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EHRESMANN CONNECTIONS FOR A FOLIATION ON A MANIFOLD WITH BOUNDARY

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Abstract. R. A. Blumenthal and J.J. Hebda defined the notion of an Ehresmann connection for a foliation on a manifold without boundary. In this paper, we define an analogous notion for a foliation on a manifold with boundary and prove some results related to it.

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§1. Introduction.

R.A. Blumenthal and J.J. Hebda defined the notion of an Ehresmann connection for a foliation on a manifold without boundary as a complementary distribution to the foliation satisfying a certain condition, and proved some results related to it (cf. [1]~[4]). In this paper, we define an analogous notion for a foliation on a manifold with boundary and also call it an Ehresmann connection (see §2). In §3, we investigate some properties of a foliation admitting an Ehresmann connection. For example, we obtain relations between the foliation and boundary components of the base manifold. In §4, we obtain sufficient conditions for the orthogonal complementary distribution of a foliation on a Riemannian manifold with boundary to be an Ehresmann connection.

Throughout this paper, unless otherwise mentioned, we assume that all objects are smooth (C^∞) and all manifolds are connected ones with boundary. Also, we identify a foliation with the tangent bundle of the foliation and conversely an integrable distribution with the foliation whose leaves are the maximal integral manifolds of the integrable distribution.

§2. Rectangles.

Let M be an n-dimensional manifold with boundary $\partial M = \bigcup_{\lambda\in\Lambda} B_\lambda$ (disjoint union), where B_λ ($\lambda \in \Lambda$) are components of ∂M, and F an m-dimensional foliation on M ($0 < m < n$). Denote by L_x^F the leaf of F through x. We call a boundary component B_λ where F is tangential (resp. transversal) the *F-tangential boundary component* (resp. *F-transversal boundary component*).

Assume that each boundary component of M is F-tangential or F-transversal. Namely, we do not treat a foliation as in Fig. 2.1.

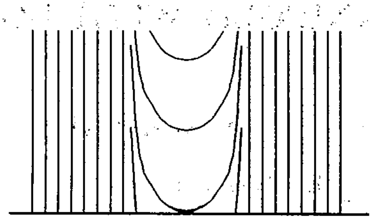

Fig. 2.1.

Let $(U, \phi = (u_1, \cdots, u_n))$ be a foliated coordinate neighbourhood, that is, a coordinate neighbourhood such that each leaf of $F|_U$ is given by $\{\phi^{-1}(u_1, \cdots, u_n) \mid u_{m+1} = c_{m+1}, \cdots, u_n = c_n\}$, where $c_{m+1}, \cdots, c_n$ are constants decided by the leaf. Then we call $f_\phi = (u_{m+1}, \cdots, u_n) : U \to R^{n-m}$ a *local submersion corresponding to* (U, ϕ). A foliated coordinate neighbourhood (U, ϕ) is said to be *distinguished* if $\phi(U) = (-1,1)^n$,$(-1,1)^{n-1} \times [0,1)$ or $[0,1) \times (-1,1)^{n-1}$ and there exists a foliated coordinate neighbourhood $(\tilde{U}, \tilde{\phi})$ with $\bar{U} \subset \tilde{U}$ and $\tilde{\phi}|_U = \phi$, where $\bar{U}$ is the closure of U in M. Let D be a complementary distribution to F satisfying the following condition $(*)$:*D is tangent to each F-transversal boundary component.* Such a distribution D can be constructed as follows. Let $\{B_{\lambda'}\}_{\lambda' \in \Lambda'}$ be the set of all F-transversal boundary components, $V_{\lambda'}$ a collar neighbourhood of $B_{\lambda'}$ $(\lambda' \in \Lambda')$ and W an open set of M with $M = (\bigcup_{\lambda' \in \Lambda'} V_{\lambda'}) \cup W$ and $W \cap B_{\lambda'} = \emptyset$ $(\lambda' \in \Lambda')$. Take Riemannian metrics $g_{\lambda'}$ on $V_{\lambda'}$ such that F is orthogonal to $B_{\lambda'}$ with respect to $g_{\lambda'}$ $(\lambda' \in \Lambda')$ and a Riemannian metric g_W on W. Let $\{\phi_{\lambda'} \mid \lambda' \in \Lambda'\} \cup \{\phi_W\}$ be a partition of unity subordinate to the open covering $\{V_{\lambda'} \mid \lambda' \in \Lambda'\} \cup \{W\}$ of M. Define a Riemannian metric g on M by $g = \sum_{\lambda' \in \Lambda'} \phi_{\lambda'} g_{\lambda'} + \phi_W g_W$. Let $F^\perp$ be the orthogonal complementary distribution of F with respect to g. This distribution $F^\perp$ is a desirable distribution. We call a piecewise smooth curve in M whose velocity vector field lies in F (resp. D) an *F-curve* (resp. a *D-curve*), where a piecewise smooth curve is a piecewise smooth map from $[0,1]$ to M. Let $\delta : [0,1] \times [0,1] \to M$ be a piecewise smooth map such that, for every fixed s_0, the curve $\delta_{\cdot s_0} := \delta(\cdot, s_0)$ is a D-curve, and, for every fixed t_0, the curve $\delta_{t_0 \cdot} := \delta(t_0, \cdot)$ is an F-curve. Such a piecewise smooth map δ is called a *rectangle*. Also, the curves $\delta_{0\cdot}$, $\delta_{1\cdot}$, $\delta_{\cdot 0}$ and $\delta_{\cdot 1}$ are called the *initial F-edge*, the *terminal F-edge*, the *initial D-edge* and the *terminal D-edge* of δ, respectively. It is easy to show that two rectangles with the same initial edges coincide. So the following notion can be defined. We call the rectangle whose initial edges are an F-curve α and a D-curve β the *rectangle associated to α and β*, and denote it by $\delta_{\alpha\beta}$. From the above condition $(*)$, we can prove the following proposition for the existence of a rectangle by imitating the method of the proof of Proposition 2.1 and 2.2 in [7].

