

# Equations of Specific Cubic Surfaces and Automorphisms

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**Abstract.** If a cubic surface has a non-trivial automorphism, it specifies the structure of the cubic surface. When the automorphism group is big enough the isomorphic class is uniquely determined. In this paper, we provide equations of specific cubic surfaces and generators of automorphism groups explicitly.

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## §1. Introduction

There are many ways to parameterize a family of non-singular cubic surfaces [Cay], [N], [S]. In [H], we used the moduli of configurations of six points in a general position on the projective plane to parameterize non-singular cubic surfaces. For a configuration of six points, we determined the automorphism group of the cubic surface associated with the configuration as a subgroup of the Weyl group  $W(E_6)$  of type  $E_6$ . The largest order of automorphism groups is 648. The surface defined by the equation;

$$y_1^3 + y_2^3 + y_3^3 + y_4^3 = 0$$

is a cubic surface with an automorphism group of order 648. If a cubic surface has an automorphism group of order 648 it is isomorphic to the surface above. Also for  $n = 8, 108, 120$ , the isomorphic class of a cubic surface with an automorphism group of order  $n$  is unique.

Here we provide an equation of such cubic surface and generators of the automorphism group as a subgroup of the projective general linear group of the projective space.

In §2, for a configuration of six points, we take a birational map from the projective plane onto a cubic surface in the projective space in concrete form so that the equation of the image is acquired.

In §3, we proceed in the following way. Let  $S$  be a cubic surface with  $\text{Aut}(S) \neq \{id_S\}$ . We may assume the equation of  $S$  is one derived in §2. An automorphism of  $S$  preserves the configuration of lines on  $S$ . By virtue of the above rational map and the equation of  $S$ , we derive a projective transformation preserving the equation of  $S$  from an automorphism of  $S$ .

In §4, we provide an equation of a cubic surface with a maximal automorphism group and generators of the automorphism group as projective linear transformations.

## §2. Equations of Cubic Surfaces

We shall fix an algebraically closed field of characteristic  $= 0$  as a ground field. We follow the notation in [H]. Put

$$U = \{((a_1 : a_2 : a_3), (b_1 : b_2 : b_3)) | gh \prod_i a_i b_i \prod_{i < j} (a_i - a_j)(b_i - b_j) \prod_{i < j} (a_i b_j - a_j b_i) \neq 0\},$$

where  $g = a_1(b_2 - b_3) + a_2(b_3 - b_1) + a_3(b_1 - b_2)$  and  $h = a_1 b_1(a_2 b_3 - a_3 b_2) + a_2 b_2(a_3 b_1 - a_1 b_3) + a_3 b_3(a_1 b_2 - a_2 b_1)$ ,  $P_1 = (1 : 0 : 0)$ ,  $P_2 = (0 : 1 : 0)$ ,  $P_3 = (0 : 0 : 1)$  and  $P_4 = (1 : 1 : 1)$ . Then for a pair  $P = (P_5, P_6) \in \mathbb{P}^2 \times \mathbb{P}^2$ , the six points  $P_i$  ( $i = 1, \dots, 6$ ) are in a general position if and only if  $P \in U$ .

Let  $l_{ij}$  be the line passing through  $P_i$  and  $P_j$ . Among fifteen such lines, we need the following lines and equations.

$$\begin{aligned} l_{12}: & f_{12} = x_3 = 0 \\ l_{14}: & f_{14} = -x_2 + x_3 = 0 \\ l_{16}: & f_{16} = -b_3 x_2 + b_2 x_3 = 0 \\ l_{23}: & f_{23} = x_1 = 0 \\ l_{25}: & f_{25} = a_3 x_1 - a_1 x_3 = 0 \\ l_{34}: & f_{34} = -x_1 + x_2 = 0 \\ l_{36}: & f_{36} = -b_2 x_1 + b_1 x_2 = 0 \\ l_{45}: & f_{45} = (-a_2 + a_3)x_1 + (a_1 - a_3)x_2 + (-a_1 + a_2)x_3 = 0 \\ l_{56}: & f_{56} = (-a_3 b_2 + a_2 b_3)x_1 + (a_3 b_1 - a_1 b_3)x_2 + (-a_2 b_1 + a_1 b_2)x_3 = 0. \end{aligned}$$

