学位申請論文

Studies on the Hochschild cohomology ring of integral cyclic algebras and on the Galois extensions arising from algebraic tori

(整係数巡回多元環のホッホシルトコホモロジーと代数的トーラスによって引き起こされるガロア拡大の研究)

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Contents

Chapter 1

Introduction

In this thesis we make a study of Hochschild cohomology and Galois extensions, and we get two main results.

The first result is concerned with the Hochschild cohomology of integral cyclic algebras [14].

Hochschild cohomology was introduced by Hochschild [7]. In particular, it is known that the Hochschild cohomology ring $HH^*(\Gamma)$ is a graded commutative ring, that is if $\alpha \in HH^m(\Gamma)$ and $\beta \in HH^n(\Gamma)$, then we have $\alpha\beta = (-1)^{mn}(\beta\alpha)$. It is also known that the Hochschild cohomology ring of an algebra over a commutative ring is an invariant under an equivalence of bounded derived categories [10].

Let $\mathbb{Z}$ be the ring of rational integers, p a prime integer and ζ a primitive p-th root of unity. Let a and b be any nonzero rational integers. We let Γ be the integral cyclic R-algebra

$$\Gamma = \bigoplus_{0\le k,l\le p-1} Ri^k j^l \quad \text{such that} \quad i^p = a,\ j^p = b,\ ji = \zeta ij.$$

In particular, in the case $p = 2$, Γ is just the generalized quaternion algebra over the ring of rational integers $\mathbb{Z}$.

In this paper, we consider the Hochschild cohomology group $HH^m(\Gamma) = \mathrm{Ext}^m_{\Gamma^e}(\Gamma, \Gamma)$ and the Hochschild cohomology ring $HH^*(\Gamma) = \bigoplus_{m\ge 0} HH^m(\Gamma)$ of Γ, where Γ^e denotes the enveloping algebra $\Gamma \otimes_R \Gamma^{op}$ of Γ.

The case of $p = 2$, the $\mathbb{Z}$-module structure of $HH^m(\Gamma)$ and $HH^*(\Gamma)$ have already been studied in [4] and [12]. In this paper, we will generalize these results to the case of any prime number p.

First, we determine the R-module structure of $HH^m(\Gamma)$ as follows.

Theorem 1.1. *(See Theorem 2.2) We set $R = \mathbb{Z}[\zeta]$, and denote $1 - \zeta$ by ω. Let a and b be any nonzero rational integers and d the greatest common divisor of a and b. Then the Hochschild cohomology group $HH^m(\Gamma)$ is given by*

$$HH^m(\Gamma) \cong \begin{cases} R \quad \textit{for } m = 0, \\ (R/dpR)^{(m-1)/2} \oplus (R/d\omega R)^{(m+1)/2} \oplus (R/\omega R)^{(p^2-2)(m+1)/2} \\ \qquad \textit{for } m \textit{ odd}, \\ (R/dpR)^{(m-2)/2} \oplus (R/d\omega R)^{m/2} \oplus (R/\omega R)^{(p^2-2)m/2} \oplus (R/apR) \\ \qquad \oplus (R/bpR) \quad \textit{for } m (\neq 0) \textit{ even.} \end{cases}$$

We prove this theorem by giving a projective bimodule resolution of Γ, and applying the functor $\mathrm{Hom}_{\Gamma^e}(-, \Gamma)$ to the resolution.

By calculating the cup product of generators of the Hochschild cohomology group $HH^m(\Gamma)$, we determine the ring structure of $HH^*(\Gamma)$ as follows.

Theorem 1.2. *(See Theorem 2.3 and Theorem 2.5)*

(i) Let p be an odd prime and a, b nonzero integers. Then the Hochschild cohomology ring $HH^*(\Gamma)$ is generated by at most $p^2 + 4p - 3$ elements from $HH^1(\Gamma)$, $HH^2(\Gamma)$ and $HH^3(\Gamma)$. In particular, if $|a| = |b| = 1$, then the Hochschild cohomology ring $HH^*(\Gamma)$ is generated by $p^2 + 2$ elements from $HH^1(\Gamma)$ and $HH^2(\Gamma)$.

(ii) Let $p = 2$ and a, b any nonzero integers. Then the Hochschild cohomology ring $HH^*(\Gamma)$ is generated by at most eight elements. In particular, if $p = 2$ and $|a| = |b| = 1$, then we get

$$HH^*(\Gamma) \cong \mathbb{Z}[x, y, z]/(2x, 2y, 2z, x^2 + y^2 + z^2), \quad x, y, z \in HH^1(\Gamma).$$

This theorem generalizes the results in [4], [12] and [16].

The second result is concerned with the ramification in cyclic extensions of number fields [13].

By classical Kummer theory, if K contains n-th roots of unity, then all cyclic extensions of degree n over K have the form $K(\sqrt[n]{a})$ for some $a \in K^\times$. Hecke [1] described the ramification in the extension $K(\sqrt[n]{a})/K$ when n is prime.

On the other hand, the Kummer theory is generalized by Kida [8] using algebraic tori even when K does not contain roots of unity. This generalization of Kummer theory is described as follows: let T be an algebraic torus defined over K, if there exists a self-isogeny λ on T which satisfies $\mathrm{Ker}\ (\lambda) \subset T(K)$, then any cyclic extensions over K can be written as $K(\lambda^{-1}(P))$ with $P \in T(K)$, where $T(K)$ is the group of K-rational points on T.

We describe the ramification in the extension $K(\lambda^{-1}(P))/K$ in our main result. Since the problem is obviously local, we assume that K is a finite extension over the p-adic field. If the degree of a self-isogeny λ is p, then the extension $K(\lambda^{-1}(P))/K$ is a cyclic extension of degree p. The following result gives the conductor of $K(\lambda^{-1}(P))/K$, which measures the wildness of the ramification in this extension.

Theorem 1.3. *(See Theorem 3.15) Let p be a fixed odd prime. Let $\mathbb{Q}_p$ be the field of p-adic numbers and k an unramified extension of $\mathbb{Q}_p$ of degree n, and $k_z = k(\zeta_p)$ where ζ_p is a primitive p-th root of unity. Let K be an intermediate field of k_z/k and m the degree of the extension K/k, and T the Weil restriction $R_{k_z/K}\mathbb{G}_m$ of the multiplicative group $\mathbb{G}_m$ to K. We assume that there exists a self-isogeny λ on T which satisfies $\mathrm{Ker}\ (\lambda) \subset T(K)$. Let $\mathbb{G}_{m,k_z}$ be a multiplicative group defined over k_z and $\widehat{T} = \mathrm{Hom}(T, \mathbb{G}_{m,k_z})$ the group of characters of T. For each $i \geq 1$, we define*

$$U^{(i)}(k_z) = \{u \in k_z \mid v_{k_z}(u - 1) \geq i\}$$

and

$$T^{(i)}(K) = \mathrm{Hom}_{\mathrm{Gal}(k_z/K)}(\widehat{T}, U^{(i)}(k_z)).$$

If $P \in T^{(jd+1)}(K)$ *and* $P \notin T^{(jd+2)}(K)$ *for some* $0 \leq j \leq m$, *then the conductor* $\mathfrak{f}$ *of* $K(\lambda^{-1}(P))/K$ *satisfies*

$$v_K(\mathfrak{f}) = \begin{cases} m - j + 1 & 0 \leq j < m, \\ 0 & j = m. \end{cases}$$

In particular, $K(\lambda^{-1}(P))/K$ *is an unramified extension if and only if* $P \in T^{(p)}(K)$.

This result generalizes a classical theorem by Hecke describing the ramification of Kummer extensions $K(\sqrt[n]{\alpha})/K$. We prove the theorem by studying the structure of the group of units of local fields as Galois modules in detail. As an application of the theorem, we can calculate the number of cyclic extensions of degree p over K with given conductor $\mathfrak{f}$ up to isomorphism as follows.

Theorem 1.4. *(See Theorem 3.16) Let* p *be an odd prime. Then, for each* $0 \leq j < m$, *the number of cyclic extensions of* $K \subset \overline{\mathbb{Q}_p}$ *of degree* p *whose conductor* $\mathfrak{f}$ *satisfies* $v_K(\mathfrak{f}) = m - j + 1$ *is* $p^{(m-(j+1))n+1} \cdot (p^n - 1)/(p - 1)$ *up to isomorphism.*

The outline of this paper is as follows. In Chapter 2, we determine the ring structure of the Hochschild cohomology $HH^*(\Gamma)$ of an integral cyclic algebra Γ by giving a projective bimodule resolution of Γ and calculating cup product by means of a diagonal approximation map. In Chapter 3, in the context of the generalized Kummer theory arising from algebraic tori, we generalize the classical theorem by Hecke.

Chapter 2

On the Hochschild cohomology ring of integral cyclic algebras

2.1 Introduction

Let $\mathbb{Z}$ be the ring of rational integers, p a prime integer and ζ a primitive p-th root of unity. We set $R = \mathbb{Z}[\zeta]$, $\omega_n = 1 - \zeta^n$ for any $n \in \mathbb{Z}$ and we denote $\omega_1 = 1 - \zeta$ by ω. We note that $pR = \omega^{p-1}R$, so $R/\omega R \cong \mathbb{Z}/p\mathbb{Z}$, and that ω_k/ω_l is a unit in R for any k, l with $k, l \not\equiv 0 \mod p$.

Let a and b be any nonzero rational integers and d the greatest common divisor of a and b. We let Γ be the integral cyclic R-algebra

$$\Gamma = \bigoplus_{0 \leq k, l \leq p-1} Ri^k j^l \quad \text{such that} \quad i^p = a,\ j^p = b,\ ji = \zeta ij.$$

In particular, in the case $p = 2$, Γ is just the generalized quaternion algebra over the ring of rational integers $\mathbb{Z}$.

In this chapter, we consider the Hochschild cohomology group $HH^m(\Gamma) = \mathrm{Ext}^m_{\Gamma^e}(\Gamma, \Gamma)$ and the Hochschild cohomology ring $HH^*(\Gamma) = \bigoplus_{m \geq 0} HH^m(\Gamma)$ of Γ, where Γ^e denotes the enveloping algebra $\Gamma \otimes_R \Gamma^{op}$ of Γ. Unless otherwise stated, $\otimes$ denotes $\otimes_R$.

Although there is basically a small number of studies about the Hochschild cohomology for algebras over a commutative ring, the Hochschild cohomology of quaternion algebras or cyclic algebras appearing as orders in semisimple algebras over a field are studied in, for example, Hayami's works [3], [4], [5], [6] and [12], [15], [16] etc. However, Hochschild cohomology is an important tool for investigating module categories of algebras. In fact it is known that the Hochschild cohomology ring of an algebra over a commutative ring is an invariant under an equivalence of bounded derived categories as triangulated categories (cf.[10, Chapter 6]).

Concerning the integral cyclic algebra Γ above, in the case a is any nonzero integer and $b = -1$, the module structure of $HH^m(\Gamma)$ was already given in [12] using spectral sequence. In the case $p = 2$, a is any nonzero integer and $b = -1$, the ring structure of the Hochschild cohomology $HH^*(\Gamma)$ was also calculated in [16] using spectral sequence. In the case $p = 2$, a and b are any nonzero integers, that is, Γ is a generalized quaternion algebra, the ring structure of $HH^*(\Gamma)$ was determined in [4]. In this chapter, we will generalize these results to the case of any prime number p.

In Section 2.2, we give a projective bimodule resolution of Γ, and applying the functor $\mathrm{Hom}_{\Gamma^e}(-,\Gamma)$ to the resolution, we have a double complex which gives the Hochschild cohomology group $HH^m(\Gamma)$. In Section 2.3, we determine the R-module structure of $HH^m(\Gamma)$ (Theorem 2.2):

$$HH^m(\Gamma) \cong \begin{cases} R \quad \text{for } m=0, \\ (R/dpR)^{(m-1)/2} \oplus (R/d\omega R)^{(m+1)/2} \oplus (R/\omega R)^{(p^2-2)(m+1)/2} \\ \qquad \text{for } m \text{ odd}, \\ (R/dpR)^{(m-2)/2} \oplus (R/d\omega R)^{m/2} \oplus (R/\omega R)^{(p^2-2)m/2} \oplus (R/apR) \\ \qquad \oplus (R/bpR) \quad \text{for } m (\neq 0) \text{ even.} \end{cases}$$

In Section 2.4, we determine the ring structure of $HH^*(\Gamma)$. First, in Subsection 2.4.1, we define a 'diagonal approximation map' for the projective bimodule resolution of Γ in order to calculate the cup product on $HH^*(\Gamma)$. In Subsection 2.4.2, by calculating the cup products of generators of the Hochoschild cohomology groups $HH^m(\Gamma)$ for $m \geq 0$, we give a system of generators of the Hochschild cohomology ring $HH^*(\Gamma)$ as an R-algebra in Theorem 2.3. As a result, if $p \geq 3$, then the Hochschild cohomology ring $HH^*(\Gamma)$ is generated by the elements of $HH^1(\Gamma)$, $HH^2(\Gamma)$ and $HH^3(\Gamma)$. Furthermore, in that section, we present the relations that the generators of $HH^*(\Gamma)$ satisfy. In addition, we study the special case $|a| = |b| = 1$. In Section 2.5, we consider the ring structure of $HH^*(\Gamma)$ in the case $p = 2$.