Proposition 2.1. *Let α and β be an F-curve and a D-curve with $\alpha(0) = \beta(0)$. Then*

(i) for a sufficiently small positive number ε, $\delta_{\alpha|_{[0,\varepsilon]}\beta}$ exists,

(ii) for a sufficiently small positive number ε, $\delta_{\alpha\beta|_{[0,\varepsilon]}}$ exists.

A pair (α, β) in Fig. 2.2 shows that this proposition does not hold without the condition $(*)$.

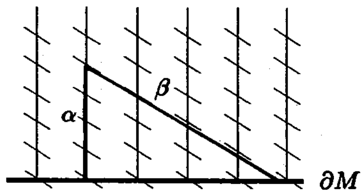

Fig. 2.2.

Let α and β be an F-curve and a D-curve with $\alpha(0) = \beta(0)$. We call a piecewise smooth map $\delta : [0,1] \times [0,1) \to M$ such that $\delta|_{[0,1]\times[0,s_1]}$ is the rectangle associated to $\alpha|_{[0,s_1]}$ and β for every $s_1 \in [0,1)$ the *F-open rectangle associated to α and β*, and a piecewise smooth map $\delta : [0,1) \times [0,1] \to M$ such that $\delta|_{[0,t_1]\times[0,1]}$ is the rectangle associated to α and $\beta|_{[0,t_1]}$ for every $t_1 \in [0,1)$ the *D-open rectangle associated to α and β*. Denote by $\overset{\circ}{\delta}{}^F_{\alpha\beta}$ (resp. $\overset{\circ}{\delta}{}^D_{\alpha\beta}$) the F-open (resp. D-open) rectangle associated to α and β. By imitating the method of the proof of Lemma 3.5 in [7], we can prove the following lemma.

Lemma 2.2. *If $\lim_{s\to 1-0}(\overset{\circ}{\delta}{}^F_{\alpha\beta})_{t\cdot}(s)$ exists for every $t \in [0,1]$, then $\overset{\circ}{\delta}{}^F_{\alpha\beta}$ extends to the rectangle associated to α and β.*

Proof. Define a map $\tilde{\delta} : [0,1] \times [0,1] \to M$ by $\tilde{\delta}|_{[0,1]\times[0,1)} = \delta$ and $\tilde{\delta}(t,1) := \lim_{s\to 1-0} \delta(t,s)$ for $t \in [0,1]$. Fix $t_0 \in [0,1]$. Take a distinguished foliated coordinate neighbourhood (U,ϕ) about $\tilde{\delta}(t_0,1)$. Let ε_1 and ε_2 be sufficiently small positive numbers such that $\tilde{\delta}_{t_0\cdot}([1-\varepsilon_1,1]) \subset U$ and $\tilde{\delta}_{\cdot 1-\varepsilon_1}((t_0-\varepsilon_2, t_0+\varepsilon_2)) \subset U$. Set $\bar{N} := f_\phi(\tilde{\delta}_{\cdot 1-\varepsilon_1}((t_0-\varepsilon_2, t_0+\varepsilon_2)))$, $N := f_\phi^{-1}(\bar{N})$ and $\gamma := f_\phi \circ \tilde{\delta}_{\cdot 1-\varepsilon_1}|_{[t_0, t_0+\varepsilon_2/2]}$. Let D_N be a 1-dimensional foliation on N defined by $(D_N)_y := D_y \cap T_yN$ for $y \in N$. Take a positive number ε_2' such that $L^{D_N}_{\tilde{\delta}(t_0,s)} \cap f_\phi^{-1}(\gamma(t)) \neq \emptyset$ for every $s \in [1-\varepsilon_1, 1]$ and every $t \in [0, \varepsilon_2']$. Note that the existence of ε_2' is assured by the condition $(*)$ in the case where $\tilde{\delta}(t_0,1)$ is a point of an F-transversal boundary component. Define a continuous map $\hat{\delta} : [0,1] \times [0,1] \to M$ by $\hat{\delta}(t,s) := L^{D_N}_{\tilde{\delta}(t_0,(1-\varepsilon_1)+\varepsilon_1 s)} \cap f_\phi^{-1}(\gamma(\varepsilon_2' t))$ for

