

ON NEWTON'S METHOD

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(Received October 8, 1986)

§0. Introduction.

The differential calculus has been investigated in a linear space where neighborhoods of the origin are symmetric.

The author has investigated the differential calculus in a linear space whose convergence structure is defined by neighborhoods of the origin, that are not necessarily symmetric. (cf. [3]). One of such spaces is the set of all real numbers, where neighborhoods of a point x are $[x, x + \varepsilon)$ for $\varepsilon > 0$. The convergence structure is constructed by the method of ranked space originated with K. Kurugi. (cf. [1]). In such convergence space, the author study the Newton's method, which can be applied to obtain the solution of a nonlinear functional equation, and if the space is usual Banach space, it coincides with the known Newton's method in the Banach space. In §1, we define an abstract convergence space used in this paper by the method of ranked space. In §2 and §3, the main result and its corollaries are proved. The author would like to thank professors T. SHIBATA and H. UMEGAKI of their important comments.

§1. Linear ranked space.

An abstract convergence space is constructed by the method of ranked space. (cf. [3]).

Let E be a linear space over the real field \mathbb{R} . Suppose that a sequence $\{\mathfrak{B}_n\}_{n=0}^{\infty}$ of families of subsets in E is given to satisfy the following condition (E.1):

(E.1) $0 \in V$ for any $V \in \mathfrak{B}_n = \bigcup_{n=0}^{\infty} \mathfrak{B}_n$, $E \in \mathfrak{B}_0$; and for any $V \in \mathfrak{B}$ and for any integer $n \geq 0$, there are another integer $m > n$ and $U \in \mathfrak{B}_m$ such that $U \subset V$.

Sets in \mathfrak{B}_n are called preneighborhoods of the origin 0 with rank n .

A sequence $\{x_k + V_k\}_{k=0}^{\infty}$ of subsets of E is called a fundamental sequence, if

$$(1) x_1 + V_1 \supset x_2 + V_2 \supset \dots \supset x_k + V_k \supset \dots,$$

$$(2) V_k \in \mathfrak{B}_{n_k} (k=1,2,\dots) \text{ with } n_1 \leq n_2 \leq \dots \leq n_k \leq \dots \rightarrow \infty,$$

$$(3) \text{ for any integer } i (i \geq 1), \text{ there are } k (k > i) \text{ and } V (V \in \mathfrak{B}_m, n_k < m \leq n_{k+1})$$

such that $x_k + V_k \supset x_k + V \supset x_{k+1} + V_{k+1}$,

moreover, in this paper we suppose the following additional condition,

$$(4) V_1 \supset V_2 \supset \dots \supset V_k \supset \dots.$$

If $x_1 = x_2 = \dots = x_k = \dots = 0$ in given fundamental sequence $\{x_k + V_k\}$, we simply call it 0-f.s. $\{V_k\}$. Given a 0-f.s. $\{V_k\}$, let

$$E(\{V_k\}) = \{x \in E \mid \text{for each } k, \text{ there is } \lambda_k > 0 \text{ such that } x \in \lambda_k V_k\},$$

and

$$E^*(\{V_k\}) = \text{Span } E(\{V_k\}).$$

If $(E, \{\mathfrak{B}_n\})$ satisfies the following conditions (E.2, 3, 4', 5') in addition to (E.1), then $E = (E, \{\mathfrak{B}_n\})$ is called a linear ranked space:

(E.2) For any two 0-f.s. $\{V_k\}$ and $\{U_k\}$, there is another 0-f.s. $\{W_k\}$ such that $V_k + U_k \subset W_k$ for each k .

(E.3) For any 0-f.s. $\{V_k\}$ and $\lambda > 0$, there are integers $1 \leq m(1) \leq m(2) \leq \dots \rightarrow \infty$ and $k_0 \geq 1$ such that $\lambda V_k \subset V_{m(k)}$ for $k \geq k_0$.

(E.4') For any $V \in \mathfrak{B}$ and $0 \leq \lambda < 1$, $\lambda V \subset V$.

(E.5') For any $x \in E$, there is a 0-f.s. $\{V_k\}$ such that $x \in E^*(\{V_k\})$.

A linear ranked space E is said to be $(\pi - T_1)$; if for any f.s. $\{x_k + V_k\}$, $\cap_k (x_k + V_k) \neq \emptyset$ is not empty, then it is a set of one point.

DEFINITION 1. A linear ranked space E is said to be complete if $\cap_k (x_k + V_k) \neq \emptyset$ for any f.s. $\{x_k + V_k\}$.

DEFINITION 2. Given a 0-f.s. $\{V_k\}$, a sequence $\{x_n\}$ in E is said to be $R - \{V_k\}$ -convergent to $x \in E$, in symbols $x_n \rightarrow x(R - \{V_k\})$, if for any k there is n_k such that $n \geq n_k$ implies $x_n \in x + V_k$. Given a f.s. $\{y_k + V_k\}$, a sequence $\{x_n\}$ in E is said to be para- $\{V_k\}$ -convergent to $x \in E$, in symbols $x_n \rightarrow x(P - \{V_k\})$, if for each k there is n_k such that $n \geq n_k$ implies $x_n \in y_k + V_k$ and $x \in y_k + V_k$ for each k . Also, $\{x_n\}$ is said to be R (resp. para)-convergent to x , in symbols $x_n \rightarrow x(R)$ (resp. (P)), if $x_n \rightarrow x(R - \{V_k\})$ (resp. $(P - \{V_k\})$) for some 0-f.s. $\{V_k\}$. And x is called an R (resp. P)-limit of $\{x_n\}$.