2.2 Projective resolution of Γ

First, we will give a Γ^e-projective resolution $(P_m, \Delta_m, \varepsilon)$ of Γ referring to [4]:

$$P_m = (\Gamma \otimes \Gamma)^{m+1} := (\Gamma \otimes \Gamma) \oplus (\Gamma \otimes \Gamma) \oplus \cdots \oplus (\Gamma \otimes \Gamma),$$

$$\Delta_m = \sum_{s+t=m} (\partial_{s,t} + \delta_{s,t}) \text{ for every integer } m \geq 0, \ \varepsilon \text{ is the augmentation.}$$

Here, for $s, t \geq 0$ with $m = s + t$, we define an element $c_{s,t} \in P_m$ by

$$c_{s,t} := (0, \ldots, 0, 1 \overset{t}{\check{\otimes}} 1, 0, \ldots, 0).$$

Then $P_m = \bigoplus_{s+t=m} \Gamma_{s,t}$, where we set $\Gamma_{s,t} := \Gamma c_{s,t} \Gamma$. We define Γ^e-homomorphisms $\partial_{s,t} : \Gamma_{s,t} \longrightarrow \Gamma_{s-1,t}$ and $\delta_{s,t} : \Gamma_{s,t} \longrightarrow \Gamma_{s,t-1}$ by

$$\partial_{s,t} = \begin{cases} \left.\begin{array}{l} \partial_1 : c_{s,t} \mapsto i c_{s-1,t} - c_{s-1,t} i \text{ for } s \text{ odd} \\ \partial_2 : c_{s,t} \mapsto \displaystyle\sum_{k=0}^{p-1} i^{p-1-k} c_{s-1,t} i^k \text{ for } s \text{ even} \end{array}\right\} \text{for } t \text{ even}, \\ \left.\begin{array}{l} \partial'_1 : c_{s,t} \mapsto i c_{s-1,t} - \zeta^{-1} c_{s-1,t} i \text{ for } s \text{ odd} \\ \partial'_2 : c_{s,t} \mapsto \displaystyle\sum_{k=0}^{p-1} \zeta^{-k} i^{p-1-k} c_{s-1,t} i^k \text{ for } s \text{ even} \end{array}\right\} \text{for } t \text{ odd}, \end{cases}$$

$$\delta_{s,t} = \begin{cases} \left.\begin{array}{l} \delta_1 : c_{s,t} \mapsto j c_{s,t-1} - c_{s,t-1} j \text{ for } t \text{ odd} \\ \delta_2 : c_{s,t} \mapsto \displaystyle\sum_{k=0}^{p-1} j^{p-1-k} c_{s,t-1} j^k \text{ for } t \text{ even} \end{array}\right\} \text{for } s \text{ even}, \\ \left.\begin{array}{l} \delta'_1 : c_{s,t} \mapsto (-1)(\zeta^{-1} j c_{s,t-1} - c_{s,t-1} j) \text{ for } t \text{ odd} \\ \delta'_2 : c_{s,t} \mapsto (-1) \displaystyle\sum_{k=0}^{p-1} \zeta^{-(p-1-k)} j^{p-1-k} c_{s,t-1} j^k \text{ for } t \text{ even} \end{array}\right\} \text{for } s \text{ odd.} \end{cases}$$

It is easy to see that the following equations hold:

$$\delta_{s,t-1}\circ\delta_{s,t}=0,\quad \partial_{s-1,t}\circ\partial_{s,t}=0,\quad \partial_{s,t-1}\circ\delta_{s,t}+\delta_{s-1,t}\circ\partial_{s,t}=0.$$

Hence, setting each $\Gamma_{s,t}$ on each lattice point on the first quadrant, we have the following double complex:

$$(\Gamma_{s,t},\partial_{s,t},\delta_{s,t}):\quad \begin{array}{ccccccc}
\downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle \delta'_1} & & \downarrow{\scriptstyle \delta_1} & & \\
\Gamma_{0,2} & \xleftarrow[\partial_1]{} & \Gamma_{1,2} & \xleftarrow[\partial_2]{} & \Gamma_{2,2} & \xleftarrow[\partial_1]{} & \\
\downarrow{\scriptstyle \delta_2} & & \downarrow{\scriptstyle \delta'_2} & & \downarrow{\scriptstyle \delta_2} & & \\
\Gamma_{0,1} & \xleftarrow[\partial'_1]{} & \Gamma_{1,1} & \xleftarrow[\partial'_2]{} & \Gamma_{2,1} & \xleftarrow[\partial'_1]{} & \\
\downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle \delta'_1} & & \downarrow{\scriptstyle \delta_1} & & \\
\Gamma_{0,0} & \xleftarrow[\partial_1]{} & \Gamma_{1,0} & \xleftarrow[\partial_2]{} & \Gamma_{2,0} & \xleftarrow[\partial_1]{} & .
\end{array}$$

Then, we show the Γ^e-projective resolution of Γ in the following proposition.

Proposition 2.1. *By taking the total complex of the above complex, we have the Γ^e-projective resolution of Γ:*

$$\cdots \xrightarrow{\Delta_3} P_2 \xrightarrow{\Delta_2} P_1 \xrightarrow{\Delta_1} P_0 \xrightarrow{\varepsilon} \Gamma \longrightarrow 0,$$

where $\Delta_m=\sum_{s+t=m}(\partial_{s,t}+\delta_{s,t})$ and ε is the multiplication map.

Proof. The exactness of the sequence is verified by giving a contracting homotopy. We define the following maps $T_{-1}:\Gamma\longrightarrow P_0$ and $T_m:P_m\longrightarrow P_{m+1}$ for $m\geq 0$ by

$$T_{-1}(\gamma)=c_{0,0}\gamma\quad(\gamma\in\Gamma);$$

for any even m,

$$T_m(i^uj^vc_{m,0})=\begin{cases}0 & \text{for } u=0 \text{ and } v=0,\\ \displaystyle\sum_{k=0}^{v-1} j^{v-1-k}c_{m,1}j^k & \text{for } u=0 \text{ and } v\neq 0,\\ \displaystyle\sum_{k=0}^{u-1} i^{u-1-k}c_{m+1,0}i^k & \text{for } u\neq 0 \text{ and } v=0,\\ \displaystyle\sum_{k=0}^{u-1} i^{u-1-k}c_{m+1,0}i^kj^v+i^u\sum_{k=0}^{v-1} j^{v-1-k}c_{m,1}j^k & \text{for } u\neq 0 \text{ and } v\neq 0,\end{cases}$$

$$T_m(i^uj^vc_{s,t})=\begin{cases}0 & \text{for } v=0 \text{ and } t\,(\neq 0) \text{ even},\\ \displaystyle i^u\sum_{k=0}^{v-1} j^{v-1-k}c_{s,t+1}j^k & \text{for } v\neq 0 \text{ and } t\,(\neq 0) \text{ even},\\ 0 & \text{for } v\neq p-1 \text{ and } t \text{ odd},\\ -\zeta^{-1}i^uc_{s,t+1} & \text{for } v=p-1 \text{ and } t \text{ odd};\end{cases}$$

and for any odd m,

$$T_m(i^u j^v c_{m,0}) = \begin{cases} 0 & \text{for } u \neq p-1 \text{ and } v = 0, \\ -\zeta i^u \displaystyle\sum_{k=0}^{v-1} j^{v-1-k} c_{m,1} j^k & \text{for } u \neq p-1 \text{ and } v \neq 0, \\ c_{m+1,0} & \text{for } u = p-1 \text{ and } v = 0, \\ \zeta^v c_{m+1,0} j^v - \zeta i^{p-1} \displaystyle\sum_{k=0}^{v-1} j^{v-1-k} c_{m,1} j^k & \text{for } u = p-1 \text{ and } v \neq 0, \end{cases}$$

$$T_m(i^u j^v c_{s,t}) = \begin{cases} 0 & \text{for } v = 0 \text{ and } t\,(\neq 0) \text{ even}, \\ -\zeta i^u \displaystyle\sum_{k=0}^{v-1} \zeta^k j^{v-1-k} c_{s,t+1} j^k & \text{for } v \neq 0 \text{ and } t\,(\neq 0) \text{ even}, \\ 0 & \text{for } v \neq p-1 \text{ and } t \text{ odd}, \\ i^u c_{s,t+1} & \text{for } v = p-1 \text{ and } t \text{ odd}. \end{cases}$$

Then T_m's satisfy the equalities

$$\begin{aligned} &\Delta_1 \circ T_0 + T_{-1} \circ \varepsilon = id_{P_0}, \\ &\Delta_{m+1} \circ T_m + T_{m-1} \circ \Delta_m = id_{P_m} \text{ for } m \geq 0. \end{aligned}$$

That is, $\{T_m\}$ is a contracting homotopy. □

We remark that the exactness above is also verified by using spectral sequence.

Next, we will define a complex giving the Hochschild cohomology of Γ. Applying the functor $\mathrm{Hom}_{\Gamma^e}(-, \Gamma)$ to the double complex above, we have the following double complex on the third quadrant:

$$\left(\Gamma^{s,t}, \partial^{s,t}, \delta^{s,t}\right) : \quad \begin{array}{ccccccc} \xleftarrow[\widetilde{\partial_1}]{} & \Gamma^{2,0} & \xleftarrow[\widetilde{\partial_2}]{} & \Gamma^{1,0} & \xleftarrow[\widetilde{\partial_1}]{} & \Gamma^{0,0} \\ & \downarrow \widetilde{\delta_1} & & \downarrow \widetilde{\delta_1'} & & \downarrow \widetilde{\delta_1} \\ \xleftarrow[\widetilde{\partial_1'}]{} & \Gamma^{2,1} & \xleftarrow[\widetilde{\partial_2'}]{} & \Gamma^{1,1} & \xleftarrow[\widetilde{\partial_1'}]{} & \Gamma^{0,1} \\ & \downarrow \widetilde{\delta_2} & & \downarrow \widetilde{\delta_2'} & & \downarrow \widetilde{\delta_2} \\ \xleftarrow[\widetilde{\partial_1}]{} & \Gamma^{2,2} & \xleftarrow[\widetilde{\partial_2}]{} & \Gamma^{1,2} & \xleftarrow[\widetilde{\partial_1}]{} & \Gamma^{0,2} \\ & \downarrow \widetilde{\delta_1} & & \downarrow \widetilde{\delta_1'} & & \downarrow \widetilde{\delta_1} \end{array}$$

where we set $\Gamma^{s,t} := \mathrm{Hom}_{\Gamma^e}(\Gamma_{s,t}, \Gamma) \cong \Gamma$ and we identify $\Gamma^{s,t}$ with Γ. So $\partial^{s,t} := \mathrm{Hom}(\partial_{s+1,t}, \iota) : \Gamma^{s,t} \longrightarrow \Gamma^{s+1,t}$ and $\delta^{s,t} := \mathrm{Hom}(\iota, \delta_{s,t+1}) : \Gamma^{s,t} \longrightarrow$

$\Gamma^{s,t+1}$ are explicitly given by

$$\partial^{s,t} = \begin{cases} \left.\begin{array}{l} \widetilde{\partial}_1 : x \mapsto ix - xi \quad \text{for } s \text{ even} \\ \widetilde{\partial}_2 : x \mapsto \displaystyle\sum_{k=0}^{p-1} i^{p-1-k} x i^k \quad \text{for } s \text{ odd} \end{array}\right\} \text{for } t \text{ even}, \\ \left.\begin{array}{l} \widetilde{\partial}'_1 : x \mapsto ix - \zeta^{-1} xi \quad \text{for } s \text{ even} \\ \widetilde{\partial}'_2 : x \mapsto \displaystyle\sum_{k=0}^{p-1} \zeta^{-k} i^{p-1-k} x i^k \quad \text{for } s \text{ odd} \end{array}\right\} \text{for } t \text{ odd}, \end{cases}$$

$$\delta^{s,t} = \begin{cases} \left.\begin{array}{l} \widetilde{\delta}_1 : x \mapsto jx - xj \quad \text{for } t \text{ even} \\ \widetilde{\delta}_2 : x \mapsto \displaystyle\sum_{k=0}^{p-1} j^{p-1-k} x j^k \quad \text{for } t \text{ odd} \end{array}\right\} \text{for } s \text{ even}, \\ \left.\begin{array}{l} \widetilde{\delta}'_1 : x \mapsto (-1)(\zeta^{-1} jx - xj) \quad \text{for } t \text{ even} \\ \widetilde{\delta}'_2 : x \mapsto (-1)\displaystyle\sum_{k=0}^{p-1} \zeta^{-(p-1-k)} j^{p-1-k} x j^k \quad \text{for } t \text{ odd} \end{array}\right\} \text{for } s \text{ odd} \end{cases}$$

for $x \in \Gamma^{s,t}$. Therefore, putting $Q^m := \bigoplus_{s+t=m} \Gamma^{s,t} \cong \Gamma^{m+1}$ and $\Delta^m := \sum_{s+t=m} (\partial^{s,t} + \delta^{s,t})$, we have the total complex of the above complex:

$$\cdots \xleftarrow{\Delta^2} Q^2 \xleftarrow{\Delta^1} Q^1 \xleftarrow{\Delta^0} Q^0 \longleftarrow 0.$$

2.3 Module structure of $HH^m(\Gamma)$

In this section, we determine the module structure of $HH^m(\Gamma) = \mathrm{Ext}^m_{\Gamma^e}(\Gamma, \Gamma)$. First, we present any element of Γ by a matrix in $M_p(R)$. If x is any element in $\Gamma^{s,t}$, then there uniquely exist $x_{kl} \in R$ $(k, l = 1, 2, \ldots, p)$ such that

$$x = \begin{pmatrix} 1 & i & \cdots & i^{p-1} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{p1} & x_{p2} & \cdots & x_{pp} \end{pmatrix} \begin{pmatrix} 1 \\ j \\ \vdots \\ j^{p-1} \end{pmatrix}.$$