We also put

$$\begin{aligned} y_1 &= f_{12} f_{34} f_{56} \\ y_2 &= f_{23} f_{45} f_{16} \\ y_3 &= f_{12} f_{36} f_{45} \\ y_4 &= f_{16} f_{25} f_{34} \\ y_5 &= f_{14} f_{25} f_{36} \\ y_6 &= f_{14} f_{23} f_{56}. \end{aligned}$$

**Lemma 2.1.** 1.  $y_1, y_2, y_3, y_4$  are linearly independent.

$$2. (a_1 - a_2)b_3y_5 = (a_1 - a_3)b_2y_1 + a_3(b_1 - b_2)y_2 + (-a_3b_2 + a_1b_3)y_3 + (-a_2b_1 + a_3b_1 + a_1b_2 - a_3b_2)y_4$$

$$3. (-a_1 + a_2)b_3y_6 = (-a_1b_2 + a_3b_2 + a_1b_3 - a_2b_3)y_1 + (-a_3b_1 + a_3b_2 + a_1b_3 - a_2b_3)y_2 + (a_3b_2 - a_2b_3)y_3 + (a_2b_1 - a_3b_1 - a_1b_2 + a_3b_2 + a_1b_3 - a_2b_3)y_4$$

*Proof.*  $y_1$  and  $y_3$  are linearly independent and  $y_1$  and  $y_3$  form a linear system  $\Lambda_1$  with a fixed component  $l_{12}$ .  $y_2$  and  $y_4$  are linearly independent and  $y_2$  and  $y_4$  form a linear system  $\Lambda_2$  with a fixed component  $l_{16}$ . Therefore  $\Lambda_1 \cap \Lambda_2 = \{0\}$  so that  $y_1, y_2, y_3, y_4$  are linearly independent.

It is straightforward to verify the relations (2) and (3).  $\square$

Since  $y_1, y_2, y_3, y_4$  are linearly independent,

$$\Psi_P = (y_1 : y_2 : y_3 : y_4) : \mathbb{P}^2 \cdots \rightarrow S_P \subset \mathbb{P}^3$$

is a birational mapping onto the closed image  $S_P$ . By the trivial relation  $y_1y_2y_5 = y_3y_4y_6$ , we derive the equation of  $S_P$ .

**Proposition 2.2.** *The equation of the cubic surface  $S_P$  is*

$$F_P = (a_1 - a_3)b_2y_1^2y_2 + a_3(b_1 - b_2)y_1y_2^2 + (-a_3b_2 + a_1b_3)y_1y_2y_3 + (-a_2b_1 + a_3b_1 + a_1b_2 - a_3b_2)y_1y_2y_4 + (-a_1b_2 + a_3b_2 + a_1b_3 - a_2b_3)y_1y_3y_4 + (-a_3b_1 + a_3b_2 + a_1b_3 - a_2b_3)y_2y_3y_4 + (a_3b_2 - a_2b_3)y_3^2y_4 + (a_2b_1 - a_3b_1 - a_1b_2 + a_3b_2 + a_1b_3 - a_2b_3)y_3y_4^2 = 0.$$

### §3. Cubic Surface with a non-trivial automorphism

Let  $E_i = \Psi_P(P_i)$  ( $i = 1, \dots, 6$ ) be the exceptional curves. Let  $F_j$  be the proper transform of the non-singular conic passing through all  $P_i$  but  $P_j$ , and  $L_{ij}$  be the proper transform of the line  $l_{ij}$ . Then the set  $\Gamma = \{E_i, F_j, L_{ij}\}$  is all lines on the cubic surface  $S_P$ . The automorphism group  $G = \text{Aut } \Gamma$  of the configuration  $\Gamma$  is isomorphic to the Weyl group  $W(E_6)$  of type  $E_6$  [M]. In [H], we defined an action of  $G$  on  $U$  [H, Proposition 3.1] and we derived that the automorphism group  $\text{Aut } S_P$  is isomorphic to the stabilizer subgroup  $G_P$  of the point  $P$  in  $U$  [H, Theorem 3.2].

Now let  $S$  be a cubic surface with a non-trivial automorphism. Then  $S \cong S_P$  for some  $P \in U$ . By [H, Theorem 5.1],  $G_P \cap (4A_1) \neq \emptyset$  where  $(4A_1)$  is the set of all elements in  $G$  which belong to the conjugacy class  $4A_1$  [Car].