$(t,s) \in [0,1] \times [0,1]$. It is easy to show that $\hat{\delta}_{\cdot s} = \tilde{\delta}_{\cdot (1-\varepsilon_1)+\varepsilon_1 s}|_{[t_0, t_0 + \frac{\varepsilon_2 \varepsilon_2'}{2}]}$ for every $s \in [0,1)$. Hence, for every $t \in [0,1]$, we have

$$\begin{aligned} \hat{\delta}_{\cdot 1}(t) = \hat{\delta}(t,1) &= \lim_{s \to 1-0} \hat{\delta}(t,s) \\ &= \lim_{s \to 1-0} \tilde{\delta}(t_0 + \frac{\varepsilon_2 \varepsilon_2' t}{2}, (1-\varepsilon_1) + \varepsilon_1 s) \\ &= \tilde{\delta}_{\cdot 1}(t_0 + \frac{\varepsilon_2 \varepsilon_2' t}{2}). \end{aligned}$$

Thus $\tilde{\delta}_{\cdot 1}|_{[t_0, t_0 + \frac{\varepsilon_2 \varepsilon_2'}{2}]} = \hat{\delta}_{\cdot 1}$ holds, that is, $\tilde{\delta}_{\cdot 1}|_{[t_0, t_0 + \frac{\varepsilon_2 \varepsilon_2'}{2}]}$ is a D-curve. Similarly, it is shown that $\tilde{\delta}_{\cdot 1}|_{[t_0 - \varepsilon_2'', t_0]}$ is a D-curve for a sufficiently small positive number ε_2''. Therefore, by the arbitrarity of t_0, $\tilde{\delta}_{\cdot 1}$ is a D-curve. By Proposition 2.1, the rectangle $\bar{\delta}$ associated to $\alpha^{-1}|_{[0,\varepsilon]}$ and $\tilde{\delta}_{\cdot 1}$ exists for a sufficiently small positive number ε. It is clear that $\bar{\delta}(t,s) = \tilde{\delta}(t, 1-\varepsilon s)$ holds for every $(t,s) \in [0,1] \times [0,1]$. This fact implies that $\tilde{\delta}$ is a piecewise smooth map, that is, $\tilde{\delta}$ is the rectangle associated to α and β. □

The distribution D is called an *Ehresmann connection* for F if for every F-curve α and every D-curve β with $\alpha(0) = \beta(0)$, the rectangle associated to α and β exists. For an Ehresmann connection, the following facts are shown as in case of $\partial M = \emptyset$.

(A) If F admits an Ehresmann connection D, then arbitrary two leaves of F are joined by a D-curve.

(B) If F admits an Ehresmann connection, then the universal coverings of leaves of F are diffeomorphic to one another.

(C) If D is an integrable Ehresmann connection for F, then there exists a covering map $\pi : \hat{L}_x^F \times \hat{L}_x^D \to M$ such that $\pi_*(T\hat{L}_x^F) = F$ and $\pi_*(T\hat{L}_x^D) = D$, where $\hat{L}_x^F$ (resp. $\hat{L}_x^D$) is the universal covering of L_x^F (resp. L_x^D), π_* is the differential of π and $T\hat{L}_x^F$ (resp. $T\hat{L}_x^D$) is a foliation on $\hat{L}_x^F \times \hat{L}_x^D$ whose leaves are fibres of the natural projection of $\hat{L}_x^F \times \hat{L}_x^D$ onto $\hat{L}_x^D$ (resp. $\hat{L}_x^F$).

Let (α, β) be a pair of an F-curve α and a D-curve β such that $\mathring{\delta}_{\alpha\beta}^F$ and $\mathring{\delta}_{\alpha\beta}^D$ exist but $\delta_{\alpha\beta}$ does not exist. We shall call such a pair an *strange pair*. Then, by Lemma 2.2, we see that $\lim_{s \to 1-0} (\mathring{\delta}_{\alpha\beta}^F)_{1\cdot}(s)$ does not exist. Hence the following two type of strange pairs are considered:

(I) both $\lim_{s \to 1-0} (\mathring{\delta}_{\alpha\beta}^F)_{1\cdot}(s)$ and $\lim_{t \to 1-0} (\mathring{\delta}_{\alpha\beta}^D)_{\cdot 1}(t)$ do not exist,

(II) $\lim_{s \to 1-0} (\mathring{\delta}_{\alpha\beta}^F)_{1\cdot}(s)$ does not exist but $\lim_{t \to 1-0} (\mathring{\delta}_{\alpha\beta}^D)_{\cdot 1}(t)$ exists.