By corresponding $x \in \Gamma^{s,t}$ to the matrix $X = (x_{kl}) \in M_p(R)$ above, $\partial^{s,t}(X)$ and $\delta^{s,t}(X)$ are given by

$$\widetilde{\partial}_1(X) = \begin{pmatrix} 0 & a\omega x_{p2} & \cdots & a\omega_{p-1} x_{pp} \\ 0 & \omega x_{12} & \cdots & \omega_{p-1} x_{1p} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \omega x_{p-12} & \cdots & \omega_{p-1} x_{p-1p} \end{pmatrix}, \quad \widetilde{\partial}_2(X) = \begin{pmatrix} apx_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ apx_{p1} & 0 & \cdots & 0 \\ px_{11} & 0 & \cdots & 0 \end{pmatrix},$$

$$\widetilde{\partial}'_1(X) = \begin{pmatrix} a\omega_{p-1} x_{p1} & 0 & a\omega x_{p3} & \cdots & a\omega_{p-2} x_{pp} \\ \omega_{p-1} x_{11} & 0 & \omega x_{13} & \cdots & \omega_{p-2} x_{1p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega_{p-1} x_{p-11} & 0 & \omega x_{p-13} & \cdots & \omega_{p-2} x_{p-1p} \end{pmatrix},$$

$$\widetilde{\partial}'_2(X) = \begin{pmatrix} 0 & apx_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & apx_{p2} & 0 & \cdots & 0 \\ 0 & px_{12} & 0 & \cdots & 0 \end{pmatrix};$$

$$\widetilde{\delta_1}(X) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ -b\omega x_{2p} & -\omega x_{21} & \cdots & -\omega x_{2p-1} \\ \vdots & \vdots & \ddots & \vdots \\ -b\omega_{p-1} x_{pp} & -\omega_{p-1} x_{p1} & \cdots & -\omega_{p-1} x_{pp-1} \end{pmatrix},$$

$$\widetilde{\delta_2}(X) = \begin{pmatrix} bpx_{12} & \cdots & bpx_{1p} & px_{11} \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$\widetilde{\delta'_1}(X) = \begin{pmatrix} b\omega_{p-1} x_{1p} & \omega_{p-1} x_{11} & \cdots & \omega_{p-1} x_{1p-1} \\ 0 & 0 & \cdots & 0 \\ b\omega x_{3p} & \omega x_{31} & \cdots & \omega x_{3p-1} \\ \vdots & \vdots & \ddots & \vdots \\ b\omega_{p-2} x_{pp} & \omega_{p-2} x_{p1} & \cdots & \omega_{p-2} x_{pp-1} \end{pmatrix},$$

$$\widetilde{\delta'_2}(X) = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ -bpx_{22} & \cdots & -bpx_{2p} & -px_{21} \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix}.$$

For $s + t = m \, (s, t \geq 0)$, we define $c^{s,t} \in Q^m$ by

$$c^{s,t} := (0, \ldots, 0, \overset{t}{\check{1}}, 0, \ldots, 0).$$

Using above expressions, we obtain the R-module structure of the Hochschild cohomology group $HH^m(\Gamma)$. In fact, we directly calculate Ker Δ^m and Im Δ^{m-1}. We present those R-modules only in the case m is even.

$$\begin{aligned}
\text{Ker } \Delta^m =& \bigoplus_{t=0}^{m} Rc^{m-t,t} \oplus \bigoplus_{\substack{1 \leq t \leq m-1, \text{ odd;} \\ 2 \leq k,l \leq p-1}} Ri^k j^l c^{m-t,t} \\
&\oplus \bigoplus_{\substack{1 \leq t \leq m-1, \text{ odd;} \\ 2 \leq l \leq p-1}} Rj^l c^{m-t,t} \oplus \bigoplus_{\substack{1 \leq t \leq m-1, \text{ odd;} \\ 2 \leq k \leq p-1}} Ri^k c^{m-t,t} \\
&\oplus \bigoplus_{\substack{0 \leq t \leq m-2, \text{ even;} \\ 0 \leq k' (\neq 1) \leq p-1}} R\left(\frac{p}{\omega_{p-1+k'}} i^{p-1+k'} c^{m-t,t} + i^{k'} j c^{m-t-1,t+1}\right) \\
&\oplus \bigoplus_{\substack{1 \leq t \leq m-1, \text{ odd;} \\ 0 \leq l' (\neq 1) \leq p-1}} R\left(i j^{l'} c^{m-t,t} + \frac{p}{\omega_{p-1+l'}} j^{p-1+l'} c^{m-t-1,t+1}\right),
\end{aligned}$$

$$\begin{aligned}
\text{Im } \Delta^{m-1} =& apRc^{m,0} \oplus \bigoplus_{1 \leq t \leq m-1, \text{ odd}} d\omega Rc^{m-t,t} \oplus \bigoplus_{2 \leq t \leq m-2, \text{ even}} dpRc^{m-t,t} \\
&\oplus bpRc^{0,m} \oplus \bigoplus_{\substack{1 \leq t \leq m-2, \text{ odd;} \\ 2 \leq k,l \leq p-1}} \omega Ri^k j^l c^{m-t-1,t-1} \\
&\oplus \bigoplus_{\substack{1 \leq t \leq m-1, \text{ odd;} \\ 2 \leq l \leq p-1}} \omega Rj^l c^{m-t,t} \oplus \bigoplus_{\substack{1 \leq t \leq m-1, \text{ odd;} \\ 2 \leq k \leq p-1}} \omega Ri^k c^{m-t,t}
\end{aligned}$$

$$\oplus \bigoplus_{\substack{0\le t\le m-2,\text{ even};\\ 0\le k'(\ne 1)\le p-1}} \omega R\left(\frac{p}{\omega_{p-1+k'}} i^{p-1+k'} c^{m-t,t} + i^{k'} j c^{m-t-1,t+1}\right)$$

$$\oplus \bigoplus_{\substack{1\le t\le m-1,\text{ odd};\\ 0\le l'(\ne 1)\le p-1}} \omega R\left(i j^{l'} c^{m-t,t} + \frac{p}{\omega_{p-1+l'}} j^{p-1+l'} c^{m-t-1,t+1}\right).$$

In the above calculation, we note that $\omega R = \omega_{p-1+k'} R$ for $0 \le k'(\ne 1) \le p-1$.

Theorem 2.2. *Let $\mathbb{Z}$ be the ring of rational integers, a, b any nonzero rational integers and d the greatest common divisor of a and b. Let p be a prime and ζ a primitive p-th root of unity. We set $R = \mathbb{Z}[\zeta]$ and put $\omega = 1 - \zeta$. Then the R-module structure of the Hochschild cohomology group of Γ is as follows:*

$$HH^m(\Gamma) \cong \begin{cases} R \quad \text{for } m = 0, \\ (R/dpR)^{(m-1)/2} \oplus (R/d\omega R)^{(m+1)/2} \oplus (R/\omega R)^{(p^2-2)(m+1)/2} \\ \qquad\qquad\qquad \text{for } m \text{ odd}, \\ (R/dpR)^{(m-2)/2} \oplus (R/d\omega R)^{m/2} \oplus (R/\omega R)^{(p^2-2)m/2} \\ \qquad\qquad \oplus (R/apR) \oplus (R/bpR) \quad \text{for } m(\ne 0) \text{ even.} \end{cases}$$

For the later use, we list the system of generators of each $HH^m(\Gamma)$ as an R-module represented by elements in $Q^m = \Gamma^{m+1}$ as follows, where we set $a' = a/d, b' = b/d$:

For $m = 1$,

$ij^l c^{1,0}$ for $1 \le l \le p-1$,

$i^k j c^{0,1}$ for $1 \le k \le p-1$,

$i^{k+1} j^l c^{1,0} - \dfrac{\omega_k}{\omega_l} i^k j^{l+1} c^{0,1}$ for $1 \le k, l \le p-1$ with $(k,l) \ne (p-1, p-1)$,

$a' j^{p-1} c^{1,0} - b' i^{p-1} c^{0,1}$.

For $m \ge 2$ even,

$c^{m-t,t}$ for $0 \le t \le m$,

$i^k j^l c^{m-t,t}$ for $2 \le k, l \le p-1$ and t odd,

$i^k c^{m-t,t}$ for $2 \le k \le p-1$ and t odd,

$j^l c^{m-t,t}$ for $2 \le l \le p-1$ and t odd,

$\dfrac{p}{\omega_{p-1+k}} i^{p-1+k} c^{m-t,t} + i^k j c^{m-t-1,t+1}$ for $0 \le k(\ne 1) \le p-1$ and t even,

$ij^l c^{m-t,t} + \dfrac{p}{\omega_{p-1+l}} j^{p-1+l} c^{m-t-1,t+1}$ for $0 \le l(\ne 1) \le p-1$ and t odd.

For $m \ge 3$ odd,

$ij^l c^{m-t,t}$ for $1 \le l \le p-1$ and t even,

$i^k j c^{m-t,t}$ for $1 \le k \le p-1$ and t odd,

$i^{k+1} j^l c^{m-t,t} - \omega_k/\omega_l i^k j^{l+1} c^{m-t-1,t+1}$

for $1 \le k, l \le p-1$ with $(k,l) \ne (p-1, p-1)$ and t even,

$a' j^{p-1} c^{m-t,t} - b' i^{p-1} c^{m-t-1,t+1}$ for t even,

$a' j c^{m-t,t} - b' i c^{m-t-1,t+1}$ for $0 \le t < m$ odd.

2.4 The ring structure of $HH^*(\Gamma)$

In this section, we will determine the ring structure of $HH^*(\Gamma) = \bigoplus_{m\geq 0} HH^m(\Gamma)$.

2.4.1 Diagonal approximation and cup product

First, we define a map $\Phi_{s,t;s',t'} : \Gamma_{s+t,s'+t'} \longrightarrow \Gamma_{s,t} \otimes_\Gamma \Gamma_{s',t'}$ of Γ^e-modules by the map sending $c_{s+t,s'+t'}$ to

$$
\begin{cases}
\displaystyle\sum_{\substack{u+v+w=p-2,\\ u'+v'+w'=p-2}} \zeta^{(v+1)(v'+1)+2-uw'} i^u j^{u'} c_{s,t} i^v j^{v'} \otimes_\Gamma c_{s',t'} i^w j^{w'} & \text{for } s,t,s',t' \text{ odd},\\
\displaystyle -\zeta \sum_{u+v+w=p-2} \zeta^u i^u c_{s,t} i^v \otimes_\Gamma c_{s',t'} i^w & \text{for } s \text{ odd}, t \text{ odd}, s' \text{ odd}, t' \text{ even},\\
\displaystyle\sum_{u'+v'+w'=p-2} \zeta^{-u'} j^{u'} c_{s,t} j^{v'} \otimes_\Gamma c_{s',t'} j^{w'} & \text{for } s \text{ odd}, t \text{ odd}, s' \text{ even}, t' \text{ odd},\\
\displaystyle\sum_{u+v+w=p-2} i^u c_{s,t} i^v \otimes_\Gamma \zeta^{-w} c_{s',t'} i^w & \text{for } s \text{ odd}, t \text{ even}, s' \text{ odd}, t' \text{ odd},\\
\displaystyle -\zeta \sum_{u'+v'+w'=p-2} j^{u'} c_{s,t} j^{v'} \otimes_\Gamma \zeta^{w'} c_{s',t'} j^{w'} & \text{for } s \text{ even}, t \text{ odd}, s' \text{ odd}, t' \text{ odd},\\
\displaystyle\sum_{u+v+w=p-2} i^u c_{s,t} i^v \otimes_\Gamma c_{s',t'} i^w & \text{for } s \text{ odd}, t \text{ even}, s' \text{ odd}, t' \text{ even},\\
\displaystyle\sum_{u'+v'+w'=p-2} j^{u'} c_{s,t} j^{v'} \otimes_\Gamma c_{s',t'} j^{w'} & \text{for } s \text{ even}, t \text{ odd}, s' \text{ even}, t' \text{ odd},\\
-\zeta^{-1} c_{s,t} \otimes_\Gamma c_{s',t'} & \text{for } s \text{ even}, t \text{ odd}, s' \text{ odd}, t' \text{ even},\\
c_{s,t} \otimes_\Gamma c_{s',t'} & \text{otherwise}.
\end{cases}
$$

Then, $\Phi = \{\Phi_{s,t;s',t'}\}$ satisfies the following relations:

$$
\begin{aligned}
&\Phi_{s,t;s',t'} \circ \partial_{s+s'+1,t+t'}\\
&\qquad = \partial_{s+1,t} \otimes \iota \circ \Phi_{s+1,t;s',t'} + (-1)^{s+t} \iota \otimes \partial_{s'+1,t'} \circ \Phi_{s,t;s'+1,t'},\\
&\Phi_{s,t;s',t'} \circ \delta_{s+s',t+t'+1}\\
&\qquad = \delta_{s,t+1} \otimes \iota \circ \Phi_{s,t+1;s',t'} + (-1)^{s+t} \iota \otimes \delta_{s',t'+1} \circ \Phi_{s,t;s',t'+1},\\
&\varepsilon \otimes \varepsilon \circ \Phi_{0,0;0,0} = \varepsilon.
\end{aligned}
$$

Therefore, $\Phi_{m,n} := \sum_{s+t=m,s'+t'=n} \Phi_{s,t;s't'}$ is a 'diagonal approximation', that is, this satisfies

$$
\begin{aligned}
&\Phi_{m,n} \circ \Delta_{m+n+1} = (\Delta_{m+1} \otimes \iota) \circ \Phi_{m+1,n} + (-1)^m (\iota \otimes \Delta_{n+1}) \circ \Phi_{m,n+1},\\
&(\varepsilon \otimes \varepsilon) \circ \Phi_{0,0} = \varepsilon.
\end{aligned}
$$