Put  $g = s(1, 1, 1, 1, 1)r[72] = ((a_3b_2 : a_2b_3 : a_3b_3), (a_3b_1 : a_1b_3 : a_3b_3))$ , then  $g \in (4A_1)$  (see [H, 6.4(2)]). Therefore we may assume  $P \in U_g$ , that is  $g \in \text{Aut } S_P$ , where  $U_g$  is the set of invariant locus of  $g$  in  $U$ .

**Lemma 3.1.** *The action of  $g$  on  $\Gamma = \{E_i, F_j, L_{ij}\}$  is as follows*

$$\begin{aligned} &g(E_1, E_2, E_3, E_4, E_5, E_6, F_1, F_2, F_3, F_4, F_5, F_6, \\ &\quad L_{12}, L_{13}, L_{14}, L_{15}, L_{16}, L_{23}, L_{24}, L_{25}, L_{26}, L_{34}, L_{35}, L_{36}, L_{45}, L_{46}, L_{56}) \\ &= (F_6, F_5, F_4, F_3, F_2, F_1, E_6, E_5, E_4, E_3, E_2, E_1, \\ &\quad L_{56}, L_{46}, L_{36}, L_{26}, L_{16}, L_{45}, L_{35}, L_{25}, L_{15}, L_{34}, L_{24}, L_{14}, L_{23}, L_{13}, L_{12}) \end{aligned}$$

*Proof.* Straightforward. Tedious but simple.  $\square$

If there is a linear transformation  $T$  such that  $T(L) = g(L)$  for all  $L \in \Gamma$  then  $T$  preserves  $S_P$  and the restriction of  $T$  on  $S_P$  coincide with  $g$ . Seeking for such a linear transformation, it is enough to only take specific five points in a general position each of which is an intersection of two lines in  $\Gamma$ .

Here we take  $Q_1 = L_{12} \cap L_{34}$ ,  $Q_2 = L_{34} \cap L_{56}$ ,  $Q_3 = L_{12} \cap L_{56}$ ,  $Q_4 = L_{16} \cap L_{23}$ ,  $Q_5 = L_{15} \cap L_{24}$ . Then by Lemma 3.1,  $Q'_1 = g(Q_1) = L_{34} \cap L_{56}$ ,  $Q'_2 = g(Q_2) = L_{12} \cap L_{34}$ ,  $Q'_3 = g(Q_3) = L_{12} \cap L_{56}$ ,  $Q'_4 = g(Q_4) = L_{16} \cap L_{45}$ ,  $Q'_5 = g(Q_5) = L_{26} \cap L_{35}$ .

Put  $P = (P_5, P_6) = ((1 : a : b), (1 : c : bc)) \in U_g$ .

**Lemma 3.2.**  $Q_1 = (0 : 1 : 0 : 0)$ ,  $Q_2 = (0 : c - ac : 1 - 2c + ac : 0)$ ,  $Q_3 = (0 : a - b - c + 2bc - abc : 0 : b(1 - 2c + ac))$ ,  $Q_4 = ((1 - a)b : 0 : 1 - 2ab : 0)$ ,  $Q_5 = ((1 - a)bc : (1 - a)bc : a - bc : (-1 + a)bc)$ ,  $Q'_1 = (0 : (1 - a)c : 1 - 2c + ac : 0)$ ,  $Q'_2 = (0 : 1 : 0 : 0)$ ,  $Q'_3 = (0 : a - b - c + 2bc - abc : 0 : b(1 - 2c + ac))$ ,  $Q'_4 = (1 : 0 : 0 : 0)$ ,  $Q'_5 = (1 : 1 : -1 : -1)$ .

*Proof.*  $Q_1 = \Psi_P(l_{12} \cap l_{34}) = \Psi_P(1 : 1 : 0) = (f_{12}f_{34}f_{56} : f_{23}f_{45}f_{16} : f_{12}f_{36}f_{45} : f_{16}f_{25}f_{34})(1 : 1 : 0) = (0 : (1 - a)(c - cb) : 0 : 0) = (0 : 1 : 0 : 0)$ . The others are obtained similarly.  $\square$

Let  $T_P$  be the linear transformation with  $T_P(Q_i) = Q'_i$  ( $i = 1, \dots, 5$ ).