It is clear that there does not exist a strange pair if D is an Ehresmann connection. Conversely, we can prove the following proposition.

Proposition 2.3. *If D is not an Ehresmann connection, then there exists a strange pair.*

Proof. Assume that D is not an Ehresmann connection. Then there exists a pair (α, β) of an F-curve α and a D-curve β such that $\alpha(0) = \beta(0)$ and $\delta_{\alpha\beta}$ does not exist. Set $s_0 := \sup\{s \in [0,1] \mid {}^\exists\delta_{\alpha|_{[0,s]}\beta}\}$ and $t_0 := \sup\{t \in [0,1] \mid {}^\exists\delta_{\alpha|_{[0,s_0]}\beta|_{[0,t]}}\}$. It is clear that $\overset{\circ}{\delta}{}^F_{\alpha|_{[0,s_0]}\beta|_{[0,t_0]}}$ and $\overset{\circ}{\delta}{}^D_{\alpha|_{[0,s_0]}\beta|_{[0,t_0]}}$ exist. On the other hand, from Proposition 2.1, we see that $\delta_{\alpha|_{[0,s_0]}\beta|_{[0,t_0]}}$ does not exist. Therefore, $(\alpha|_{[0,s_0]}, \beta|_{[0,t_0]})$ is a strange pair. □

§3. Ehresmann foliations.

Let F be a foliation on a manifold M such that each boundary component of M is F-tangential or F-transversal, and D a complementary distribution to F satisfying the condition $(*)$ in the previous section. In the sequel, we shall call a foliation admitting an Ehresmann connection an *Ehresmann foliation.* From the fact (B) in the previous section, we can prove the following result.

Proposition 3.1. *An Ehresmann foliation is tangential to all boundary components or transversal to all ones.*

Proof. Let F be an Ehresmann foliation. Suppose that there exist both an F-tangential boundary component B and an F-transversal boundary component B'. Let $x \in B$ and $x' \in B'$. Since $\partial L^F_x = \emptyset$ (resp. $\partial L^F_{x'} \neq \emptyset$), we have $\partial \hat{L}^F_x = \emptyset$ (resp. $\partial \hat{L}^F_{x'} \neq \emptyset$). Hence $\hat{L}^F_x \not\cong \hat{L}^F_{x'}$. This contradicts (B). This completes the proof. □

For an Ehresmann foliation transversal to boundary, the following proposition holds.

Proposition 3.2. *Let F be an Ehresmann foliation transversal to boundary. Then each leaf of F meets all boundary components.*

Proof. Let L be an arbitrary leaf of F and B an arbitrary boundary component. We have only to show $L \cap B \neq \emptyset$. Let D be an Ehresmann connection for F. Take $y \in B$. According to the fact (A) in the previous section, there exists a D-curve β with $\beta(0) \in L^F_y$ and $\beta(1) \in L$. Let α be an F-curve with $\alpha(0) = \beta(0)$ and $\alpha(1) = y$. Then $\delta_{\alpha\beta}(1,1) \in L \cap B$. Thus $L \cap B \neq \emptyset$. □

Especially, we can obtain the following proposition.

Proposition 3.3. *If there exists a 1-dimensional Ehresmann foliation on a manifold M transversal to boundary, then the number of boundary components of M is at most two.*

For a manifold with a 1-dimensional Ehresmann foliation transversal to boundary, we can prove the following two theorems.

Theorem 3.4. *Let M be a manifold with two boundary components B and B'. Assume that there exists a 1-dimensional Ehresmann foliation F on M transversal to boundary. Then*
(i) B and B' are diffeomorphic to each other,
(ii) M is homeomorphic to $B \times [0,1]$.

Proof. According to Proposition 3.2, each leaf of F intersects with B and B' at a point. Hence we have $M = \bigcup_{x \in B} L_x^F$ and $L_x^F \cong [0,1]$ $(x \in B)$. Define a map $\eta : B \to B'$ by $\eta(x) = L_x^F \cap B'$ for $x \in B$. It is easy to show that η is a diffeomorphism. Thus the statement (i) is shown. Take a Riemannian metric g on M. Let α_x $(x \in B)$ be a regular F-curve with $\alpha_x(0) = x, \alpha_x(1) = \eta(x)$ and $l(\alpha_x|_{[0,t]}) = tl(\alpha_x)$ for every $t \in [0,1]$, where $l(\cdot)$ is the length of a curve with respect to g. Define a map $\psi : B \times [0,1] \to M$ by $\psi(x,t) := \alpha_x(t)$ for $(x,t) \in B \times [0,1]$. It is clear that ψ is a homeomorphism. Thus the statement (ii) is shown. □

Theorem 3.5. *Let M be a manifold with one boundary component B. If there exists a 1-dimensional Ehresmann foliation F on M transversal to boundary, then M is homeomorphic to $B \times [0,\infty)$ or covered doubly by $B \times [0,1]$.*