Using Φ, we define the cup product

$$HH^m(\Gamma) \otimes HH^n(\Gamma) \overset{\smile}{\longrightarrow} HH^{m+n}(\Gamma); \quad \alpha \otimes \beta \mapsto a \smile \beta$$

by

$$\alpha \smile \beta = (\alpha \otimes_\Gamma \beta) \circ \Phi_{s,t;s',t'} : \Gamma_{s+t,s'+t'} \to \Gamma_{s,t} \otimes_\Gamma \Gamma_{s',t'} \to \Gamma \otimes_\Gamma \Gamma = \Gamma.$$

for $\alpha \in \Gamma^{s,t}$ with $s+t = m$ and $\beta \in \Gamma^{s',t'}$ with $s'+t' = n$. Hence $\Gamma^{s,t} \otimes \Gamma^{s',t'} \overset{\smile}{\longrightarrow} \Gamma^{s+t,s'+t'}$ is explicitly presented by

$$\alpha \smile \beta =$$

$$
\begin{cases}
\displaystyle\sum_{\substack{u+v+w=p-2,\\ u'+v'+w'=p-2}} \zeta^{(v+1)(v'+1)+2-uw'} i^u j^{u'} \alpha i^v j^{v'} \beta i^w j^{w'} & \text{for } s,t,s',t' \text{ odd},\\
-\zeta \displaystyle\sum_{u+v+w=p-2} \zeta^u i^u \alpha i^v \beta i^w & \text{for } s \text{ odd}, t \text{ odd}, s' \text{ odd}, t' \text{ even},\\
\displaystyle\sum_{u'+v'+w'=p-2} \zeta^{-u'} j^{u'} \alpha j^{v'} \beta j^{w'} & \text{for } s \text{ odd}, t \text{ odd}, s' \text{ even}, t' \text{ odd},\\
\displaystyle\sum_{u+v+w=p-2} i^u \alpha i^v \zeta^{-w} \beta i^w & \text{for } s \text{ odd}, t \text{ even}, s' \text{ odd}, t' \text{ odd},\\
-\zeta \displaystyle\sum_{u'+v'+w'=p-2} j^{u'} \alpha j^{v'} \zeta^{w'} \beta j^{w'} & \text{for } s \text{ even}, t \text{ odd}, s' \text{ odd}, t' \text{ odd},\\
\displaystyle\sum_{u+v+w=p-2} i^u \alpha i^v \beta i^w & \text{for } s \text{ odd}, t \text{ even}, s' \text{ odd}, t' \text{ even},\\
\displaystyle\sum_{u'+v'+w'=p-2} j^{u'} \alpha j^{v'} \beta j^{w'} & \text{for } s \text{ even}, t \text{ odd}, s' \text{ even}, t' \text{ odd},\\
-\zeta^{-1} \alpha\beta & \text{for } s \text{ even}, t \text{ odd}, s' \text{ odd}, t' \text{ even},\\
\alpha\beta & \text{otherwise}.
\end{cases}
$$

for $\alpha \in \Gamma^{s,t}$ and $\beta \in \Gamma^{s',t'}$. In the above, we identify $\Gamma^{s,t}$ with Γ and so on. As long as there is no confusion, we often denote $\alpha \smile \beta$ by $\alpha\beta$ for simplicity. It is well known that the anti-commutativity $\alpha\beta = (-1)^{mn}\beta\alpha$ holds for $\alpha \in HH^m(\Gamma)$ and $\beta \in HH^n(\Gamma)$. That is, the Hochschild cohomology ring $HH^*(\Gamma)$ is a graded commutative ring.

2.4.2 Generators of $HH^*(\Gamma)$ as an R-algebra and the relations

In this subsection, we determine the ring structure of the Hochschild cohomology ring $HH^*(\Gamma)$ using cup product on generators of $HH^m(\Gamma)$. By the way, the ring structure of the Hochschild cohomology ring $HH^*(\Gamma)$ in the case $p = 2$ was already known in [4]. So, we mainly treat the case $p \geq 3$.

We denote the representatives of each element of $HH^m(\Gamma)$ by $(*, *, \dots, *) \in Q^m = \Gamma^{m,0} \oplus \Gamma^{m-1,1} \oplus \cdots \oplus \Gamma^{0,m}$. Then, referring to Theorem 2.2, generators of $HH^m(\Gamma)$ for $m = 1, 2, 3$ as an R-module are as follows including the case $p = 2$:

Generators of $HH^1(\Gamma)$:

$$
\begin{aligned}
\sigma_l &:= (ij^l, 0) \quad \text{for } 1 \leq l \leq p-1,\\
\tau_k &:= (0, i^k j) \quad \text{for } 1 \leq k \leq p-1,\\
\theta_{k,l} &:= (i^{k+1} j^l, -\frac{\omega_k}{\omega_l} i^k j^{l+1}) \quad \text{for } 1 \leq k, l \leq p-1 \text{ with } (k,l) \neq (p-1, p-1),\\
\pi &:= (a' j^{p-1}, -b' i^{p-1}).
\end{aligned}
$$

Generators of $HH^2(\Gamma)$:

$$
\begin{aligned}
\varphi &:= (1, 0, 0),\\
\psi &:= (0, 1, 0),\\
\chi &:= (0, 0, 1),
\end{aligned}
$$

$$\rho_k := (\frac{p}{\omega_{p-1+k}} i^{p-1+k}, i^k j, 0) \text{ for } 0 \le k (\neq 1) \le p-1,$$
$$\eta_l := (0, ij^l, \frac{p}{\omega_{p-1+l}} j^{p-1+l}) \text{ for } 0 \le l (\neq 1) \le p-1,$$
$$\mu_{k,l} := (0, i^k j^l, 0) \text{ for } 0 \le k, l (\neq 1) \le p-1 \text{ with } (k,l) \neq (0,0).$$

Generators of $HH^3(\Gamma)$:

$$(ij^l, 0, 0, 0) \text{ for } 1 \le l \le p-1,$$
$$(0, i^k j, 0, 0) \text{ for } 1 \le k \le p-1,$$
$$(0, 0, ij^l, 0) \text{ for } 1 \le l \le p-1,$$
$$(0, 0, 0, i^k j) \text{ for } 1 \le k \le p-1,$$
$$(i^{k+1} j^l, -\frac{\omega_k}{\omega_l} i^k j^{l+1}, 0, 0) \text{ for } 1 \le k, l \le p-1 \text{ with } (k,l) \neq (p-1, p-1),$$
$$(0, 0, i^{k+1} j^l, -\frac{\omega_k}{\omega_l} i^k j^{l+1}) \text{ for } 1 \le k, l \le p-1 \text{ with } (k,l) \neq (p-1, p-1),$$
$$(a' j^{p-1}, -b' i^{p-1}, 0, 0),$$
$$(0, 0, a' j^{p-1}, -b' i^{p-1}),$$
$$\kappa := (0, a' j, -b' i, 0).$$

Let $x = (x^{m,0}, \ldots, x^{0,m}) \in HH^m(\Gamma)$. Then, it is easy to check that the elements $(x^{m,0}, \ldots, x^{0,m}, 0, 0)$ and $(0, 0, x^{m,0}, \ldots, x^{0,m}) \in HH^{m+2}(\Gamma)$ are given by $x\varphi$ and $x\chi$ respectively. In particular, if x is a generator, then $x\varphi$ and $x\chi$ are also generators. Therefore, we see that the generators of $HH^m(\Gamma)$ for any $m \ge 3$ except κ are given by the cup products of the generators above of $HH^1(\Gamma)$ and $HH^2(\Gamma)$ and $\kappa \in HH^3(\Gamma)$. On the other hand, the relation $\sigma_l \tau_k = \mu_{k+1,l+1}$ holds for $1 \le k, l < p-1$.

Therefore we have the following main theorem.

Theorem 2.3. *Let p be an odd prime and a, b nonzero integers, and set $d = \gcd(a,b)$, $a' = a/d$, $b' = b/d$. Then the Hochschild cohomology ring $HH^*(\Gamma)$ is the graded commutative ring generated by at most the following $p^2 + 4p - 3$ elements:*

$\sigma_l, \tau_k, \theta_{k',l'}, \pi \in HH^1(\Gamma)$ *for* $1 \le k, k', l, l' \le p-1$ *with* $(k', l') \neq (p-1, p-1)$,
$\varphi, \psi, \chi, \mu_{k,0}, \mu_{0,l}, \rho_{k'}, \eta_{l'} \in HH^2(\Gamma)$ *for* $2 \le k, l \le p-1, 0 \le k', l' (\neq 1) \le p-1$,
$\kappa \in HH^3(\Gamma)$.

The list of the relations of the generators above is as follows:
The relations in $HH^1(\Gamma)$:

$$\omega\tau_k = \omega\sigma_l = d\omega\pi = \omega\theta_{k',l'} = 0.$$

The relations in $HH^2(\Gamma)$:

$$ap\varphi = d\omega\psi = bp\chi = \omega\rho_{k'} = \omega\eta_{l'} = \omega\mu_{k,0} = \omega\mu_{0,l} = \pi\pi = 0.$$

$$\tau_{k'}\tau_k = \begin{cases} \frac{p}{\omega_k} \zeta^k ab\chi & \text{if } k + k' = p, \\ 0 & \text{if } k + k' \neq p. \end{cases}$$

$$\sigma_l\sigma_{l'} = \begin{cases} \frac{p}{\omega_l}\zeta^l ab\varphi & \textit{if } l+l'=p, \\ 0 & \textit{if } l+l'\neq p. \end{cases}$$

$$\tau_k\pi = \begin{cases} -\zeta^{-1}a'b\eta_0 & \textit{if } k=1, \\ -\zeta^{-1}a'b\mu_{k,0} & \textit{if } 1<k. \end{cases}$$

$$\sigma_l\pi = \begin{cases} -\zeta^{-1}b'a\rho_0 & \textit{if } l=1, \\ -\zeta^{-l}b'a\mu_{0,l} & \textit{if } 1<l. \end{cases}$$

$$\sigma_l\tau_k = \begin{cases} b\mu_{k+1,0} & \textit{if } k<p-1 \textit{ and } l=p-1, \\ a\mu_{0,l+1} & \textit{if } k=p-1 \textit{ and } l<p-1, \\ ab\psi & \textit{if } k=p-1 \textit{ and } l=p-1. \end{cases}$$

$$\theta_{k,l}\theta_{k',l'} =$$

$$\begin{cases} (\frac{\omega_k}{\omega_l}\zeta^{k'+l}-\frac{\omega_{k'}}{\omega_{l'}})\zeta^{k'l}\sigma_{l+l'}\tau_{k+k'} & \textit{if } 0<k+k'<p-1 \\ & \quad \textit{and } 0<l+l'<p-1, \\ (\frac{\omega_k}{\omega_l}\zeta^{k'+l}-\frac{\omega_{k'}}{\omega_{l'}})\zeta^{k'l}b\mu_{k+k'+1,0} & \textit{if } 0<k+k'<p-1 \\ & \quad \textit{and } l+l'=p-1, \\ \frac{\omega_{k+k'}}{\omega_l}\zeta^{l(k'+1)}b\rho_{k+k'+1} & \textit{if } 0<k+k'<p-1 \textit{ and } l+l'=p, \\ (\frac{\omega_k}{\omega_l}\zeta^{k'+l}-\frac{\omega_{k'}}{\omega_{l'}})\zeta^{k'l}b\sigma_{l+l'-p}\tau_{k+k'} & \textit{if } 0<k+k'<p-1 \textit{ and } p<l+l', \\ (\frac{\omega_k}{\omega_l}\zeta^{k'+l}-\frac{\omega_{k'}}{\omega_{l'}})\zeta^{k'l}a\mu_{0,l+l'+1} & \textit{if } k+k'=p-1 \\ & \quad \textit{and } 0<l+l'<p-1, \\ (\frac{\omega_k}{\omega_l}\zeta^{k'+l}-\frac{\omega_{k'}}{\omega_{l'}})\zeta^{k'l}ab\psi & \textit{if } k+k'=p-1 \textit{ and } l+l'=p-1, \\ \frac{\omega_{k+k'}}{\omega_l}\zeta^{l(k'+1)}ab\rho_0 & \textit{if } k+k'=p-1 \textit{ and } l+l'=p, \\ (\frac{\omega_k}{\omega_l}\zeta^{k'+l}-\frac{\omega_{k'}}{\omega_{l'}})\zeta^{k'l}ab\mu_{0,l+l'+1-p} & \textit{if } k+k'=p-1 \textit{ and } p<l+l', \\ \frac{\omega_k\omega_{l+l'}}{\omega_l\omega_{l'}}\zeta^{-k(l+1)}a\eta_{l+l'+1} & \textit{if } k+k'=p \textit{ and } 0<l+l'<p, \\ \frac{\omega_k\omega_{l+l'}}{\omega_l\omega_{l'}}\zeta^{-k(l+1)}ab\eta_0 & \textit{if } k+k'=p \textit{ and } l+l'=p-1, \\ \frac{p}{\omega_l\omega_{l'}}\zeta^{kl'}ab(\omega_{l'}a\varphi+\omega_{k'}b\chi) & \textit{if } k+k'=p \textit{ and } l+l'=p, \\ \frac{\omega_k\omega_{l+l'}}{\omega_l\omega_{l'}}\zeta^{-k(l+1)}ab\eta_{l+l'+1-p} & \textit{if } k+k'=p \textit{ and } p<l+l', \\ (\frac{\omega_k}{\omega_l}\zeta^{k'+l}-\frac{\omega_{k'}}{\omega_{l'}})\zeta^{k'l}a\sigma_{l+l'}\tau_{k+k'-p} & \textit{if } p<k+k' \textit{ and } 0<l+l'<p, \\ (\frac{\omega_k}{\omega_l}\zeta^{k'+l}-\frac{\omega_{k'}}{\omega_{l'}})\zeta^{k'l}ab\mu_{k+k'+1-p,0} & \textit{if } p<k+k' \textit{ and } l+l'=p-1, \\ \frac{\omega_{k+k'}}{\omega_l}\zeta^{l(k'+1)}ab\rho_{k+k'+1-p} & \textit{if } p<k+k' \textit{ and } l+l'=p, \\ (\frac{\omega_k}{\omega_l}\zeta^{k'+l}-\frac{\omega_{k'}}{\omega_{l'}})\zeta^{k'l}ab\sigma_{l+l'-p}\tau_{k+k'-p} & \textit{if } p<k+k' \textit{ and } p<l+l'. \end{cases}$$

$$\pi\theta_{k,l} = \begin{cases} \frac{p}{\omega_1}ab(-\zeta^{-1}a'\varphi+b'\chi) & \textit{if } k=1 \textit{ and } l=1, \\ \frac{\omega_{l-1}}{\omega_l}a'b\eta_l & \textit{if } k=1 \textit{ and } 1<l, \\ -\frac{\omega_{k-1}}{\omega_1}\zeta^{-k}b'a\rho_k & \textit{if } 1<k \textit{ and } l=1, \\ (\zeta^{-1}-\frac{\omega_k}{\omega_l}\zeta^{-k})a'b'd\sigma_{l-1}\tau_{k-1} & \textit{if } 1<k \textit{ and } 1<l. \end{cases}$$

