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AFFINITY INTEGRAL MANIFOLDS OF LINEAR SINGULARLY PERTURBED SYSTEMS OF IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract. In the present paper sufficient conditions for the existence of affinity integral manifolds of linear singularly perturbed systems of impulsive differential equations are obtained.

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1. Introduction

Let R^n be the n-dimentional Euclidean space with norm $\|\cdot\|$ and let $I = [0,\infty)$. Consider the linear singularly perturbed system

$$(1)\qquad \begin{cases} \dfrac{dx}{dt} = A(t)x + B(t)y, & t \neq \tau_k, \\ \mu\dfrac{dy}{dt} = C(t)x + D(t)y, & t \neq \tau_k, \\ \Delta x = A_k x + B_k y, & t = \tau_k, \\ \Delta y = C_k x + D_k y, & t = \tau_k,\ k = 1, 2, \dots \end{cases}$$

where $\mu > 0$ is small parameter, and $x: I \to R^n$, $y: I \to R^m$, $\Delta x = x(t+0) - x(t-0)$, $\Delta y = y(t+0) - y(t-0)$, $A: I \to R^{m+n}$, $B: I \to R^{m+n}$, $C: I \to R^{m+n}$, $D: I \to R^{n+n}$, $0 < \tau_1 < \tau_2 < \dots$, $\lim_{k\to\infty} \tau_k = \infty$, E_n is unit $n \times n$ matrix, and the constants matrices A_k, B_k, C_k, D_k, $k = 1, 2, \dots$ are $m \times m$, $n \times m$, $m \times n$, $n \times n$ dimensional respectively.

The system (1) is characterized as follows:

1. At the moments $t \neq \tau_k$, $t \in I$, $k = 1, 2, \dots$ the solution $(x(t), y(t))$ of (1) is defined by the differential equation

$$\frac{dx}{dt} = A(t)x + B(t)y,$$
$$\mu\frac{dy}{dt} = C(t)x + D(t)y.$$

2. At the moments $t = \tau_k$, $k = 1, 2, \dots$ the mapping point (t, x, y) (undergoing short period forces as a hit, an impulse etc.) moves from the position $(t, x(t), y(t))$ in the position $(t, x(t) + A_k x(t) + B_k y(t), y(t) + C_k x(t) + D_k y(t))$ "instantly". We assume that the solutions of system (1) are left continuous at the moments of jump i.e.

$$\begin{aligned} x(\tau_k - 0) &= x(\tau_k), \quad y(\tau_k - 0) = y(\tau_k), \\ x(\tau_k + 0) &= x(\tau_k) + A_k x(\tau_k) + B_k y(\tau_k), \\ y(\tau_k + 0) &= y(\tau_k) + C_k x(\tau_k) + D_k y(\tau_k). \end{aligned}$$

2. Preliminary notes.

Definition 1. An arbitrary manifold J in the extended phase space of the system (1) is said to be an integral manifold of (1), if for arbitrary solution $(x(t), y(t))$ from $(t_0, x(t_0), y(t_0)) \in J$, $t_0 > 0$ it follows that $(t, x(t), y(t)) \in J$, $t \geq t_0$.

Definition 2. The integral manifold J is said to be affinity integral manifold of (1) if J is graph of the function $\varphi: I \times R^m \to R^n$, $\varphi(t, x) = Q(t)x + \eta(t, x)$, for which

a) $Q(t)$ is piecewise continuous matrix function with a dimensional $n \times m$ and with points of discontinuities of the first kind at the moments $t = \tau_k$, $k = 1, 2, \dots$ at which is continuous from the left.

b) $\eta: I \times R^m \to R^n$ is a bounded function which is continuous at the variable x and for $t = \tau_k$, $k = 1, 2, \dots$ have discontinuities of the first kind and is continuous from the left.

Definition 3. The function $\varphi(t, x)$ definited on Definition 2 is said to be a parameter function to the integral manifold.

Introduce the following conditions

H1. The matrix $A(t)$ is piecewise continuous with discontinuities of the first kind at the points $t = \tau_k$, $k = 1, 2, \dots$.

H2. $\det(E_m + A_k) \neq 0$, $k = 1, 2, \dots$.

