p)\cap\{\|x-y\|_p>\delta\}} (u(x)-u(y))(v(x)-v(y))\rho_k(x,y)\, d\mu_k(x)d\mu_k(y)$$

with domain $\mathcal{C}_k$ and

$$\mathcal{E}^{(\delta)}(u,v) = \frac{1}{2}\iint_{(\mathbb{Z}_p\times\mathbb{Z}_p)\cap\{\|x-y\|_p>\delta\}} (u(x)-u(y))(v(x)-v(y))\rho(x,y)\, d\mu_0(x)d\mu_0(y)$$

with domain $\mathcal{C}$. Then, $\mathcal{E}_k$ and $\mathcal{E}$ admit similar expressions obtained by taking $\delta = 0$.

In discussing the convergence of $L^2(\mathbb{Z}_p;\mu_k)$ to $L^2(\mathbb{Z}_p;\mu_0)$ in the sense of Kuwae and Shioya [27], we can take the identity map as the linear operator $\Phi_k : \mathcal{C} \to L^2(\mathbb{Z}_p;\mu_k)$ for any k. This is because the vague convergence of $\{\mu_k\}$ to μ_0 implies $\lim_{k\to\infty} \|u\|_{L^2(\mathbb{Z}_p;\mu_k)} = \|u\|_{L^2(\mathbb{Z}_p;\mu_0)}$ for any $u \in \mathcal{C}$.

For our situation, we can restate the definition of the KS-strongly convergence in [27] as follows: A sequence $\{v_k\}_{k=1}^{\infty}$ of functions $v_k \in L^2(\mathbb{Z}_p; \mu_k)$ is said to be KS-strongly convergent to a function $v \in L^2(\mathbb{Z}_p; \mu_0)$, if there exists a sequence $\{\varphi_j\}_{j=1}^{\infty}$ with $\varphi_j \in \mathcal{C}$ such that $\lim_{j\to\infty} \limsup_{k\to\infty} \|\varphi_j - v_k\|_{L^2(\mathbb{Z}_p;\mu_k)} = 0$ and $\lim_{j\to\infty} \|\varphi_j - v\|_{L^2(\mathbb{Z}_p;\mu_0)} = 0$. Then we recall that a sequence $\{u_k\}_{k=1}^{\infty}$ of functions $u_k \in L^2(\mathbb{Z}_p; \mu_k)$ is said to be KS-weakly convergent to a function $u \in L^2(\mathbb{Z}_p; \mu_0)$, if $\lim_{k\to\infty}(u_k, v_k)_{L^2(\mathbb{Z}_p;\mu_k)} = (u, v)_{L^2(\mathbb{Z}_p;\mu_0)}$ for any KS-strongly convergent sequence $\{v_k\}_{k=1}^{\infty}$ with $v_k \in L^2(\mathbb{Z}_p; \mu_k)$ and the KS-strong limit v.

Let us take a KS-weakly convergent sequence $\{u_k\}_{k=1}^{\infty}$ with a KS-weak limit in $L^2(\mathbb{Z}_p; \mu_0)$ and fix a positive integer ℓ satisfying $p^{-\ell} \le \delta$. For a given locally integrable function u on $\mathbb{Z}_p$ with respect to μ_k, we define a locally constant function on $\mathbb{Z}_p$ by

$$(u)_{\mu_k,\ell}(x) = \frac{1}{\mu_k(B(x, p^{-\ell}))} \int_{B(x,p^{-\ell})} u(y)\, d\mu_k(y).$$

Lemma 4.4 (i) $\sup_k \|u_k\|_{L^2(\mathbb{Z}_p;\mu_k)} < \infty$,

(ii) $k \ge \ell$ *implies* $\int_{\mathbb{Z}_p} (u_k)_{\mu_k,\ell}(x)\, d\mu_k(x) = \int_{\mathbb{Z}_p} u_k(x)\, d\mu_k(x)$,

(iii) $\sup_{\ell} \sup_{k\ge\ell} \|(u_k)_{\mu_k,\ell}\|_{L^2(\mathbb{Z}_p;\mu_k)} < \infty$,

(iv) *there exists a sequence* $\{K(\ell)\}_{\ell=1}^{\infty}$ *of non-negative integers such that* $\sup_{\ell} \sup_{k\ge K(\ell)} \|(u_k)_{\mu_k,\ell}\|_{L^2(\mathbb{Z}_p;\mu_0)} < \infty$.

For any pair B, B' of disjoint balls with the same radius $p^{-\ell}$ with $p^{-\ell} \le \delta$, we denote the values $\rho(x, x')$ and $\rho_k(x, x')$ determined independently of the choice $x \in B, x' \in B'$ by $\rho(B, B')$ and by $\rho_k(B, B')$, respectively, and then we see $\rho_k(B, B') \ge (1 - \delta)\rho(B, B')$ for sufficiently large k owing to the convergence $\lim_{k\to\infty} \rho_k(x, y) = \rho(x, y)$ for any distinct $x, y \in \mathbb{Z}_p$.

Since one can observe that

$$\begin{aligned}
&|(u)_{\mu_k,\ell}(x) - (u)_{\mu_k,\ell}(y)| \\
&= \frac{1}{\mu_k(B(x, p^{-\ell}))\mu_k(B(y, p^{-\ell}))} \\
&\quad \times \Big| \int_{B(x,p^{-\ell})} \int_{B(y,p^{-\ell})} (u(w) - u(z))\, d\mu_k(w) d\mu_k(z) \Big| \\
&\le \Big(\frac{1}{\mu_k(B(x, p^{-\ell}))\mu_k(B(y, p^{-\ell}))} \\
&\quad \times \int_{B(x,p^{-\ell})} \int_{B(y,p^{-\ell})} (u(w) - u(z))^2\, d\mu_k(w)\, d\mu_k(z) \Big)^{1/2}
\end{aligned}$$

with the right-hand side independent of $x \in B, y \in B'$ for any balls B, B' with the radius $p^{-\ell}$

and that $\mu_k(B) \geq (1-\delta)\mu_0(B)$ for any ball B with the radius $p^{-\ell}$ and sufficiently large k,

$$\begin{aligned}
&\mathcal{E}_k(u,u)\\
&\geq \mathcal{E}_k^{(\delta)}(u,u)\\
&\geq \iint \frac{1_{\{\|x-y\|_p>\delta\}}}{\mu_k(B(x,p^{-\ell}))\mu_k(B(y,p^{-\ell}))}\\
&\quad\times \int_{B(x,p^{-\ell})}\int_{B(y,p^{-\ell})} (u(w)-u(z))^2\rho_k(z,w)\,d\mu_k(w)d\mu_k(z)d\mu_k(x)d\mu_k(y)\\
&\geq \iint_{\{\|x-y\|_p>\delta\}} ((u)_{\mu_k,\ell}(x)-(u)_{\mu_k,\ell}(y))^2\rho_k(x,y)\,d\mu_k(x)d\mu_k(y)\\
&= \sum_{B\cap B'\neq\emptyset,\mathrm{diam}(B)=\mathrm{diam}(B')=p^{-\ell}} ((u)_{\mu_k,\ell}(B)-(u)_{\mu_k,\ell}(B'))^2\rho_k(B,B')\mu_k(B)d\mu_k(B')\\
&\geq (1-\delta)^3\sum_{B\cap B'\neq\emptyset} ((u)_{\mu_k,\ell}(B)-(u)_{\mu_k,\ell}(B'))^2\rho(B,B')\mu_0(B)d\mu_0(B')\\
&\geq (1-\delta)^3\iint_{\{\|x-y\|_p>\delta\}} ((u)_{\mu_k,\ell}(x)-(u)_{\mu_k,\ell}(y))^2\rho(x,y)\,d\mu_0(x)d\mu_0(y),
\end{aligned}$$

where $(u)_{\mu_k,\ell}(B)$ and $(u)_{\mu_k,\ell}(B')$ stand for the values $(u)_{\mu_k,\ell}(x)$ and $(u)_{\mu_k,\ell}(y)$ determined independently of the choice $x \in B, y \in B'$ respectively.

For any KS-weakly convergent sequence $\{u_k\}_{k=1}^{\infty}$, by taking sufficiently large k, we have

$$\begin{aligned}
&\mathcal{E}_k(u_k,u_k)\\
&\geq \mathcal{E}_k^{(\delta)}(u_k,u_k)\\
&\geq \iint \frac{1_{\{\|x-y\|_p>\delta\}}}{\mu_k(B(x,p^{-\ell}))\mu_k(B(y,p^{-\ell}))}\\
&\quad\times \int_{B(x,p^{-\ell})}\int_{B(y,p^{-\ell})} (u_k(w)-u_k(z))^2\rho_k(z,w)\,d\mu_k(w)d\mu_k(z)d\mu_k(x)d\mu_k(y)\\
&\geq (1-\delta)^3\iint_{\{\|x-y\|_p>\delta\}} ((u_k)_{\mu_k,\ell}(x)-(u_k)_{\mu_k,\ell}(y))^2\rho(x,y)\,d\mu_0(x)d\mu_0(y)\\
&\geq (1-\delta)^3\mathcal{E}^{(\delta)}((u_k)_{\mu_k,\ell},(u_k)_{\mu_k,\ell}).
\end{aligned}\tag{4.3}$$

We may assume that $\sup_k \mathcal{E}_k(u_k,u_k) < \infty$ without losing general setting in our discussion on the generalized Mosco-convergence. Accordingly, the boundedness of the sequence $\{(u_k)_{\mu_k,\ell}\}_{k\geq K(\ell)}$ with respect to the norm $(\mathcal{E}^{(\delta)}(u,u)+\|u\|^2_{L^2(\mathbb{Z}_p;\mu_0)})^{1/2}$ is obtained.