Now, we consider the following additional assumptions (A.1, 2', 4) for $E = (E, \{\mathfrak{B}_n\})$, which will be assumed frequently in this paper:

(A.1) For each 0-f.s. $\{V_k\}$, there is k_0 such that $V_{k_0} \subset \bar{E}(\{V_k\})$.

(A.2') Let $\{V_k\}$ be a 0-f.s.. If $x_n \rightarrow x$ (R) (resp. (P)), $\{x_n\} \subset E^*(\{V_k\})$ and $x \in E^*(\{V_k\})$, then $x_n \rightarrow x$ (R- $\{V_k\}$) (resp. (P- $\{V_k\}$)).

(A.4) Let $\{V_k\}$ be a 0-f.s. and $y \in E$. If $x_k \in y + V_k$ for each k , then there is $\{m(k)\}$ such that $\{-x_{m(k)} + V_{m(k)}\}$ is a f.s. and for $n \geq m(k+1)$, $-x_n + V_n \subset -x_{m(k)} + V_{m(k)}$.

EXAMPLE 1. R_1 : The real space R , where neighborhoods of 0 are $\{x \in R \mid 0 \leq x < \varepsilon\}$ for $\varepsilon > 0$. This space is non-metrizable. Let $V(\varepsilon) = \{x \in R \mid 0 \leq x < \varepsilon\}$ for $\varepsilon > 0$. Put $\mathfrak{B}_0 = \{V(\varepsilon) \mid \varepsilon > 1\} \cup \{R\}$, $\mathfrak{B}_n = \{V(\varepsilon) \mid (n+1)^{-1} < \varepsilon \leq n^{-1}\}$ ($n = 1, 2, \dots$). Then $(R, \{\mathfrak{B}_n\})$ is a $(\pi - T_1)$ linear ranked space, which satisfy (A.1, 2', 4).

LEMMA 1. Let E be $(\pi - T_1)$ linear ranked space satisfying (A.4) and $\{V_k\}$ be a 0-f.s. in E . Then $x \in V_k - V_k$ for each k implies $x = 0$.

PROOF. Since $x \in V_k - V_k$ for each k , there is some $y_k \in V_k$ such that $x \in V_k - y_k$ for each k . Thus we have $y_k \in V_k$ and $y_k \in -x + V_k$ for each k . Hence $y_k \rightarrow 0$ (R) and $y_k \rightarrow -x$ (R). Since an R-limit of $\{y_k\}$ is unique, we have $x = 0$. (cf. [3] Proposition 1.1)

LEMMA 2. If S is a convex set and $\{V_k\}$ is a convex 0-f.s., then $\bar{S}(R - \{V_k\})$ and $\bar{S}(P - \{V_k\})$ are both convex. (We say that 0-f.s. $\{V_k\}$ is convex, if each V_k is convex set.)

PROOF. Let $T = \bar{S}(R - \{V_k\})$ and $x, y \in T$. Then we see that for each k $x \in S - V_k$ and $y \in S - V_k$. And so for $a, b \geq 0$ and $a + b = 1$, $ax + by \in aS + bS - aV_k - bV_k \subset S - V_k$. Thus $ax + by \in \bar{S}(R - \{V_k\}) = T$. Hence $aT + bT \subset T$. Also we can prove that $\bar{S}(P - \{V_k\})$ is convex.

PROPOSITION 3. If $\{V_k\}$ is a convex 0-f.s., then $(\overline{V_k - V_k})^P \subset \lambda(V_k - V_k)$ for any $\lambda > 1$ and each k . (For simplicity, we put $\bar{S}^P = \bar{S}(P - \{V_k\})$.)

PROOF. By (E.3), for any $\lambda > 1$ and k , there is k' such that

$$(\lambda - 1)V_k \supset V_{k'}.$$

If $x \in (\overline{V_k - V_k})^p$, then for each m $x \in V_k - V_k + V_m - V_m$. Thus for $m > k'$,
 $V_k - V_k + V_m - V_m \subset V_k - V_k + (\lambda - 1)V_k - (\lambda - 1)V_k = \lambda(V_k - V_k)$.

DEFINITION 3. Given a 0-f.s. $\{V_k\}$, a sequence $\{x_n\}$ is called an R (resp. P)-Cauchy sequence by $\{V_k\}$, if for each k there is $n(k)$ such that $m \geq n \geq n(k)$ implies $x_m - x_n \in V_k$ (resp. $x_m - x_n \in V_k - V_k$).