Let $U_k(t, s)$, $k = 1, 2, \dots$, $t \in (\tau_{k-1}, \tau_k]$ is Cauchy's matrix of the linear system

$$\frac{dx}{dt} = A(t)x, \quad (\tau_{k-1} < t \leq \tau_k)$$

and the conditions H1, H2 are met.

Definition 4 ([3]). The matrix $W(t,s)$, where

(2)

$$W(t,s) = \begin{cases} U_k(t,s), \quad t,s \in (\tau_{k-1},\tau_k], \\ U_{k+1}(t,\tau_k+0)(E_m+A_k)U_k(\tau_k,s), \quad \tau_{k-1} < s \le t < \tau_{k+1}, \\ U_k(t,\tau_k)(E_m+A_k)^{-1}\prod_{j=k}^{i+1}(E_m+A_j)U_j(\tau_j,\tau_{j-1}+0)(E_m+A_i)U_i(\tau_i,s), \\ \qquad \text{for } \tau_{i-1} < s \le \tau_i < \tau_k < t \le \tau_{k+1}, \\ U_i(t,\tau_i)\prod_{j=i}^{k-1}(E_m+A_j)^{-1}U_{j+1}(\tau_j+0,\tau_{j+1})(E_m+A_k)^{-1}U_{k+1}(\tau_k+0,s), \\ \qquad \text{for } \tau_{i-1} < t \le \tau_i < \tau_k < s \le \tau_{k+1}. \end{cases}$$

is said to be Cauchy's matrix of the system:

$$\begin{cases} \dfrac{dx}{dt} = A(t)x, \quad t \ne \tau_k, \\ \Delta x = A_k x \qquad t = \tau_k,\ k = 1,2,\dots. \end{cases} \tag{3}$$

It is easily to verify that the following relations are hold

$$\begin{aligned} &W(t,t) = E_m, \\ &W(\tau_k - 0,\tau_k) = W(\tau_k,\tau_k - 0) = E_m, \\ &W(\tau_k + 0, s) = (E_m + A_k)W(\tau_k, s), \\ &W(s,\tau_k + 0) = W(s,\tau_k)(E_m + A_k), \\ &\frac{\partial W}{\partial t} = A(t)W(t,s), \quad (t \ne \tau_k), \\ &\frac{\partial W}{\partial s} = -W(t,s)A(s), \quad (s \ne \tau_k). \end{aligned} \tag{4}$$

Introduce the following condition:

H3. $\det(E_n + D_k) \ne 0$.

H4. The matrix $D(t)$ is piecewise continuous with discontinuities of the first kind at the points $t = \tau_k$, $k = 1,2,\dots$.

With $Y(t,\mu)$, $Y(t_0,\mu) = E_n$, $\mu > 0$ and $t_0 \in I$ we denote the fundamental matrix of the linear system

$$\begin{cases} \mu\dfrac{dy}{dt} = D(t)y, \quad t \ne \tau_k, \\ \Delta y = D_k y \qquad t = \tau_k,\ k = 1,2,\dots. \end{cases} \tag{5}$$

Definition 5. Let P is projector ($P^2 = P$) in the space R^n. The function

$$G(t,s,\mu) = \begin{cases} Y(t,\mu)PY^{-1}(s,\mu), & t \geq s \geq 0, \\ Y(t,\mu)(P - E_n)Y^{-1}(s,\mu), & s \geq t \geq 0 \end{cases}$$

is said to be Green's function of the system (5).

It is easily to verify that the following relations are valid

(6)

$$\begin{aligned}
&G(\tau_k + 0, t, \mu) = (E_n + D_k)G(\tau_k, t, \mu), \quad t \neq \tau_k, \\
&G(t, \tau_k + 0, \mu) = G(t, \tau_k, \mu)(E_n + D_k)^{-1}, \quad t \neq \tau_k, \\
&G(t + 0, t, \mu) - G(t - 0, t, \mu) = E_n, \quad t \neq \tau_k, \\
&G(t, t + 0, \mu) - G(t, t - 0, \mu) = -E_n, \quad t \neq \tau_k, \\
&G(\tau_k + 0, \tau_k + 0, \mu) = (E_n + D_k)G(\tau_k, \tau_k + 0, \mu) + E_n, \quad k = 1, 2, \dots, \\
&\mu \frac{\partial G(t,s,\mu)}{\partial t} = D(t)G(t,s,\mu), \quad t \neq s, \\
&\frac{\partial G(t,s,\mu)}{\partial s} = -G(t,s,\mu)D(s), \quad t \neq s.
\end{aligned}$$

