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A SIMPLE PROOF OF NASH–WILLIAMS' FORMULA ON THE ARBORICITY OF A GRAPH

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ABSTRACT. Let G be a finite multigraph without loops. A subset S of $V(G)$ is called r-sparse if the number of edges joining vertices in S is at most $r \cdot (|S| - 1)$. Nash-Williams proved that $E(G)$ can be decomposed into r forests if and only if every nonempty subset of $V(G)$ is r-sparse. In this paper, we give a simple proof of this result.

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In this paper, we consider finite undirected graphs that may contain parallel edges, but no loops. That is, a graph $G = (V(G), E(G), \varphi_G)$ consists of the vertex set $V(G)$, the edge set $E(G)$, and the map φ_G from $E(G)$ to $\binom{V(G)}{2}$, where $\binom{V(G)}{2}$ denotes the set of all the unordered pairs of vertices. For a vertex x, $N_G(x)$ denotes the set of vertices adjacent to x, and $E_G(x)$ denotes the set of edges incident to x, i.e.,

$$E_G(x) := \{e \in E(G) | x \in \varphi_G(e)\},$$

and $d_G(x) := |E_G(x)|$ is the degree of x in G. The minimum degree $\delta(G)$ is defined as

$$\delta(G) := \min\{d_G(x) | x \in V(G)\}.$$

For two vertices x and y, $[x, y]_G$ denotes the set of edges joining x and y, i.e.,

$$[x, y]_G = \{e \in E(G) | \varphi_G(e) = \{x, y\}\}.$$

For a subset S of $V(G)$, $\langle S\rangle_G$ denotes the subgraph induced by S. That is,

$$\langle S\rangle_G := (S, E_G(S), \varphi|_{E_G(S)}),$$

where $E_G(S) := \{e \in E(G) | \varphi_G(e) \subseteq S\}$. Similarly, for a subset F of $E(G)$,

$$\langle F \rangle_G := (V(G), F, \varphi_G|_F).$$

A decomposition

$$E(G) = F_1 \cup F_2 \cup \cdots \cup F_r$$

is called a *forest decomposition* of G if $\langle F_i \rangle_G$ is a forest for $1 \leq i \leq r$. The *arboricity* of G, denoted by $a(G)$, is the minimum number of forests that decompose $E(G)$. For disjoint subsets $S_1, \cdots, S_m$ $(m \geq 2)$,

$$\begin{aligned} E_G(S_1, \cdots, S_m) &:= \left\{ e \in E(G) \,\middle|\, \begin{array}{c} \varphi_G(e) \subseteq S_1 \cup \cdots \cup S_m \\ |\varphi_G(e) \cap S_i| \leq 1 \end{array} \right\} \\ &= E_G(S_1 \cup \cdots \cup S_m) - \bigcup_{i=1}^{m} E_G(S_i) \end{aligned}$$

i.e., the set of edges joining vertices in different S_is. A subset S of $V(G)$ is called *r-sparse* if $|E_G(S)| \leq r \cdot (|S| - 1)$, and if the equality holds, S is called *r-critical.* For a real number z, $\lceil z \rceil$ denotes the least integer not less than z.

Tutte [4] and Nash-Williams [2] independently proved the folloing theorem.

Theorem 1. *A graph G contains r edge-disjoint spanning trees if and only if*

$$|E_G(S_1, \cdots, S_m)| \geq (m - 1) \cdot r$$

for any partition $V(G) = \bigcup_{i=1}^{m} S_i$.

Using Theorem 1, Nash-Williams [3] proved the following theorem.

Theorem 2. *The arboricity of a graph G is at most r if and only if every nonempty subset of $V(G)$ is r-sparse, i.e.,*

$$a(G) = \max \left\{ \left\lceil \frac{|E_G(S)|}{|S| - 1} \right\rceil \,\middle|\, \begin{array}{c} S \subseteq V(G) \\ |S| \geq 2 \end{array} \right\}.$$

In this paper, we give a simple self-contained proof of Theorem 2.

For a subset S of $V(G)$, let v_S be a new vertex not contained in $V(G)$, and let

$$V' := (V(G) - S) \cup \{v_S\},$$
$$E' := E(G) - E_G(S),$$
$$\varphi'(e) := \begin{cases} (\varphi_G(e) - S) \cup \{v_S\} & \text{if } |\varphi_G(e) \cap S| = 1 \\ \varphi_G(e) & \text{if } \varphi_G(e) \cap S = \emptyset. \end{cases}$$

Then (V', E', φ'), denoted by G/S, is called the graph obtained from G by contracting S.