PROPOSITION 4. Let E be a complete linear ranked space satisfying (A.4), $\{V_k\}$ be a convex 0-f.s. in E and $\{x_k\}$ be a sequence in E . Suppose $V_n - V_n + x_n \supset V_{n+1} - V_{n+1} + x_{n+1}$ for $n \geq 1$, then there is a point W in E such that $x_n \rightarrow W$ ($P - \{V_k\}$).

PROOF. Since $\{V_k\}$ is a convex 0-f.s., there are two subsequences of natural numbers, $\{k(n)\}$, $\{h(n)\}$ such that $k(n) < h(n) < k(n+1)$ and $V_{k(n)} \supset V_{h(n)} + V_{h(n)}$ for each n (cf. [3] Lemma 1.1.). Thus there are two sequences $\{y_n\}$ and $\{z_n\}$ such that $x_{h(n)} = y_n - z_n$ ($n \geq 3$) and that $\{y_{n+1} + V_{h(n)}\}$ and $\{z_{n+1} + V_{h(n)}\}$ are f.s.'s. Because, since $V_{h(2)} - V_{h(2)} + x_{h(2)} \supset V_{h(3)} - V_{h(3)} + x_{h(3)}$, we have $V_{h(2)} - V_{h(2)} + x_{h(2)} \ni x_{h(3)}$. Hence there are y_3 and z_3 such that $x_{h(3)} = y_3 - z_3$, $x_{h(2)} + V_{h(2)} \ni y_3$ and $V_{h(2)} \ni z_3$. Suppose there are $\{y_n\}_{n=3}^m$ and $\{z_n\}_{n=3}^m$ such that $x_{h(n)} = y_n - z_n$ ($n = 3, 4, \dots, m$) and

$$y_4 + V_{h(3)} \supset y_5 + V_{h(4)} \supset \dots \supset y_m + V_{h(m-1)}$$

and

$$z_4 + V_{h(3)} \supset z_5 + V_{h(4)} \supset \dots \supset z_m + V_{h(m-1)}.$$

Then, since $V_{h(m)} - V_{h(m)} + x_{h(m)} \supset V_{h(m+1)} - V_{h(m+1)} + x_{h(m+1)}$, $V_{h(m)} - V_{h(m)} + y_m - z_m \ni x_{h(m+1)}$. Thus there are y_{m+1} and z_{m+1} such that $x_{h(m+1)} = y_{m+1} - z_{m+1}$, $y_m + V_{h(m)} \ni y_{m+1}$ and $z_m + V_{h(m)} \ni z_{m+1}$. Hence

$y_m + V_{h(m-1)} \supset y_m + V_{h(m)} \supset y_m + V_{h(m)} + V_{h(m)} \supset y_{m+1} + V_{h(m)}$, also $z_m + V_{h(m-1)} \supset z_m + V_{h(m)} \supset z_{m+1} + V_{h(m)}$. Thus, $\{y_{n+1} + V_{h(n)}\}$ and $\{z_{n+1} + V_{h(n)}\}$ are f.s.'s. Since E is complete, there are u and v such that

$u \in \cap (y_{n+1} + V_{h(n)})$ and $v \in \cap (z_{n+1} + V_{h(n)})$. Then we have $V_{h(n)} - V_{h(n)} + (u - v) \ni x_{h(n+1)}$. Hence, for $m \geq h(n+2)$,

$$\begin{aligned} V_{h(n)} - V_{h(n)} + (u - v) &\supset V_{h(n+1)} - V_{h(n+1)} + V_{h(n+1)} - V_{h(n+1)} + (u - v) \\ &\supset V_{h(n+2)} - V_{h(n+2)} + x_{h(n+2)} \ni x_m. \end{aligned}$$

Thus, we see $x_m \rightarrow (u-v) \ (P-\{V_k\})$. (cf. [3] Lemma 1.2(c))

PROPOSITION 5. Let E be a complete linear ranked space satisfying (A.4), $\{V_k\}$ be a convex 0-f.s. in E and $\{x_k\}$ be an R (resp. P)-Cauchy sequence by $\{V_k\}$. Then there is $z \in E$ such that $x_k \rightarrow z \ (P-\{V_k\})$.

PROOF. Since $\{V_k\}$ is a convex 0-f.s., there is a subsequence of natural numbers $\{k(n)\}$ such that $V_{k(n)} \supset V_{k(n+1)} + V_{k(n+1)}$ for each n . (cf. [3] Lemma 1.1.) Let $\{x_k\}$ be a P -Cauchy sequence by $\{V_k\}$, then for each $k(n)$ there is $h(n)$ such that $k(n) < h(n) < h(n+1)$ and $x_m - x_l \in V_{k(n)} - V_{k(n)}$ for $m \geq l \geq h(n)$. Thus

$$\begin{aligned} V_{k(n)} - V_{k(n)} + x_{h(n+1)} &\supset V_{k(n+1)} + V_{k(n+1)} - V_{k(n+1)} - V_{k(n+1)} + x_{h(n+1)} \\ &\supset V_{k(n+1)} - V_{k(n+1)} + x_{h(n+2)}. \end{aligned}$$