Introduce the following conditions:

H5. $0 < t_0 < \tau_1$ and there exist a constants $p > 0$ and $\varepsilon > 0$ such that uniformly at $t \in I$ and $s \in I$ the following inequality is valid

$$i(s,t) \leq p(t - s) + \varepsilon,$$

where by $i(s,t)$ we have denoted the number of the pointes τ_k in the interval $(s,t]$.

H6. The following inequalities are valid

$$\|W(t,s)\| \leq Ke^{\alpha|t-s|}, \quad t \in I,\ s \in I,$$

$$\|G(t,s,\mu)\| \leq Ne^{-\frac{\beta}{\mu}|t-s|}, \quad t \in I,\ s \in I,$$

where $K > 0$, $N > 0$, $\alpha > 0$ and $\beta > 0$.

Lemma 1 ([1]). *Let the following inequality hold:*

$$u(t) \leq \int_{t_0}^{t} u(s)v(s)\,ds + F(t) + \sum_{t_0<\tau_k<t} \gamma_k u(\tau_k) + \sum_{t_0<\tau_k<t} \delta_k(t),$$

where the function $u(t)$ is piecewice continuous with discontinuity of the first kind at the points τ_k, $k = 1, 2, \dots$, $v(t)$ is locally integrable function, $F(t)$ and $\delta_k(t)$ non decreasing for $t \in (t_0, \infty)$, $\delta_k(t) \geq 0$, $\gamma_k \geq 0$, $k = 1, 2, \dots$.

Then

$$u(t) \leq \Bigg(F(t) + \sum_{t_0<\tau_k<t} \delta_k(t)\Bigg) \prod_{t_0<\tau_k<t} (1 + \gamma_k) \exp\bigg(\int_{t_0}^{t} v(s)\,ds\bigg).$$

3. Main results

Let J is affinity integral manifold of (1) in the form

$$J = \{(t, x, y) \colon y = Q(t)x, t \in [t_0, \infty), x \in R^m\}. \tag{7}$$

Along with J we consider the system

$$\begin{cases} Q' + QA + \frac{1}{\mu}QBQ = \frac{1}{\mu}DQ + C, \quad t \neq \tau_k, \\ \Delta Q(\tau_k) + Q(\tau_k + 0)A_k + \frac{1}{\mu}Q(\tau_k + 0)B_kQ(\tau_k) = \mu C_k + D_kQ(\tau_k), \\ \qquad k = 1, 2, \dots . \end{cases} \tag{8}$$

Lemma 2. *THe manifold* (7) *is affinity integral manifold of* (1) *if and only if* $Q(t)$ *is bounded solution of* (8).

Proof. Lemma 2 is proved by straightforward calculations.

Theorem 1. *Let the following conditions hold:*

1. *The conditions H1–H6 are met.*
2. *The relations* $B(t) = 0$, $t \in I$ *and* $B_k = 0$, $k = 1, 2, \dots$ *are hold.*
3. *There exist a positive constant* δ *such that*

$$\sup_{t \in I} \|D(t)\| \leq \delta, \qquad \sup_{k=1,2,\dots} \|D_k\| \leq \delta,$$

where $\delta = \delta(\mu)$, $\delta(\mu) \to 0$ *at* $\mu \to 0$.

