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A NOTE ON INTEGRAL REPRESENTATIONS OF ALTERNATING GROUPS

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Abstract. This paper presents the set of all classes of integral representations rationally equivalent to a certain $\mathbf{Z}$-representatinon of A_{n+1}.

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§1. Introduction.

For the symmetric group S_{n+1} of degree $n+1$,

$$\mathbf{F}_n((k,k+1)) = \mathbf{E}_{k,k-1} + 2\mathbf{E}_{k,k} + \mathbf{E}_{k,k+1} - \mathbf{I}_n, \quad 1 \le k \le n$$

gives a $\mathbf{Z}$-representation of degree n, which is absolutely irreducible by Young's representation theory. Define the matrix $\mathbf{U}_n(d)$ by

$$\mathbf{U}_n(d) = \sum_{i=1}^{n-1}(d\mathbf{E}_{ii} + (-1)^{n-i+1} i\mathbf{E}_{in}) + \mathbf{E}_{nn}$$

for a positive integer d. The following result was shown by M.Craig.

THEOREM (CRAIG[1]). *The* $\mathbf{F}_n^d = \mathbf{U}_n(d)^{-1}\mathbf{F}_n\mathbf{U}_n(d)$ *for* $d \mid n+1$ *are* $\mathbf{Z}$*-inequivalent* $\mathbf{Z}$*-representations of* S_{n+1}*, and give all classes of integral representations rationally equivalent to* $\mathbf{F}_n$.

By the way, $\mathbf{F}_n$ which is regarded as a representation of the alternating group A_{n+1} of degree $n+1$ is as follows

$$\begin{cases} \mathbf{F}_n((1,2,3)) - \mathbf{I}_n = -\mathbf{E}_{1,1} + \mathbf{E}_{1,2} + \mathbf{E}_{1,3} - \mathbf{E}_{2,1} - 2\mathbf{E}_{2,2} - \mathbf{E}_{2,3}, \\ \mathbf{F}_n((1,2)(k,k+1)) - \mathbf{I}_n = -2\mathbf{E}_{1,1} - \mathbf{E}_{1,2} - \mathbf{E}_{k,k-1} - 2\mathbf{E}_{k,k} - \mathbf{E}_{k,k+1} \\ \qquad\qquad\qquad\qquad\qquad\qquad (3 \le k \le n). \end{cases}$$

It is well known that, above $\mathbf{F}_n$ for $n \ge 3$ is an absolutely irreducible representation. In this paper, we will determine the set of all classes of integral representations rationally equivalent to the representation $\mathbf{F}_n$ of A_{n+1}.

§2. Preliminaries.

Let G be a finite group of order g and $\mathbf{F} : G \longrightarrow GL_n(\mathbf{Z})$ be an absolutely irreducible representation of degree n. For $\mathbf{U} = (u_{ij}) \in GL_n(\mathbf{Q})$,

$$\mathbf{\Phi}(h) = \mathbf{U}^{-1}\mathbf{F}(h)\mathbf{U}, \quad h \in G$$

is a $\mathbf{Q}$-representation of G which is $\mathbf{Q}$-equivalent to $\mathbf{F}$. We may clearly suppose the u_{ij} to be integers, such that

$$\text{G.C.D.}(u_{ij}) = 1.$$

We will make use of the following five results which are due to Craig [1].

LEMMA 2.1. *If $\mathbf{\Psi} = \mathbf{V}^{-1}\mathbf{F}\mathbf{V}$ is a second such representation, then $\mathbf{\Phi}, \mathbf{\Psi}$ are $\mathbf{Z}$-equivalent iff for some unimodular $\mathbf{W}$, we have $\mathbf{V} = \mathbf{U}\mathbf{W}$.*

For $\mathbf{A} \in M_n(\mathbf{R})$, we denote the i-th column by $\mathbf{a}_i$ and define

$$\langle \mathbf{A} \rangle = \mathbf{Z}\mathbf{a}_1 + \cdots + \mathbf{Z}\mathbf{a}_n.$$

The action of G as a group of linear transformations of $\mathbf{R}^n$ is given by

$$(h, \mathbf{x}) \longrightarrow \mathbf{F}(h)\mathbf{x}, \quad \text{for } h \in G, \mathbf{x} \in \mathbf{R}^n.$$