Hence, by Proposition 4, there is $z \in E$ such that $x_{h(n)} \rightarrow z \ (P-\{V_k\})$. Thus, by [3] Lemma 1.2 (c), for $k(n)$ there is $h^*(n)$ ($h^*(n) > k(n)$) such that for any $k^*(n) > h^*(n)$

$$x_{k^*(n)} \in z + V_{k(n)} - V_{k(n)}.$$

Then
$$\begin{aligned} V_{k(n-1)} - V_{k(n-1)} + z &\supset V_{k(n)} + V_{k(n)} - V_{k(n)} - V_{k(n)} + z \\ &\supset V_{k(n)} - V_{k(n)} + x_{k^*(n)} \ni x_m \quad \text{for } m > k^*(n) \geq h(n). \end{aligned}$$

Thus, we see $x_m \rightarrow z \ (P-\{V_k\})$.

DEFINITION 4. Let E and F be linear subspaces in linear ranked spaces X and Y , respectively. Then E and F are said to be isomorphic if there exists a bijective linear mapping $T: E \rightarrow F$ as the following: for each 0-f.s. $\{V_k\}$ in X there is a 0-f.s. $\{U_k\}$ in Y such that $T(E \cap V_k) \subset U_k$ for each k and for any 0-f.s. $\{U_k\}$ in Y there is a 0-f.s. $\{V_k\}$ in X such that $T^{-1}(F \cap U_k) \subset V_k$ for each k . In this case, T is called an isomorphism of E onto F .

DEFINITION 5. Let D be a subset of a linear ranked space E , $x \in E$ and $\{V_k\}$ be a 0-f.s. in E . Then x is said to be an R (resp. P)- $\{V_k\}$ -interior point in D if $x \in E^*(\{V_k\})$ and there is some h such that $V_h \subset D - x$

(resp. $V_h - V_h \subset D - x$). And if all points of D are R (resp. P) - $\{V_k\}$ -interior in D , D is said to be R (resp. P) - $\{V_k\}$ -open.

LEMMA 6. Let E and F be linear ranked spaces and $\{V_k\}$ be a 0-f.s. in E . Then a continuous linear mapping $\ell: E^*(\{V_k\}) \rightarrow F$ is R (resp. P)-differentiable at an R (resp. P) - $\{V_k\}$ -interior point x in $E^*(\{V_k\})$.

PROOF. Since x is an R (resp. P) - $\{V_k\}$ -interior point in $E^*(\{V_k\})$, there is a V_n such that $x + V_n \subset E^*(\{V_k\})$ (resp. $x + V_n - V_n \subset E^*(\{V_k\})$). Thus for any $h \in V_n$, we have $\gamma(h) = \ell(x + h) - \ell(x) - \ell(h) = 0$. Hence, ℓ is R (resp. P)-differentiable at x .

§2. Newton's method.

Let E and F be $(\pi - T_1)$ complete linear ranked spaces satisfying (A.1, 2', 4). Suppose that D is a subset in E , $\{V_k\}$ is a convex 0-f.s. in E and D is P - $\{V_k\}$ -open. Let $f: D \rightarrow F$ be continuous and R -differentiable at every $x \in D$. Moreover, suppose there is a point $x_0 \in D$ such that $f'(x_0): E^*(\{V_k\}) \rightarrow f'(x_0)E^*(\{V_k\})$ is isomorphism.

THEOREM 1. If for $g = f' \circ f$ and some $\lambda \geq 1$, the following (1) ~ (4) are satisfied;

- (1) there is ℓ such that $V_\ell \subset E(\{V_k\})$ and $x_0 + \lambda W_\ell^\alpha \subset D$ (for simplicity, we put $W_k = V_k - V_k$ and $W_\ell^\alpha = \overline{W}_\ell(P - \{V_k\})$, $\overline{W}_k^P = \overline{W}_k(P - \{V_k\})$,
- (2) $g(x_0) - g(x_0 + \lambda W_\ell^\alpha) \subset E^*(\{V_k\})$,
- (3) for each k , there is L_k ($0 < L_k < 1/2$) such that

$$[I - g'(x_0 + x)](V_k) \subset L_k W_k \quad \text{for } x \in \lambda W_\ell^\alpha,$$

- (4) $-g(x_0) \in \lambda(1 - 2L_\ell)W_\ell^\alpha$,

then, there is a unique point z in $x_0 + \lambda W_\ell^\alpha$ such that

$$f(z) = 0.$$

PROOF. We divide the proof into several steps.

- (a) We have $-g(x_0) \in E^*(\{V_k\})$, $g(x_0 + \lambda W_\ell^\alpha) \subset E^*(\{V_k\})$ and $g'(x_0 + x)E^*(\{V_k\}) \subset E^*(\{V_k\})$ for $x \in \lambda W_\ell^\alpha$.