Lemma 3. *Suppose that every nonempty subset of a graph G is r-sparse, and that S is an r-critical subset of $V(G)$. Then every nonempty subset of $V(G/S)$ is r-sparse.*

Proof. Let T be a nonempty subset of G/S. If $v_S \notin T$, then $E_{G/S}(T) = E_G(T)$. Hence T is r-sparse. If $v_S \in T$, then $E_{G/S}(T) = E_G((T - \{v_S\}) \cup S) - E_G(S)$. Hence

$$\begin{aligned} |E_{G/S}(T)| &\le r \cdot (|(T - \{v_S\}) \cup S| - 1) - r \cdot (|S| - 1) \\ &= r \cdot (|T| - 1). \quad \square \end{aligned}$$

Proof of Theorem 2. Suppose

$$E(G) = F_1 \cup \cdots \cup F_r$$

is a forest decomposition, and S a nonempty subset of $V(G)$. Let

$$F_i' := F_i \cap E_G(S).$$

Then (S, F_i') is a forest, and so $|F_i'| \le |S| - 1$. Hence

$$\begin{aligned} |E_G(S)| &= \Big|\bigcup_{i=1}^{r} F_i'\Big| \\ &\le r \cdot (|S| - 1). \end{aligned}$$

In the rest of the proof, we assume that every nonempty subset of $V(G)$ is r-sparse, and prove that $a(G) \le r$. We use induction on $|V(G)|$. The conclusion is obvious if $|V(G)| = 1$ or $|V(G)| = 2$. Hence we may assume that $|V(G)| \geq 3$.

Claim 1. $|[x,y]_G| \leq r$ for any $\{x,y\} \in \binom{V(G)}{2}$.

Proof. Let $S := \{x,y\}$. Then

$$|E_G(S)| = |[x,y]_G| \leq r \cdot (|S| - 1) = r,$$

since S is r-sparse. □

Claim 2. We may assume that $d_G(x) > r$ for all $x \in V(G)$.

Proof. Suppose $E_G(x) = \{e_1, \cdots, e_s\}$, $s \leq r$. Let $H := \langle V(G) - \{x\}\rangle_G$. Since every nonempty subset of $V(H)$ is r-sparse, $E(H)$ can be decomposed into r forests by induction. Let

$$E(H) = F_1 \cup \cdots \cup F_r$$

be a forest decomposition. Define

$$F_i' := \begin{cases} F_i \cup \{e_i\} & 1 \leq i \leq s \\ F_i & s < i \leq r. \end{cases}$$

Then

$$E(G) = F_1' \cup \cdots \cup F_r'$$

is a forest decomposition of $E(G)$. □

Claim 3. $\delta(G) < 2r$.

Proof. Since $V(G)$ itself is r-sparse,

$$|E(G)| \leq r \cdot (|V(G)| - 1).$$

On the other hand,

$$|E(G)| = \frac{1}{2} \sum_{x \in V(G)} d_G(x) \geq \frac{1}{2}\delta(G) \cdot |V(G)|.$$

Hence

$$\delta(G) \leq \frac{2r(|V(G)| - 1)}{|V(G)|} < 2r. \quad \square$$

Choose any vertex x of degree less than $2r$, and let

$$N_G(x) = \{y_1, \cdots, y_t\},$$
$$E_G(x) = \{e_1, \cdots, e_{r+s}\},$$

where $r + s = d_G(x)$.
Claim 4. We may assume that $\varphi(e_i) \neq \varphi(e_{r+i})$ for $1 \leq i \leq s$.
Proof. We arrange the edges incident to x as

$$
\begin{aligned}
[x, y_1]_G &= \{e_1, \cdots, e_{j_1}\}, \\
[x, y_2]_G &= \{e_{j_1+1}, \cdots, e_{j_2}\}, \\
&\vdots \\
[x, y_i]_G &= \{e_{j_{i-1}+1}, \cdots, e_{j_i}\}, \\
&\vdots \\
[x, y_t]_G &= \{e_{j_{t-1}+1}, \cdots, e_{j_t}\}.
\end{aligned}
$$

Then $j_i - j_{i-1} \leq r$ by Claim 1. This implies the conclusion of Claim 4. □

Let $\psi(e_i)$ be the end vertex of e_i other than x, that is, $\varphi_G(e_i) = \{x, \psi(e_i)\}$, and

$$G_i := (V(G), E(G), \varphi_i),$$

where

$$
\varphi_i(e) := \begin{cases} \varphi_G(e) & \text{if } i = 0 \\ \varphi_{i-1}(e) & \text{if } i > 0 \text{ and } e \neq e_i \\ \{\psi(e_i), \psi(e_{r+i})\} & \text{if } e = e_i. \end{cases}
$$

That is, $G_0 = G$ and G_i is obtained from G_{i-1} by removing the edge e_i and adding an edge joining $\psi(e_i)$ and $\psi(e_{r+i})$.
Case 1. Every nonempty subset of $V(G_s)$ is r-sparse.

In this case, let $S := V(G) - \{x\}$.
Case 2. Every nonempty subset of $V(G_i)$ is r-sparse, but a nonempty subset S of $V(G_{i+1})$ is not r-sparse for some $i \leq s - 1$.

