

## ON THE BESICOVITCH COVERING THEOREM

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**Abstract.** A relatively simple proof is given for the case of a finite dimensional normed vector space of a covering theorem first established for the plane by A. S. Besicovitch and subsequently proved for special metric spaces by H. Federer. The proof here also establishes an extension of the covering theorem for more general metric spaces.

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We give a proof of a covering theorem first established for the plane by A. S. Besicovitch [1] and subsequently proved for special metric spaces by H. Federer [3]. The Besicovitch covering theorem is used in analysis to establish maximal inequalities (e.g., [4]) and to obtain general forms of the Lebesgue density theorem (see, for example, [2].) In this note, we modify and reduce Federer's proof in [3] to the case of an arbitrary finite dimensional normed vector space  $(X, \|\cdot\|)$ . The proof is much simpler than that in [3]. The constant associated with the covering theorem and the use of ordinals in the proof are new. We will also establish a covering theorem for more general metric spaces.

For a finite dimensional normed vector space  $(X, \|\cdot\|)$ ,  $\rho(x, y)$  will denote the distance  $\|x - y\|$ . When  $r > 0$ ,  $B(a, r)$  will denote the closed ball  $\{x \in X : \rho(a, x) \leq r\}$ . We will let  $N(X, \|\cdot\|, d)$  or simply  $N(d)$  denote the maximum possible cardinality of a set  $S$  contained in the unit sphere  $\{x \in X : \|x\| = 1\}$  such that for all distinct points  $x, y \in S$ ,  $\|x - y\| \geq d$ . Let  $K(X, \|\cdot\|)$  be the least integer in the set

$$\{N(2 - u) + N(1 - u^{-1}) : u \in \mathbf{R}, 1 < u < 2\}$$

**THEOREM 1(Besicovitch).** *Let  $(X, \|\cdot\|)$  be a finite dimensional normed vector space. Fix an arbitrary nonempty set  $A \subseteq X$  and  $p > 0$ . With*

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each  $a \in A$ , associate a ball  $B(a, r)$  with  $r \leq p$ . From this covering of  $A$ , one can extract  $M$  non-intersecting subcollections  $\mathcal{F}_i$  so that  $\mathcal{F} \equiv \bigcup_{i=1}^M \mathcal{F}_i$  also covers  $A$ , in each  $\mathcal{F}_i$  the balls are disjoint, and  $M \leq K(X, \|\cdot\|) + 1$ .

**COROLLARY.** Each  $a \in A$  is in at most  $M$  balls from the collection  $\mathcal{F}$ .

Note that the upper bound to  $M$  in the theorem depends only on the space and the norm. We can even replace the use of  $N(X, \|\cdot\|, d)$  with the use of a larger constant that depends only on the dimension of  $X$ . One can not allow balls of unlimited size in the theorem since otherwise one could fix  $x_0 \in X$  and have each ball contain  $x_0$ . The theorem's proof is based on an analysis of the following configuration of balls.

**DEFINITION.** Fix a ball  $B(a, r)$  and a constant  $\tau > 1$ . A finite, ordered set of balls  $\{B(b_i, s_i) : 1 \leq i \leq n\}$  is said to be in  $\tau$  **satellite configuration** with respect to  $B(a, r)$  if the following conditions hold for  $1 \leq i \leq j \leq n$ :

$$\begin{aligned} & i) \rho(a, b_i) \leq r + s_i, \quad ii) r < \tau \cdot s_i, \quad iii) \max(r, s_i) < \tau \cdot \rho(a, b_i), \\ & iv) \rho(b_i, b_j) > s_i > s_j / \tau \text{ for } i < j. \end{aligned}$$

In this definition, one takes  $\tau$  close to 1. This gives a set of balls, all intersecting one of the balls  $B(a, r)$ , for which each center is outside or almost outside every other ball and  $r$  is almost the minimum radius.

**LEMMA.** For any  $d > 0$ , there is an  $\epsilon > 0$  such that  $N(d - \epsilon) = N(d)$ .

**PROOF:** Assume no such  $\epsilon$  exists. Then for each  $n \in \mathbb{N}$  with  $1/n < d$ , there is a set  $A_n = \{a_i^n : 1 \leq i \leq N(d) + 1\}$  in the unit sphere with each point in  $A_n$  a distance at least  $d - 1/n$  from any other point in  $A_n$ . By taking a subsequence, we may assume that  $a_i^n \rightarrow a_i$ . The set  $A = \{a_i\}$  has  $N(d) + 1$  members and each point is a distance at least  $d$  from any other point of  $A$ . This contradicts the definition of  $N(d)$ . ■

**PROPOSITION.** Given a finite dimensional normed vector space  $(X, \|\cdot\|)$ , there is a constant  $\delta > 0$  such that for each  $\tau$  with  $1 < \tau < 1 + \delta$ , there can be at most  $K(X, \|\cdot\|)$  balls in  $\tau$  satellite configuration with respect to any ball.