Proof of (a): By Proposition 3, $W_\ell^\alpha \subset 2W_\ell = 2(V_\ell - V_\ell)$. Thus we have $-g(x_0) \in 2\lambda(1 - 2L_\ell)(V_\ell - V_\ell) \subset E^*(\{V_k\})$. By (2) $g(x_0 + \lambda W_\ell^\alpha) \subset E^*(\{V_k\}) + g(x_0)$, then $g(x_0 + \lambda W_\ell^\alpha) \in E^*(\{V_k\})$. By (3), we see that $g'(x_0 + x)(z) \in E^*(\{V_k\}) + z$

for $x \in \lambda W_\ell^\alpha$ and $z \in E^*(\{V_k\})$. Thus $g'(x_0 + x)E^*(\{V_k\}) \subset E^*(\{V_k\})$.

(b) If $v + tu \in \lambda W_\ell^\alpha$ for $0 \leq t \leq 1$, and $u \in \mu_k V_k$ ($\mu_k > 0; k = 1, 2, \dots$), then $H(t) = g(x_0 + v + tu)$ is R -differentiable at t ($0 \leq t < 1$) in R_1 , where R_1 is all real numbers as in Example 1. And

$$H'(t) = g'(x_0 + v + tu)(u) = f'(x_0)^{-1} \circ f'(x_0 + v + tu)(u).$$

Proof of (b): Put, for $0 \leq t < 1$, $0 < s$ and $t + s < 1$,

$$\begin{aligned} \gamma(s) &= H(t+s) - H(t) - g'(x_0 + v + tu)(su) \\ &= g(x_0 + v + tu + su) - g(x_0 + v + tu) - g'(x_0 + v + tu)(su) \\ &= f'(x_0)^{-1} \{f(x_0 + v + tu + su) - f(x_0 + v + tu) - f'(x_0 + v + tu)(su)\} \end{aligned}$$

and

$$\gamma_1(w) = f(x_0 + v + tu + w) - f(x_0 + v + tu) - f'(x_0 + v + tu)(w).$$

Since f is R -differentiable at every point in D and $u \in E(\{V_k\})$, we see that $t_n^{-1} \gamma_1(t_n u) \rightarrow 0(p)$ for each sequence $\{t_n\}$ ($t_n > 0, t_n \rightarrow 0$). And so, since $f'(x_0)$ is isomorphism, $t_n^{-1} \gamma(t_n) \rightarrow 0(p)$.

Since by (a), $t_n^{-1} \gamma(t_n) \in E^*(\{V_k\})$, we see that $t_n^{-1} \gamma(t_n) \rightarrow 0$ ($P - \{V_k\}$) by (A.2'). Hence $H(t)$ is R -differentiable at t .

(c) For $x \in \lambda W_\ell^\alpha$, we have

$$\{I - g'(x_0 + x)\} \bar{W}_k^P \subset 2L_k \bar{W}_k^P \quad (k = 1, 2, \dots).$$

Proof of (c): If $z \in \bar{W}_k^P$, then for each n $z \in W_k + V_n - V_n$. Thus there are some $y, y' \in V_n$ and some $u, u' \in V_k$ such that $z = u - u' + y - y'$. Hence by (3), for each n

$$\begin{aligned} \{I - g'(x_0 + x)\}(z) &= \{I - g'(x_0 + x)\}(u) - \{I - g'(x_0 + x)\}(u') \\ &\quad + \{I - g'(x_0 + x)\}(y) - \{I - g'(x_0 + x)\}(y') \in L_k W_k - L_k W_k + L_n W_n - L_n W_n \\ &\subset 2L_k W_k + V_n - V_n. \end{aligned}$$

Thus we see that $\{I - g'(x_0 + x)\}(z) \in 2L_k \bar{W}_k^P$. (cf. [3] Lemma 1.6 (a))

(d) If $v + tu \in \lambda W_\ell^\alpha$ for $0 \leq t \leq 1$ and $u \in \mu_k \bar{W}_k^P$ ($\mu_k > 0; k = 1, 2, \dots$), then $u - g(x_0 + v + u) + g(x_0 + v) \in 2L_k \mu_k \bar{W}_k^P$.

Proof of (d): Put $F(t) = v + tu - g(x_0 + v + tu) + g(x_0 + v)$ for $0 \leq t \leq 1$, then $F(t) \in E^*(\{V_k\})$ by (a). By Lemma 6 and (b), we have

$F^*(t) = \{I - g^*(x_0 + v + tu)\}(u)$ for $0 \leq t < 1$. And so, we see $F^*(t) \in 2L_k \mu_k \bar{w}_k^P$ by (c). Thus by the mean value theorem (cf. [3]), we have

$$F(1) - F(0) \in 2L_k \mu_k \bar{w}_k^P.$$

Since $F(1) - F(0) = u - g(x_0 + v + u) + g(x_0 + v)$, the Proof is complete.

(e) Put $T(y) = y - g(y)$ for $y \in x_0 + \lambda W_\ell^\alpha$, then $T(x_0 + \lambda W_\ell^\alpha) \subset x_0 + \lambda W_\ell^\alpha$.