$$\tau_{k'}\theta_{k,l} = \begin{cases} -\zeta^k \sigma_l \tau_{k+k'} & \textit{if } 0 < k+k' < p \textit{ and } l < p-1, \\ -\zeta^k b\mu_{k+k'+1,0} & \textit{if } 0 < k+k' < p \textit{ and } l = p-1, \\ -\zeta^k a\eta_{l+1} & \textit{if } k+k' = p \textit{ and } l < p-1, \\ -\zeta^k ab\eta_0 & \textit{if } k+k' = p \textit{ and } l = p-1, \\ -\zeta^k a\sigma_l\tau_{k+k'-p} & \textit{if } p < k+k' \textit{ and } l < p-1, \\ -\zeta^k ab\mu_{k+k'+1-p,0} & \textit{if } p < k+k' \textit{ and } l = p-1. \end{cases}$$

$$\sigma_{l'}\theta_{k,l} = \begin{cases} -\frac{\omega_k}{\omega_l}\zeta^{kl'}\sigma_{l+l'}\tau_k & \textit{if } 0 < l+l' < p \textit{ and } k < p-1, \\ -\frac{\omega_{p-1}}{\omega_l}\zeta^{-l'}a\mu_{0,l+l'+1} & \textit{if } 0 < l+l' < p \textit{ and } k = p-1, \\ \frac{\omega_k}{\omega_{l'}}\zeta^{l'(k+1)}b\rho_{k+1} & \textit{if } l+l' = p \textit{ and } k < p-1, \\ \frac{\omega_{p-1}}{\omega_{l'}}ab\rho_0 & \textit{if } l+l' = p \textit{ and } k = p-1, \\ -\frac{\omega_k}{\omega_l}\zeta^{kl'}b\sigma_{l+l'-p}\tau_k & \textit{if } p < l+l' \textit{ and } k < p-1, \\ -\frac{\omega_{p-1}}{\omega_l}\zeta^{-l'}ab\mu_{0,l+l'+1-p} & \textit{if } p < l+l' \textit{ and } k = p-1. \end{cases}$$

The relations in $HH^3(\Gamma)$:

$$dp\kappa = \pi\psi = 0.$$

$$\tau_k\psi = \begin{cases} \frac{p}{\omega_1}\sigma_{p-1}\chi & \textit{if } k = 1, \\ 0 & \textit{if } 1 < k. \end{cases}$$

$$\sigma_l\psi = \begin{cases} \frac{p}{\omega_1}\tau_{p-1}\varphi & \textit{if } l = 1, \\ 0 & \textit{if } 1 < l. \end{cases}$$

$$\tau_k\rho_{k'} = \begin{cases} -\frac{p}{\omega_1}d\kappa & \textit{if } k+k'-1 = 0 \; (\textit{i.e. } k=1, k'=0), \\ \frac{p}{\omega_{p-1+k'}}\zeta^{k'-1}a\tau_{k+k'-1}\varphi & \textit{if } 0 < k+k'-1 < p, \\ -\frac{p}{\omega_k}ad\kappa & \textit{if } k+k'-1 = p, \\ \frac{p}{\omega_{k'-1}}\zeta^{k'-1}a^2\tau_{k+k'-1-p}\varphi & \textit{if } p < k+k'-1. \end{cases}$$

$$\sigma_l\eta_{l'} = \begin{cases} \frac{p}{\omega_1}\zeta d\kappa & \textit{if } l+l'-1 = 0 \; (\textit{i.e. } l=1, l'=0), \\ \frac{p}{\omega_{p-1+l'}}b\sigma_{l+l'-1}\chi & \textit{if } 0 < l+l'-1 < p, \\ \frac{p}{\omega_l}\zeta^l bd\kappa & \textit{if } l+l'-1 = p, \\ \frac{p}{\omega_{p-1+l'}}b^2\sigma_{l+l'-1-p}\chi & \textit{if } p < l+l'-1. \end{cases}$$

$$\sigma_l\rho_k = \begin{cases} \frac{p}{\omega_{p-1}}\zeta^{-l}\theta_{p-1,l}\varphi & \textit{if } k = 0, \\ \frac{p}{\omega_{k-1}}\zeta^{l(k-1)}a\theta_{k-1,l}\varphi & \textit{if } 0 < k. \end{cases}$$

$$\tau_k\eta_l = \begin{cases} -\frac{p}{\omega_k}\theta_{k,p-1}\chi & \textit{if } l = 0, \\ -\frac{p}{\omega_k}b\theta_{k,l-1}\chi & \textit{if } 0 < l. \end{cases}$$

$$\tau_k\mu_{k',0} = \begin{cases} \frac{p}{\omega_k}a\sigma_{p-1}\chi & \textit{if } k+k'-1 = p \; (\textit{i.e. } k=1, k'=0), \\ 0 & \textit{if } k+k'-1 \neq p. \end{cases}$$

$$\sigma_l\mu_{0,l'} = \begin{cases} \frac{p}{\omega_l}b\tau_{p-1}\varphi & \textit{if } l+l'-1 = p \; (\textit{i.e. } l=1, l'=0), \\ 0 & \textit{if } l+l'-1 \neq p. \end{cases}$$

$$\tau_k\mu_{0,l} = \begin{cases} \frac{p}{\omega_1}b\sigma_{l-1}\chi & \textit{if } k = 1, \\ 0 & \textit{if } 1 < k. \end{cases}$$

$$\sigma_l\mu_{k,0} = \begin{cases} \frac{p}{\omega_1}\zeta^k a\tau_{k-1}\varphi & \textit{if } l=1, \\ 0 & \textit{if } 1<l. \end{cases}$$

$$\pi\mu_{k,0} = \begin{cases} \frac{p}{\omega_1}\zeta^{-1}a'b\sigma_{p-2}\chi & \textit{if } k=2, \\ 0 & \textit{if } 2<k. \end{cases}$$

$$\pi\mu_{0,l} = \begin{cases} -\frac{p}{\omega_1}\zeta ab'\tau_{p-2}\varphi & \textit{if } k=2, \\ 0 & \textit{if } 2<l. \end{cases}$$

$$\pi\rho_k = \begin{cases} \frac{p}{\omega_{p-1}}\zeta a'\theta_{p-2,p-1}\varphi & \textit{if } k=0, \\ \frac{p}{\omega_1}\zeta^{-1}a'(a\sigma_{p-1}\varphi + b\sigma_{p-1}\chi) & \textit{if } k=2, \\ \frac{p}{\omega_{k-1}}\zeta^{1-k}aa'\theta_{k-2,p-1}\varphi & \textit{if } 2<k. \end{cases}$$

$$\pi\eta_l = \begin{cases} \frac{p}{\omega_{p-1}}\frac{\omega_{p-2}}{\omega_{p-1}}b'\theta_{p-1,p-2}\chi & \textit{if } l=0, \\ -\frac{p}{\omega_1}b'(a\zeta\tau_{p-1}\varphi + b\tau_{p-1}\chi) & \textit{if } l=2, \\ \frac{p}{\omega_{p-1}}\frac{\omega_{l-2}}{\omega_{p-1}}bb'\theta_{p-1,l-2}\chi & \textit{if } 2<l. \end{cases}$$

$$\theta_{k,l}\mu_{k',0} = \begin{cases} \frac{p}{\omega_1}\zeta^{k'}a\tau_{k+k'-1}\varphi & \textit{if } 1<k+k'-1<p \textit{ and } l=1, \\ 0 & \textit{if } 1<k+k'-1<p \textit{ and } 1<l, \\ \frac{p}{\omega_1}\zeta^{-(k-1)}ad\kappa & \textit{if } k+k'-1=p \textit{ and } l=1, \\ -\frac{p}{\omega_l}\zeta^{-l(k-1)}ab\sigma_{l-1}\chi & \textit{if } k+k'-1=p \textit{ and } 1<l, \\ \frac{p}{\omega_1}\zeta^{k'}a^2\tau_{k+k'-1-p}\varphi & \textit{if } p<k+k'-1 \textit{ and } l=1, \\ 0 & \textit{if } p<k+k'-1 \textit{ and } 1<l. \end{cases}$$

$$\theta_{k,l}\mu_{0,l'} = \begin{cases} -\frac{p}{\omega_l}b\sigma_{l+l'-1}\chi & \textit{if } 1<l+l'-1<p \textit{ and } k=1, \\ 0 & \textit{if } 1<l+l'-1<p \textit{ and } 1<k, \\ \frac{p}{\omega_l}bd\kappa & \textit{if } l+l'-1=p \textit{ and } k=1, \\ \frac{p}{\omega_l}ab\tau_{k-1}\varphi & \textit{if } l+l'-1=p \textit{ and } 1<k, \\ -\frac{p}{\omega_l}b^2\sigma_{l+l'-1-p}\chi & \textit{if } p<l+l'-1 \textit{ and } k=1, \\ 0 & \textit{if } p<l+l'-1 \textit{ and } 1<k. \end{cases}$$

$$\theta_{k,l}\rho_{k'} = \begin{cases} \frac{p}{\omega_{p-1}}\zeta^{-l}a\sigma_l\varphi - \frac{p}{\omega_l}b\sigma_l\chi & \textit{if } k+k'-1=0 \\ & \quad (\textit{i.e. } k=1, k'=0), \\ \frac{p}{\omega_{k'-1}}\zeta^{l(k'-1)}a\theta_{k+k'-1,l}\varphi & \textit{if } 0<k+k'-1<p, \\ \frac{p}{\omega_{k'-1}}\zeta^{l(k'-1)}a^2\sigma_l\varphi - \frac{p}{\omega_l}\zeta^{k'l}ab\sigma_l\chi & \textit{if } k+k'-1=p, \\ \frac{p}{\omega_{k'-1}}\zeta^{l(k'-1)}a^2\theta_{k+k'-1-p,l}\varphi & \textit{if } p<k+k'-1. \end{cases}$$

$$\theta_{k,l}\eta_{l'} = \begin{cases} \frac{p}{\omega_1}\zeta d(a'\tau_k\varphi + \frac{\omega_k}{\omega_1}b'\tau_k\chi) & \textit{if } l+l'-1=0 \ (\textit{i.e. } l=1, l'=0), \\ \frac{p}{\omega_{p-1+l'}}\frac{\omega_{l+l'-1}}{\omega_l}b\theta_{k,l+l'-1}\chi & \textit{if } 0<l+l'-1<p, \\ \frac{p}{\omega_l}\zeta^l bd(a'\tau_k\varphi + \frac{\omega_k}{\omega_l}b'\tau_k\chi) & \textit{if } l+l'-1=p, \\ \frac{p}{\omega_{p-1+l'}}\frac{\omega_{l+l'-1}}{\omega_l}b^2\theta_{k,l+l'-1-p}\chi & \textit{if } p<l+l'-1. \end{cases}$$

$$\theta_{k,l}\psi = \begin{cases} \frac{p}{\omega_1}d\kappa & \textit{if } k=1 \textit{ and } l=1, \\ -\frac{p}{\omega_l}b\sigma_{l-1}\chi & \textit{if } k=1 \textit{ and } 1<l, \\ \frac{p}{\omega_1}a\tau_{k-1}\varphi & \textit{if } 1<k \textit{ and } l=1, \\ 0 & \textit{if } 1<k \textit{ and } 1<l. \end{cases}$$

The relations in $HH^4(\Gamma)$:

$\psi\psi = \psi\mu_{k,0} = \psi\mu_{0,l} = \mu_{k,0}\mu_{k',0} = \mu_{0,l}\mu_{0,l'} = 0.$
$\pi\kappa = a'b'(a\psi\varphi - \zeta^{-1}b\psi\chi).$

$$\tau_k\kappa = \begin{cases} b'\rho_{k+1}\chi & \text{if } k < p-1, \\ b'a\rho_0\chi & \text{if } k = p-1. \end{cases}$$

$$\sigma_l\kappa = \begin{cases} a'\eta_{l+1}\varphi & \text{if } l < p-1, \\ a'b\eta_0\varphi & \text{if } l = p-1. \end{cases}$$

$$\theta_{k,l}\kappa = \begin{cases} a'\sigma_l\tau_k\varphi - \frac{\omega_k}{\omega_l}\zeta^l b'\sigma_l\tau_k\chi & \text{if } k < p-1 \text{ and } l < p-1, \\ a'b\mu_{k+1,0}\varphi - \frac{\omega_k}{\omega_{p-1}}\zeta^{p-1}b'b\mu_{k+1,0}\chi & \text{if } k < p-1 \text{ and } l = p-1, \\ aa'\mu_{0,l+1}\varphi - \frac{\omega_{p-1}}{\omega_l}\zeta^l ab'\mu_{0,l+1}\chi & \text{if } k = p-1 \text{ and } l < p-1. \end{cases}$$

$$\psi\rho_k = \begin{cases} \frac{p}{\omega_{p-1}}\mu_{p-1,0}\varphi & \text{if } k = 0, \\ \frac{p}{\omega_1}a\eta_0\varphi & \text{if } k = 2, \\ \frac{p}{\omega_{k-1}}a\mu_{k-1,0}\varphi & \text{if } 2 < k. \end{cases}$$

$$\psi\eta_l = \begin{cases} \frac{p}{\omega_{p-1}}\mu_{0,p-1}\chi & \text{if } l = 0, \\ \frac{p}{\omega_1}b\rho_0\chi & \text{if } l = 2, \\ \frac{p}{\omega_{l-1}}b\mu_{0,l-1}\chi & \text{if } 2 < l. \end{cases}$$

$$\rho_k\mu_{k',0} = \begin{cases} \frac{p}{\omega_{p-1}}a\eta_0\varphi & \text{if } k+k'-2 = 0 \ (i.e.\ k = 0, k' = 2), \\ \frac{p}{\omega_{p-1}}a\mu_{k+k'-1,0}\varphi & \text{if } 0 < k+k'-2 < p, \\ \frac{p}{\omega_{k-1}}a^2\eta_0\varphi & \text{if } k+k'-2 = p, \\ \frac{p}{\omega_{p-1}}a^2\mu_{k+k'-1-p,0}\varphi & \text{if } p < k+k'-2. \end{cases}$$