In this case, S is r-critical in G_i, because S is not r-sparse in G_{i+1} and

$$|E_{G_{i+1}}(S)| \leq |E_{G_i}(S)| + 1.$$

If x is contained in S, then $|E_{G_{i+1}}(S)| = |E_G(S)|$. This contradicts the assumption. Hence x is not contained in S. By renumbering the edges incident to x, if necessary, we may assume that $\varphi_i(e_j)$ is contained in S for $1 \leq j \leq i$.

In case 1, let $i := s$. Then in either case, every nonempty subset of G_i/S is r-sparce. Also, every nonempty subset of $\langle S \rangle_{G_i}$ is r-sparse. Hence both $E(\langle S \rangle_{G_i})$ and $E(G_i/S)$ can be decomposed into r forests. Let

$$E(\langle S \rangle_{G_i}) = F_1 \cup \cdots \cup F_r$$

be a forest decomposition of $\langle S\rangle_{G_i}$. We may assume that

$$F_p \cap \{e_1, \cdots, e_i\} \neq \emptyset \text{ for } 1 \leq p \leq u$$

and

$$F_p \cap \{e_1, \cdots, e_i\} = \emptyset \text{ for } u < p \leq r.$$

By renumbering the edges, we may assume that $e_p \in F_p$ for $1 \leq p \leq u$. Let

$$E(G_i/S) = F'_1 \cup \cdots \cup F'_r$$

be a forest decomposition of G_i/S. Note that e_{r+j} $(1 \leq j \leq s)$ are parallel edges joining x and v_S in G_i/S. Hence

$$|F'_j \cap \{e_{r+1}, \cdots, e_{r+s}\}| \leq 1$$

for $1 \leq j \leq r$. So, we may assume that $e_{r+j} \in F'_j$ $(1 \leq j \leq s)$. Let $F''_j := F_j \cup F'_j$ for $1 \leq j \leq r$. Then F''_j is a forest in G_i. Furthermore, F''_j $(u < j \leq r)$ is a forest in G, because it contains no e_p $(1 \leq p \leq i)$. Suppose F''_j $(1 \leq j \leq u)$ contains a cycle C in G_{q-1} but no cycles in G_q. Then C passes through e_q. In $\langle F''_j - \{e_q\}\rangle_{G_{q-1}}$, x and $\psi(e_q)$ are in the same component, and $\psi(e_q)$ and $\psi(e_{r+q})$ are in different components. Note that this in particular implies $e_{q+r} \notin F'_j$, and hence $q > u$. Thus $(F''_j - \{e_q\}) \cup \{e_{q+r}\}$ and $(F''_q - \{e_{q+r}\}) \cup \{e_q\}$ are forests in G_{q-1}. Continuing this way, we may assume that F''_j is a forest in G, by interchanging the roles of e_p and e_{p+r}, if necessary, for some p with $u < p \leq i$.

This completes the proof of Theorem 2. □

We can prove Theorem 1 using Theorem 2. So, this gives a simple self-contained proof of Theorem 1.

Proof of Theorem 1. It is easily seen that if G contains r edge-disjoint spanning trees, then $|E_G(S_1, \cdots, S_m)| \geq (m-1) \cdot r$ holds. So, suppose $|E_G(S_1, \cdots, S_m)| \geq (m-1) \cdot r$ for any partition $V(G) = \bigcup_{i=1}^{m} S_i$. We may assume that G is edge-minimal. That is,

$$|E_G(S_1, \cdots, S_m)| = (m-1) \cdot r$$

for some $S_1, \cdots, S_m$ with $m \geq 2$. If $|S_i| \geq 2$ for some i, we can apply induction to G/S_i and $\langle S_i\rangle_G$. Hence we may assume that $|S_i| = 1$ for $1 \leq i \leq m$. This means that

$$|E(G)| = r \cdot (|V(G)| - 1),$$

that is, $V(G)$ is r-critical. Let S be any nonempty subset of $V(G)$, and suppose $V(G) - S = \{x_1, \cdots, x_k\}$. Let

$$T_i := \begin{cases} \{x_i\} & \text{for } 1 \leq i \leq k \\ S & \text{for } i = k+1. \end{cases}$$

Then

$$|E_G(S_1, \cdots, S_{k+1})| = |E(G) - E_G(S)| \geq kr.$$

Hence

$$\begin{aligned} |E_G(S)| &\leq |E(G)| - kr \\ &= r \cdot (|V(G)| - 1) - r \cdot (|V(G)| - |S|) \\ &= r \cdot (|S| - 1). \end{aligned}$$

This proves that every nonempty subset of $V(G)$ is r-sparse. By Theorem 2, we can decompose $E(G)$ into r forests. Since $V(G)$ is r-critical, each forest contains $|V(G)| - 1$ edges. This means that it is a tree. This completes the proof of Theorem 1. □

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