Proof of (e): Put $G(t) = t(y - x_0) - g\{x_0 + t(y - x_0)\} + g(x_0)$ for $y \in x_0 + \lambda W_\ell^\alpha$ and $0 \leq t \leq 1$, then $G(t) \in E^*(\{V_k\})$ by (a). Thus $G^*(t) = [I - g^*\{x_0 + t(y - x_0)\}](y - x_0)$ for $0 \leq t < 1$, and so, $G^*(t) \in 2L_\ell \lambda W_\ell^\alpha$ for $0 \leq t < 1$. Hence by the mean value theorem, we have

$$G(1) - G(0) = (y - x_0) - g(y) + g(x_0) = \{y - g(y)\} - \{x_0 - g(x_0)\} \in 2L_\ell \lambda W_\ell^\alpha.$$

And so, $T(y) \in x_0 - g(x_0) + 2L_\ell \lambda W_\ell^\alpha \subset x_0 + \lambda(1 - 2L_\ell)W_\ell^\alpha + 2L_\ell \lambda W_\ell^\alpha$, by (4).

Since W_ℓ^α is convex, we see $T(y) \in x_0 + \lambda W_\ell^\alpha$. Thus, $T(x_0 + \lambda W_\ell^\alpha) \subset x_0 + \lambda W_\ell^\alpha$.

(f) Define $\{y_n\}$ by $y_1 = T(x_0)$ and $y_{n+1} = T(y_n)$ ($n = 1, 2, \dots$). Then for each k ($k = 1, 2, \dots$) there is $\alpha_k > 0$ such that $y_{n+1} - y_n \in (2L_k)^n \alpha_k \bar{w}_k^P$ ($n = 1, 2, \dots$).

Proof of (f): Since $-g(x_0) \in E^*(\{V_k\})$ by (a), there is $\alpha_k > 0$ for each k such that $-g(x_0) \in \alpha_k W_k$. Now by induction we shall prove

$$(5) \quad t(y_{n+1} - x_0) \in \lambda W_\ell^\alpha \quad 0 \leq t \leq 1; \quad n = 0, 1, 2, \dots,$$

$$(6) \quad y_{n+1} - y_n \in (2L_k)^n \alpha_k \bar{w}_k^P \quad n = 1, 2, \dots; k = 1, 2, \dots.$$

Since $y_1 - x_0 = T(x_0) - x_0 = -g(x_0) \in \alpha_k W_k$, ($k = 1, 2, \dots$),

$t(y_1 - x_0) = -tg(x_0) \in t\lambda(1 - 2L_\ell)W_\ell^\alpha \subset \lambda W_\ell^\alpha$ for $0 \leq t \leq 1$ and

$y_2 - y_1 = y_1 - x_0 - g\{x_0 + (y_1 - x_0)\} + g(x_0)$, (d) implies that $y_2 - y_1 \in 2L_k \alpha_k \bar{w}_k^P$,

and so (6) holds with $n = 1$. Since $y_1 - x_0 \in \lambda W_\ell^\alpha$, (e) implies

$y_2 = T(y_1) \in x_0 + \lambda W_\ell^\alpha$, and so $y_2 - x_0 \in \lambda W_\ell^\alpha$. Then, for $0 \leq t \leq 1$,

$t(y_2 - x_0) \in t\lambda W_\ell^\alpha \subset \lambda W_\ell^\alpha$. Hence (5) holds for $n = 0, 1$. Suppose (5) and (6)

hold for $n = 1, 2, \dots, m$. Then, since

$(y_m - x_0) + t(y_{m+1} - y_m) = t(y_{m+1} - x_0) + (1 - t)(y_m - x_0) \in \lambda W_\ell^\alpha$ for $0 \leq t \leq 1$ and

$y_{m+1} - y_m \in (2L_k)^m \alpha_k \bar{w}_k^P$, (d) implies

$y_{m+2} - y_{m+1} = y_{m+1} - y_m - g\{x_0 + (y_m - x_0) + (y_{m+1} - y_m)\} + g\{x_0 + (y_m - x_0)\}$
 $\in (2L_k)^{m+1} \alpha_k \bar{w}_k^P$. Thus (6) holds with $n = m + 1$. Moreover, since

$y_{m+1} - x_0 \in \lambda W_\ell^\alpha$ by (5) for $n = m$, (e) implies $y_{m+2} = T(y_{m+1}) \in x_0 + \lambda W_\ell^\alpha$ and so

$y_{m+2} - x_0 \in \lambda W_\ell^\alpha$. Then for $0 \leq t \leq 1$, $t(y_{m+2} - x_0) \in t\lambda W_\ell^\alpha \subset \lambda W_\ell^\alpha$. Hence (5)

holds with $n = m + 1$. Thus we prove (5) and (6) by induction. By (6), we

see that for $n < m$ and each k ,

$$y_m - y_n = \sum_{h=n}^{m-1} (y_{h+1} - y_h) \in \sum_{h=n}^{m-1} (2L_k)^h \alpha_k \bar{w}_k^P \subset \sum_{h=n}^{m-1} (2L_k)^h \alpha_k \cdot 2W_k$$

$$\subset (2L_k)^n (1 - 2L_k)^{-1} \cdot 2\alpha_k W_k.$$

Thus for sufficiently large n such that $2\alpha_k (2L_k)^n (1 - 2L_k)^{-1} < 1$, we have $y_m - y_n \in W_k$. Hence $\{y_k\}$ is a P -Cauchy sequence by $\{V_k\}$. Since E is complete, by Proposition 5, there is $z \in E$ such that $y_n \rightarrow z$ ($P - \{V_k\}$). Also, since $y_n \in x_0 + \lambda W_k^\alpha$ by (5), we see $z \in \overline{(x_0 + \lambda W_k^\alpha)} (P - \{V_k\}) = x_0 + \lambda W_k^\alpha \subset D$. On the other hand since g is continuous, T is continuous. Hence by $y_{n+1} = T(y_n)$, we have $z = T(z) = z - g(z)$. Then $g(z) = 0$. Thus $f(z) = 0$ because $f(z) = f''(x_0)g(z)$.