**PROOF:** Fix a constant  $u \in (1, 2)$  and an  $\epsilon > 0$  such that

$$K(X, \|\cdot\|) = N(2 - u - \epsilon) + N(1 - u^{-1} - \epsilon).$$

Fix  $\delta > 0$  so that if  $1 < \tau < 1 + \delta$ , then  $\tau \cdot u < u + \epsilon$ , and  $\tau(\tau - 1) < \epsilon$ ; fix  $\tau$  with  $1 < \tau < 1 + \delta$ . Choose a ball  $B(a, r)$ , and a finite set  $\{B(b_i, s_i) :$

$1 \leq i \leq n$  in  $\tau$  satellite configuration with respect to  $B(a, r)$ . Partition the set  $\{b_i : 1 \leq i \leq n\}$  into two sets

$$S_1 = \{b_i : \rho(a, b_i) \leq u \cdot r\}, S_2 = \{b_i : u \cdot r < \rho(a, b_i)\}.$$

Let  $B(b, s)$  and  $B(c, t)$  denote any two of the balls  $B(b_i, s_i)$  for  $1 \leq i \leq n$ . We will assume that  $\sigma \equiv \|a - c\| \leq \|a - b\|$ . Fix the point

$$x = a + \sigma \cdot \frac{b - a}{\|b - a\|}$$

on the ray from  $a$  to  $b$  so that  $\|a - x\| = \|a - c\|$ . Since

$$\|b - c\| \leq \|b - x\| + \|x - c\| = \|a - b\| - \|a - c\| + \|x - c\|,$$

we have the "Bow and Arrow Inequality"

$$\|x - c\| \geq \|a - c\| + \|c - b\| - \|a - b\|.$$

If  $b$  and  $c$  are in  $S_1$ , then since  $\|c - b\| > \min(s, t) > r/\tau$  and  $\tau \cdot \sigma > r$ ,

$$\|x - c\| \geq \sigma + r/\tau - u \cdot r \geq [1 + (1 - u \cdot \tau)] \cdot \sigma \geq (2 - u - \epsilon) \cdot \sigma.$$

If  $b$  and  $c$  are in  $S_2$ , then  $\|a - b\| \leq r + s$ , either  $\|c - b\| > s$  or  $\|c - b\| > t > s/\tau$ ,  $u \cdot r < \sigma$ , and  $t < \tau \cdot \sigma$ . We therefore have

$$\begin{aligned} \|x - c\| &\geq \|a - c\| - r + \|c - b\| - s \geq \sigma - u^{-1} \cdot \sigma + t - \tau t \\ &\geq [1 - u^{-1} - \tau(\tau - 1)] \cdot \sigma \geq (1 - u^{-1} - \epsilon) \cdot \sigma. \end{aligned}$$

In the first case we have

$$\left\| \frac{b - a}{\|b - a\|} - \frac{c - a}{\|c - a\|} \right\| = \left\| \frac{x}{\sigma} - \frac{c}{\sigma} \right\| \geq 2 - u - \epsilon.$$

Thus, the image of  $S_1$  under the injective mapping

$$y \rightarrow \frac{y - a}{\|y - a\|}$$

is a set  $S$  in the unit sphere with each point of  $S$  a distance at least  $2 - u - \epsilon$  from any other. It follows that there are at most  $N(2 - u - \epsilon)$  points in  $S_1$ . Similarly, there are at most  $N(1 - u^{-1} - \epsilon)$  points in  $S_2$ . ■

EXAMPLE. Consider  $\mathbf{R}^2$  with the norm  $\|(x, y)\| = |x| + |y|$ . Relaxing requirements for strict inequality and using the notation of the proposition, we let  $a = (0, 0)$ ,  $b = (0, u)$ ,  $c = (1 - u/2, u/2)$ , and  $r = s = t = 1$ . Here,  $b$  and  $c$  are in  $S_1$ ,  $x = (0, 1)$  and  $\rho(x, c) = 2 - u = (2 - u) \cdot \|c\|$ . On the other hand, if we change  $b$  and  $c$  so  $b = (0, 2)$  and  $c = (u/2 - 1/2, u/2 + 1/2)$ , then  $b$  and  $c$  are in  $S_2$ ,  $x = (0, u)$ , and  $\rho(x, c) = u - 1 = (1 - 1/u) \cdot \|c\|$ .

PROOF OF THEOREM 1: Fix  $\tau > 1$  so that for any ball  $B(a, r)$ , there can be at most  $K(X, \|\cdot\|)$  balls in  $\tau$  satellite configuration with respect to  $B(a, r)$ . Let  $\mathcal{G}$  denote the original collection of balls, and let  $T$  be a choice function on the set of nonempty subcollections  $Q$  of  $\mathcal{G}$  such that  $T(Q)$  is a ball  $B(a, r) \in Q$  with  $\tau r > \sup\{t : B(c, t) \in Q\}$ .