LEMMA 2.2. *The condition for $\mathbf{\Phi}$ to be a $\mathbf{Z}$-representation, is that $\langle \mathbf{U} \rangle$ should be a G-submodule (invariant sublattice) of $\langle \mathbf{I}_n \rangle$.*

COROLLARY. *Equivalently, $\mathbf{\Phi}$ is integral iff all columns of $\mathbf{F}(h)\mathbf{U}$ lie in $\langle \mathbf{U} \rangle$, for all $h \in G_0$, where G_0 denotes a fixed set of generators for G. (That is, instead of testing $\mathbf{F}(h)\mathbf{x} \in \langle \mathbf{U} \rangle$ for all $h \in G$ and $\mathbf{x} \in \langle \mathbf{U} \rangle$, it is enough to examine the action of generators for G upon generators for $\langle \mathbf{U} \rangle$.)*

LEMMA 2.3. *If $\mathbf{\Phi}$ is a $\mathbf{Z}$-representation, then $\langle \mathbf{U} \rangle$ has $\langle (g/n)\mathbf{I}_n \rangle$ as a sublattice.*

LEMMA 2.4. *Suppose $m = p_1^{a_1} \ldots p_r^{a_r}$ is the factorization of $m = g/n$ in powers of distinct primes, and set $q_i = m/p_i^{a_i}$. Then the G-invariant lattices $\langle \mathbf{U} \rangle$ with $\langle \mathbf{I}_n \rangle \supset \langle \mathbf{U} \rangle \supset \langle m\mathbf{I}_n \rangle$, are precisely the $\langle \mathbf{U} \rangle = \sum_1^r q_i \langle \mathbf{U}_i \rangle$, where for each i, $\langle \mathbf{U}_i \rangle$ denote a G-invariant lattice such that $\langle \mathbf{I}_n \rangle \supset \langle \mathbf{U}_i \rangle \supset \langle p^{a_i}\mathbf{I}_n \rangle$.*

§3. All Classes of Integral Representations Rationnally Equivalent to $\mathbf{F}_n$.

In this section our purpose is to establish:

THEOREM. *The* $\mathbf{F}_n^d = \mathbf{U}_n(d)^{-1}\mathbf{F}_n\mathbf{U}_n(d)$ *for* $d \mid n+1$ *are* $\mathbf{Z}$*-inequivalent* $\mathbf{Z}$*-representations of* A_{n+1}*, and give all classes of integral representations rationally equivalent to* $\mathbf{F}_n$.

Our proof will be based on the treatment given by Craig[1]. The integral representations $\mathbf{Q}$-equivalent to $\mathbf{F}_n$ will be determined by finding the A_{n+1}-invariant lattices $\langle \mathbf{U}_n \rangle \supset \langle m\mathbf{I}_n \rangle$, where $m = (n+1)!/2n = p_1^{a_1} \cdots p_r^{a_r}$. We first determine those $\langle \mathbf{U}_n \rangle \supset \langle p^a \mathbf{I}_n \rangle$, where p is any prime and $a > 0$. Because $\langle \mathbf{U}_n \rangle = \langle \mathbf{U}_n \mathbf{W} \rangle$ for any $\mathbf{W} \in GL_n(\mathbf{Z})$, we may assume that $u_{ii} > 0$ for every i, and $u_{ij} = 0$ for $i > j$. Hence $\mathbf{U}_n$ can be expressed as

$$\mathbf{U}_n = \begin{pmatrix} \mathbf{U}_{n-1}^* & \mathbf{v} \\ 0 & u_{nn} \end{pmatrix}.$$

LEMMA 3.1. $\mathbf{U}_n$ *(normalized as described above) has the property that*

$$u_{ij} \equiv 0 \pmod{u_{nn}}, \quad 1 \le i, j \le n.$$

Proof. Clearly, $\mathbf{F}_n((1,2)(k,k+1))\mathbf{U}_n$ has columns in $\langle \mathbf{U}_n \rangle$ if and only if $(\mathbf{F}_n((1,2)(k,k+1)) - \mathbf{I}_n)\mathbf{U}_n$ has. Therefore, by the simple calculation, it is easily shown that the statement is correct in the case $n = 3$. Suppose inductively, that every $\mathbf{U}_{n-1}$ has above property.