$$\eta_l\mu_{0,l'} = \begin{cases} -\frac{p}{\omega_{p-1}}b\rho_0\chi & \text{if } l+l'-2 = 0 \ (i.e.\ l = 0, l' = 2), \\ -\frac{p}{\omega_{p-1}}b\mu_{0,l+l'-1}\varphi & \text{if } 0 < l+l'-2 < p, \\ -\frac{p}{\omega_{l-1}}b^2\rho_0\chi & \text{if } l+l'-2 = p, \\ -\frac{p}{\omega_{p-1}}b^2\mu_{0,l+l'-1-p}\varphi & \text{if } p < l+l'-2. \end{cases}$$

$$\rho_k\mu_{0,l} = \begin{cases} \frac{p}{\omega_{p-1}}\sigma_{l-1}\tau_{p-2}\varphi & \text{if } k = 0, \\ \frac{p}{\omega_1}a\eta_l\varphi & \text{if } k = 2, \\ \frac{p}{\omega_{k-1}}a\sigma_{l-1}\tau_{k-2}\varphi & \text{if } 2 < k. \end{cases}$$

$$\eta_l\mu_{k,0} = \begin{cases} \frac{p}{\omega_{p-1}}\zeta^{-k}\sigma_{p-2}\tau_{k-1}\chi & \text{if } l = 0, \\ \frac{p}{\omega_1}\zeta^k b\rho_k\chi & \text{if } l = 2, \\ \frac{p}{\omega_{l-1}}\zeta^{k(l-1)}b\sigma_{l-2}\tau_{k-1}\chi & \text{if } 2 < l. \end{cases}$$

$$\rho_k\rho_{k'} = \begin{cases} \frac{p}{\omega_{p-1}}\frac{\omega_{p-2}}{\omega_{p-1}}\rho_{p-1}\varphi & \text{if } k = k' = 0, \\ -(\frac{p}{\omega_{k-1}}a)^2\zeta^{k-1}\varphi\varphi + \frac{p(p-1)}{2}\frac{p}{\omega_{k-1}}ab\varphi\chi & \text{if } k+k'-2 = 0, \\ \frac{p}{\omega_{p-1+k}}\frac{\omega_{p-2+k+k'}}{\omega_{p-1+k'}}a\rho_{k+k'-1}\varphi & \text{if } 0 < k+k'-2 < p-1, \\ \frac{p}{\omega_{p-1+k}}\frac{\omega_{p-1}}{\omega_{p-k}}a^2\rho_0\varphi & \text{if } k+k'-2 = p-1, \\ -(\frac{p}{\omega_{k-1}}a)^2\zeta^{k-1}a\varphi\varphi + \frac{p(p-1)}{2}\frac{p}{\omega_{k-1}}a^2b\varphi\chi & \text{if } k+k'-2 = p, \\ \frac{p}{\omega_{p-1+k}}\frac{\omega_{p-2+k+k'}}{\omega_{p-1+k'}}a^2\rho_{k+k'-1-p}\varphi & \text{if } p < k+k'-2. \end{cases}$$

$$\rho_k\eta_l = \begin{cases} \frac{p}{\omega_{p-1}}a\psi\varphi + \frac{p}{\omega_{p-1}}b\psi\chi & \textit{if } k=0 \textit{ and } l=0, \\ \frac{p}{\omega_{p-1}}a\mu_{0,l}\varphi + \frac{p}{\omega_{l-1}}b\mu_{0,l}\chi & \textit{if } k=0 \textit{ and } 0<l, \\ \frac{p}{\omega_{k-1}}a\mu_{k,0}\varphi + \frac{p}{\omega_{p-1}}b\mu_{k,0}\chi & \textit{if } 0<k \textit{ and } l=0, \\ \frac{p}{\omega_{k-1}}a\sigma_{l-1}\tau_{k-1}\varphi + \frac{p}{\omega_{l-1}}b\sigma_{l-1}\tau_{k-1}\chi & \textit{if } 0<k \textit{ and } 0<l. \end{cases}$$

$$\eta_l\eta_{l'} = \begin{cases} \frac{p}{\omega_{p-1}}\frac{\omega_{p-2}}{\omega_{p-1}}\eta_{p-1}\chi & \textit{if } l=l'=0, \\ -\frac{p(p-1)}{2}\frac{p}{\omega_{l-1}}\zeta^{l-1}ab\varphi\chi + (\frac{p}{\omega_{l-1}}b)^2\zeta^{l-l}\chi\chi & \textit{if } l+l'-2=0, \\ \frac{p}{\omega_{p-1+l}}\frac{\omega_{p-2+l+l'}}{\omega_{p-1+l'}}b\eta_{l+l'-1}\chi & \textit{if } 0<l+l'-2<p-1, \\ \frac{p}{\omega_{p-1+l}}\frac{\omega_{p-1}}{\omega_{p-l}}b^2\eta_0\chi & \textit{if } l+l'-2=p-1, \\ -\frac{p(p-1)}{2}\frac{p}{\omega_{l-1}}\zeta^{l-1}ab^2\varphi\chi + (\frac{p}{\omega_{l-1}}b)^2\zeta^{l-1}b\chi\chi & \textit{if } l+l'-2=p, \\ \frac{p}{\omega_{p-1+l}}\frac{\omega_{p-2+l+l'}}{\omega_{p-1+l'}}b^2\eta_{l+l'-1-p}\chi & \textit{if } p<l+l'-2. \end{cases}$$

$$\mu_{k,0}\mu_{0,l} = \begin{cases} \frac{p^2}{\omega_1^2}\zeta ab\varphi\chi & \textit{if } k=2 \textit{ and } l=2, \\ 0 & \textit{if } 2<k \textit{ or } 2<l. \end{cases}$$

The relations in $HH^5(\Gamma)$:

$$\psi\kappa = -\frac{p}{\omega_1}\zeta\varphi\chi\pi.$$

$$\rho_k\kappa = \begin{cases} \frac{p}{\omega_{p-1}}a'\tau_{p-1}\varphi\varphi + \frac{p(p-1)}{2}b'\tau_{p-1}\varphi\chi & \textit{if } k=0, \\ \frac{p}{\omega_{k-1}}aa'\tau_{k-1}\varphi\varphi + \frac{p(p-1)}{2}ab'\tau_{k-1}\varphi\chi & \textit{if } 0<k. \end{cases}$$

$$\eta_l\kappa = \begin{cases} \frac{p(p-1)}{2}a'\sigma_{p-1}\varphi\chi + \frac{p}{\omega_1}b'\sigma_{p-1}\chi\chi & \textit{if } l=0, \\ \frac{p(p-1)}{2}ba'\sigma_{l-1}\varphi\chi + \frac{p}{\omega_{1-l}}bb'\sigma_{l-1}\chi\chi & \textit{if } 0<l. \end{cases}$$

$$\mu_{k,0}\kappa = \frac{p}{\omega_{k-1}}a'\theta_{k-1,p-1}\varphi\chi.$$

$$\mu_{0,l}\kappa = \frac{p}{\omega_{p-1}}b'\theta_{p-1,l-1}\varphi\chi.$$

The relation in $HH^6(\Gamma)$:

$$\kappa\kappa = \frac{p(p-1)}{2}a'b'\varphi\chi(a\varphi + b\chi).$$

Last, we consider the Hochschild cohomology ring $HH^*(\Gamma)$ in the special case $|a| = |b| = 1$.

For example, if $p \geq 3$ and $a = b = 1$, then we have the following relations from Theorem 2.3:

$$\begin{aligned} \sigma_{p-1}\tau_k &= \mu_{k+1,0} \quad \text{for } 1 \leq k < p-1, \\ \sigma_l\tau_{p-1} &= \mu_{0,l+1} \quad \text{for } 1 \leq l < p-1, \\ \sigma_{p-1}\tau_{p-1} &= \psi, \\ \sigma_k\theta_{k,p-k} &= \zeta^{k(k+1)}\rho_{k+1} \quad \text{for } 1 \leq k < p-1, \\ \sigma_{p-1}\theta_{p-1,1} &= \rho_0, \\ \tau_{p-1}\theta_{1,l} &= -\zeta\eta_{l+1} \quad \text{for } 1 \leq l < p-1, \\ \tau_{p-1}\theta_{1,p-1} &= -\zeta\eta_0. \end{aligned}$$

Hence, we have the following corollary:

Corollary 2.4. *Let $p \geq 3$ be a prime number and $|a| = |b| = 1$. Then the Hochschild cohomology ring $HH^*(\Gamma)$ is the graded commutative ring generated by the following $p^2 + 2$ elements:*

$\sigma_l, \tau_k, \theta_{k',l'}, \pi \in HH^1(\Gamma)$ *for* $1 \leq k, k', l, l' \leq p-1$ *with* $(k', l') \neq (p-1, p-1)$,
$\varphi, \chi \in HH^2(\Gamma), \quad \kappa \in HH^3(\Gamma)$.

2.5 The ring structure of $HH^*(\Gamma)$ in the case $p = 2$

In the last section, we deal with the case $p = 2$. Then Γ is a generalized quaternion algebra over $\mathbb{Z}$:

$$\Gamma = \mathbb{Z}1 \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}ij, \quad i^2 = a, j^2 = b, ji = -ij \ (a, b \in \mathbb{Z}, \neq 0).$$

In that case, $\zeta = -1$ and $R = \mathbb{Z}$ and the diagonal approximation map Φ is

$$\Phi_{s,t;s',t'}(c_{s+t,s'+t'}) = c_{s,t} \otimes_\Gamma c_{s',t'},$$

hence, the cup product $\smile$ is

$$\alpha \smile \beta = \alpha\beta$$

for $\alpha \in \Gamma^{s,t}$ and $\beta \in \Gamma^{s',t'}$. Furthermore, we note that the following relations hold:

$$\begin{aligned}
\pi\pi &= (a'b'a, 0, a'b'b) = a'b'a\varphi + a'b'b\chi, \\
\pi\psi &= (0, a'j, -b'i, 0) = \kappa, \\
\psi\psi &= (0, 0, 1, 0, 0),
\end{aligned}$$

where d is the greatest common divisor of a and b, and set $a' = a/d$, $b' = b/d$.

Hence we have the following theorem. This result was already known in [4], and also [16] for a special case.

Theorem 2.5 ([4, Theorem 3.8])**.** *Let $p = 2$ and a, b any nonzero integers. Then the Hochschild cohomology ring $HH^*(\Gamma)$ is the graded commutative ring generated by at most the eight elements*

$$\sigma_1, \tau_1, \pi \in HH^1(\Gamma), \quad \varphi, \psi, \chi, \eta_0, \rho_0 \in HH^2(\Gamma)$$

with the following relations.
The relations in $HH^1(\Gamma)$:

$$2\sigma_1 = 2\tau_1 = 2d\pi = 0.$$

The relations in $HH^2(\Gamma)$:

$$\begin{aligned}
&2a\varphi = 2d\psi = 2b\chi = 2\rho_0 = 2\eta_0 = 0, \\
&\sigma_1\sigma_1 = ab\varphi, \quad \sigma_1\tau_1 = ab\psi, \quad \sigma_1\pi = b'a\rho_0, \\
&\tau_1\tau_1 = ab\chi, \quad \tau_1\pi = a'b\eta_0, \quad \pi\pi = a'b'(a\varphi + b\chi).
\end{aligned}$$

The relations in $HH^3(\Gamma)$:

$$\begin{aligned}&\tau_1\varphi = \sigma_1\psi, \quad \tau_1\psi = \sigma_1\chi, \quad \tau_1\eta_0 = d\pi\chi,\\ &\tau_1\rho_0 = \sigma_1\eta_0 = d\pi\psi, \quad \sigma_1\rho_0 = d\pi\varphi, \quad \pi\rho_0 = a'\sigma_1\varphi + b'\sigma_1\chi,\\ &\pi\eta_0 = a'\tau_1\varphi + b'\tau_1\chi.\end{aligned}$$

The relations in $HH^4(\Gamma)$:

$$\begin{aligned}&\varphi\chi = \psi\psi, \quad \varphi\eta_0 = \psi\rho_0, \quad \psi\eta_0 = \chi\rho_0,\\ &\rho_0\rho_0 = a\varphi\varphi + b\psi\psi, \quad \rho_0\eta_0 = a\varphi\psi + b\psi\chi, \quad \eta_0\eta_0 = a\psi\psi + b\chi\chi.\end{aligned}$$

In particular, if $|a| = |b| = 1$, then we have the following result of [12] from Theorem 2.5:

Corollary 2.6 ([12])**.** *If* $p = 2$ *and* $|a| = |b| = 1$*, then we have the ring isomorphism*

$$HH^*(\Gamma) \cong \mathbb{Z}[x, y, z]/(2x, 2y, 2z, x^2 + y^2 + z^2).$$

Chapter 3

Ramification in Kummer extensions arising from algebraic tori

3.1 Some definitions

In this section, we give some definitions we will use in the later sections.

Definition 3.1. ([11]) Let G be a commutative algebraic group. We put

$$\widehat{G} = \mathrm{Hom}(G, \mathbb{G}_m),$$

the set of all rational homomorphisms of G in $\mathbb{G}_m$. We call $\widehat{G}$ the character group of G.

Definition 3.2. Let k be an algebraic number field, K a finite Galois extension over k of degree d . Let $\mathbb{G}_m$ be a multiplicative group defined over K and (α_i) be a basis of K as a vector space over k. The Weil restriction $R_{K/k}(\mathbb{G}_m)$ is defined to be

$$R_{K/k}(\mathbb{G}_m) = \left\{ A \in GL_d(\overline{k}) \mid a\boldsymbol{x} = \boldsymbol{x}A \, , \, a = \sum_{i=1}^{d} a_i \alpha_i \neq 0 \, , \, a_i \in \overline{k} \, , \, \boldsymbol{x} = (\alpha_1 \, \alpha_2 \, \ldots \, \alpha_d) \right\},$$

where $\overline{k}$ is an algebraic closure of k.