Then there exist a constant $\mu_0 > 0$ *such that for all* $\mu \in (0, \mu_0]$ *and* $t > t_0$, (1) *has affinity integral manifold.*

Proof. From (2) it follows that any solutions $x(t) = x(t; t_0, x_0)$ of the Cauchy's problem of the system (3) with $x(t_0) = x_0$ is the form $x(t) = W(t, t_0)x_0$. Then it is follows that the system

$$\begin{cases} \mu \dfrac{dy}{dt} = D(t)y + C(t)W(t, t_0)x, \quad t \neq \tau_k, \\ \Delta y = D_k y + C_k W(t, t_0)x, \quad t = \tau_k,\ k = 1, 2, \dots \end{cases}$$

has only one bounded solution in the form

$$y(t) = \frac{1}{\mu} \int_{t_0}^{\infty} G(t, s, \mu)C(s)W(s, t_0)x_0 \, ds + \sum_{k=1}^{\infty} G(t, \tau_k + 0, \mu)C_k W(\tau_k, t_0)x_0.$$

If the graph of the solution $(x(t), y(t))$ is from a affinity integral manifold then

$$QW(t, s)x_0 = \frac{1}{\mu} \int_{t_0}^{\infty} G(t, s, \mu)C(s)W(s, t_0)x_0 \, ds + \sum_{k=1}^{\infty} G(t, \tau_k + 0, \mu)C_k W(\tau_k, t_0)x_0.$$

We shall proof Theorem 1 if we proof that

$$(9)\quad Q(t)=\frac{1}{\mu}\int_{t_0}^{\infty}G(t,s,\mu)C(s)W(s,t)x_0\,ds+\sum_{k=1}^{\infty}G(t,\tau_k+0,\mu)C_kW(\tau_k,t)x_0$$

is bounded solutions of the system (1) such that $B(t)=0$ at $t\in I$, $B_k=0$ at $k=1,2,\dots$.

From (4) and (6) at $t\neq\tau_k$, $t>t_0$ we obtain

(10)

$$\begin{aligned}
&\frac{d}{dt}Q(t)\\
&=\frac{d}{dt}\left(\frac{1}{\mu}\int_{t_0}^{t}G(t,s,\mu)C(s)W(s,t)\,ds+\frac{1}{\mu}\int_{t}^{\infty}G(t,s,\mu)C(s)W(s,t)\,ds\right)\\
&\quad+\frac{d}{dt}\left(\sum_{k=1}^{\infty}G(t,\tau_k+0,\mu)C_kW(\tau_k,t)\right)\\
&=\frac{1}{\mu}G(t,t-0,\mu)C(t)W(t-0,t)-\frac{1}{\mu}G(t,t+0,\mu)C(t)W(t+0,t)\\
&\quad+\frac{1}{\mu^2}\int_{t_0}^{\infty}D(t)G(t,s,\mu)C(t)W(s,t)\,ds\\
&\quad-\frac{1}{\mu}\int_{t_0}^{\infty}G(t,s,\mu)C(s)W(s,t)A(t)\,ds\\
&\quad+\frac{1}{\mu}\sum_{k=1}^{\infty}D(t)G(t,\tau_k+0,\mu)C_kW(\tau_k,t)\\
&\quad-\sum_{k=1}^{\infty}G(t,\tau_k+0,\mu)C_kW(\tau_k,t)A(t)\\
&=C(t)+\frac{1}{\mu}D(t)Q(t)-Q(t)A(t)
\end{aligned}$$

and at $t=\tau_i$, $i=1,2,\dots$ it follows that

(11)

$$\begin{aligned}
&\Delta Q(\tau_i)+Q(\tau_k+0)A_k\\
&=\frac{1}{\mu}\int_{t_0}^{\infty}G(\tau_i+0,s,\mu)W(s,\tau_i+0)(E_m+A_i)\,ds\\
&\quad+\sum_{k=1}^{\infty}G(\tau_i+0,\tau_k+0,\mu)C_kW(\tau_k,\tau_i+0)(E_m+A_i)\\
&\quad-\frac{1}{\mu}\int_{t_0}^{\infty}G(\tau_i,s,\mu)C(s)W(s,\tau_i)\,ds-\sum_{k=1}^{\infty}G(\tau_i,\tau_k+0,\mu)C_kW(\tau_i,\tau_k)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\mu}\int_{t_0}^{\infty}(E_n+D_i)G(\tau_i,s,\mu)C(s)W(s,\tau_i)\,ds \\
&\quad + \sum_{k=1}^{\infty}(E_n+D_i)G(\tau_i,\tau_k+0,\mu)C_kW(\tau_k,\tau_i) + \frac{1}{\mu}C_i \\
&\quad - \frac{1}{\mu}\int_{t_0}^{\infty}G(\tau_i,s,\mu)C(s)W(s,\tau_i)\,ds - \sum_{k=1}^{\infty}G(\tau_i,\tau_k+0,\mu)C_kW(\tau_i,\tau_k) \\
&= \frac{1}{\mu}C_i + D_iQ(\tau_i).
\end{aligned}
$$

Then (9) is solution of (8) for $B(t)=0$, $t>t_0$; $B_k=0$, $k=1,2,\dots$.