Inspection of $\mathbf{F}_n$ above, shows that for $2 \le k \le n-1$,

$$\mathbf{F}_n((1,2)(k,k+1)) = \begin{pmatrix} \mathbf{F}_{n-1}((1,2)(k,k+1)) & \mathbf{f}((1,2)(k,k+1)) \\ 0 & 1 \end{pmatrix}.$$

Hence for $2 \le k \le n-1$, and for suitable column vectors $\mathbf{w}((1,2)(k,k+1))$,

$$\begin{aligned} &\mathbf{F}_n((1,2)(k,k+1))\mathbf{U}_n \\ &= \begin{pmatrix} \mathbf{F}_{n-1}((1,2)(k,k+1))\mathbf{U}_{n-1}^* & \mathbf{w}((1,2)(k,k+1)) \\ 0 & u_{nn} \end{pmatrix}. \end{aligned}$$

Therefore, if $\mathbf{F}_n((1,2)(k,k+1))\mathbf{U}_n$ has columns in $\langle \mathbf{U}_n \rangle$, $\mathbf{F}_{n-1}((1,2)(k,k+1))\mathbf{U}_{n-1}^*$ has columns in $\langle \mathbf{U}_{n-1}^* \rangle$. Thus, there exist an A_n-invariant lattiace $\mathbf{U}_{n-1}$ and an integer λ such that

$$\mathbf{U}_{n-1}^* = \lambda \mathbf{U}_{n-1}.$$

By the inductive hypothesis, we obtain

$$u_{ij} \equiv 0 \pmod{u_{n-1,n-1}}, \quad 1 \le i, j \le n-1. \tag{3.1}$$

Next, the $n-1$-th column of $(\mathbf{F}_n((1,2)(n,n+1))-\mathbf{I}_n)\mathbf{U}_n$ belongs to $\langle \mathbf{U}_n\rangle$, so that $u_{n-1,n-1}\equiv 0 \pmod{u_{nn}}$. Thus, by (3.1), we get

$$u_{ij}\equiv 0 \pmod{u_{nn}},\quad 1\le i,j\le n-1. \tag{3.2}$$

Additionlly, for $2\le k\le n$, the n-th column of $(\mathbf{F}_n((1,2)(k,k+1))-\mathbf{I}_n)\mathbf{U}_n$ belongs to $\langle \mathbf{U}_n\rangle$, so that we conclude

$$u_{k-1,n}+2u_{k,n}+u_{k+1,n}\equiv 0 \pmod{u_{kk}}. \tag{3.3}$$

Letting k run through the values $2,3,\dots,n$ in reverse order, we obtain

$$u_{in}\equiv 0 \pmod{u_{nn}},\quad 1\le i\le n. \tag{3.4}$$

Hence, by (3.2) (3.4), we get

$$u_{ij}\equiv 0 \pmod{u_{nn}},\quad 1\le i,j\le n.$$

COROLLARY. *The coefficient u_{nn} is* 1.

EXAMPLE. We shall determine all posibilities for $\mathbf{U}_3$. The lattice $\langle \mathbf{U}_3\rangle$ contains the columns of the $(\mathbf{F}_3((1,2)(k,k+1))-\mathbf{I}_3)\mathbf{U}_3$, for $k=2,3$. Therefore, we obtain

$$\langle \mathbf{U}_3\rangle = \langle \mathbf{U}_3(d)\rangle \quad \text{for } d=1,2 \text{ or } 4.$$

(We remark that the above example proves the Lemma 3.2 in the case $n=3$.)

LEMMA 3.2. *Let $\langle \mathbf{U}_n\rangle$ be an A_{n+1}-invariant lattice containing $\langle p^a\mathbf{I}_n\rangle$, and let* G.C.D.$(u_{ij})=1$. *Then $\langle \mathbf{U}_n\rangle = \langle \mathbf{U}_n(p^b)\rangle$ for some b $(a\ge b\ge 0)$ such that $p^b \mid n+1$.*

Proof. Previously, the case n=3 was verified. Suppose inductively, that the statement is correct with $n-1$ in place of n.