In dealing with the measure μ_0, we may remove the balls with radius $p^{-\ell}$ from $\mathbb{Z}_p$ which are not charged by the measure μ_0. For a precise description, we denote the family of all balls with radius $p^{-\ell}$ contained in $\mathbb{Z}_p$ by $\{B_i \mid i=0,\dots,p^\ell-1\}$ and replace $\mathbb{Z}_p$ with $S_\ell = \bigcup_{i\in I_\ell} B_i$ by introducing $I_\ell = \{i \in \{0,\dots,p^\ell-1\} \mid \mu_0(B_i) > 0\}$ in the definition of the KS-strong convergence. For instance, we can observe the KS-strong convergence of $\{v_k 1_{S_\ell}\}_{k=0}^{\infty}$ to $v1_{S_\ell}$ when the KS-strong convergence of the sequence $\{v_k\}_{k=0}^{\infty}$ to v is obtained. Here, we introduce a positive finitely-many valued function $\{g_\ell\}$ and a sequence $\{h_{k,\ell}\}$ of finitely-many valued functions on S_ℓ by $g_\ell(x) = \sum_{i\in I_\ell} \mu_0(B_i)1_{B_i}(x)$ and $h_{k,\ell}(x) = \sum_{i\in I_\ell} \mu_k(B_i)1_{B_i}(x)$. Then the vague convergence of $\{\mu_k\}_{k=1}^\infty$ to μ_0 implies $\lim_{k\to\infty} h_{k,\ell}(x) = g_\ell(x)$ at each $x \in S_\ell$. In what follows, we will denote $\mathrm{supp}[\mu_k]$ by F_k and $\mathrm{supp}[\mu_0]$ by F. Now we are in position to discuss the following convergences:

Lemma 4.5 *For any KS-weakly convergent sequence $\{u_k\}_{k=1}^\infty$ with the KS-weak limit $u \in L^2(F;\mu_0)$,*

(i) *$\{u_k(g_\ell/h_{k,\ell})1_{S_\ell}\}_{k=1}^\infty$ is a KS-weakly convergent sequence to u,*

(ii) *there exists a subsequence $\{k_\ell\}$ satisfying*

$$\lim_{\ell\to\infty}((u_{k_\ell}1_{S_\ell})_{k_\ell,\ell}, 1_B)_{L^2(F_{k_\ell};\mu_{k_\ell})} = (u, 1_B)_{L^2(F;\mu_0)}$$

for any ball B in $\mathbb{Z}_p$.

Let $\{u_k\}$ be a KS-weakly convergent sequence with a KS-weak limit $u \in L^2(F;\mu_0)$ satisfying $u_k \in \mathcal{C}_k$ for any $k = 1, 2, \ldots$. By applying the Lemma 4.5 (ii), we can take a subsequence $\{k_\ell\}$ of $\{k\}$ such that

$$\begin{aligned}\lim_{\ell\to\infty}((u_{k_\ell})_{\mu_{k_\ell},\ell}, 1_B)_{L^2(F;\mu_0)} &= \lim_{\ell\to\infty}((u_{k_\ell}1_{S_\ell})_{\mu_{k_\ell},\ell}, 1_B)_{L^2(F;\mu_0)}\\ &= (u, 1_B)_{L^2(F;\mu_0)}\end{aligned}$$

for any ball B with sufficiently small radius, where the first identity follows from the identity $(u_{k_\ell})_{\mu_{k_\ell},\ell} = (u_{k_\ell}1_{S_\ell})_{\mu_{k_\ell},\ell}$ on S_ℓ and $\mathrm{supp}[\mu_0] = F \subset S_\ell$ for any ℓ.

4.4 Convergence of Hunt processes

In this section, we will validate the conditions (a) and (b) listed in Definition 2.1 (iv) of [14]. For that purpose, we consider the KS-weakly convergent sequence $\{u_k\}$ with the KS-weak limit u and its subsequence $\{(u_{k_\ell})_{\mu_k,\ell}\}$ taken in the previous section. We may assume that $\{(u_{k_\ell})_{\mu_k,\ell}\}$ is extracted from $\{u_k\}$ so that $\{u_{k_\ell}\}$ satisfies $\lim_{\ell\to\infty} \mathcal{E}_{k_\ell}(u_{k_\ell}, u_{k_\ell}) = \liminf_{k\to\infty} \mathcal{E}_k(u_k, u_k)$. The Banach-Saks theorem shows that some subsequence $\{(u_{k_{\ell'}})_{\ell'}\}$ of $\{(u_{k_\ell})_{\mu_k,\ell}\}$ is extracted so as to converge weakly to some element v in $L^2(F;\mu_0)$ and satisfy

$$\lim_{n\to\infty}\left\|\frac{1}{n}\sum_{\ell'=1}^{n}(u_{k_{\ell'}})_{\ell'} - v\right\|_{L^2(F;\mu_0)} = 0$$

and

$$\lim_{n\to\infty}\mathcal{E}^{(\delta)}\Big(\frac{1}{n}\sum_{\ell'=1}^{n}(u_{k_{\ell'}})_{\ell'} - v, \frac{1}{n}\sum_{\ell'=1}^{n}(u_{k_{\ell'}})_{\ell'} - v\Big) = 0.$$

On the other hand, we have seen that $\lim_{\ell\to\infty}((u_{k_\ell})_{\mu_k,\ell}, 1_B)_{L^2(F;\mu_0)} = (u, 1_B)_{L^2(F;\mu_0)}$ for any ball B with sufficiently small radius. Any $u \in L^2(F;\mu_0)$ gives the functional $T(\varphi) = \int_F u(x)\varphi(x)\, d\mu_0(x)$ defined for the restriction $\varphi|_F$ of locally constant function φ on $\mathbb{Z}_p$ to F

and the fact that T vanishes if and only if u is the zero element in $L^2(F;\mu_0)$ implies the family $\mathcal{C}(F)$ of continuous functions on F is a dense subfamily of $L^2(F;\mu_0)$. Therefore, we conclude that $u = v$. Similarly to Section 5 in [14], we can prove

$$\liminf_{\ell'\to\infty} \mathcal{E}_{k_{\ell'}}(u_{k_{\ell'}}, u_{k_{\ell'}}) \geq \mathcal{E}(u,u),$$

as required in the generalized Mosco-convergence. In fact, by applying

$$\mathcal{E}^{(\delta)}\Big(\frac{1}{n}\sum_{\ell'=1}^{n}(u_{k_{\ell'}})_{\ell'}, \frac{1}{n}\sum_{\ell'=1}^{n}(u_{k_{\ell'}})_{\ell'}\Big)^{1/2} \leq \frac{1}{n}\sum_{\ell'=1}^{n}\mathcal{E}^{(\delta)}((u_{k_{\ell'}})_{\ell'}, (u_{k_{\ell'}})_{\ell'})^{1/2},$$

we can derive

$$\begin{aligned}
\liminf_{k\to\infty} \mathcal{E}_k(u_k,u_k) &\geq \lim_{\ell'\to\infty} \mathcal{E}^{(\delta)}_{k_{\ell'}}(u_{k_{\ell'}}, u_{k_{\ell'}}) \\
&\geq (1-\delta)^3 \limsup_{\ell'\to\infty} \mathcal{E}^{(\delta)}((u_{k_{\ell'}})_{\ell'}, (u_{k_{\ell'}})_{\ell'}) \\
&\geq (1-\delta)^3 \lim_{n\to\infty} \mathcal{E}^{(\delta)}\Big(\frac{1}{n}\sum_{\ell'=1}^{n}(u_{k_{\ell'}})_{\ell'}, \frac{1}{n}\sum_{\ell'=1}^{n}(u_{k_{\ell'}})_{\ell'}\Big) \\
&\geq (1-\delta)^3 \mathcal{E}^{(\delta)}(u,u)
\end{aligned}$$

from (4.3).

Since these inequalities hold valid for any $\delta > 0$, the required inequality for the condition (a) has been validated.

The rest of the section is devoted to verify the other condition for the generalized Mosco-convergence. Namely, we will show that, for any $u \in L^2(F;\mu_0)$, there exists a KS-strongly convergent sequence $\{u_k\}$ with the KS-strong limit u satisfying $\limsup_{k\to\infty} \mathcal{E}_k(u_k,u_k) \leq \mathcal{E}(u,u)$ as in (iv)-(b) in Definition 2.1 in [14].

We first note that we may assume u in $\mathcal{F}$. By the definition of the smallest closed extension $\mathcal{E}$ of $\mathcal{E}_0$, for any $u \in \mathcal{F}$, we can take the sequence $\{\varphi_n\}$ in the family of locally constant function on $\mathbb{Z}_p$,

$$\lim_{n\to\infty} \mathcal{E}(\varphi_n,\varphi_n) = \mathcal{E}(u,u) \text{ and } \lim_{n\to\infty} \|\varphi_n|_F - u\|_{L^2(F;\mu_0)} = 0. \tag{4.4}$$

By Proposition 4.1, Corollary 4.2 and the vague convergence of $\{\mu_k\}$ to μ_0, there exists some $k_1 > 0$ such that for $k > k_1$

$$|\,\|\varphi_1 - \varphi_0\|_{L^2(\mathbb{Z}_p;\mu_k)} - \|\varphi_1 - \varphi_0\|_{L^2(F;\mu_0)}\,| < \frac{1}{2}$$

and

$$|\mathcal{E}_k(\varphi_1,\varphi_1) - \mathcal{E}(\varphi_1,\varphi_1)| < \frac{1}{2}. \tag{4.5}$$

Here and the sequel, the domain of the measure μ_0 is viewed as the family of topological Borel sets in $\mathbb{Z}_p$. For some $k_2 > 0$, $k > k_2$ implies that

$$|\varphi_2 - \varphi_0{}_{L^2(\mathbb{Z}_p;\mu_k)} - \|\varphi_2 - \varphi_0\|_{L^2(F;\mu_0)}\,| < \frac{1}{2^2},$$

$$|\,\|\varphi_2 - \varphi_1\|_{L^2(\mathbb{Z}_p;\mu_k)} - \|\varphi_2 - \varphi_1\|_{L^2(F;\mu_0)}\,| < \frac{1}{2^2}$$

and

$$|\mathcal{E}_k(\varphi_2, \varphi_2) - \mathcal{E}(\varphi_2, \varphi_2)| < \frac{1}{2^2}.$$

By repeating this procedure, for any positive integer j, there exists k_j such that $k > k_j$ implies

$$|\,\|\varphi_j - \varphi_i\|_{L^2(\mathbb{Z}_p;\mu_k)} - \|\varphi_j - \varphi_i\|_{L^2(F;\mu_0)}\,| < \frac{1}{2^j}$$

for any positive integer i with $i < j$ and

$$|\mathcal{E}_k(\varphi_j, \varphi_j) - \mathcal{E}(\varphi_j, \varphi_j)| < \frac{1}{2^j}$$

for any integer k with $k > k_j$. For any K, there exists i_K such that $i, j \geq i_K$ implies $\|\varphi_j - \varphi_i\|_{L(\mathbb{Z}_p;\mu_0)} < \frac{1}{2^K}$. Accordingly, one sees that $j \geq i_K$ implies

$$\|\varphi_j - \varphi_{i_K}\|_{L^2(F;\mu_0)} \leq \frac{1}{2^K},$$

which shows that

$$\|\varphi_j - \varphi_{i_K}\|_{L^2(\mathbb{Z}_p;\mu_k)} \leq \|\varphi_j - \varphi_{i_K}\|_{L^2(F;\mu_0)} + \frac{1}{2^j} \leq \frac{1}{2^K} + \frac{1}{2^j},$$

for any integers j with $j > i_K$ and $k > k_j$.