Proof. According to the fact (B) in the previous section and Proposition 3.2, the following two cases are considered:
(i) each leaf of F intersects with B at one point and hence is diffeomorphic to $[0,\infty)$,
(ii) each leaf of F intersects with B at two points and hence is diffeomorphic to $[0,1]$.
First we consider the case (i). Let V be a collar neighbourhood of B such that $L_x^F \not\subset V$ and $L_x^F \cap V$ is connected for every $x \in B$, and W an open set of M such that $M = V \cup W$ and $W \cap B = \emptyset$. Since W is a manifold without boundary, there exists a complete Riemannian metric g_W on W. Let g_V be a Riemannian metric on V and $\{\phi_V, \phi_W\}$ a partition of unity subordinate to an open covering $\{V, W\}$ of M. Define a Riemannian metric g on M by $g := \phi_V g_V + \phi_W g_W$. Since $g = g_W$ on $M \setminus V$, g_W is complete, $L_x^F \not\subset V$ and $L_x^F \cap V$ is connected $(x \in B)$, we see that the length of L_x^F $(x \in B)$ with respect to g is infinite. Let $\alpha_x : [0,\infty) \to L_x^F$ $(x \in B)$ be a smooth map with $\alpha_x(0) = x$ and $l(\alpha_x|_{[0,t]}) = t$ for $t \in [0,\infty)$, where $l(\cdot)$ is the length of a curve

$\cdot$ with respect to g. Define a map $\psi : B \times [0, \infty) \to M$ by $\psi(x,t) := \alpha_x(t)$ for every $(x,t) \in B \times [0,\infty)$. It is clear that ψ is a homeomorphism. Thus M is homeomorphic to $B \times [0,\infty)$. Next we consider the case (ii). Take a Riemannian metric g' on M. Let $\alpha'_x : [0,1] \to M$ $(x \in B)$ be a regular F-curve with $\alpha'(0) = x, \alpha'_x(1) \in B$ and $l'(\alpha'_x|_{[0,t]}) = tl'(\alpha'_x)$ for every $t \in [0,1]$. Define a map $\psi' : B \times [0,1] \to M$ by $\psi'(x,t) := \alpha'_x(t)$ for $(x,t) \in B \times [0,1]$, where $l'(\cdot)$ is the length of a curve $\cdot$ with respect to g'. It is clear that ψ' is a (topological) double covering of $B \times [0,1]$ onto M (cf. Fig. 3.1). $\square$

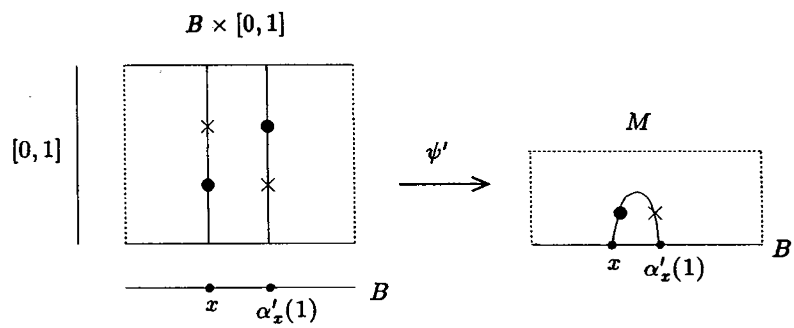

Fig. 3.1.

Here we shall give an example of a 1-dimensional Ehresmann foliation as the case (ii) in the proof of Theorem 3.5.

Example. Let F be a foliation on an annulus $S^1 \times [0,1]$ whose leaves are fibres of the natural projection of $S^1 \times [0,1]$ onto S^1 and ϕ a diffeomorphism of $S^1 \times [0,1]$ defined by $\phi(z,t) := (-z, 1-t)$, where we regard S^1 as a unit circle in a complex plane. It is clear that ϕ is involutive and preserves F. Denote by $< \phi >$ the cyclic group generated by ϕ. Then we see that a foliation on the orbit space $M := S^1 \times [0,1]/ < \phi >$ induced from F is a 1-dimensional Ehresmann foliation as the case (ii) in the proof of Theorem 3.5. Here note that M is homeomorphic to a Möbius band.

Denote by ∂F a foliation on ∂M induced from F and ∂D a distribution on ∂M induced from D. The following lemma is directly deduced from the definition of an Ehresmann connection.

Lemma 3.6. *Assume that D is an Ehresmann connection for F. Then ∂D is an Ehresmann connection for ∂F on each boundary component.*

For the existences of a codimension-1 Ehresmann foliation transversal to boundary and a codimension-2 Ehresmann foliation tangential to boundary, we can prove the following theorem.