(i) Suppose that p does not divide n. Therefore $\langle \mathbf{U}^*_{n-1}\rangle = \langle p^b\mathbf{I}_{n-1}\rangle$, where $a\ge b\ge 0$. The last column $(\mathbf{F}_n((1,2,3))-\mathbf{I}_n)\mathbf{U}_n$ is element of $\langle \mathbf{U}_n\rangle$. Thus,

$$2u_{1n}+u_{2n}\equiv 0 \pmod{p^b}.$$

Hence, by (3.3), we have

$$u_{k-1,n}+2u_{k,n}+u_{k+1,n}\equiv 0 \pmod{p^b},\quad 1\le k\le n-1. \tag{3.5}$$

By induction on k, (3.5) gives

$$u_{k+1,n} \equiv (-1)^k (k+1) u_{1n} \pmod{p^b}, \quad 0 \le k \le n-1. \tag{3.6}$$

In particular,

$$u_{n-1,n} \equiv (-1)^{n-2}(n-1)u_{1n} \pmod{p^b},$$

$$1 \equiv (-1)^{n-1} n u_{1n} \pmod{p^b}, \tag{3.7}$$

so that, we have

$$u_{1n} \equiv (-1)^{n-1}(u_{n-1,n}+1) \pmod{p^b}. \tag{3.8}$$

Because $\langle \mathbf{U}_n \rangle$ contains the last column of $(\mathbf{F}_n((1,2)(n,n+1)) - \mathbf{I}_n)\mathbf{U}_n$, we get $u_{1n}(u_{n-1,n}+2) \equiv 0 \pmod{p^b}$. Multiplying by $(-1)^{n-1}n$ and using (3.7), we conclude that $u_{n-1,n} \equiv -2 \pmod{p^b}$. Thus, by (3.8), we obtain

$$u_{1n} \equiv (-1)^n \pmod{p^b}. \tag{3.9}$$

Hence, (3.6) and (3.9) imply $\langle \mathbf{U}_n \rangle = \langle \mathbf{U}_n(p^b) \rangle$.

(ii) Suppose $p \mid n$. By the inductive hypothesis,

$$\langle \mathbf{U}^*_{n-1} \rangle = p^{b-c} \langle \mathbf{U}_{n-1}(p^c) \rangle,$$

for some b, c where $a \ge b \ge c \ge 0$. As before (cf. (3.7)), we obtain that $(-1)^{n-1} n u_{1n} \equiv 1 \pmod{p^{b-c}}$. This implies G.C.D.$(n, p^{b-c}) = 1$. Therefore, $p \mid n$ shows $b = c$, so that $\langle \mathbf{U}^*_{n-1} \rangle = \langle \mathbf{U}_{n-1}(p^c) \rangle$. In particular, $u_{n-1,n-1} = 1$, hence we may assume $u_{n-1,n} = 0$. Setting $k = 1$ in (3.5), we see $2u_{1n} + u_{2n} \equiv 0 \pmod{p^c}$. The last column of $(\mathbf{F}_n((1,2)(n-1,n)) - \mathbf{I}_n)\mathbf{U}_n$ belongs to $\langle \mathbf{U}_n \rangle$. This implies

$$u_{n-2,n} + 1 \equiv 0 \pmod{p^c}. \tag{3.10}$$

Additionally, $\langle \mathbf{U}_n \rangle$ contains the $n-1$-th column of $(\mathbf{F}_n((1,2)(n,n+1)) - \mathbf{I}_n)\mathbf{U}_n$, that is $-\mathbf{e}_n$, so that, we find

$$u_{n-2,n} \equiv 0 \pmod{p^c}. \tag{3.11}$$

Hence, (3.10) and (3.11) give that $1 \equiv 0 \pmod{p^c}$. This shows that

$$\langle \mathbf{U}_n \rangle = \langle \mathbf{I}_n \rangle.$$

Lemma 3.3. *Let $\langle \mathbf{U}_n \rangle$ be invariant over $\langle m\mathbf{I}_n \rangle$, where G.C.D.$(u_{ij}) = 1$ and $m = (n+1)!/2n$. Then $\langle \mathbf{U}_n \rangle = \langle \mathbf{U}_n(d) \rangle$ for some $d \ge 1$ such that $d \mid n+1$.*

Proof. See(Craig [1]).

COROLLARY. *Every* $\langle \mathbf{U}_n(d) \rangle$ *for which* $d \mid n+1$ *is indeed invariant over* $\langle m\mathbf{I}_n \rangle$.

Proof. Straightforward.

LEMMA 3.4. *Suppose, for divisors* d_1, d_2 *of* $n+1$, *the representations* $\mathbf{F}_n^{d_1}, \mathbf{F}_n^{d_2}$ *are* $\mathbf{Z}$*-equivalent. Then* $d_1 = d_2$.

Proof. See(Craig [1]).

This completes the proof of the theorem.

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