On the other hand for $t>t_0$ it is follows that

$$
\|Q(t)\| \le \frac{1}{\mu}\int_{t_0}^{\infty} KNe^{-(\frac{\beta}{\mu}-\alpha)|t-s|}\delta\,ds + \sum_{k=1}^{\infty}KNe^{-(\frac{\beta}{\mu}-\alpha)|t-\tau_k|}\delta. \tag{12}
$$

From H5 it is follows that there exist $\mu_0>0$ such that for all $\mu\in(0,\mu_0]$ the following inequality is valid

$$
\sum_{k=1}^{\infty}KNe^{-(\frac{\beta}{\mu}-\alpha)|t-\tau_k|} < v_\mu, \tag{13}
$$

where v_μ depend only from μ, $\mu\in(0,\mu_0]$ and the sequence $\{\tau_k\}_{k=1}^{\infty}$.

From (12) and (13) it is follows that $Q(t)$ is bounded solution of (8) for $B(t)=0$, $t>t_0$; $B_k=0$, $k=1,2,\dots$.

Theorem 2. *Let the following conditions hold:*

1. The conditions H1–H6 are met.

2. There exist a positive constant δ such that the following inequalities hold:

$$
\begin{aligned}
&\sup_{t\in I}\|B(t)\|\le\delta, \qquad \sup_{k=1,2,\dots}\|B_k\|\le\delta,\\
&\sup_{t\in I}\|C(t)\|\le\delta, \qquad \sup_{k=1,2,\dots}\|C_k\|\le\delta,
\end{aligned}
$$

where $\delta=\delta(\mu)$, $\delta(\mu)\to 0$ at $\mu\to 0$.

Then there exist a positive constant μ^* such that for all $\mu\in(0,\mu^*]$ the system (1) has affinity integral manifold in the form (7) at $t>t_0$.

Proof. The parameter function from (7) we shall obtain by the method of consistent approach.

Set

$$
\begin{aligned}
&\varphi_0 = 0,\\
&\varphi_n = Q_n(t)x, \quad n=1,2,\dots,
\end{aligned}
$$

where

(14)

$$Q_n(t) = \frac{1}{\mu}\int_t^\infty G(t,s,\mu)C(s)W_{n-1}(s,t)\,ds + \sum_{k=1}^\infty G(t,\tau_k+0,\mu)C_kW_{n-1}(\tau_k,t),$$

and $W_{n-1}(t,s)$ is Cauchy's matrix of the system

$$\begin{cases} \dfrac{dx}{dt} = [A(t)+B(t)Q_{n-1}(t)]x, & t \neq \tau_k, \\ \Delta x = [A_k + B_kQ_{n-1}(t)]x, & t = \tau_k,\ k = 1,2,\dots. \end{cases} \tag{15}$$

We consider the system

$$\begin{cases} \dfrac{dx}{dt} = [A(t)+B(t)Q_{n-1}(t)]x, & t \neq \tau_k, \\ \mu\dfrac{dy}{dt} = C(t)x + D(t)y, & t \neq \tau_k, \\ \Delta x = [A_k + B_kQ_{n-1}(t)]x, & t = \tau_k, \\ \Delta y = C_kx + D_ky, & t = \tau_k,\ k = 1,2,\dots. \end{cases} \tag{16}$$

We shall proof that $\{Q_n(t)\}$ is uniformly bounded sequence.

For $n = 1$ the system (16) satisfies the conditions of Theorem 1. Then there exists the constat $q > 0$ such that

$$\|Q_1(t)\| \leq q.$$

Let $\|Q_n(t)\| \leq q$ for arbitrary $n \geq 1$.