We denote φ_{i_K} by ψ_K and define $\{u_k\}$ by $u_k = \varphi_j$ for $k_{j+1} \geq k > k_j$. Then for sufficiently large k,

$$\|u_k - \psi_K\|_{L^2(\mathbb{Z}_p;\mu_k)} \leq \frac{1}{2^j} + \frac{1}{2^K},$$

which shows

$$\limsup_{k\to\infty} \|u_k - \psi_K\|_{L^2(\mathbb{Z}_p;\mu_k)} \leq \frac{1}{2^K}.$$

Since $\limsup_{K\to\infty} \|u - \psi_K\|_{L^2(F;\mu_0)} = 0$, the sequence $\{u_k\}$ converges KS-strongly to u. By combining this with $\limsup_{k\to\infty} \mathcal{E}_k(u_k, u_k) = \mathcal{E}(u, u)$ obtained by (4.4) and (4.5), the generalized Mosco-convergence of the Dirichlet forms is verified.

We shall shift our attention to the tightness of the Hunt processes associated with the Dirichlet spaces. We see that our Dirichlet form $\mathcal{E}_k$ is covered by the framework in [18] with Kolmogorov's equation. In fact, $\mathbb{Z}_p$ is described as the disjoint union of finitely many balls $\{K_i^{(m)}\}$ with radius p^{-m} and the value $\mathcal{E}_k(1_{K_i^{(m)}}, 1_{K_j^{(m)}})$ of the symmetric bilinear form for distinct balls $K_i^{(m)}$, $K_j^{(m)}$ is given by $\mu_k(K_i^{(m)}) \int_{K_j^{(m)}} \rho_k(x, y)\, d\mu_k(y)$ determined independently by choice of $x \in K_i^{(m)}$. This shows that the Hunt process $\{X_t^{(k)}\}$ generated by the Dirichlet form $\mathcal{E}_k$ on $L^2(\mathbb{Z}_p; \mu_k)$ admits a characterization by Kolmogorov's equations

$$\begin{cases} \dfrac{d}{dt} P_{K_f^{(m)}, K_i^{(m)}}(t) = -\widetilde{a}_k(K_f^{(m)}) P_{K_f^{(m)}, K_i^{(m)}}(t) + \displaystyle\sum_{j\neq f}^{\infty} \widetilde{u}_k(K_f^{(m)}, K_j^{(m)}) P_{K_j^{(m)}, K_i^{(m)}}(t), \\ \dfrac{d}{dt} P_{K_f^{(m)}, K_i^{(m)}}(t) = -\widetilde{a}_k(K_i^{(m)}) P_{K_f^{(m)}, K_i^{(m)}}(t) + \displaystyle\sum_{j\neq i}^{\infty} P_{K_f^{(m)}, K_j^{(m)}}(t) \widetilde{u}_k(K_j^{(m)}, K_i^{(m)}), \end{cases}$$

with the coefficients

$$\widetilde{u}_k(K_f^{(m)}, K_j^{(m)}) = \int_{K_j^{(m)}} \rho_k(x, y)\, d\mu_k(y)$$

and

$$\widetilde{a}_k(K_f^{(m)}) = \sum_{j \neq f}^{\infty} \widetilde{u}_k(K_f^{(m)}, K_j^{(m)}) = \int_{K_f^{(m)c}} \rho_k(x, y)\, d\mu_k(y)$$

determined independently by the choice of $x \in K_f^{(m)}$.

This also shows that the exponential holding time of $\{X_t^{(k)}\}$ for the transition out of state $K_f^{(m)}$ is determined by the parameter $\widetilde{a}_k(K_f^{(m)})$. Namely, the exit time $\tau_{k,K_f^{(m)}}$ of $\{X_t^{(k)}\}$ from the ball $K_f^{(m)}$ is exponentially distributed with the parameter $\widetilde{a}_k(K_f^{(m)})$ under P_x with $x \in K_f^{(m)}$. Accordingly, it admits the following estimate

$$P_x(\tau_{k,K_f^{(m)}} < \delta) = 1 - \exp(-\widetilde{a}_k(K_f^{(m)})\delta) \leq \delta \int_{K_f^{(m)c}} \rho_k(x, y)\, d\mu_k(y)$$

for any $x \in K_f^{(m)}$ and $\delta > 0$ from which we can derive that

$$\begin{aligned} \sum_{K_f^{(m)} \subset \mathbb{Z}_p} \int_{K_f^{(m)}} P_x(\tau_{k,K_f^{(m)}} < \delta)\, d\mu_k(x) &\leq \delta \sum_{K_f^{(m)} \subset \mathbb{Z}_p} \int_{K_f^{(m)}} \int_{K_f^{(m)c}} \rho_k(x, y)\, d\mu_k(x) d\mu_k(y) \\ &= \frac{\delta}{2} \sum_{K_f^{(m)} \subset \mathbb{Z}_p} \mathcal{E}_k(1_{K_f^{(m)}}, 1_{K_f^{(m)}}). \end{aligned}$$

Since the right-hand side does not exceed an arbitrarily given real number $\varepsilon > 0$ for sufficiently small $\delta > 0$ independently of the choice of sufficiently large $k$as long as the radius p^{-m} is fixed, the tightness of $\{X_t^{(k)}\}$ on any finite time interval follows from the general framework on Markov processes, as discussed in Section 6 of [14].

For the main assertions in what follows, we replace the measure μ_k with $\mu_k(\cdot)/\mu_k(\mathbb{Z}_p)$ and denote the replaced measure again by μ_k for each $k = 0, 1, 2, \ldots$ under the condition $\mu_0(\mathbb{Z}_p) > 0$. In fact, all results obtained so far are valid after these replacements which can be performed with a positive probability in the cases based on the random fractal due to the results in [33]. When $\mu_0(\mathbb{Z}_p) = 0$, we conventionally redefine μ_0 as the trivial measure vanishing on $\mathbb{Z}_p$.

By applying the methods in the proofs of Theorem 2.1 and subsequent Corollary 2.1 in [14], we can obtain assertions similar to those in [14] on the resolvent $\{G_\lambda^{(k)}\}$ and the semi-group $\{P_t^{(k)}\}$ associated with the Dirichlet space $(\mathcal{E}_k, \mathcal{F}_k)$ for $k = 1, 2, \ldots$. Since the sequence of random Dirichlet forms has originally triggered our discussion, we can summarize the assertions in the following statements:

Theorem 4.6

(i) *The sequence* $\{\mathcal{E}_k\}$ *of Dirichlet forms is generalized Mosco-convergent to* $\mathcal{E}$ *as* $k \to \infty$ *with probability one under the condition* $\mu_0(\mathbb{Z}_p) > 0$,

(ii) *for any* $\lambda > 0$, *the sequence* $\{G_\lambda^{(k)}\}$ *of the resolvent operators is KS-convergent to* G_λ *as* $k \to \infty$ *with probability one under the condition* $\mu_0(\mathbb{Z}_p) > 0$,

(iii) *for any* $t > 0$, *the sequence* $\{P_t^{(k)}\}$ *of the semi-group operators is KS-convergent to* P_t *as* $k \to \infty$ *with probability one under the condition* $\mu_0(\mathbb{Z}_p) > 0$.

Corollary 4.7 *The finite dimensional distribution of* $\{X_t^{(k)}\}$ *with initial distribution* μ_k *generated by the Dirichlet space* $(\mathcal{E}_k, \mathcal{F}_k)$ *weakly converges to the one of the Hunt process* $\{X_t^{(0)}\}$ *with initial distribution* μ_0 *generated by the Dirichlet space* $(\mathcal{E}, \mathcal{F})$, *i.e., for any positive real numbers* $t_1, t_2, \ldots, t_\ell$ *with* $t_1 < t_2 < \cdots < t_\ell$ *and any real-valued continuous function* u *on* $\mathbb{Z}_p^\ell$

$$\lim_{k\to\infty} E^{\mu_k}[u(X^{(k)}(t_1), \ldots, X^{(k)}(t_\ell))] = E^{\mu_0}[u(X^{(0)}(t_1), \ldots, X^{(0)}(t_\ell))]$$

with probability one under the condition $\mu_0(\mathbb{Z}_p) > 0$.

Let us denote the space of $\mathbb{Z}_p$-valued right continuous function on $[0, t]$ with left limits by $\mathbf{D}_{\mathbb{Z}_p}([0, t])$. Then we have the following assertion similarly to [14]:

Theorem 4.8 *The probability law of the Hunt process* $\{X_t^{(k)}\}$ *under* P^{μ_k} *weakly converges to the one of* $\{X_t^{(0)}\}$ *under* P^{μ_0} *in* $\mathbf{D}_{\mathbb{Z}_p}([0, t])$ *as* k *tends to* ∞ *with probability one under the condition* $\mu_0(\mathbb{Z}_p) > 0$.

Chapter 5

Orlicz norm and Sobolev-Orlicz capacity on ends of tree based on probabilistic Bessel kernels

5.1 Sobolev-Oricz capacity on ultrametric spaces

A class of Markov processes on the field of p-adic numbers constructed by Albeverio and Karwowski associates the spectral theory for spectral analysis in [1] and their method was improved so that a more general class of Markov processes on ends of tree is covered in [2]. It is noteworthy to recall that their transition semi-groups are explicitly described. This is partly because capacitary estimates have been discussed in [31] and [32] based on the kernels determined by transition probability rooted in [28], where the use of probabilistic counterpart of the Bessel kernels is proposed.

It is widely accepted that the natural random walk on the binary tree gives a reinterpretation of a Markov process on the Cantor set equivalently on the ring of 2-adic integers by restricting our attention on the displacements of the random walk while the particle is traveling on the ends of tree which are attached to the tree as geometric ideal boundary points. Historically, Baxter suggested in [3] that such relationship of random walk on tree and Markov process on the ends of tree can be discussed. Afterwards, explicitly in [21], a clearer potential theoretic relationship is discussed when the ends constitute a compact set, which covers how the harmonic extension into the tree is determined by the boundary values on the ends. In [22], it is suggested that the complete orthonormal system in the family of the square integrable functions on the ends of tree plays an important role for a construction of Markov process on the ends of tree and the harmonic extension into the tree is considered without assuming compactness of the ends of tree.

As for capacity on the ends of tree, any estimate on it has not been discussed persistently based on the complete orthonormal system. The main objective of this chapter is rebuild a scheme on capacity theory based on the complete orthonormal system as taken in [22] and [34]. We will look at the Sobolev-Orlicz capacity on the ends of tree in the same spirit as [17] based on probabilistic reinterpretation proposed in [28]. We will derive a sufficient condition for non-polarity of singleton on the ends of tree from a spectral analytic method, probabilistic significance of which is validated in the Dirichlet space theory.