Definition 3.3. Let T be an algebraic torus. We say a homomorphism $\lambda : T \to T$ is a self-isogeny if λ is surjective with finite kernel.

3.2 Introduction

Let p be a fixed odd prime. Let $\mathbb{Q}_p$ be the field of p-adic numbers and $\overline{\mathbb{Q}_p}$ an algebraic closure of $\mathbb{Q}_p$. We assume that any algebraic extensions of $\mathbb{Q}_p$ are contained in $\overline{\mathbb{Q}_p}$. Let l be an odd prime and denote by ζ_l a primitive l-th root of unity in $\overline{\mathbb{Q}_p}$. Let k be an unramified extension of $\mathbb{Q}_p$ of degree n, and $k_z = k(\zeta_l)$. Let K be an intermediate field of k_z/k, and T the Weil restriction $R_{k_z/K}\mathbb{G}_m$ of multiplicative group $\mathbb{G}_m$ to K. We assume that there exists a self-isogeny λ on T of degree l whose kernel $\mathrm{Ker}\lambda$ is contained in the group $T(K)$ of K-rational points of T. Several conditions for the existence of such λ are given in [8]. Also some examples of λ are found in [8]. Under this assumption, we have a following isomorphism

$$\kappa_K : T(K)/\lambda T(K) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(\overline{K}/K), \mathrm{Ker}\lambda(\overline{K}))$$

proved by M. Kida in [8]. Here $\overline{K}$ is an algebraic closure of K in $\overline{\mathbb{Q}_p}$ and the right hand side is the group of continuous homomorphisms. In this theorem, if $K = k_z$, then we get the classical Kummer theory. Hence this is a generalization of the Kummer theory for fields without roots of unity. In particular, any cyclic extensions of degree l over K can be written as $K(\lambda^{-1}(P))$ with $P \in T(K)$. In this chapter, we determine the ramification in $L = K(\lambda^{-1}(P))$ over K.

In the case where K is a finite extension of $k = \mathbb{Q}(\zeta_l + \zeta_l^{-1})$, the ramification in the cyclic extension L/K is studied by T. Komatsu in [9] using an algebraic torus of dimension 1 which consists of kernel of norm map in a quadratic extension. We shall generalize his result to the case $\zeta_l + \zeta_l^{-1} \notin K$. Since the problem is obviously local, we assume that base field K is a local field.

The following notations will be used throughout this chapter. Let v_{k_z} (resp. v_K) be the discrete valuation of k_z (resp. K), normalized by $v_{k_z}(k_z^\times) = \mathbb{Z}$ (resp. $v_K(K^\times) = \mathbb{Z}$). Let $U(k_z)$ be the group of units in k_z defined by

$$U(k_z) = \{u \in k_z \,|\, v_{k_z}(u) = 0\}\,, \tag{3.1}$$

and $U^{(i)}(k_z)$ the groups of higher principal units defined by

$$U^{(i)}(k_z) = \{u \in k_z \,|\, v_{k_z}(u-1) \geq i\} \quad i \in \mathbb{N}\,. \tag{3.2}$$

Our main theorem is stated as follows.

Theorem 3.4 (See Theorem 3.15)**.** *Let $p = l$ be an odd prime. Let m be the degree of the extension K/k. Let $\widehat{T} = \mathrm{Hom}(T, \mathbb{G}_{m,k_z})$ be the group of characters of T. For each $i \geq 1$ we set $T^{(i)}(K) = \mathrm{Hom}_{\mathrm{Gal}(k_z/K)}(\widehat{T}, U^{(i)}(k_z))$.*

If $P \in T^{(jd+1)}(K)$ and $P \notin T^{(jd+2)}(K)$ for some $0 \leq j \leq m$, then the conductor $\mathfrak{f}$ of $K(\lambda^{-1}(P))/K$ satisfies

$$v_K(\mathfrak{f}) = \begin{cases} m - j + 1 & 0 \leq j < m, \\ 0 & j = m. \end{cases}$$

In particular, $K(\lambda^{-1}(P))/K$ is an unramified extension if and only if $P \in T^{(l)}(K)$.

Using this theorem, we can calculate the number of cyclic extensions of degree l over K with given conductor $\mathfrak{f}$ up to isomorphism in $\overline{\mathbb{Q}_p}$ (see Theorem 3.16).

The outline of the chapter is as follows. In Section 3.3, we discuss the $\mathrm{Gal}(k_z/K)$-module structure of $S_l(k_z^\times) = k_z^\times/(k_z^\times)^l$, and determine the structure of S_1^K which is a certain eigenspace of $S_l(k_z^\times)$. In Section 3.4, we prove the main theorem using Hecke's theorem[1], which describes the ramification in a cyclic extensions of k_z.

Remark 3.5. In the case of $l \mid p^n - 1$, we can use the classical Kummer theory since $K = k_z$. Therefore we may assume the condition $l \nmid p^n - 1$. Theorem 3.4 deals with the difficult case $p = l$. For the easier case with $p \neq l$, see Proposition 3.19.

3.3 Galois module structure of $S_l(k_z^\times)$

Let $p = l$ be an odd prime and k an unramified extension of $\mathbb{Q}_l$ of degree n. We denote by k_z the field $k(\zeta_l)$ as above. Let K be an intermediate field of k_z/k of degree m over k. Set $d = (l-1)/m$. The Galois groups $\mathrm{Gal}(k_z/k)$ and $\mathrm{Gal}(k_z/K)$ act naturally on the group $S_l(k_z^\times) = k_z^\times/(k_z^\times)^l$. In this section, we consider the structure of $S_l(k_z^\times)$ as Galois modules.

Let τ be a fixed generator of $\mathrm{Gal}(k_z/k)$. Then we have $\mathrm{Gal}(k_z/k) = \langle\tau\rangle$ and $\mathrm{Gal}(k_z/K) = \langle\tau^m\rangle$. Let g be a primitive root modulo l such that $\tau(\zeta_l) = \zeta_l^g$. For $1 \le i \le l-1$, set

$$e_i(k_z/k) := \frac{1}{l-1}\sum_{1\le j\le l-1}(g^m)^{-ij}\tau^j,$$

and for $1 \le i \le d$, set

$$e_i(k_z/K) := \frac{1}{d}\sum_{1\le j\le d}(g^m)^{-ij}(\tau^m)^j.$$

It is known that $e_i(k_z/k)$'s (resp. $e_i(k_z/K)$'s) are orthogonal idempotents in the group ring $\mathbb{F}_l[\mathrm{Gal}(k_z/k)]$ (resp. $\mathbb{F}_l[\mathrm{Gal}(k_z/K)]$) over the finite field $\mathbb{F}_l$ of l elements. Therefore we can write

$$S_l(k_z^\times) = \bigoplus_{1\le i\le l-1} e_i(k_z/k)S_l(k_z^\times).$$

We set S_i^k as the eigenspace corresponding to $e_i(k_z/k)$, that is,

$$S_i^k := e_i(k_z/k)S_l(k_z^\times) = \left\{e_i(k_z/k)(x)\,|\,x \in S_l(k_z^\times)\right\}. \tag{3.3}$$

Similarly, we define S_i^K by the following formula

$$S_i^K := e_i(k_z/K)S_l(k_z^\times) = \left\{e_i(k_z/K)(x)\,|\,x \in S_l(k_z^\times)\right\}. \tag{3.4}$$

If λ is the self-isogeny on T of degree l inducing the Kummer duality κ_K, then S_1^K and $T(K)/\lambda T(K)$ are closely related to each other as follows.

Proposition 3.6. *Subgroups in S_1^K are in one-to-one correspondence to those in $T(K)/\lambda T(K)$.*

Proof. Since T is an algebraic torus over K, we have an isomorphism $\psi : T(\overline{K}) \cong (\overline{K}^\times)^d$. If we set $P \in T(K)$, then $\psi(P) = (\alpha_1, \ldots, \alpha_d)$ for some $\alpha_i \in k_z^\times$. First we define a map φ_K from $T(K)/\lambda T(K)$ to S_1^K. We note that if $P \in \lambda T(K)$, then $K(\lambda^{-1}(P)) = K$. So we assume that $P \in T(K)$ does not belong to $\lambda T(K)$, then $K(\lambda^{-1}(P))$ is a cyclic extension of K of degree l [8, Theorem 1.1], and $K(\lambda^{-1}(P))(\zeta_l) = k_z(\sqrt[l]{\alpha_1^{e_1}})$ [8, Proposition 6.3], which $\alpha_1^{e_1}$ is $e_1(k_z/k)(\alpha_1)$. Also we know that if $u = 1$ in S_1^K, then $k_z(\sqrt[l]{u}) = k_z$, so assume that $u \in S_1^K$ is not the identity, then $k_z(\sqrt[l]{u})$ is a cyclic extension of k_z of degree l, and there exists a cyclic extension L of K of degree l such that $L(\zeta_l) = k_z(\sqrt[l]{u})$ [1, Theorem 5.3.5]. On the other hand, it is known that the fields $k_z(\sqrt[l]{u^i})$ $(1 \le i \le l-1)$ are mutually equal by

Kummer theory, for $1 \leq i \leq l-1$. Hence we define $\varphi_K(P) = \langle \alpha_1^{e_1} \rangle$ such that $K(\lambda^{-1}(P))(\zeta_l) = k_z(\sqrt[l]{\alpha_1^{e_1}})$. It is easy to check that φ_K is a surjective map.

Next, we assume that $\varphi_K(P) = \langle \alpha \rangle$, $\varphi_K(Q) = \langle \beta \rangle$ and $k_z(\sqrt[l]{\alpha}) = k_z(\sqrt[l]{\beta})$ for $P, Q \in T(K) \backslash \lambda T(K)$. Let L_1 (resp. L_2) be a cyclic extension of degree l over K which satisfies $L_1(\zeta_l) = k_z(\sqrt[l]{\alpha})$ (resp. $L_2(\zeta_l) = k_z(\sqrt[l]{\beta})$). Since $k_z(\sqrt[l]{\alpha}) = k_z(\sqrt[l]{\beta})$, we have $L_1 = L_2$. Therefore, we show $\langle P \rangle = \langle Q \rangle$ in $T(K)/\lambda T(K)$, that is, φ_K is bijective map. □

For simplicity, we shall identify an element of S_1^K with a coset of $S_l(k_z^\times)$ which contains the element in the following discussion.

By Proposition 3.6, we may study the structure of S_1^K instead of that of $T(K)/\lambda T(K)$. Thus we consider the Galois module structures of $S_l(k_z^\times)$, S_i^k and S_i^K. A basis of $U^{(1)}(k_z)$ as a $\mathbb{Z}_l$-module is given in [2]. Let ξ be a primitive $(l^n - 1)$-th root of unity in k.

Proposition 3.7 ([2, I (6.4)])**.** *The $(l-1)n+1$ elements*

$$u_l := 1 + \eta \pi^l,$$
$$u_{i,j} := 1 + \xi^i \pi^j$$

constitute a $\mathbb{Z}_l$-basis of $U^{(1)}(k_z)$, where i and j run over $0 \leq i \leq n-1$ and $1 \leq j \leq l-1$. Here π is a prime element of k_z, and η is ξ^i for some $i \geq 0$ such that $1 + \xi^i \pi^l$ is not an l-th power in $U^{(1)}(k_z)$.

The structure of the multiplicative group $k_z^\times$ is given by $k_z^\times \cong \langle \pi \rangle \times \langle \xi \rangle \times U^{(1)}(k_z)$. Noting that we have $\langle \xi \rangle / (\langle \xi \rangle)^l = 1$ since $(\, l^n - 1, l\,) = 1$, we readily get the following proposition.

Proposition 3.8. *If $l = p$, then $(l-1)n+2$ elements π, u_l, and $u_{i,j}$ constitute an $\mathbb{F}_l$-basis of $S_l(k_z^\times)$, where i and j run over $0 \leq i \leq n-1$ and $1 \leq j \leq l-1$.*

In the following, we fix a prime element $\pi = \zeta_l - 1$, and we consider the action of $\tau \in \mathrm{Gal}(k_z/K)$ on $S_l(k_z^\times)$.

Lemma 3.9. *The matrix X of τ with respect to the basis*

$$(\pi, u_l, u_{n-1,l-1}, u_{n-2,l-1}, \ldots, u_{0,1})$$

is given by the following formula:

$$X = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ * & g^l & * & \cdots & * \\ * & 0 & A_{l-1} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & * \\ * & 0 & \cdots & 0 & A_1 \end{pmatrix}.$$

Here for $1 \leq j \leq l-1$, A_j are $n \times n$ matrices written by

$$A_j = \begin{pmatrix} g^j & 0 & 0 & 0 & 0 \\ 0 & g^j & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & g^j & 0 \\ 0 & 0 & 0 & 0 & g^j \end{pmatrix}.$$

Proof. To avoid heavy notation, we rename the basis $(v_1, \ldots, v_{n(l-1)+2}) = (\pi, u_l, u_{n-1,l-1}, \ldots, u_{0,1})$ in this proof. Then we have $u_{i,j} = v_{(l-j)n+2-i}$. We set $X = (x_{st})$. First we show that the first columm is 0 except x_{11}. Recall that g is the chosen primitive root satisfying $\tau(\zeta_l) = \zeta_l^g$. We define

$$\omega := \frac{\zeta_l^g - 1}{\zeta_l - 1} = \zeta_l^{g-1} + \cdots + \zeta_l + 1,$$

then we can write $\tau(\pi) = \omega\pi$. Since ω is a unit element, we have $v_{k_z}(\tau(\pi)) = 1$. Moreover we get $v_{k_z}(\tau(v_i)) = 0$ for all $i \geq 2$ because

$$\tau(v_2) = \tau(1 + \eta\pi^l) = 1 + \eta(\omega\pi)^l$$

and

$$\tau(v_i) = \tau(u_{a,b}) = \tau(1 + \xi^a\pi^b) = 1 + \xi^a(\omega\pi)^b$$

for all $i \geq 3$ and some a and b. Hence the first columm of X satisfies $x_{1\,1} = 1$ and $x_{1\,t} = 0$ for all $t \geq 2$.