Then from (14) it follows that

$$\begin{aligned} \|Q_{n+1}(t)\| \leq \frac{1}{\mu}\int_{t_0}^\infty \|G(t,s,\mu)\|\,\|C(s)\|\,\|W_n(s,t)\|\,ds \\ + \sum_{k=1}^\infty \|G(t,\tau_k+0,\mu)\|\,\|C_k\|\,\|W_n(\tau_k,s)\|. \end{aligned} \tag{17}$$

From (15) for $t > s$; $t \in I$, $s \in I$ it is follows that

$$\begin{aligned} W_n(t,s) = W(t,s) + \int_s^t W(t,\tau)B(\tau)Q_n(\tau)W_n(\tau,s)\,d\tau \\ + \sum_{s<\tau_k<t} W(t,\tau_k)B_kQ_n(\tau_k)W_n(\tau_k,s). \end{aligned}$$

Then

$$\begin{aligned} \|W_n(t,s)\| \leq Ke^{\alpha(t-s)} + \int_s^t Kq\delta e^{\alpha(t-\tau)}\|W_n(\tau,s)\|\,d\tau \\ + \sum_{s<\tau_k<t} Kq\delta e^{\alpha(t-\tau_k)}\|W_n(\tau_k,s)\|. \end{aligned}$$

Set

$$u(t) = e^{-\alpha t}\|W_n(t,s)\|, \quad v(t) = Kq\delta,$$
$$F(t) = Ke^{-\alpha t}, \quad \gamma_k = Kq\delta,$$
$$\delta_k(t) \equiv 0.$$

From Lemma 1 we obtain that

$$\begin{aligned}\|W_n(t,s)\| &\le Ke^{\alpha(t-s)} \prod_{s<\tau_k<t}(1+Kq\delta)e^{Kq\delta(t-s)} \\ &\le Ke^{\alpha(t-s)}(1+Kq\delta)^{p(t-s)+\varepsilon}e^{Kq\delta(t-s)} \\ &= K(1+Kq\delta)^{\varepsilon}e^{[\alpha+Kq\delta+p\ln(1+Kq\delta)](t-s)}.\end{aligned} \tag{18}$$

For $s > t$ the proof is analogouly.

From (17) and (18) we obtain that

$$\begin{aligned}\|Q_{n+1}(t)\| \le \frac{1}{\mu}\int_{t_0}^{\infty} NK\delta(1+Kq\delta)^{\varepsilon}e^{-\sigma|t-s|}\,ds \\ + \sum_{k=1}^{\infty} NK\delta(1+Kq\delta)^{\varepsilon}e^{-\sigma|t-\tau_k|},\end{aligned} \tag{19}$$

where $\sigma = \frac{\beta}{\mu} - \big(\alpha + Kq\delta + p\ln(1+Kq\delta)\big)$.

It is easily to verify that there exist a positive constat μ_0, $\mu_0 < \beta\big(\alpha+Kq\delta+p\ln(1+Kq\delta)\big)^{-1}$ such that for all $\mu \in (0,\mu_0]$, σ is positive. Then from (19) it follows that

$$\|Q_{n+1}(t)\| \le NK\delta(1+Kq\delta)^{\varepsilon}\left(\frac{1}{\mu\sigma} + v_\mu\right). \tag{20}$$

Hence it is follows that $Q_n(t)$ is bounded at $t > t_0$.

On the other hand

(21)

$$\begin{aligned}Q_{n+1}(t) - Q_n(t) = \frac{1}{\mu}\int_{t_0}^{\infty} G(t,s,\mu)C(s)\big(W_n(s,t) - W_{n-1}(s,t)\big)\,ds \\ + \sum_{k=1}^{\infty} G(t,\tau_k+0,\mu)C_k\big(W_n(\tau_k,t) - W_{n-1}(\tau_k,t)\big).\end{aligned}$$

It is immediately verified that $V(t) = W_n(t,s) - W_{n-1}(t,s)$ is solution of the system:

$$\begin{cases} \dfrac{dV}{dt} = \big(A(t) + B(t)Q(t)\big)V + B(t)\big(Q_{n-1}(t) - Q_n(t)\big)W_{n-1}, \quad t \ne \tau_k, \\ \Delta V = \big(A_k + B_kQ(t)\big)V + B_K\big(Q_{n-1}(t) - Q_n(t)\big)W_{n-1}, \\ \qquad t = \tau_k,\ k = 1,2,\dots. \end{cases}$$