**Lemma 3.3.**

$$T_P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1-2b+ab}{(1-a)b} & \frac{1-2c+ac}{(1-a)c} & -1 & \frac{-a+b+c-2bc+abc}{(1-a)bc} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

**Theorem 3.4.** *If  $S$  is a non-singular cubic surface with a non-trivial automorphism then  $S$  is isomorphic to  $S_P$  for some  $P \in U$  where  $P = ((1 : a : b), (1 : c : bc))$  and  $S_P$  is the cubic surface defined by the equation;*

$$F_P = (1 - b)cy_1^2y_2 + b(1 - c)y_1y_2^2 + (-a + b + c - bc)y_1y_2y_4 + (-1 + 2b - ab)cy_1y_3y_4 + (-1 + 2c - ac)by_2y_3y_4 + (1 - a)bcy_3^2y_4 + (a - b - c + 2bc - abc)y_3y_4^2 = 0.$$

*The linear transformation  $T_P$  in Lemma 3.3 preserves the equation  $F_P = 0$  and induces a non-trivial automorphism of  $S_P$ .*

#### §4. Cubic Surface with maximal automorphism group

Let  $S$  be a non-singular cubic surface. If the order of the automorphism group of  $S$  is 648 or 120 or 108 or 8 the isomorphic class of  $S$  is unique [H]. Here we provide an equation of  $S$  and generators of  $\text{Aut } S$ . Since the case of order 648 is well known, we treat only the other three cases.

##### 4.1. The case of order 120

Take the point

$$P = \left( \left( 1 : \frac{-1 + \sqrt{5}}{2} : \frac{3 - \sqrt{5}}{2} \right), \left( 1 : \frac{3 - \sqrt{5}}{2} : \frac{-1 + \sqrt{5}}{2} \right) \right).$$

Then the equation of the cubic surface  $S_P$  is

$$F_P = y_1^2 y_2 + y_1 y_2^2 + 2y_1 y_2 y_3 - y_3^2 y_4 + y_3 y_4^2 = 0.$$

The automorphism group  $G_P = \text{Aut } S_P$  is of order 120 and generated by the following four elements  $g_1, g_2, g_3, g_4$ .

$$g_1 = \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$g_2 = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$g_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & -1 \\ -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$g_4 = \begin{pmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ -1 & -1 & -2 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix}.$$

The action of  $g_i$  on the configuration  $\Gamma$  is as follows.

$$\begin{aligned} &g_1(E_1, E_2, E_3, E_4, E_5, E_6, F_1, F_2, F_3, F_4, F_5, F_6, \\ &\quad L_{12}, L_{13}, L_{14}, L_{15}, L_{16}, L_{23}, L_{24}, L_{25}, L_{26}, L_{34}, L_{35}, L_{36}, L_{45}, L_{46}, L_{56}) \\ &= (F_6, F_4, F_5, F_2, F_3, F_1, E_6, E_4, E_5, E_2, E_3, E_1, \end{aligned}$$

$$\begin{aligned}
& L_{46}, L_{56}, L_{26}, L_{36}, L_{16}, L_{45}, L_{24}, L_{34}, L_{14}, L_{25}, L_{35}, L_{15}, L_{23}, L_{12}, L_{13}) \\
g_2(E_1, E_2, E_3, E_4, E_5, E_6, F_1, F_2, F_3, F_4, F_5, F_6, \\
& L_{12}, L_{13}, L_{14}, L_{15}, L_{16}, L_{23}, L_{24}, L_{25}, L_{26}, L_{34}, L_{35}, L_{36}, L_{45}, L_{46}, L_{56}) \\
= & (F_6, F_3, F_2, F_5, F_4, F_1, E_6, E_3, E_2, E_5, E_4, E_1, \\
& L_{36}, L_{26}, L_{56}, L_{46}, L_{16}, L_{23}, L_{35}, L_{34}, L_{13}, L_{25}, L_{24}, L_{12}, L_{45}, L_{15}, L_{14}) \\
g_3(E_1, E_2, E_3, E_4, E_5, E_6, F_1, F_2, F_3, F_4, F_5, F_6, \\
& L_{12}, L_{13}, L_{14}, L_{15}, L_{16}, L_{23}, L_{24}, L_{25}, L_{26}, L_{34}, L_{35}, L_{36}, L_{45}, L_{46}, L_{56}) \\
= & (F_5, F_6, F_4, F_3, F_1, F_2, E_5, E_6, E_4, E_3, E_1, E_2, \\
& L_{56}, L_{45}, L_{35}, L_{15}, L_{25}, L_{46}, L_{36}, L_{16}, L_{26}, L_{34}, L_{14}, L_{24}, L_{13}, L_{23}, L_{12}) \\
g_4(E_1, E_2, E_3, E_4, E_5, E_6, F_1, F_2, F_3, F_4, F_5, F_6, \\
& L_{12}, L_{13}, L_{14}, L_{15}, L_{16}, L_{23}, L_{24}, L_{25}, L_{26}, L_{34}, L_{35}, L_{36}, L_{45}, L_{46}, L_{56}) \\
= & (F_4, F_6, F_5, F_1, F_3, F_2, E_4, E_6, E_5, E_1, E_3, E_2, \\
& L_{46}, L_{45}, L_{14}, L_{34}, L_{24}, L_{56}, L_{16}, L_{36}, L_{26}, L_{15}, L_{35}, L_{25}, L_{13}, L_{12}, L_{23})
\end{aligned}$$