(g) A point z such that $f(z) = 0$ and $z \in x_0 + \lambda W_k^\alpha$, is only one.

Proof of (g): Suppose $z_i = T(z_i)$ and $z_i \in x_0 + \lambda W_k^\alpha$ ($i = 1, 2$).

Then, since $z_i - x_0 \in \lambda W_k^\alpha \subset E^*(\{V_k\})$, there are some positive numbers β_k such that $z_i - x_0 \in \beta_k W_k$, ($k = 1, 2, \dots$). For $\{y_n\}$ defined in (f), we shall prove by induction that

$$(7) \quad z_i - y_n \in (2L_k)^n \beta_k \overline{W}_k^P, \quad n = 1, 2, \dots; k = 1, 2, \dots.$$

Since $t(z_i - x_0) \in \lambda W_k^\alpha$ for $0 \leq t \leq 1$ and $z_i - x_0 \in \beta_k W_k$ ($k = 1, 2, \dots$), (d) implies $z_i - y_1 = z_i - x_0 - g\{x_0 + (z_i - x_0)\} + g(x_0) \in 2L_k \beta_k \overline{W}_k^P$. Then, (7) holds with $n=1$. Next, suppose (7) holds for $n = 1, 2, \dots, m$. Then, since by (5)

$y_m - x_0 + t(z_i - y_m) = (1-t)(y_m - x_0) + t(z_i - x_0) \in \lambda W_k^\alpha$ for $0 \leq t \leq 1$ and

$z_i - y_m \in (2L_k)^m \beta_k \overline{W}_k^P$ ($k = 1, 2, \dots$), (d) implies

$z_i - y_{m+1} = z_i - y_m - g\{x_0 + (y_m - x_0) + (z_i - y_m)\} + g\{x_0 + (y_m - x_0)\} \in (2L_k)^{m+1} \beta_k \overline{W}_k^P$.

And so, (7) holds with $n = m+1$. Hence we prove (7). Then we have for sufficiently large n such that $4(2L_k)^n \beta_k < 1$, by (7) and Proposition 3

$$\begin{aligned} z_1 - z_2 &= (z_1 - y_n) - (z_2 - y_n) \in (2L_k)^n \beta_k (\overline{W}_k^P - \overline{W}_k^P) \\ &\subset (2L_k)^n \beta_k (2W_k - 2W_k) \subset V_k - V_k \quad (k = 1, 2, \dots). \end{aligned}$$

Hence, by Lemma 1, we see $z_1 = z_2$.

§3. Corollaries of the main theorem.

Let E and F be complete real countably semi-normed Hausdorff linear spaces and $f: D \rightarrow F$ (D is an open subset of E) be Fréchet differentiable. (cf. [6]). Suppose there is a point x_0 in D such that $f'(x_0)$ is a linear homeomorphism of E onto F . Let $\{P_n\}$ be the system of all semi-norms in E such that $P_n < P_{n+1}$. Put $V_n = \{x \in E \mid P_n(x) \leq n^{-1}\}$.

COROLLARY 1. If for $g = f''(x_0)^{-1} \circ f$ and $\lambda \geq 1$, the following (1)~(3)

are satisfied;

(1) there is m such that $x_0 + 2\lambda V_m \subset D$,

(2) for each k ($k=1, 2, \dots$) there is M_k ($0 < M_k < 1$) such that

$$P_k[\{I - g^*(x_0 + x)\}(y)] \leq M_k \cdot k^{-1} \quad \text{for } x \in 2\lambda V_m \text{ and } y \in V_k,$$

(3) $P_m\{g(x_0)\} \leq \lambda(1 - M_m)2m^{-1}$,

then, there is a unique point z in $x_0 + 2\lambda V_m$ such that

$$f(z) = 0.$$

REMARK 1: A countably semi-normed Hausdorff linear space E is regarded as linear ranked space by the following. Let $\{P_n\}$ be the system of all semi-norms in E such that $P_n < P_{n+1}$, and let $V(n, \varepsilon) = \{x \in E \mid P_n(x) \leq \varepsilon\}$ for $\varepsilon > 0$. Put $\mathfrak{B}_0 = \{V(n, \varepsilon) \mid n=1, 2, \dots; \varepsilon > 1\} \cup \{E\}$, $\mathfrak{B}_n = \{V(n, \varepsilon) \mid (n+1)^{-1} < \varepsilon \leq n^{-1}\}$ ($n=1, 2, \dots$). Then $(E, \{\mathfrak{B}_n\})$ is a $(\pi - T_1)$ linear ranked space. Both R - and P -convergences coincide with the convergence with respect to the original topology of E and any P -Cauchy sequence is a Cauchy sequence with respect to the original topology of E . Moreover, $V(n, \varepsilon)$ is both R - and P -closed.