In Section 2, we will recall the notions of directed tree and ends of tree as in [22] equivalently as in [34]. In Section 3 and 4, we will recapitulate the notions of Luxemburg norm and its equivalent norms described as in [17], one of which will admits a tighter relationship with Dirichlet form. In Section 5, we will take the class of Dirichlet spaces established in [34] to obtain strongly continuous contraction semi-groups of linear operators determined by transition probability semi-groups via the Dirichlet space theory. We will see that the Sobolev-Orlizc capacities on the ends of tree can be given thanks to the natural counterpart of the Bessel kernel proposed in [28]. In the final section, we will show estimate on a Sobolev-Orlicz capacity, probabilistic importance of which will be unveiled by combining our observation with probabilistic results in the Dirichlet space theory as in [12], in a particular case that the capacity is coherently given to the Dirichlet space theory. We will discuss upper and lower estimates on another Sobolev-Orlicz capacity which is viewed as a natural probabilistic counterpart of [17].

5.2 Tree and its ends

We take a set T consisting of countably infinite vertices and a map $\mathcal{A} : T \times T \to \{0,1\}$ satisfying $\mathcal{A}(x,x) = 0$ for any $x \in T$ and $\mathcal{A}(x,y) = \mathcal{A}(y,x)$ for any $x, y \in T$. Each element in T will be called a node and the set $\{x,y\}$ of two distinct nodes x, y satisfying $\mathcal{A}(x,y) = 1$ will be called an edge. A sequence $(a_0, a_1, \dots, a_n)$ of nodes in T is called a path, if $\mathcal{A}(a_i, a_{i+1}) = 1$ is satisfied for any $i = 0, \dots, n-1$. If any pair of distinct elements y, z in T admits a path $(a_0, a_1, \dots, a_n)$ with $a_0 = y$ and $a_n = z$, the pair $(T, \mathcal{A})$ is called a non-directed tree. If a sequence $(a_0, a_1, \dots, a_n)$ satisfies $a_i \neq a_j$ for any distinct integers i, j, the sequence is said to be simple. The set $V(x)$ of nodes directly tied with $x \in T$ is given by $V(x) = \{y \mid \mathcal{A}(x,y) = 1\}$. Throughout the chapter, we suppose that a non-directed tree $(T, \mathcal{A})$ satisfying the following properties is given as in [34]:

(i) the tree does not admit any path $(a_0, a_1, \dots, a_n)$ satisfying $a_0 = a_n$ with distinct edges $\{a_0, a_1\}, \{a_1, a_2\}, \dots, \{a_{n-1}, a_n\}$,

(ii) $V(x)$ is a finite set.

We introduce the notion of end of the tree in the next similarly to [34]. An infinite sequence $(a_0, a_1, \dots)$ of nodes is called a geodesic ray if any finite subsequence of $(a_0, a_1, \dots, a_n, \dots)$ is simple path. We denote the set of geodesic rays by $\mathcal{R}$ and introduce an equivalence relation "$\sim$" on $\mathcal{R}$ defined by

$$(a_0, a_1, \dots) \sim (b_0, b_1, \dots) \Leftrightarrow \begin{cases} \text{there exists an integer } k \text{ satisfying} \\ a_{k+m} = b_m \text{ for any } m \geq 0 \end{cases}.$$

We restrict our attention to the case that there exists a node $o \in T$ such that any element in the quotient space $\mathcal{R}/\sim$ admits representative element $(o, a_1, a_2, \dots)$ and that for any

$x \in T$ is connected with o by a unique simple path $(o, \dots, x, \dots)$. We will denote $\mathcal{R}/\sim$ by Σ^+.

Then, the map $\pi : T \setminus \{o\} \to T$ is defined by

$$\pi(x) = \begin{cases} x' & \text{if } (o, \dots, x', x) \text{ is a simple path connecting } o \text{ and } x, \\ o & \text{if } (o, x) \text{ is a simple path connecting } o \text{ and } x. \end{cases}$$

T is represented as the disjoint union of its subsets T_m with $m = 0, 1, 2, \dots$ defined by $T_m = \{x \in T \mid \pi^m(x) = o\}$ for positive integer m and $T_0 = \{o\}$. Then, it turns out that $\pi(T_m) = T_{m-1}$ for any positive integer m. We consider the case that the property

(iii) $\#(\pi^{-1}(\{x\})) \geq 2$ for any $x \in T$

is satisfied as in [22].

Let us define

$$S_x = \{y \in T \mid \pi^k(y) = x \text{ for some non-negative integer } k\} \quad \text{for any } x \in T$$

and

$$\Sigma^+_x = \left\{ \eta \in \Sigma^+ \;\middle|\; \begin{array}{l} \eta \text{ admits a geodesic ray } (a_0, a_1, \dots) \\ \text{as a representative sequence of } \eta \\ \text{satsfying } a_0, a_1, \dots \in S_x \end{array} \right\}.$$

We can introduce a topology on Σ^+. As a matter of fact, the family $\{\Sigma^+_S \mid S \subset T\}$ of subsets $\Sigma^+_S = \cup_{x \in S} \Sigma^+_x$ determined by $S \subset T$ satisfies the axioms for open sets on Σ^+. We will regard Σ^+ as a topological space equipped with the family of open sets. It is easy to see that Σ^+ is compact.

Example 5.1 (A tree $T_{\mathbb{Z}_{p_0}}$ associated with the ring $\mathbb{Z}_{p_0}$ of p_0-adic integers) .
Let p_0 be a prime number and $T_{\mathbb{Z}_{p_0}}$ be the set consisting of all balls in $\mathbb{Z}_{p_0}$ and denote the radius of ball B by $\mathrm{r}(B)$. Then we define $\mathcal{A}_{\mathbb{Z}_{p_0}}(B, B')$ for $B, B' \in T_{\mathbb{Z}_{p_0}}$ by

$$\mathcal{A}_{\mathbb{Z}_{p_0}}(B, B') = \begin{cases} 1 & \text{if either } B \subset B',\, p_0\mathrm{r}(B) = \mathrm{r}(B') \text{ or } B' \subset B,\, p_0\mathrm{r}(B') = \mathrm{r}(B), \\ 0 & \text{otherwise.} \end{cases}$$

Then it is not difficult to see the pair $(T_{\mathbb{Z}_{p_0}}, \mathcal{A}_{\mathbb{Z}_{p_0}})$ is a tree satisfying condition (i) and (ii). Take the ball Δ_0 centered at the origin 0 and with the radius 1. Then Δ_0 is coincide with $\mathbb{Z}_{p_0}$ and any geodesic ray is represented as $(\Delta_0, \Delta_1, \dots)$ with elements Δ_i in $T_{\mathbb{Z}_{p_0}}$ satisfying $\Delta_i \subset \Delta_{i+1}$ and $\mathrm{r}(\Delta_i) = p_0^i$ for any $i = 0, 1, 2, \dots$. The map π is defined by $\pi(B) = B'$ with the ball B' characterized by $B \subset B'$ and $p_0\mathrm{r}(B) = \mathrm{r}(B')$ and in addition a homeomorphism between Σ^+ and $\mathbb{Z}_{p_0}$ is obtained. In fact, any end $\eta \in \Sigma^+$ admits a geodesic ray $(B_0, B_1, \dots)$ represented by a sequence of balls satisfying $B_0 \supsetneq B_1 \supsetneq \cdots$, which determines a singleton $\{a\} \subset \mathbb{Z}_{p_0}$ by $\{a\} = \cap_i B_i$. The map $\eta \mapsto a$ gives a bijection from Σ^+ to $\mathbb{Z}_{p_0}$ which is viewed as a homeomorphism.

The main assertions in [21] show that a Dirichlet form on the Cantor set is constructed so as to be a natural counterpart of the classical Douglas integral on the unit circle, where functions on the unit circle are replaced with ones on the Cantor set and the standard Brownian motion on the unit disk is replaced with a random walk on $T_{\mathbb{Z}_2}$ for accommodating the Dirichlet form to the generalization of the classical Douglas integral based on its probabilistic reinterpretation as in [6]. The results in [21] can be viewed as this sort of reconsideration natually arising from the case the unit circle is replaced with $\mathbb{Z}_2$. A more general scheme founded on the same motives is built in [22]. A relationship between random walks on tree and a capacity on the ends of tree is discussed in [35].

5.3 Young function, Luxemburg norm and Orlicz space

Throughout the chapter, we take a finite Radon measure μ on Σ^+ with the support Σ^+. For any Borel measurable set Ω, we denote the family of Borel measurable functions taking finite value a.e. with respect to μ by $\mathcal{M}_0(\Omega)$. Namely,

$$\mathcal{M}_0(\Omega) = \left\{ f \;\middle|\; \begin{array}{l} f \text{ is a Borel measurable function,} \\ f \text{ takes finite value a.e. with respect to } \mu \end{array} \right\}.$$

We will denote the indicator function for a Borel measurable subset $\Omega \subset \Sigma^+$ by

$$1_\Omega(x) = \begin{cases} 1 & x \in \Omega, \\ 0 & x \notin \Omega. \end{cases}$$

Definition 5.2 For any function $f \in \mathcal{M}_0(\Omega)$, the distribution function of f is defined as function $m_f : [0,\infty) \to [0,\infty]$ given by

$$m_f(\lambda) = \mu\{x \in \Omega \mid |f(x)| > \lambda\}.$$

Definition 5.3 The decreasing rearrangement of $f \in \mathcal{M}_0(\Omega)$ is given as the function $f^* : [0,\infty) \to [0,\infty]$ determined by

$$f^*(t) = \inf\{\lambda \mid m_f(\lambda) \le t\},$$

with the conventional assignment $\inf \emptyset = \infty$.

Definition 5.4 For $f \in \mathcal{M}_0(\Omega)$, the maximal function $f^{**} : [0,\infty) \to [0,\infty]$ is defined by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds.$$

Proposition 5.5 *Let $f, g \in \mathcal{M}_0(\Omega)$ and let $\{f_n\}_{n=1}^\infty$ be a sequence of functions in $\mathcal{M}_0(\Omega)$. Then,*

(i) *f^{**} is a non-negative and non-decreasing on $(0,\infty)$ satisfying $f = 0$ on $\Sigma^+ \Leftrightarrow f^{**} \equiv 0$,*

(ii) *$f^* \le f^{**}$ on $(0,\infty)$,*

(iii) $|g| \leq |f|$ *a.e. with respect to* μ *implies* $g^{**}(t) \leq f^{**}(t)$ *on* $(0,\infty)$,

(iv) $(cf)^{**} = |c|f^{**}$ *for any* $c \in \mathbb{R}$,

(v) $(f+g)^{**} \leq f^{**} + g^{**}$ *on* $(0,\infty)$,

(vi) $0 \leq f_1 \leq \cdots \leq f_n \leq f_{n+1} \leq \cdots$ *a.e. with respect to* μ *and* $\lim_{n\to\infty} f_n = f$ *a.e. with respect to* μ *on* Σ^+ *imply* $0 \leq f_1^{**} \leq \cdots \leq f_n^{**} \leq f_{n+1}^{**} \leq \cdots$ *and* $\lim_{n\to\infty} f_n^{**} = f^{**}$ *on* $(0,\infty)$.