Form a one-to-one correspondence between an initial segment of the ordinal numbers and a subcollection of  $\mathcal{G}$  as follows. Set  $Q_1 = \mathcal{G}$  and  $B_1 = T(Q_1)$ . Having chosen  $B_\alpha$  for  $\alpha < \beta$ , let  $Q_\beta$  be the set of balls in  $\mathcal{G}$  with centers not in  $\cup_{\alpha < \beta} B_\alpha$ . If  $Q_\beta \neq \phi$ , set  $B_\beta = T(Q_\beta)$ . There exists a first ordinal  $\gamma$  for which  $Q_\gamma = \phi$ , that is,  $A \subseteq \cup_{\alpha < \gamma} B_\alpha$ . We set  $\mathcal{F} = \{B_\alpha : \alpha < \gamma\}$ . Note that for any pair  $B(a, r) = B_\alpha$  and  $B(b, s) = B_\beta$  from  $\mathcal{F}$  with  $\alpha < \beta$ ,  $\rho(a, b) > r > s/\tau$  since  $B_\beta \in Q_\alpha$ . Let  $T_f(Q)$  denote the first element in  $Q$  for each nonempty subcollection  $Q$  of  $\mathcal{F}$ .

Given any nonempty subcollection  $\mathcal{H}$  of  $\mathcal{F}$ , form a one-to-one correspondence between an initial segment of the ordinal numbers and a pairwise disjoint subcollection  $P(\mathcal{H})$  of  $\mathcal{H}$  as follows. Set  $Q_1 = \mathcal{H}$  and  $B(1) = T_f(Q_1)$ . Having chosen  $B(\alpha)$  for  $\alpha < \beta$ , let  $Q_\beta$  be the set of balls in  $\mathcal{H}$  that are contained in  $X - \cup_{\alpha < \beta} B(\alpha)$ . If  $Q_\beta \neq \phi$ , set  $B(\beta) = T_f(Q_\beta)$ . There exists a first ordinal  $\gamma$  for which  $Q_\gamma = \phi$ . Let  $P(\mathcal{H}) = \{B(\alpha) : \alpha < \gamma\}$ . For all  $B(a, r) \in \mathcal{H} - P(\mathcal{H})$ , there is a  $B(b, s) \in P(\mathcal{H})$  with  $\rho(a, b) \leq r + s$ . If  $B(b, s)$  is the first such ball with respect to the ordering, then  $s > r/\tau$ .

Now form the families  $\mathcal{F}_i$  by induction as follows. Set  $\mathcal{F}_1 = P(\mathcal{F})$ . Having chosen  $\mathcal{F}_i$  for  $1 \leq i \leq n$ , let  $\mathcal{H}_n = \mathcal{F} - \cup_{i=1}^n \mathcal{F}_i$ . If  $\mathcal{H}_n \neq \phi$ , set  $\mathcal{F}_{n+1} = P(\mathcal{H}_n)$ . Note that for  $B(a, r) \in \mathcal{H}_n$ , there are balls  $B(b_i, s_i) \in \mathcal{F}_i$ ,  $1 \leq i \leq n$ , such that  $\rho(a, b_i) \leq r + s_i$  and  $s_i > r/\tau$ . Since, after a reordering, these balls are in  $\tau$  satellite configuration with respect to  $B(a, r)$ ,  $\mathcal{H}_n = \phi$  for some  $n \leq K(X, \|\cdot\|) + 1$ . ■

One can simplify the proposition's proof somewhat by replacing the constant  $K(X, \|\cdot\|)$  with the constant  $2 \cdot N(1/4)$ . One must then make the same replacement in Theorem 1. With this replacement, one avoids the lemma by fixing  $u = 3/2$  and working with  $N(1/2) + N(1/3)$ . If one does not have the conditions necessary to establish the proposition, the proof of Theorem 1 yields the following result.

**THEOREM 2.** Fix a metric space  $(X, \rho)$ , a nonempty set  $A \subseteq X$ , and constants  $p > 0$  and  $\tau > 1$ . Let  $S = \{B(a, r) \subset X : a \in A, r \leq p\}$ . Assume there is an upper bound  $N(X, \rho, A, p, \tau)$  to the cardinality of any finite ordered set of balls in  $S$  that is in  $\tau$  satellite configuration with respect to any other ball in  $S$ . With each  $a \in A$ , associate a ball  $B(a, r)$  with  $r \leq p$ . From this covering of  $A$ , one can extract  $M$  non-intersecting subcollections  $\mathcal{F}_i$  so that  $\mathcal{F} \equiv \cup_{i=1}^M \mathcal{F}_i$  also covers  $A$ , in each  $\mathcal{F}_i$  the balls are disjoint, and  $M \leq N(X, \rho, A, p, \tau) + 1$ .

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