Then for $t > s$,

$$V(t) = \int_s^t W_n(t,\theta)B(\theta)\big(Q_{n-1}(\theta) - Q_n(\theta)\big)W_{n-1}(\theta,s)\,d\theta + \sum_{s<\tau_k<t} W_n(t,\tau_k)B_k\big(Q_{n-1}(\tau_k) - Q_n(\tau_k)\big)W_{n-1}(\tau_k,s).$$

(22)

$$\begin{aligned}\|V(t)\| \le {} & \left[\int_s^t \big(K(1+Kq\delta)^{\varepsilon}\big)^2 \delta e^{(\alpha+Kq\delta+p\ln(1+Kq\delta))(t-s)}\,d\theta\right] \\ & \times \sup_{t\in I}\|Q_{n-1}(s) - Q_n(s)\| \\ & + \left(\sum_{s<\tau_k<t} \big(K(1+Kq\delta)^{\varepsilon}\big)^2 \delta e^{(\alpha+Kq\delta+p\ln(1+Kq\delta))(t-\tau_k)}\right) \\ & \times \sup_{t\in I}\|Q_{n-1}(t) - Q_n(t)\| \\ = {} & \big(K(1+Kq\delta)^{\varepsilon}\big)^2 \delta e^{(\alpha+Kq\delta+p\ln(1+Kq\delta))(t-s)}\big((1+p)(t-s)+\varepsilon\big) \\ & \times \sup_{t\in I}\|Q_n(t) - Q_{n-1}(t)\|.\end{aligned}$$

At $t < s$ the proof is analogously.

Then from (21) and (22) we obtain that

$$\begin{aligned}&\|Q_{n+1}(t) - Q_n(t)\| \\ &\le \left[\frac{1}{\mu}\int_{t_0}^{\infty} N\big(K(1+Kq\delta)^{\varepsilon}\big)^2 \delta^2\big((1+p)|t-s|+\varepsilon\big)e^{-\sigma|t-s|}\,ds\right] \\ &\quad \times \sup_{t\in I}\|Q_n(t) - Q_{n-1}(t)\| \\ &\quad + \left[\sum_{k=1}^{\infty} N\big(K(1+Kq\delta)^{\varepsilon}\big)^2 \delta^2\big((1+p)|t-\tau_k|+\varepsilon\big)e^{-\sigma|t-\tau_k|}\right] \\ &\quad \times \sup_{t\in I}\|Q_n(t) - Q_{n-1}(t)\|.\end{aligned}$$

From H4 and (13) it is imediately that there exist $\mu_1 > 0$ such that for all $\mu \in (0, \mu_1]$ the following inequality is valid

$$\sum_{k=1}^{\infty} |t-\tau_k| e^{-\sigma|t-\tau_k|} < \lambda_k,$$

where λ_k depended only of the sequence $\{\tau_k\}_{k=1}^{\infty}$ and μ.

Then

(23)

$$\|Q_{n+1}(t) - Q_n(t)\| \le \left\{ N\big(K(1+Kq\delta)^{\varepsilon}\big)^2 \delta^2 \left[(1+p)\left(\frac{2}{\sigma^2} - \frac{1}{\sigma^2} e^{-\sigma(t-t_0)} - \frac{1}{\sigma} e^{-\sigma(t-t_0)} + \lambda_k \right) \right] \right\} \times \sup_{t\in I} \|Q_n(t) - Q_{n-1}(t)\| + \left\{ N\big(K(1+Kq\delta)^{\varepsilon}\big)^2 \delta^2 \left[\left(\frac{2}{\sigma^2} - \frac{1}{\sigma} e^{-\sigma(t-t_0)} + \gamma_k \right) \right] \right\} \times \sup_{t\in I} \|Q_n(t) - Q_{n-1}(t)\|.$$

From (23) follows that there exist μ^*, $\mu^* < \min\{\mu_0, \mu_1\}$ such that for all μ, $\mu \in (0, \mu^*]$ the sequence $\{Q_n(t)\}_{n=1}^{\infty}$ is uniformly convergent to $Q(t)$.

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