We recall a measurable function $f : \Omega \to \mathbb{R}$ is viewed as an element in the Lebesgue space $L^p(\Omega;\mu)$, if and only if

$$\int_\Omega |f(x)|^p \, d\mu(x) < \infty.$$

As the generalized family of function of the so-called L^p space, we introduce the Orlicz space based on the notion of Luxemburg norm given by a Young function according to the results stated in [17]. In fact, we can establish those notions based on the space Σ^+.

Definition 5.6 If a strictly increasing convex function $\Phi : [0,\infty) \to [0,\infty)$ satisfies

$$\lim_{t\to 0+} \frac{\Phi(t)}{t} = \lim_{t\to\infty} \frac{t}{\Phi(t)} = 0,$$

then Φ is called a Young function.

Throughout the chapter, we deal only with a Young function satisfying the following condition called Δ_2-condition:

$$t \geq 0 \Rightarrow \Phi(2t) \leq C\Phi(t).$$

In [17], it is shown that Φ satisfies Δ_2-condition if and only if for any $l > 1$ there exists a constant $C(l)$ with $C(l) \geq 1$ such that $t \geq 0$ implies $\Phi(lt) \leq C(l)\Phi(t)$.

Definition 5.7 Let Φ be a Young function. For any Borel measurable set Ω in Σ^+, the family of functions called Orlicz class is defined as follows:

$$\widetilde{L}^\Phi(\Omega) = \left\{ f : \Omega \to \mathbb{R} \;\middle|\; f \text{ is a measurable function satisfying } \int_\Omega \Phi(|f(x)|) \, d\mu(x) < \infty \right\}.$$

Definition 5.8 Let Φ be a Young function and Ω be any Borel measurable set in Σ^+. Then, for any $f \in \mathcal{M}_0(\Omega)$, the Luxemburg norm $\|f\|_{L^\Phi(\Omega)}$ of f is defined by

$$\|f\|_{L^\Phi(\Omega)} = \inf \left\{ \theta > 0 \;\middle|\; \int_\Omega \Phi\left(\frac{|f(x)|}{\theta}\right) d\mu(x) \leq 1 \right\}.$$

We recall the notion of Banach function space based on which we will define Orlicz space $L^\Phi(\Omega)$.

Definition 5.9 If the map $\rho : \mathcal{M}_0(\Omega) \to [0,\infty]$ satisfies the following conditions, then ρ is called Banach function norm:

(i) $\rho(f) = 0 \iff f = 0$ a.e. with respect to μ,

(ii) for any non-negative real number a and $f, g \in \mathcal{M}_0$, $\rho(af) = a\rho(f)$ and $\rho(f+g) \leq \rho(f) + \rho(f)$,

(iii) if $0 \leq g \leq f$ a.e. with respect to μ, then $\rho(g) \leq \rho(g)$,

(iv) if $\{f_n\} \subset \mathcal{M}_0(\Omega)$ is a sequence of functions satisfying $0 \leq f_1 \leq \cdots \leq f_n \leq f_{n+1} \leq \cdots$ a.e. with respect to μ and $\lim\limits_{n\to\infty} f_n = f$ a.e. with respect to μ, then $\rho(f_n)$ is non-decreasing and $\lim\limits_{n\to\infty} \rho(f_n) = \rho(f)$,

(v) if E is a Borel measurable subset of Ω satisfying $\mu(E) < \infty$, then $\rho(1_E) < \infty$,

(vi) if E is a Borel measurable subset of Ω satisfying $\mu(E) < \infty$ and if f is non-negative, then these exists a positive constant C_E possibly depending on E such that

$$\int_E f(x)dx \leq C_E\rho(f).$$

The result on f^{**} in the Proposition 5.5 (v) will be applied to shed light on a probabilistic significance of the Orlicz norm.

Definition 5.10 For any Banach function norm ρ, the subfamily B_ρ of $\mathcal{M}_0(\Omega)$ is defined by

$$B_\rho = \{f \in \mathcal{M}_0(\Omega) \mid \rho(|f|) < \infty\}$$

and for any $f \in B_\rho$, $\|f\|_{B_\rho}$ is defined by

$$\|f\|_{B_\rho} = \rho(|f|).$$

It is shown in [Theorem 3.4.16, p. 99, 6] that the Luxemburg norm is a Banach function norm. Hence, the functions with finite Luxemburg norm constitute a Banach function space as a subfamily of $\mathcal{M}_0(\Omega)$. We are now in position to define Orlicz space.

Definition 5.11 Let Ω be a Borel measurable subset of Σ^+ and Φ be a Young function. The Orlicz space is defined as the Banach function space consisting of functions $f \in \mathcal{M}_0(\Omega)$ satisfying $\|f\|_{L^\Phi(\Omega)} < \infty$. The Orlicz space will be denoted by $L^\Phi(\Omega)$.

Remark 5.12 It is known that any Young function Φ satisfying Δ_2-condition provides us with the linear space $\widetilde{L}^\Phi(\Omega)$ which is exactly same as $L^\Phi(\Omega)$. We will employ the notation $L^\Phi(\Omega)$ instead of $\widetilde{L}^\Phi(\Omega)$ for the linear space.

Proposition 5.13 *If a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ satisfying $\Phi(0) = 0$, then*

$$\int_\Omega \Phi(|f(x)|)\, d\mu(x) = \int_0^{\mu(\Omega)} \Phi(f^*(t))\, dt,$$

for any $f \in \mathcal{M}_0(\Omega)$.

In what follows, we restrict our attention to the case that $\Phi(t) = t^p\varphi(t)$ with $\varphi : [0,\infty) \to [1,\infty)$ satisfying the following conditions:

(C 3.1) there exists a positive constant C such that $\varphi(t^2) \le C\varphi(t)$ for any $t \in [0,\infty)$.

(C 3.2) there exists a positive constant $\varepsilon < 1$ such that

$$\int_0^1 t^{-\varepsilon}\varphi\Big(\frac{1}{t}\Big)\,dt < \infty.$$

In such situation, for any $p \in [1,\infty)$ and $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(t) = t^p\varphi(t)$ gives a Young function Φ satisfying Δ_2-condition. By combining (C 3.1) and increasing property of $\varphi(t) \ge 1$ with respect to t, it turns out that there exists a positive constant C such that $s, t \in [0,\infty)$ imply

$$\varphi(st) \le C\varphi(s)\varphi(t).$$

In what follows, we assume that φ admits the expression $\Phi(t) = t^p\varphi(t)$ with φ satisfying (C 3.1), (C 3.2).

5.4 Equivalent norms

In this section, we will recall equivalent norms of $\|f\|_{L^\Phi(\Omega)}$ as indicated in [17], which will play an important role for a capacitary estimate directly related to the Dirichlet space theory. We start this section with some results which is proved by taking a similar procedure to [17] by assuming similar to [17] that Φ is a Young function satisfying Δ_2-condition. Throughout this section, we discuss equivalent norms to the Luxemburg norm on a Borel measurable subset Ω of Σ^+ satisfying $\mu(\Omega) = 1$. First we state three propositions substantially proved in [17].

Proposition 5.14 *For any $f \in \mathcal{M}_0(\Omega)$, $f \in L^\Phi(\Omega)$ if and only if*

$$\Big(\int_0^1 f^*(t)^p\varphi\Big(\frac{1}{t}\Big)\,dt\Big)^{1/p} < \infty.$$

In what follows, we take a real number p with $p \in (1,\infty)$.

Lemma 5.15 *For any $f \in \mathcal{M}_0(\Omega)$,*

$$\Big(\int_0^1 f^{**}(t)^p\varphi\Big(\frac{1}{t}\Big)\,dt\Big)^{1/p} \le \frac{p}{p-1}\Big(\int_0^1 f^*(t)^p\varphi\Big(\frac{1}{t}\Big)\Big)^{1/p} dt.$$

Proposition 5.16 *For any $f \in L^\Phi(\Omega)$, $f^*(\cdot)\varphi(\frac{1}{\cdot})^{1/p}$ and $f^{**}(\cdot)\varphi(\frac{1}{\cdot})^{1/p}$ are both belong to $L^p(0,1)$. For any $f \in L^\Phi(\Omega)$, there exist positive constants c_1 and c_2 such that*

$$c_1\Big(\int_0^1 f^{**}(t)^p\varphi\Big(\frac{1}{t}\Big)\,dt\Big)^{1/p} \le \Big(\int_0^1 f^*(t)^p\varphi\Big(\frac{1}{t}\Big)\,dt\Big)^{1/p} \le c_2\Big(\int_0^1 f^{**}(t)^p\varphi\Big(\frac{1}{t}\Big)\,dt\Big)^{1/p}.$$

In the next, we will see an equivalent Banach function norm is obtained by replacing f^* with f^{**}.

Lemma 5.17 *The map $\rho : \mathcal{M}_0(\Omega) \to [0, \infty]$ given by*

$$\rho(f) = \left(\int_0^1 f^{**}(t)^p \varphi\Big(\frac{1}{t}\Big)\, dt \right)^{1/p}$$

is a Banach space norm.

As pointed out in [17], we can derive the following fact from Corollary 1.1.9 in [4].

Lemma 5.18 *If two Banach function spaces consist of the same set of functions over an identical measure space, then their norms are equivalent.*

By taking the same p as in the previously given description $\Phi(t) = t^p \varphi(t)$, we see the following assertion as already pointed out in [17]:

Theorem 5.19 *For any $f \in \mathcal{M}_0(\Omega)$, $f \in L^\Phi(\Omega)$ if and only if*

$$\left(\int_0^1 f^{**}(t)^p \varphi\Big(\frac{1}{t}\Big)\, dt \right)^{1/p} < \infty$$

and there exist some positive constants c_3 and c_4 such that

$$c_3 \|f\|_{L^\Phi(\Omega)} \le \left(\int_0^1 f^{*}(t)^p \varphi\Big(\frac{1}{t}\Big)\, dt \right)^{1/p} \le c_4 \|f\|_{L^\Phi(\Omega)}$$

for any $f \in L^\Phi(\Omega)$.

In the case that $\Sigma^+ = \Omega$, such choice is allowed when $\mu(\Sigma^+) = 1$,

$$\left(\int_0^1 f^{*}(t)^p \varphi\Big(\frac{1}{t}\Big)\, dt \right)^{1/p}$$

will be denoted simply by $\|f\|_{L^\Phi}$.