PROOF. E and F are regarded as linear ranked spaces by the manner in Remark 1. Let $\{x_n\}$ be an R - $\{V_k\}$ q.b.s. in linear ranked space E , then $\{x_n\}$ is bounded with respect to the original topology of E . Given $\{h_n\}$ ($h_n > 0$, $h_n \rightarrow 0$), for any $y \in \{x_n\}$, $\{h_n y\}$ converges uniformly to 0 with respect to the original topology of E . Since f is Fréchet differentiable to every $x \in D$, we see $\lim_{h \rightarrow 0} h^{-1} \gamma(h_n y) = 0$ with respect to the original topology of E , where $\gamma(y) = f(x+y) - f(x) - f^*(x)y$. Since the R -convergence coincides with the original convergence in E , $f: D \rightarrow F$ is R -differentiable at every $x \in D$. Thus $f^*(x)$ is the R -derivative of f at x . If we put $m = \ell$ and $M_k = 2L_k$, then Corollary 1 is proved by Theorem 1.

COROLLARY 2. Let E and F be real Banach spaces. Let $f: B(x_0, \gamma) \rightarrow F$ be Fréchet differentiable at every $x \in B(x_0, \gamma)$, where $B(x_0, \gamma)$ is a sphere of E , and $f^*(x_0)$ be a linear homeomorphism of E onto F . If there are ℓ ($0 < \ell < \gamma$) and M ($0 < M < 1$) such that

$$\| \{I - f^*(x_0)^{-1} \circ f^*(x)\}(y) \| \leq M \|y\|$$

for any $x \in \overline{B}(x_0, \ell)$ and any $y \in E$, and $\|f'(x_0)^{-1}f(x_0)\| \leq (1-M)\ell$, then there is unique $z \in \overline{B}(x_0, \ell)$ such that $f(z) = 0$.

PROOF. Put $V_k = \{x \in E \mid \|x\| \leq k^{-1}\}$ for natural number k , and $g = f'(x_0) \circ f$. If we put $\ell = 2m^{-1} \cdot \lambda$ for some $\lambda \geq 1$ and some natural number m , then we have

$$(1) \quad x_0 + 2\lambda V_m \subset B(x_0, \gamma)$$

$$(2) \quad \|\{I - g'(x_0 + x)\}(y)\| \leq M k^{-1} \text{ for } x \in 2\lambda V_m \text{ and } y \in V_k$$

$$(3) \quad \|g(x_0)\| \leq (1-M)\ell = \lambda(1-M)2m^{-1}.$$

Thus, by Corollary 1, there is $z \in x_0 + 2\lambda V_m$, that is, $\|z - x_0\| \leq 2m^{-1} \cdot \lambda = \ell$, such that $f(z) = 0$.

COROLLARY 3. (cf. [5]). Let E and F be real Banach spaces. Let $f: B(x_0, \gamma) \rightarrow F$ be Fréchet differentiable at every $x \in B(x_0, \gamma)$, where $B(x_0, \gamma)$ is a sphere of E . Suppose that for constant m ,

$$\|f'(x_1) - f'(x_2)\| \leq m\|x_1 - x_2\| \quad (x_1, x_2 \in B(x_0, \gamma))$$

$\|f'(x_0)^{-1}\| = L$, $\|f'(x_0)^{-1}f(x_0)\| = k$ and $h = Lkm < 1/4$. Let t_0 be a smaller root of the equation $ht^2 - t + 1 = 0$, and $kt_0 < \gamma$. Then there is unique $z \in \overline{B}(x_0, kt_0)$ such that $f(z) = 0$.

PROOF. By the condition for t_0 , we see $t_0 = (2h)^{-1}(1 - \sqrt{1 - 4h})$. Put $\ell = kt_0$ and $M = mL\ell$. Then $M = t_0h$ because $h = Lkm$. Moreover, $k = \ell t_0^{-1} = \ell \cdot 2h(1 - \sqrt{1 - 4h})^{-1} = \ell(1 - t_0h) = \ell(1 - M)$ and $M = t_0h = 2^{-1}(1 - \sqrt{1 - 4h}) < 2^{-1}$. Thus, there are $\ell(0 < \ell < \gamma)$ and $M(0 < M < 1)$ such that for $\|x - x_0\| \leq \ell$ and any $y \in E$, $\|\{I - f'(x_0)^{-1} \circ f'(x)\}(y)\| \leq \ell Lm\|y\| = M\|y\|$. And $\|f'(x_0)^{-1}f(x_0)\| = k = (1-M)\ell$. Thus by Corollary 2, there is unique $z \in \overline{B}(x_0, \ell)$ such that $f(z) = 0$.

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