Theorem 3.7. *(i) If M is compact and the fundamental group $\pi_1(M)$ is finite, then there does not exist a codimension-1 Ehresmann foliation on M transversal to boundary.*

(ii) If M posseses a compact boundary component B with finite fundamental group, then there do not exist a codimension-1 Ehresmann foliation on M transversal to boundary and a codimension-2 Ehresmann foliation on M tangential to boundary.

Proof. First we shall prove the statement (i). Suppose that there exists a codimension-1 Ehresmann foliation F on M transversal to boundary. Let D be an Ehresmann connection for F. According to the fact (C) in the previous section, the universal covering of M is diffeomorphic to $\hat{L}_x^F \times \hat{L}_x^D$ $(x \in M)$. Since L_x^D is a 1-dimensional manifold without boundary, we have $\hat{L}_x^D \cong R$. Hence the universal covering of M is non-compact. This contradicts the fact that M is compact and $\pi_1(M)$ is finite. Therefore, there does not exist a codimension-1 Ehresmann foliation on M transversal to boundary.

Next we shall prove the statement (ii). Suppose that there exists a codimension -1 Ehresmann foliation F on M transversal to boundary. Let D be an Ehresmann connection for F. According to Lemma 3.6, the restriction $\partial D|_B$ of ∂D to B is an Ehresmann connection for $\partial F|_B$. Hence, the universal covering of B is diffeomorphic to $\hat{L}_x^{\partial F|_B} \times \hat{L}_x^{\partial D|_B}$ $(\approx \hat{L}_x^{\partial F|_B} \times R)$ $(x \in B)$, that is, non-compact. This contradicts the fact that B is compact and $\pi_1(B)$ is finite. Therefore, there does not exist a codimension-1 Ehresmann foliation on M transversal to boundary. Similarly, it is shown that there does not exist a codimension-2 Ehresmann foliation on M tangential to boundary. □

§4. Sufficient conditions to be an Ehresmann connection.

Let F be a foliation on a Riemannian manifold (M, g) tangential (or transversal) to boundary. Assume that F is orthogonal to boundary in the case where F is transversal to boundary. Hence the orthogonal complementary distribution $F^\perp$ of F satisfies the condition $(*)$ in §2. Let α (resp. β) be an F-curve (resp. an $F^\perp$-curve). Denote by $Rec(\alpha, \cdot)$ the set of all rectangles whose initial F-edge is α and initial $F^\perp$-edge is a regular curve, and $Rec(\cdot, \beta)$ the set of all rectangles whose initial $F^\perp$-edge is β and initial F-edge is a regular curve. Define a function $G_\alpha^\perp$ on $Rec(\alpha, \cdot)$ by $G_\alpha^\perp(\delta) := \dfrac{l(\delta_{\cdot 1})}{l(\delta_{\cdot 0})}$ for $\delta \in Rec(\alpha, \cdot)$, a function G_β^T on $Rec(\cdot, \beta)$ by $G_\beta^T(\delta) := \dfrac{l(\delta_{1\cdot})}{l(\delta_{0\cdot})}$ for $\delta \in Rec(\cdot, \beta)$, where $l(\cdot)$ is the length of a curve $\cdot$ with respect to g. Note that $G_\beta^T = 1$ for every $F^\perp$-curve β in the case where F is totally geodesic, and $G_\alpha^\perp = 1$ for every F-curve α in the case where g is bundle-like for F. Now we shall obtain sufficient conditions for $F^\perp$ to be an Ehresmann connection.

Theorem 4.1. *Let F be a foliation on a Riemannian manifold (M, g) tangential or orthogonal to boundary. Assume that the following conditions (i) and (ii) hold:*

(i) the induced Riemannian metrics on leaves of F are complete,

(ii) $\sup G^T_\beta < \infty$ for every $F^\perp$-curve β.

Then $F^\perp$ is an Ehresmann connection.

Proof. Suppose that $F^\perp$ is not an Ehresmann connection. Then, according to Proposition 2.3, there exists a strange pair (α, β). Since $\lim_{s\to 1-0}(\mathring{\delta}^F_{\alpha\beta})_{1\cdot}(s)$ does not exist, $\lim_{s\to 1-0} l((\mathring{\delta}^F_{\alpha\beta})_{1\cdot}|_{[0,s]}) = \infty$ is deduced from the completeness of the induced Riemannian metric on $L^F_{\beta(1)}$. Hence we obtain $\lim_{s\to 1-0} G^T_\beta(\delta_{\alpha|_{[0,s]}\beta}) = \infty$. This contradicts the condition (ii). Therefore, $F^\perp$ is an Ehresmann connection. $\square$

We prepare the following lemma to obtain another sufficient condition.