5.5 Hunt process on ends of tree and Sobolev-Orlicz capacity

A function taking constant on every Σ_y^+ for some disjoint open cover $\{\Sigma_y^+\}_{y \in S}$ of Σ^+ determined by some $S \subset T$ is said to be locally constant. The family of locally constant functions taking constant on every Σ_y^+ with $y \in T_{m+1}$ will be denoted by $\mathcal{C}^m(\Sigma^+)$. The family of locally constant functions vanishing outside Σ_x^+ will be denoted by $\mathcal{C}(\Sigma_x^+)$ for every $x \in T$. The Stone-Weierstrass theorem shows that $\mathcal{C}(\Sigma_x^+)$ is contained densely in the family of continuous functions with support in Σ_x^+. In what follows, the intersection $\mathcal{C}(\Sigma_x^+) \cap \mathcal{C}^m(\Sigma^+)$ given by $x \in T_m$ will play an important role and will be denoted by $\mathcal{C}_x$. In this chapter, a node $x \in T$

will be called the confluent node for $y, z \in T$, if there exist positive integers m, ℓ such that $\pi^\ell(y) = \pi^m(z) = x$ and $\pi^{\ell-1}(y) \neq \pi^{m-1}(z)$. The confluent node for $y, z \in T$ will be denoted x by $[y, z]$.

We will denote $\#(\pi^{-1}(\{x\})) - 1$ by $n(x)$ for any $x \in T \setminus \{o\}$ and $\#(\pi^{-1}(\{o\}))$ by $n(o)$. Since we assume the condition (iii) in Section 2, we see that $n(x)$ is given as a positive integer for each $x \in T \setminus \{o\}$ and $n(o) \geq 2$. Throughout this chapter, we restrict our attention to the case that Σ^+ admits a Radon measure μ on Σ^+ satisfying $\mu(\Sigma^+) = 1$ with the support Σ^+ and a complete orthonormal system $\mathcal{V}$ of $L^2(\Sigma^+; \mu)$ are given so that $\mathcal{V}$ is divided into orthonormal systems $\{\mathcal{V}_x\}_{x \in T}$ each of which is assigned by

$$\mathcal{V}_x = \{v \in \mathcal{V} \cap \mathcal{C}_x \mid (v, 1_{\Sigma^+})_{L^2(\Sigma^+;\mu)} = 0\} \quad \text{for each } x \in T \setminus \{o\}$$

and

$$\mathcal{V}_o = \{v \in \mathcal{V} \cap \mathcal{C}_o \mid (v, 1_{\Sigma^+})_{L^2(\Sigma^+;\mu)} = 0\} \cup \{1_{\Sigma^+}\}.$$

The condition (iii) in Section 2 implies that each $\mathcal{V}_x$ consists of $n(x)$ elements. The existence of such complete orthonormal system is explicitly assumed in [34] and is substantially ensured in [2] along with the coincidence with the system of the eigenfunctions. In what follows, the linear span of $\mathcal{V}_x$ will be denoted by $\mathcal{C}_{x,0}$ and the orthogonal projection to $\mathcal{C}_{x,0}$ by P_x for each $x \in T$.

In this section, we start with a family $\{(\mathcal{E}_x, \mathcal{C}_x)\}_{x \in T}$ of the Dirichlet spaces satisfying $\mathcal{E}_x(u, v) = 0$ for any u in the orthogonal subspace of $\mathcal{C}_{x,0}$ in $L^2(\Sigma^+; \mu)$ and any $v \in \mathcal{C}_x$. Thanks to the property, we can denote $\mathcal{E}_x(P_x u, P_x v)$ again by $\mathcal{E}_x(u, v)$ without confusions.

A family of the Dirichlet specs $\{(\mathcal{E}_x, \mathcal{C}_x)\}_{x \in T}$ is said to be admissible, if the symmetric bilinear form

$$\mathcal{E}^x_{\pi^k(x)}(u, v) = \mathcal{E}_x(u, v) + \mathcal{E}_{\pi(x)}(u, v) + \cdots + \mathcal{E}_{\pi^{k-1}(x)}(u, v) + \mathcal{E}_{\pi^k(x)}(u, v)$$

with domain $\mathcal{C}^x_{\pi^k(x)} = \mathcal{C}_{\pi^k(x)} \oplus \cdots \oplus \mathcal{C}_x$ satisfies $\mathcal{E}^x_{\pi^k(x)}(1_{\Sigma^+_y}, 1_{\Sigma^+_z}) \leq 0$ for any distinct $y, z \in T \setminus \{o\}$ satisfying $\pi(y) = \pi(z) = x \in \cup_{m=k}^\infty T_m$ and non-negative integer k.

Let us define the symmetric bilinear form

$$\begin{aligned}\mathcal{E}^m(u, v) = \mathcal{E}_o(u, v) &\\ + \sum_{y \in T_1, \pi(y) = o} \Big(\mathcal{E}_y(u, v) + \cdots &\\ + \sum_{y'' \in T_{m-1}, \pi(y'') = y'} \Big(\mathcal{E}_{y''}(u, v) + \sum_{y''' \in T_m, \pi(y''') = y''} \mathcal{E}_{y'''}(u, v)\Big) \cdots \Big)\end{aligned}$$

for $u, v \in C^m(\Sigma^+)$.

In the sequel, $S^+(x)$ stands for $\pi^{-1}(\{x\})$ and $\Pi(\xi, \eta)$ for the node x characterized by $\{\xi, \eta\} \subset \Sigma^+_x$ and by $\{\xi, \eta\} \not\subset \Sigma^+_y$ as long as $y \in S^+(x)$ for any distinct $\xi, \eta \in \Sigma^+$.

Proposition 5.20 *If a family of the Dirichlet specs $\{(\mathcal{E}_x, \mathcal{C}_x)\}_{x\in T}$ is admissible, then there exists a function $K(x, y)$ depending only on the pair (x, y) of distinct nodes given by $\xi \in \Sigma_x^+, \eta \in \Sigma_y^+$ and $\pi(x) = \pi(y)$ such that*

$$\mathcal{E}^m(u, v) = \frac{1}{2}\iint_{\Sigma_o^+\times\Sigma_o^+\setminus\bigcup_{\pi^{m+1-k(y)}=o}\Sigma_y^+\times\Sigma_y^+}(u(\xi) - u(\eta))(v(\xi) - v(\eta))K(\Pi(\xi, \eta))\mu(d\xi)\mu(d\eta)$$

for any $u, v \in C^m(\Sigma^+)$. The sequence $\{(\mathcal{E}^m, \mathcal{C}^m(\Sigma^+))\}_{m=1}^\infty$ of Dirichlet spaces obtained is consistent in the sense that

$$\mathcal{E}^{m+1}(u, v) = \mathcal{E}^m(u, v)$$

for any $u, v \in \mathcal{C}^m(\Sigma^+)$ and any positive integer m.

Since the identity

$$\mathcal{E}^m(1_{\Sigma_z^+}, 1_{\Sigma_w^+}) = \frac{\mu(\Sigma_z^+)\mu(\Sigma_w^+)}{\mu(\Sigma_x^+)\mu(\Sigma_y^+)}\mathcal{E}_o^{[x,y]}(1_{\Sigma_x^+}, 1_{\Sigma_y^+})$$

implies

$$\mathcal{E}_x(P_x 1_{\Sigma_z^+}, P_x 1_{\Sigma_w^+}) = \mathcal{E}^m(1_{\Sigma_z^+}, 1_{\Sigma_w^+}) - \frac{\mu(\Sigma_z^+)\mu(\Sigma_w^+)}{\mu(\Sigma_x^+)\mu(\Sigma_x^+)}\mathcal{E}^m(1_{\Sigma_x^+}, 1_{\Sigma_x^+})$$

for any $z, w \in S^+(x)$ and the identity

$$\mathcal{E}(u, v) = \frac{1}{2}\iint_{\Sigma^+\times\Sigma^+\setminus\{\xi\neq\eta\}}(u(\xi) - u(\eta))(v(\xi) - v(\eta))K(\Pi(\xi, \eta))\,\mu(d\xi)\mu(d\eta)$$

is defined for $u, v \in \cup_{m=0}^\infty C^m$, as similarly to the proof of Theorem 2 in [34], a regular Dirichlet space $(\mathcal{E}, \mathcal{F})$ satisfying $\mathcal{V} \subset \mathcal{F}$ is uniquely determined:

Theorem 5.21 *If a family of the Dirichlet spaces $\{(\mathcal{E}_x, \mathcal{C}_x)\}_{x\in T}$ is admissible, then there exists a regular Dirichlet space $(\mathcal{E}, \mathcal{F})$ on $L^2(\Sigma^+; \mu)$ characterized by*

$$\mathcal{E}_x(u, v) = \mathcal{E}(u, v) - (u)_{\mu,x}(v)_{\mu,x}\mathcal{E}(1_{\Sigma_x^+}, 1_{\Sigma_x^+})$$

for any $u, v \in \mathcal{C}_x$ and at any node $x \in T$, where $(u)_{\mu,x} = \big(\frac{1}{\mu(\Sigma_x^+)}\int_{\Sigma_x^+} u(\eta)\,\mu(d\eta)\big)1_{\Sigma_x^+}$.

By applying the general theory in [12], the Hunt process associated with the Dirichlet space $(\mathcal{E}, \mathcal{F})$ is obtained by taking a strongly continuous contraction semi-group $\{T_t\}$ of Markovian linear operators on $L^2(\Sigma^+; \mu)$ determined by the Dirichlet space $(\mathcal{E}, \mathcal{F})$. The general theory in [12] also provides us with the transition probability kernel semi-group $\{p_t\}$ which can be viewed as the family of the kernels representing the semi-group $\{T_t\}$ on $L^2(\Sigma^+; \mu)$.

Since the semi-group associated with the Dirichlet form is obtained, we briefly recall a generalized notion of the Bessel kernels proposed in [28] and see a probabilistic variant of Sobolev-Orlicz capacity is introduced so that it is viewed as a natural counterpart of the one proposed in [17].

We first recall that the Bessel kernel

$$G_\alpha(x) = \frac{1}{(4\pi)^{\alpha/2}} \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\frac{\alpha-n}{2}} e^{-\frac{\pi|x|^2}{t} - \frac{t}{4\pi}} \frac{dt}{t}$$

is defined for every $\alpha > 0$ by taking the Gamma function $\Gamma(s) = \int_0^1 e^{-t} t^{s-1}\, dt$ and that, for a subset E of the ball $B^n(0, R_0)$ centered at the origin 0 with radius $R_0 > 0$ in the n-dimensional Euclidean space $\mathbb{R}^n$, the Sobolev-Orlicz capacity $\mathcal{P}_{\alpha,\Phi}(E)$ of E is defined in [17] by

$$\mathcal{P}_{\alpha,\Phi}(E) = \inf\{\|f\|_{L^\Phi(B^n(0,R_0))} \mid f \text{ is non-negative and } (G_\alpha * f)(x) \geq 1 \text{ on } E\},$$

where $\alpha = n/p$ and $\Phi(t) = t^p\varphi(t)$ stands for the Young function determined by $\varphi(t)$ satisfying conditions (C 3.1), (C 3.2).