#### 4.2. The case of order 108

Take the point

$$P = \left( (1 : \frac{\sqrt{3} - \sqrt{-1}}{2} : \frac{\sqrt{-1}(\sqrt{3} - 2 - \sqrt{-1})}{2}), (1 : \frac{\sqrt{-1}(\sqrt{3} - 2 - \sqrt{-1})}{2} : \frac{2 - \sqrt{-1} - \sqrt{3}}{2}) \right).$$

Then the equation of the cubic surface  $S_P$  is

$$F_P = y_1^2 y_2 + y_1 y_2^2 - \left( \frac{\sqrt{3}-1}{2} + \frac{(\sqrt{3}+1)\sqrt{-1}}{2} \right) y_1 y_2 y_3 - \left( \frac{\sqrt{3}-1}{2} - \frac{(\sqrt{3}+1)\sqrt{-1}}{2} \right) y_1 y_2 y_4 + \frac{(\sqrt{3}+1)(1+\sqrt{-1})}{2} y_3^2 y_4 + \frac{(\sqrt{3}+1)(1-\sqrt{-1})}{2} y_3 y_4^2 = 0.$$

The automorphism group  $G_P = \text{Aut } S_P$  is of order 108 and generated by the following two elements  $g_1, g_2$ .

$$g_1 = \begin{pmatrix} \frac{-2+\sqrt{3}+\sqrt{-1}}{2} & \frac{\sqrt{-1}}{2} & \frac{\sqrt{3}+\sqrt{-1}}{2} & \frac{-1-\sqrt{3}}{2} \\ \frac{(\sqrt{3}-1)(1-\sqrt{-1})}{2} & \frac{-1+(2-\sqrt{3})\sqrt{-1}}{2} & \frac{(\sqrt{3}-1)(1-\sqrt{-1})}{2} & \frac{-1+(2-\sqrt{3})\sqrt{-1}}{2} \\ \frac{2-\sqrt{3}-\sqrt{-1}}{2} & \frac{1-(2-\sqrt{3})\sqrt{-1}}{2} & \frac{-\sqrt{3}-\sqrt{-1}}{2} & \frac{(\sqrt{3}-1)(\sqrt{-1}-1)}{2} \\ \frac{(\sqrt{3}-1)(\sqrt{-1}-1)}{2} & \frac{-1-(2-\sqrt{3})\sqrt{-1}}{2} & \frac{(\sqrt{3}-1)(\sqrt{-1}-1)}{2} & \frac{-1-(2-\sqrt{3})\sqrt{-1}}{2} \end{pmatrix},$$

$$g_2 = \begin{pmatrix} \frac{-1-(2+\sqrt{3})\sqrt{-1}}{2} & \frac{-1-(2+\sqrt{3})\sqrt{-1}}{2} & \frac{(\sqrt{3}+1)(\sqrt{-1}-\sqrt{3})}{2} & 1 + \sqrt{3} \\ -1 & \frac{\sqrt{3}-\sqrt{-1}}{2} & \frac{\sqrt{3}-\sqrt{-1}}{2} & -1 \\ \frac{-\sqrt{3}+\sqrt{-1}}{2} & \frac{-\sqrt{3}+\sqrt{-1}}{2} & 1 + \sqrt{-1} & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

The action of  $g_i$  on the configuration  $\Gamma$  is as follows.