Next we show that any A_j are diagonal matrices whose diagonal entries are integer powers of g. Since $\omega \equiv g \pmod{\pi}$ and $\tau(\xi) = \xi$, we have

$$\tau(u_{i,j}) \equiv 1 + \xi^i\omega^j\pi^j \equiv 1 + g^j\xi^i\pi^j \equiv (1 + \xi^i\pi^j)^{g^j} \pmod{\pi^{j+1}}$$

for each j. Hence we get $\tau(u_{i,j}) \equiv {u_{i,j}}^{g^j} \pmod{\pi^{j+1}}$. Since ξ^i's are independent, any A_j are diagonal matrices. In a similar way, we can prove the assertion for $\tau(u_l)$.

Finally we show that x_{st} equals 0 for any $s > 2$ and $s > t$. To do this, pick up a j-th columm of X as $(x_1, \ldots, x_{(l-1)n+2})$, and consider the action of τ on the j-th base v_j for $j > 1$. Then we may prove $x_{j'} = 0$ for all $j' > j$. We set $v_j = u_{a,b}$ and write $\tau(v_j)$ as follows:

$$\tau(v_j) = \prod_{2 \leq i \leq (l-1)n+2} v_i^{x_i} \quad \text{in } S_l(k_z^\times). \tag{3.5}$$

Let i' be the maximal number i such that $x_i \neq 0$. Then the right hand side of (3.5) satisfies

$$\prod_{2 \leq i \leq (l-1)n+2} v_i^{x_i} = \prod_{2 \leq i \leq i'} v_i^{x_i}.$$

Hence if $v_{i'} = u_{a',b'}$, then we get

$$v_{k_z}\Big(\prod_{2 \leq i \leq i'} v_i^{x_i} - 1\Big) = b',$$

and the left hand side of (3.5) satisfies

$$\begin{aligned} v_{k_z}(\tau(v_j) - 1) &= v_{k_z}(\tau(\zeta^a\pi^b)) \\ &= a \cdot v_{k_z}(\zeta) + b \cdot v_{k_z}(\omega) + b \cdot v_{k_z}(\pi) \\ &= b. \end{aligned}$$

Thus we get $b' = b$. Moreover, since $\tau(v_j) \equiv {v_j}^{g^m} \pmod{\pi^{m+1}}$ for some $m \geq 0$, we have $a' = a$. Hence we prove $i' = j$ and $x_{j'} = 0$ for all $j' > j$. □

By Lemma 3.9, we get a formula for the dimension of S_i^k for $1 \leq i \leq l-1$.

Proposition 3.10. *For each* $1 \leq i \leq l-1$ *we see*

$$\dim_{\mathbb{F}_l} S_i^k = \begin{cases} n+1 & i = 1 \text{ or } l-1, \\ n & 1 < i < l-1. \end{cases}$$

Proof. Let X be the matrix defined in Lemma 3.9. We calculate the characteristic polynomial of X as follows:

$$\begin{aligned} &(x-1)(x-g^l)(x-g^{l-1})^n \cdots (x-g)^n \\ &= (x-1)^{n+1}(x-g)^{n+1}(x-g^2)^n \cdots (x-g^{l-2})^n; \end{aligned}$$

and the minimal polynomial of X is given by

$$(x-1)(x-g)\cdots(x-g^{l-2}).$$

Hence this polynomial doesn't have multiple roots. Thus the matrix X is diagonalizable, and the dimension of each eigenspace S_i^k coincides with the multiplicity of each eigenvalue g^i. □

Moreover, there is a following relationship between S_i^k and S_i^K.

Lemma 3.11. *For* $1 \leq i \leq d$*, we have*

$$S_i^K = \bigoplus_{\substack{1 \leq j < l-1 \\ j \equiv i \, (\mathrm{mod\, d})}} S_j^k.$$

Proof. For $u \in S_j^k$, we see $\tau^m(u) = g^{mj}u$ since $\tau(u) = g^j u$. Furthermore, any element $u' \in S_{j+d}^k$ satisfies

$$\begin{aligned} \tau^m(u') &= g^{m(j+d)}u' \\ &\equiv g^{mj}u' \pmod{l}. \end{aligned}$$

Thus we have $S_i^K \supset S_{j'}^k$ for any $j' \equiv i \pmod{d}$. Therefore we get

$$S_i^K \supset \bigoplus_{\substack{1 \leq j < l-1 \\ j \equiv i \, (\mathrm{mod}\, d)}} S_j^k. \tag{3.6}$$

However, we know

$$\bigoplus_{1 \leq i \leq d} S_i^K = S_l(k_z^\times)$$

and

$$\bigoplus_{1 \leq i \leq d} \left(\bigoplus_{\substack{1 \leq j \leq l-1 \\ j \equiv i \, (\mathrm{mod}\, d)}} S_j^k \right) = \bigoplus_{1 \leq j \leq l-1} S_j^k = S_l(k_z^\times),$$

thus the assertion follows. □

Hence we get a dimension of S_i^K for $1 \leq i \leq l-1$.

Proposition 3.12. *For each $1 \leq i \leq d$ we get*

$$\dim_{\mathbb{F}_l} S_i^K = \begin{cases} mn+1 & i = 1 \text{ or } d, \\ mn & 1 < i < d. \end{cases}$$

Finally, we determine the basis of S_1^K using Proposition 3.8 and Lemma 3.11.

Theorem 3.13. *Keep the above notations. Then $mn+1$ elements $u_{i,j}$ and u_l constitute an $\mathbb{F}_l$-basis of S_1^K, where i and j run over $0 \leq i \leq n-1$ and $1 \leq j \leq l-1$ and $j \equiv 1 \pmod{d}$.*

3.4 Proof of the Main Theorem

Let us recall the setting in Section 3.2. We have assumed that there exists a self-isogeny λ on $T = R_{k_z/K}\mathbb{G}_m$ of degree l whose kernel is contained in the group $T(K)$ of K-rational points. Let P be a K-rational point on the torus T. Then we have a cyclic extension $L = K(\lambda^{-1}(P))$ over K. In this section, we determine the ramification in L/K using the structure of S_1^K. To do this, first we describe the ramification in the Kummer extension L_z/k_z using Hecke's theorem which we recall now.

Proposition 3.14 ([1, Theorem10.2.9]). *Let π be a prime element in k_z, and $L_z = k_z(\sqrt[l]{\alpha})$ with $\alpha \in S_1^K - \{1\}$. Let $d(L_z/k_z)$ be the discriminant of L_z/k_z. Let a be the largest exponent w such that the congruence*

$$x^l \equiv \alpha \pmod{\pi^{w+v_{k_z}(\alpha)}}$$

has a solution. Then we have:

(i) l is unramified in L_z/k_z if and only if $a = l$;

(ii) l is totally ramified in L_z/k_z if and only if $a \leq l-1$; in that case we have $v_{k_z}(d(L_z/k_z)) = (l-1)(l+1-a)$.

Let $\widehat{T} = \mathrm{Hom}(T, \mathbb{G}_{m,k_z})$ be the group of characters of T, and set $T^{(i)}(K) = \mathrm{Hom}_{\mathrm{Gal}(k_z/K)}(\widehat{T}, U^{(i)}(k_z))$ for $i \geq 1$ (see [11, Section 2]), where $U^{(i)}(k_z)$ are the groups of higher principal units defined by (3.2). We note that $T^{(i)}(K)$'s are subgroups of $\mathrm{Hom}_{\mathrm{Gal}(k_z/K)}(\widehat{T}, U(k_z))$, which is the maximal compact subgroup of $T(K)$.

Now we shall prove the main theorem.

Theorem 3.15. *Let $p = l$ be an odd prime. Let K be a finite extension of k of degree m and set $d = (l-1)/m$. If $P \in T^{(jd+1)}(K)$ and $P \notin T^{(jd+2)}(K)$ for some $0 \leq j \leq m$, then the conductor $\mathfrak{f}$ of the cyclic extension $K(\lambda^{-1}(P))/K$ satisfies*

$$v_K(\mathfrak{f}) = \begin{cases} m-j+1 & 0 \leq j < m, \\ 0 & j = m. \end{cases}$$

In particular, $K(\lambda^{-1}(P))/K$ is an unramified extension if and only if $P \in T^{(l)}(K)$.

Proof. We denote $d(L/K)$ by the discriminant of L/K. Since k_z and L are intermediate fields of L_z/K, we have

$$\mathcal{N}_{L_z/K} = \mathcal{N}_{k_z/K} \circ \mathcal{N}_{L_z/k_z} = \mathcal{N}_{L/K} \circ \mathcal{N}_{L_z/L},$$

using chain rule of norm map. Using this equation, we have

$$\mathcal{N}_{k_z/K}(d(L_z/k_z)) \cdot d(k_z/K)^l = \mathcal{N}_{L/K}(d(L_z/L)) \cdot d(L/K)^d.$$

If $P \in T^{(jd+1)}(K)$ and $P \notin T^{(jd+2)}(K)$ for some $0 \leq j < m$, then $v_{k_z}(d(L_z/k_z)) = (l-1)(l-jd)$ by Proposition 3.14(ii). Now $v_K(\mathcal{N}_{k_z/K}(d(L_z/k_z)))$ equals $(l-1)(l-jd)$ since k_z/K is a totally ramified extension. Since k_z/K is a tamely extension and L_z/L is a tamely and totally extension, we have $v_K(d(k_z/K)^l) = l(d-1)$ and $v_K(\mathcal{N}_{L/K}(d(L_z/L))) = d-1$. Noting that $d(L/K) = \mathfrak{f}^{l-1}$, we get

$$(l-1)(l-jd) + l(d-1) = (d-1) + (l-1)dv_K(\mathfrak{f}).$$

Therefore, we show $v_K(\mathfrak{f}) = m - j + 1$.

For the case of $j = m$, we have $v_K(\mathfrak{f}) = 0$ since L/K is an unramified extension by Proposition 3.14(i). □

In Theorem 3.15, we calculated the conductor of $K(\lambda^{-1}(P))/K$ for some $0 \leq j \leq m$ such that $P \in T^{(jd+1)}(K)$ and $P \notin T^{(jd+2)}(K)$. Therefore, counting the number of such points P, we can essentially calculate the number of cyclic extensions of K of degree l with a fixed conductor.

Theorem 3.16. *Let $p = l$ be an odd prime. Then, for each $0 \leq j < m$, the number of cyclic extensions of $K \subset \overline{\mathbb{Q}_l}$ of degree l whose conductor $\mathfrak{f}$ satisfies $v_K(\mathfrak{f}) = m - j + 1$ is $l^{(m-(j+1))n+1} \cdot (l^n - 1)/(l-1)$ up to isomorphism in $\overline{\mathbb{Q}_l}$.*

Proof. Let r_j be the number of $u \in S_1^K$ such that $u \in U^{(jd+1)}(k_z)$ and $u \notin U^{(jd+2)}(k_z)$. We write

$$u = u_l^{a_l} \prod_{\substack{0 \leq i \leq n-1 \\ 1 \leq j \leq l-1 \\ j \equiv 1 \pmod{d}}} u_{i,j}^{a_{i,j}}$$

with $0 \leq a_{i,j}, a_l \leq l-1$. If $0 \leq j < m$, then $a_{i,j'} = 0$ for any $0 \leq j' < j$ since $v_{k_z}(u-1) = jd+1$. And at least one of $a_{0,jd+1}, \ldots, a_{n-1,jd+1}$ is nonzero. Thus we can calculate r_j as follows:

$$r_j = l^{(mn+1)-n-jn} \cdot (l^n - 1) = l^{(m-(j+1))n+1} \cdot (l^n - 1).$$

On the other hand, it is known that the fields $k_z(\sqrt[l]{u^i})$ $(1 \leq i \leq l-1)$ are mutually isomorphic by Kummer theory, for $1 \leq i \leq l-1$. By Proposition 3.6, a cyclic extension of k_z of degree l is corresponding to a cyclic extension of K of degree l. So we can calculate the number of cyclic extensions L/K by dividing r_j by $l-1$. □

Remark 3.17. In the Theorem 3.16, assuming that there exists an isogeny λ on T of degree l whose kernel is contained in $T(K)$, we calculated the number of cyclic extensions of $K \subset \overline{\mathbb{Q}_l}$ of degree l with a fixed conductor. If there exists no such λ, then there seems to be no known method of counting it.

Theorem 3.15 and 3.16 deal only with the case of $p = l$. Finally, we briefly mention to the case $l \nmid p^n - 1$ and $p \neq l$. Let k be an unramified extension of $\mathbb{Q}_p$ of degree n, $q = p^n$ and keep the above notation.

Since $(l, q-1) = 1$, we see that k_z/k is an unramified extension. And then $u \mapsto u^l$ is an isomorphism map since $v_{k_z}(l) = 0$, thus we get $(U^{(1)}(k_z))^l = U^{(1)}(k_z)$. Hence we have proved the following result.

Proposition 3.18. *If $l \nmid q - 1$ and $l \neq p$, then 2 elements p, ζ_{q-1} constitute an $\mathbb{F}_l$-basis of S_1^K.*

Let τ be a generator of $\mathrm{Gal}(k_z/k)$. Then τ acts trivially on both p and ζ_{q-1}. Thus we have $S_1^K = S_l(k_z^\times)$ for any intermediate field K of k_z/k. Consequently we obtain the following proposition.

Proposition 3.19. *Let p be an odd prime and l a prime satisfying $l \nmid q - 1$ and $p \neq l$. We set $T(U(k_z)) = \mathrm{Hom}_{\mathrm{Gal}(k_z/K)}(\widehat{T}, U(k_z))$. Then, for $P \in T(K)$ with $P \notin \lambda T(K)$, $K(\lambda^{-1}(P))/K$ is a tamely ramified extension if and only if $P \notin T(U(k_z))$; in that case the conductor $\mathfrak{f}$ satisfies $v_K(\mathfrak{f}) = 1$.*

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