In [28], it is assumed that a strongly continuous contraction semi-group $\{T_t\}$ of Markovian linear operators on $L^p(X; \mu_X)$ with $p > 1$ is given on a metric space X with a Radon measure μ_X and it is shown that the Markovian contractive operator V_r on $L^p(X; \mu_X)$ is defined when such a $\{T_t\}$ is given. Since we can obtain the transition probability semi-group kernels $\{p_t\}$ representing the strongly continuous contraction semi-group $\{T_t\}$, the natural counterpart of the Bessel kernel for fixed $r > 1$ is defined by

$$\mathrm{v}_r(x, dy) = \frac{1}{\mathit{\Gamma}(r/2)} \int_0^\infty t^{r/2-1} e^{-t} p_t(x, dy)\, dt,$$

which assigns the same operator $V_r : L^p(X; \mu_X) \to L^p(X; \mu_X)$ as in [28]. In fact, the operator is defined by $V_r f(\cdot) = \int_X f(y)\mathrm{v}_r(\cdot, dy)$.

Especially in our case, the kernel v_r is obtained and the map

$$V_r : L^2(\Sigma^+; \mu) \to L^2(\Sigma^+; \mu)$$

is reasonably used in defining a probabilistic counterpart of the Sobolev-Orlicz capacity. In fact, the Sobolev-Orlicz capacity $\mathcal{P}_{V_r,\Phi}(O)$ for any open subset O of Σ^+ by

$$\mathcal{P}_{V_r,\Phi}(O) = \inf\left\{\|f\|_{L^\Phi} \,\middle|\, \begin{array}{l} f \text{ is non-negative and } (V_r f)(x) \geq 1 \text{ a.e.} \\ \text{with respect to } \mu \text{ on } O \end{array}\right\},$$

based on the kernel V_r and the Sobolev-Orlicz capacity $\mathcal{P}_{V_r,\Phi}(E)$ of any subset E of Σ^+ by

$$\mathcal{P}_{V_r,\Phi}(E) = \inf\{\mathcal{P}_{V_r,\Phi}(O) \mid E \subset O, O \text{ is open}\},$$

where $\Phi(t) = t^p\varphi(t)$ stands for the Young function determined by $\varphi(t)$ satisfying conditions (C 3.1), (C 3.2). We introduce another Sobolev-Orlizc capacity $\widetilde{\mathcal{P}}_{V_r,\Phi}$ defined by

$$\widetilde{\mathcal{P}}_{V_r,\Phi}(O) = \inf\left\{\|V_r f\|_{L^\Phi} \,\middle|\, \begin{array}{l} f \in L^2(\Sigma^+; \mu) \text{ and } (V_r f)(x) \geq 1 \text{ a.e.} \\ \text{with respect to } \mu \text{ on } O \end{array}\right\}$$

for any open subset O of Σ^+ and by

$$\widetilde{\mathcal{P}}_{V_r,\Phi}(E) = \inf\{\widetilde{\mathcal{P}}_{V_r,\Phi}(O) \mid E \subset O, O \text{ is open}\}$$

for any subset E of Σ^+.

5.6 Comparison of Sobolev-Orlicz capacity with probabilistic capacity

In the scheme build in [34], the set $\{x\} \times \{1, \dots, n(x)\}$ is denoted by $N(x)$ and each eigenfunction associated with the Dirichlet space $(\mathcal{E}, \mathcal{F})$ is specified by the notation v_ν with some $\nu \in N(x)$ so that $\mathcal{V}_x = \{v_\nu \mid \nu \in N(x)\}$ is satisfied for any $x \in T$. We recall that the eigenvalue associated with v_ν is written as λ_ν and the orthogonal projection to the linear space spanned by v_ν is written as P_ν in the scheme.

Remark 5.22 The Dirichlet space $(\mathcal{E}, \mathcal{F})$ determined by the family $\{\lambda_\nu \mid \nu \in \cup_{x \in T} N(x)\}$ satisfying $\lambda_\nu = \lambda_{\nu'}$ for any $\nu, \nu' \in N(x)$ and any $x \in T$ is covered by the results in [22].

We will employ these notations and denote $\cup_{x \in T} N(x)$ by $\overline{T}$ as in [34]. Then, it is not difficult to see that

$$\mathcal{E}(u, v) = \sum_{x \in T} \sum_{\nu \in N(x)} \lambda_\nu (P_\nu u, P_\nu v)_{L^2(\Sigma^+;\mu)} = \sum_{\nu \in \overline{T}} \lambda_\nu (P_\nu u, P_\nu v)_{L^2(\Sigma^+;\mu)}$$

for any $u, v \in \mathcal{F}$ and

$$V_r f = \sum_{\nu \in \overline{T}} \frac{c_\nu}{(1 + \lambda_\nu)^{\frac{r}{2}}} v_\nu$$

is defined as an element of $L^2(\Sigma^+; \mu)$ for any $f = \sum_{\nu \in \overline{T}} c_\nu v_\nu \in L^2(\Sigma^+; \mu)$.

One of objectives of this section is figuring out sufficient conditions on Dirichlet space $(\mathcal{E}, \mathcal{F})$ such that the finiteness of the hitting time of singleton is derived from the non-polarity of Sobolev-Orlicz capacity of the singleton in Σ^+. For that purpose, we define $\delta_\nu = \inf\{\mu(\Sigma_y^+) \mid v_\nu$ takes constant on every Σ_y^+ for any $y \in S^+(x)\}$ for any $\nu \in \overline{T}$ and denote $\mathcal{E}(u, v) + (u, v)_{L^2(\Sigma^+;\mu)}$ by $\mathcal{E}_1(u, v)$ for any $u, v \in \mathcal{F}$.

Proposition 5.23 *If $(\lambda_\nu + 1) \geq \frac{1}{\delta_\nu} \int_0^{\delta_\nu} \varphi(\frac{1}{t})\, dt$ for any $\nu \in \overline{T}$, then there exists a positive constant C_Φ independent of the coefficients c_ν with $\nu \in \cup_{x \in T_0 \cup \cdots \cup T_n} N(x)$ and of n such that*

$$\int_0^1 \Big(\sum_{i=0}^{n} \sum_{\nu \in N(x), x \in T_i} c_\nu v_\nu \Big)^{*2} \varphi\Big(\frac{1}{t}\Big)\, dt \leq C_\Phi \mathcal{E}_1 \Big(\sum_{i=0}^{n} \sum_{\nu \in N(x), x \in T_i} c_\nu v_\nu, \sum_{i=0}^{n} \sum_{\nu \in N(x), x \in T_i} c_\nu v_\nu \Big).$$

We shall show an estimate on $\widetilde{\mathcal{P}}_{V_1,\Phi}$ so that positive hitting probability of $E \subset \Sigma^+$ with respect to the Markov process generated by $\{T_t\}$ is validated by the positivity of $\widetilde{\mathcal{P}}_{V_1,\Phi}(E)$. In fact, we take the capacity $\mathrm{C}_{1,2}$ which admits the probabilistic characterization as pointed out in [28] and apply the capacity theory in [28]. We will take the abbreviation "q.e." for quasi-everywhere as in [28].

Theorem 5.24 *If $(\lambda_\nu + 1) \geq \frac{1}{\delta_\nu} \int_0^{\delta_\nu} \varphi(\frac{1}{t})\, dt$ for any ν,*

$$\widetilde{\mathcal{P}}_{V_1,\Phi}(\{\zeta\}) \leq C_\Phi^{1/2} \sqrt{C_{1,2}(\{\zeta\})}$$

for any $\zeta \in \Sigma^+$.

For a lower estimate for $\mathcal{P}_{V_r,\Phi}(\{\zeta\})$, we introduce the following condition on ζ:

(C 6.1) for any $x \in T$ satisfying $\zeta \in \Sigma_x^+$, $\#N(x,\zeta) = 1$ and the function $v_\nu \in \mathcal{V}_x$ uniquely determined by $\nu \in N(x,\zeta)$ takes a constant on $\cup_{\zeta\notin\Sigma_y^+, y\in S^+(x)}\Sigma_y^+$.

The function $v_\nu \in \mathcal{V}_x$ in (C 6.1) will be denoted by $v_{(x,j(\zeta))}$ and the function $1_{\Sigma_x^+}$ admits the expansion

$$1_{\Sigma_x^+} = \sum_{k=0}^{n} c_{(\pi^k(x),j(\zeta))} v_{(\pi^k(x),j(\zeta))}$$

with the eigenfuctions $v_{(\pi^k(x),j(\zeta))}$ and the real coefficients $c_{(\pi^k(x),j(\zeta))}$.

First we fix $p > 1$ and define $f_0 = 1_{\Sigma^+}$. Then we see $V_r f_0 = 1_{\Sigma^+} = 1$ q.e. on Σ^+. This is because the eigenvalue λ_0 associated with the eigenfunction 1_{Σ^+} vanishes. Inductively, we find such s_n in $[0,1]$ that $\left\| \frac{s}{C_n} 1_{\Sigma_x^+} + (1-s) f_{n-1} \right\|_\Phi$ is minimized at $s = s_n$ and define $f_n = \frac{s_n}{C_n} 1_{\Sigma_x^+} + (1-s_n) f_{n-1}$ so that $V_r f_n \geq 1$ q.e. on Σ_x^+ is satisfied, where $C_n = \sum_{k=0}^n \frac{c_{(\pi^k(x),j(\zeta))}}{(\lambda_{(\pi^k(x),j(\zeta))}+1)^{r/2}} v_{(\pi^k(x),j(\zeta))}(\zeta)$ is given by taking a unique x in T_n satisfying $\zeta \in \Sigma_x^+$.