$$g_1(E_1, E_2, E_3, E_4, E_5, E_6, F_1, F_2, F_3, F_4, F_5, F_6,$$

$$\begin{aligned}
& (L_{12}, L_{13}, L_{14}, L_{15}, L_{16}, L_{23}, L_{24}, L_{25}, L_{26}, L_{34}, L_{35}, L_{36}, L_{45}, L_{46}, L_{56}) \\
= & (L_{12}, F_4, L_{23}, E_4, L_{25}, L_{26}, L_{14}, F_2, L_{34}, E_2, L_{45}, L_{46}, \\
& E_1, L_{56}, F_1, L_{36}, L_{35}, E_3, L_{24}, E_5, E_6, F_3, L_{16}, L_{15}, F_5, F_6, L_{13}) \\
g_2(E_1, E_2, E_3, E_4, E_5, E_6, F_1, F_2, F_3, F_4, F_5, F_6, \\
& L_{12}, L_{13}, L_{14}, L_{15}, L_{16}, L_{23}, L_{24}, L_{25}, L_{26}, L_{34}, L_{35}, L_{36}, L_{45}, L_{46}, L_{56}) \\
= & (E_3, L_{45}, L_{14}, L_{15}, E_2, E_6, L_{26}, F_1, F_5, F_4, L_{36}, L_{23}, \\
& L_{13}, L_{35}, L_{34}, F_6, F_2, E_4, E_5, L_{12}, L_{16}, E_1, L_{25}, L_{56}, L_{24}, L_{46}, F_3)
\end{aligned}$$

### 4.3. The case of order 8

Take the point

$$P = ((1 : 1 - \sqrt{-1} : -\sqrt{-1}), (1 : -\sqrt{-1} - \frac{1 + \sqrt{-1}}{\sqrt{2}} : (1 - \sqrt{-1})(1 + \frac{1}{\sqrt{2}}))).$$

Then the equation of the cubic surface  $S_P$  is

$$\begin{aligned}
F_P = & (1 - (1 + \sqrt{2})\sqrt{-1})y_1^2y_2 + \frac{(\sqrt{2}+1)(1-\sqrt{-1})}{\sqrt{2}}y_1y_2^2 + (2 + \sqrt{2} - (1 + \sqrt{2})\sqrt{-1})y_1y_2y_3 - \\
& (1 + \sqrt{2})\sqrt{-1}y_1y_2y_4 + (2\sqrt{-1} + \frac{1+3\sqrt{-1}}{\sqrt{2}})y_1y_3y_4 + (2 + \sqrt{2})\sqrt{-1}y_2y_3y_4 + (-1 + \\
& 2\sqrt{-1} - \frac{1-3\sqrt{-1}}{\sqrt{2}})y_3^2y_4 + (1 + 2\sqrt{-1} + \frac{1+3\sqrt{-1}}{\sqrt{2}})y_3y_4^2 = 0.
\end{aligned}$$

The automorphism group  $G_P = \text{Aut } S_P$  is of order 8 and generated by the following element  $g$ .

$$g = \begin{pmatrix} -1 & \frac{-1-(1+\sqrt{2})\sqrt{-1}}{2} & \frac{-(1+\sqrt{2})(1+\sqrt{-1})}{\sqrt{2}} & \frac{-1-\sqrt{2}+\sqrt{-1}}{2} \\ \sqrt{-1} & 0 & 0 & \frac{1+\sqrt{-1}}{\sqrt{2}} \\ -\sqrt{-1} & 0 & 0 & -\sqrt{-1} \\ 1 & \frac{1+\sqrt{-1}}{\sqrt{2}} & \frac{(1+\sqrt{2})(1+\sqrt{-1})}{\sqrt{2}} & \frac{1-\sqrt{-1}}{\sqrt{2}} \end{pmatrix}$$

The action of  $g$  on the configuration  $\Gamma$  is as follows.

$$\begin{aligned}
g(E_1, E_2, E_3, E_4, E_5, E_6, F_1, F_2, F_3, F_4, F_5, F_6, \\
& L_{12}, L_{13}, L_{14}, L_{15}, L_{16}, L_{23}, L_{24}, L_{25}, L_{26}, L_{34}, L_{35}, L_{36}, L_{45}, L_{46}, L_{56}) \\
= & (L_{34}, L_{24}, L_{23}, E_5, E_1, E_6, F_2, F_3, F_4, L_{16}, L_{56}, L_{15}, \\
& E_4, E_3, L_{25}, L_{12}, L_{26}, E_2, L_{35}, L_{13}, L_{36}, L_{45}, L_{14}, L_{46}, F_6, F_1, F_5).
\end{aligned}$$

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