Lemma 4.2. *Let (α, β) be a strange pair of type (II). There exist sequences $\{t_n\}$ and $\{s_n\}$ in $[0, 1)$ such that $\lim_{n\to\infty} t_n = 1$, $\lim_{n\to\infty} s_n = 1$ and*

$$\lim_{n\to\infty} G^\perp_{(\delta_{\alpha|_{[0,s_n]}\beta|_{[0,t_n]}})_{1\cdot}}(\mathring{\delta}^F_{\alpha\beta}|_{[t_n,1]\times[0,s_n]}) = \infty.$$

Proof. For simplicity, denote $\mathring{\delta}^F_{\alpha\beta}$ (resp. $\mathring{\delta}^{F^\perp}_{\alpha\beta}$) by $\mathring{\delta}^F$ (resp. $\mathring{\delta}^{F^\perp}$). Set $x_0 := \lim_{t\to 1-0} \mathring{\delta}^{F^\perp}_{\cdot 1}(t)$. Denote by d the distance function given by g. Since $\lim_{s\to 1-0} \mathring{\delta}^F_{1\cdot}(s)$ does not exist, there exist $\varepsilon > 0$ and a sequence $\{s_n\}$ in [0,1) such that $\lim_{n\to\infty} s_n = 1$ and $d(x_0, \mathring{\delta}^F_{1\cdot}(s_n)) > \varepsilon$. Take a sequence $\{t_n\}$ in [0,1) such that $d(x_0, \mathring{\delta}^{F^\perp}(t_n, 1)) < \frac{1}{n}$ and $l(\beta|_{[t_n,1]}) < \frac{1}{n}$. Furthermore, take a subsequence $\{s'_n\}$ of $\{s_n\}$ such that $d(\mathring{\delta}^{F^\perp}(t_n, s'_n), \mathring{\delta}^{F^\perp}(t_n, 1)) < \frac{1}{n}$. Then we have

$$\begin{aligned}
l(\mathring{\delta}^F_{\cdot s'_n}|_{[t_n,1]}) &\geq d(\mathring{\delta}^F(t_n, s'_n), \mathring{\delta}^F(1, s'_n)) \\
&\geq d(x_0, \mathring{\delta}^F(1, s'_n)) - d(x_0, \mathring{\delta}^F(t_n, s'_n)) \\
&> \varepsilon - d(x_0, \mathring{\delta}^{F^\perp}(t_n, 1)) - d(\mathring{\delta}^{F^\perp}(t_n, 1), \mathring{\delta}^{F^\perp}(t_n, s'_n)) \\
&> \varepsilon - \frac{2}{n} \quad \text{(cf. Fig. 4.1)}
\end{aligned}$$

and hence

$$\lim_{n\to\infty} G^{\perp}_{(\delta_{\alpha|_{[0,s'_n]}}\beta|_{[0,t_n]})_1}\cdot(\delta^{\mathring{F}}_{\alpha\beta}|_{[t_n,1]\times[0,s'_n]}) = \lim_{n\to\infty}\frac{l(\delta^{\mathring{F}}_{\cdot s'_n}|_{[t_n,1]})}{l(\beta|_{[t_n,1]})}$$
$$\geq \lim_{n\to\infty}\frac{\varepsilon-\frac{2}{n}}{\frac{1}{n}} = \lim_{n\to\infty}(n\varepsilon-2)=\infty.$$

Thus $\{t_n\}$ and $\{s'_n\}$ are desired sequences. $\square$

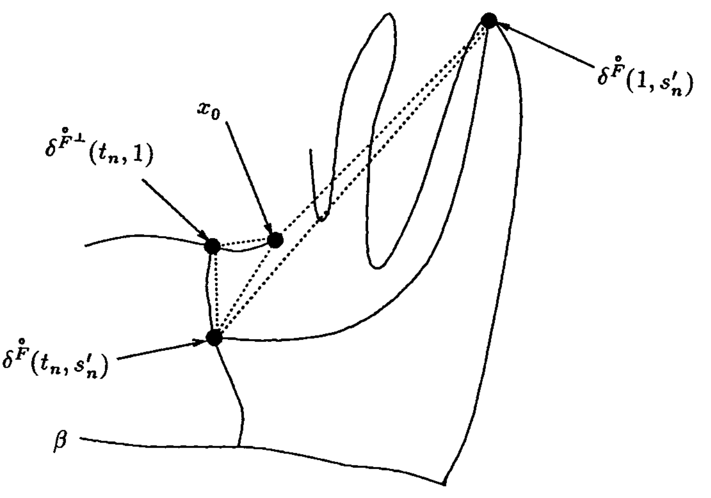

Fig. 4.1.

Let α be a piecewise smooth map from $[0,1)$ to M whose velocity vector field lies in F. We call such a curve α an F-curve without the terminal point.

Theorem 4.3. *Let F be a foliation on a Riemannian manifold (M,g) tangential or orthogonal to boundary. Assume that the following conditions (i) and (ii) hold:*

(i) (M,g) is complete,

(ii) $\sup_{s\in[0,1)} \sup G^{\perp}_{\alpha|_{[0,s]}} < \infty$ for every F-curve α without the terminal point.

Then $F^{\perp}$ is an Ehresmann connection.