We will find a lower estimate for $\min_{s\in[0,1]} \left\| \frac{s}{C_n} 1_{\Sigma_x^+} + (1-s) f_{n-1} \right\|_\Phi$, by taking the minimal value γ_n of

$$\Big(\frac{s}{C_n}\Big)^p \int_0^1 1_{[0,\mu(\Sigma_x^+)]}(t)\varphi\Big(\frac{1}{t}\Big)\,dt + (1-s)^p \int_0^1 f_{n-1}(t)^{*p}\varphi\Big(\frac{1}{t}\Big)\,dt$$

subject to the condition $0 \leq s \leq 1$. In fact, we see

$$\min_{s\in[0,1]} \left\| \frac{s}{C_n} 1_{\Sigma_x^+} + (1-s) f_{n-1} \right\|_\Phi^p \geq \gamma_n.$$

This is because it is easy to see that

$$\Big(\frac{s}{C_n} 1_{\Sigma_x^+} + (1-s) f_{n-1}\Big)^* = \frac{s}{C_n} 1_{[0,\mu(\Sigma_x^+)]} + (1-s) f_{n-1}^*$$

which implies

$$\begin{aligned}\Big(\frac{s}{C_n} 1_{\Sigma_x^+} + (1-s) f_{n-1}\Big)^{*p} &= \Big(\frac{s}{C_n} 1_{[0,\mu(\Sigma_x^+)]} + (1-s) f_{n-1}^*\Big)^p \\ &\geq \Big(\frac{s}{C_n} 1_{[0,\mu(\Sigma_x^+)]}\Big)^p + \Big((1-s) f_{n-1}^*\Big)^p.\end{aligned}$$

For the function $g(s) = \theta s^p + \gamma(1-s)^p$ defined on $[0,1]$ determined by positive real values γ, θ. It is not difficult to see that g attains its minimum $\theta\Big(\frac{\gamma^{\frac{1}{p-1}}}{\theta^{\frac{1}{p-1}}+\gamma^{\frac{1}{p-1}}}\Big)^p + \gamma\Big(\frac{\theta^{\frac{1}{p-1}}}{\theta^{\frac{1}{p-1}}+\gamma^{\frac{1}{p-1}}}\Big)^p$ at $s = \frac{\gamma^{\frac{1}{p-1}}}{\theta^{\frac{1}{p-1}}+\gamma^{\frac{1}{p-1}}}$. Therefore, $\gamma \geq \min_{t\in[0,1]} g(t) \geq \gamma\Big(\frac{\theta^{\frac{1}{p-1}}}{\theta^{\frac{1}{p-1}}+\gamma^{\frac{1}{p-1}}}\Big)^p$. By introducing a function $G(\theta,\gamma) = \gamma\Big(\frac{\theta^{\frac{1}{p-1}}}{\theta^{\frac{1}{p-1}}+\gamma^{\frac{1}{p-1}}}\Big)^p$, for any positive sequence $\{\theta_n\}_{n=0}^\infty$, we obtain the decreasing sequence $\{\gamma_n\}$ defined by

$$\gamma_0 = \theta_0 \quad \text{and} \quad \gamma_n = G(\theta_n, \gamma_{n-1}) \quad \text{for } n = 1, 2, \ldots.$$

Then, we have

$$\gamma_n \geq \theta_0 \left(\prod_{k=1}^{n} \frac{\theta_k^{\frac{1}{p-1}}}{\theta_k^{\frac{1}{p-1}} + \theta_0^{\frac{1}{p-1}}} \right)^p = \theta_0 \left(\prod_{k=1}^{n} \left(1 - \frac{\theta_0^{\frac{1}{p-1}}}{\theta_k^{\frac{1}{p-1}} + \theta_0^{\frac{1}{p-1}}} \right) \right)^p.$$

Therefore, we can drive $\liminf_{n\to\infty} \gamma_n > 0$ from the convergence of $\sum_{n=1}^{\infty} \frac{1}{\theta_n^{\frac{1}{p-1}} + \theta_0^{\frac{1}{p-1}}}$. By taking this principle into our account, we can show the following assertion:

Theorem 5.25 *If $\zeta \in \Sigma^+$ satisfies condition (C 6.1) and the sequence*

$$\theta_n = \frac{\int_0^{\delta(x,j(\zeta))} \varphi(\frac{1}{t})\, dt}{C_n^p} \quad (n = 0, 1, \dots)$$

given by $x \in T_n$ and $\zeta \in \Sigma_x^+$ satisfies $\sum_{n=1}^{\infty} \frac{1}{\theta_n^{\frac{1}{p-1}} + 1} < \infty$, then

$$\mathcal{P}_{V_r,\Phi}(\{\zeta\}) \geq \theta_0^{1/p} \prod_{k=1}^{n} \left(1 - \frac{\theta_0^{\frac{1}{p-1}}}{\theta_k^{\frac{1}{p-1}} + \theta_0^{\frac{1}{p-1}}} \right) > 0.$$

For an upper estimate for $\mathcal{P}_{V_r,\Phi}(\{\zeta\})$, we still assume on $\zeta \in \Sigma^+$ that:

(C 6.1) for any $x \in T$ satisfying $\zeta \in \Sigma_x^+$, $\#N(x, \zeta) = 1$ and the function $v_\nu \in \mathcal{V}_x$ uniquely determined by $\nu \in N(x, \zeta)$ takes a constant on $\cup_{\zeta \notin \Sigma_y^+, y \in S^+(x)} \Sigma_y^+$

and denote $x \in T_n$ satisfying $\zeta \in \Sigma_x^+$ by x_n. We denote the eigenvalue λ_ν with $\nu \in N(x_n, \zeta)$ by λ_{ν_n} and we denote the function $v_\nu \in \mathcal{V}_{x_n}$ with $\nu \in N(x_n, \zeta)$ by v_{ν_n}, which takes a constant on $\cup_{\zeta \notin \Sigma_y^+, y \in S^+(x_n)} \Sigma_y^+$ due to (C 6.1). We assume also that

(C 6.2) each $\zeta \in \Sigma^+$ yields the increasing sequence of eigenvalues given as $\lambda_{\nu_0} \leq \lambda_{\nu_1} \leq \cdots \leq \lambda_{\nu_n} \leq \cdots$.

Then, we see that

$$1_{\Sigma_{x_n}^+} = \sum_{k=0}^{n} c_{\nu_k} v_{\nu_k} \text{ with } c_{\nu_k} = (1_{\Sigma_{x_n}^+}, v_{\nu_k})_{L^2(\Sigma^+;\mu)}$$

and

$$V_r 1_{\Sigma_{x_n}^+} = \sum_{k=0}^{n} \frac{c_{\nu_k}}{(\lambda_{\nu_k} + 1)^{r/2}} v_{\nu_k} \geq \frac{1}{(\lambda_{\nu_n} + 1)^{r/2}} 1_{\Sigma_{x_n}^+}$$

which imply that

$$(\lambda_{\nu_n} + 1)^{r/2} V_r 1_{\Sigma_{x_n}^+} = (\lambda_{\nu_n} + 1)^{r/2} \sum_{k=0}^{n} \frac{c_{\nu_k}}{(\lambda_{\nu_k} + 1)^{r/2}} v_{\nu_k} \geq 1_{\Sigma_{x_n}^+}$$

for any $n = 1, 2, \dots$. For a sequence $\{b_n\}_{n=1}^{\infty}$ satisfying $0 < b_n < 1$, $n = 1, 2, \dots$, we define

$$V^{(1)} = b_1 (\lambda_{\nu_0} + 1)^{r/2} V_r 1_{\Sigma^+} + (1 - b_1)(\lambda_{\nu_1} + 1)^{r/2} V_r 1_{\Sigma_{x_1}^+}$$

and

$$V^{(n)} = b_n V^{(n-1)} + (1-b_n)(\lambda_{\nu_n}+1)^{r/2} V_r 1_{\Sigma^+_{x_n}} \quad \text{for } n = 2, 3, \dots.$$

We have

$$\begin{aligned} V^{(2)} &= b_2 V^{(1)} + (1-b_2)(\lambda_{\nu_2}+1)^{r/2} V_r 1_{\Sigma^+_{x_2}} \\ &= b_2 b_1 (\lambda_{\nu_0}+1)^{r/2} V_r 1_{\Sigma^+} + b_2(1-b_1)(\lambda_{\nu_1}+1)^{r/2} V_r 1_{\Sigma^+_{x_1}} + (1-b_2)(\lambda_{\nu_2}+1)^{r/2} V_r 1_{\Sigma^+_{x_2}} \\ &\geq 1_{\Sigma^+_{x_n}} \end{aligned}$$

which implies

$$\begin{aligned} & b_2 b_1 (\lambda_{\nu_0}+1)^{r/2} V_r 1_{\Sigma^+ \setminus \Sigma^+_{x_1}} + b_2 (\lambda_{\nu_1}+1)^{r/2} V_r 1_{\Sigma^+_{x_1} \setminus \Sigma^+_{x_2}} + (\lambda_{\nu_2}+1)^{r/2} V_r 1_{\Sigma^+_{x_2}} \\ & \geq V^{(2)} \geq 1_{\Sigma^+_{x_n}}. \end{aligned}$$

By repeating this procedure, we obtain

$$\begin{aligned} & \left(\prod_{k=1}^{n} b_k\right)(\lambda_{\nu_0}+1)^{r/2} V_r 1_{\Sigma^+_{x_0} \setminus \Sigma^+_{x_1}} + \left(\prod_{k=2}^{n} b_k\right)(\lambda_{\nu_1}+1)^{r/2} V_r 1_{\Sigma^+_{x_1} \setminus \Sigma^+_{x_2}} \\ & \qquad\qquad + \cdots + (\lambda_{\nu_n}+1)^{r/2} V_r 1_{\Sigma^+_{x_n}} \\ & \geq V^{(n)} \geq 1_{\Sigma^+_{x_n}}, \end{aligned}$$

where $\Sigma^+_{x_0}$ stands for Σ^+. From this observation, we see such value that dominates

$$\left(\int_0^1 f_n^*(t)^p \varphi\Big(\frac{1}{t}\Big)\, dt\right)^{1/p}$$

with

$$f_n = \left(\prod_{k=1}^{n} b_k\right)(\lambda_{\nu_0}+1)^{r/2} 1_{\Sigma^+_{x_0} \setminus \Sigma^+_{x_1}} + \left(\prod_{k=2}^{n} b_k\right)(\lambda_{\nu_1}+1)^{r/2} 1_{\Sigma^+_{x_1} \setminus \Sigma^+_{x_2}} + \cdots + (\lambda_{\nu_n}+1)^{r/2} 1_{\Sigma^+_{x_n}}$$

for sufficiently large n gives an upper estimate of $P_{V_r,\Phi}(\{\zeta\})$.

Theorem 5.26 *If there exits a positive constant α satisfying*

$$\frac{1+\alpha}{2\alpha(\alpha-1)} < 1 \quad \textit{and} \quad \alpha^n \geq (\lambda_{\nu_n}+1)^{r/2}$$

for any $n = 0, 1, 2, \dots,$ then

$$\mathcal{P}_{V_r,\Phi}(\{\zeta\}) \leq \liminf_{n\to\infty} \left(\epsilon^{pn} \sum_{k=1}^{n} \Big(\frac{\alpha}{\epsilon}\Big)^{p(k-1)} \mu(\Sigma^+_{\nu_{k-1}}) \varphi\Big(\frac{1}{\mu(\Sigma^+_{\nu_k})}\Big) + A_n\right)^{1/p},$$

where $\epsilon = \frac{1+\alpha}{2\alpha(\alpha-1)}$ and $A_n = \alpha^{pn} \int_0^{\mu(\Sigma^+_{x_n})} \varphi\big(\frac{1}{t}\big)\, dt$. In particular,

$$\sum_{k=1}^{\infty} \Big(\frac{\alpha}{\epsilon}\Big)^{p(k-1)} \mu(\Sigma^+_{\nu_{k-1}}) \varphi\Big(\frac{1}{\mu(\Sigma^+_{\nu_k})}\Big) < \infty$$

and $\liminf_{n\to\infty} A_n = 0$ imply $\mathcal{P}_{V_r,\Phi}(\{\zeta\}) = 0$.

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