Proof. Suppose that $F^{\perp}$ is not an Ehresmann connection. Then, according to Proposition 2.3, there exists a strange pair (α,β). Suppose that (α,β) is of type (I). Then $\lim_{t\to 1-0} l((\delta^{\mathring{F}^{\perp}}_{\alpha\beta})_{\cdot 1}|_{[0,t]}) = \infty$ is deduced from the completeness of

(M,g). Hence we obtain $\lim_{t\to 1-0} G^{\perp}_{\alpha}(\delta_{\alpha\beta}|_{[0,t]}) = \infty$. Take an F-curve $\tilde{\alpha}$ without the terminal point such that $\tilde{\alpha}|_{[0,s_0]} = \alpha$ for $s_0 \in [0,1)$. Then we obtain $\sup_{s\in[0,1)} \sup G^{\perp}_{\tilde{\alpha}|_{[0,s]}} \geq \sup G^{\perp}_{\alpha} = \infty$. This contradicts the condition (ii). Therefore, (α,β) is not of type (I). Next suppose that (α,β) is of type (II). Then, according to Lemma 4.2, there exist sequences $\{t_n\}$ and $\{s_n\}$ in $[0,1)$ such that $\lim_{n\to\infty} t_n = 1$, $\lim_{n\to\infty} s_n = 1$ and $\lim_{n\to\infty} G^{\perp}_{(\delta_{\alpha|_{[0,s_n]}\beta|_{[0,t_n]}})_{1\cdot}}(\overset{\circ}{\delta}{}^{F}_{\alpha\beta}|_{[t_n,1]\times[0,s_n]}) = \infty$. This fact deduces $\sup_{s\in[0,1)} \sup G^{\perp}_{(\overset{\circ}{\delta}{}^{F}_{\alpha\beta})_{1\cdot}|_{[0,s]}} = \infty$. This contradicts the condition (ii). Therefore, (α,β) is not of type (II). Namely, (α,β) is not a strange pair. Thus a contradiction results. Therefore, $F^{\perp}$ is an Ehresmann connection. □

Remark. For a foliation on a Riemannian manifold without boundary, results analogous to Theorem 4.1, 4.3 have been already shown in [7].

Now we illustrate that neither condition (i) nor (ii) in Theorem 4.1 and 4.3 is dispensable. Foliations on a half-disk as in Fig. 4.2 imply that the condition (i) is indispensable. Also, a foliation on a half-plane as in Fig. 4.3-(i) and a foliation on a complete Riemannian submanifold of an Euclidean plane as in Fig. 4.3-(ii) imply that the condition (ii) is indispensable.

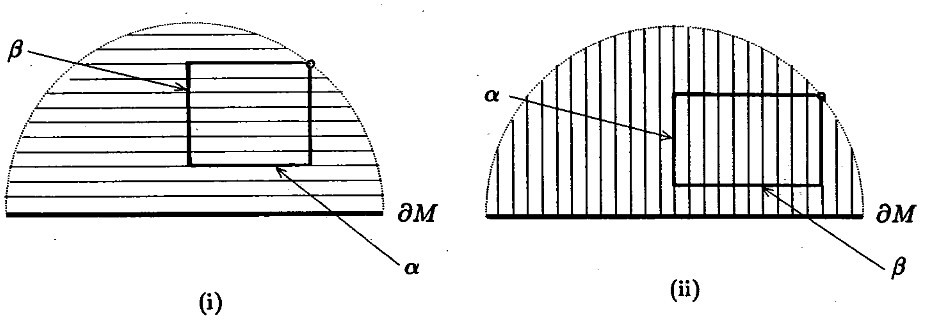

Fig. 4.2

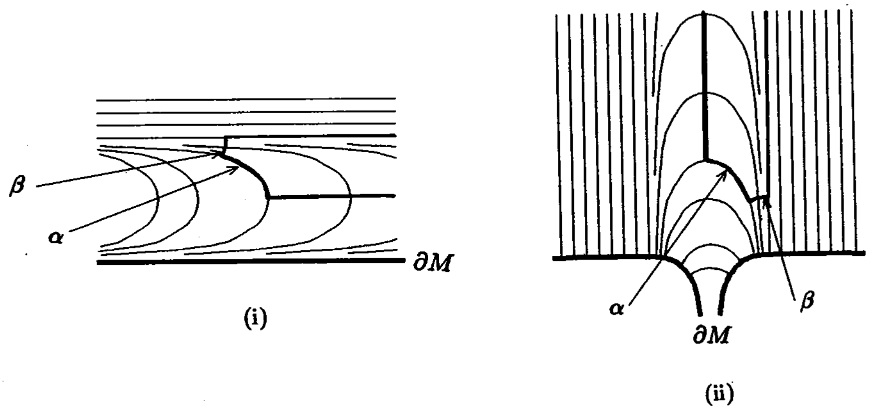

Fig. 4.3.

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