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Singular Integrals along Surfaces of Revolution with Rough Kernels

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Abstract. In this paper we continue the study of the L^p mapping properties of singular integrals along surfaces of revolution which was initiated in [KWWZ] and obtain improvements over existing results by allowing kernels to be in certain block spaces.

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§1. Introduction

Let $\mathbf{R}^n$, $n \geq 2$ be the $n-$dimensional Euclidean space and $\mathbf{S}^{n-1}$ be the unit sphere in $\mathbf{R}^n$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(\cdot)$. Let $h(\cdot)$ be a locally integrable function on $\mathbf{R}^+$ and let Ω be a homogeneous function of degree 0 on $\mathbf{R}^n$ satisfying

$$\int_{\mathbf{S}^{n-1}} \Omega\left(u\right) d\sigma\left(u\right) = 0. \tag{1.1}$$

For a suitable surface $\Gamma : \Gamma : \mathbf{R}^n \to \mathbf{R}^d$ with $d \geq n+1$, let $T_{\Gamma,h}$ be the singular integral operator along the surface Γ and let $\mathcal{M}_{\Gamma,h}$ be the related maximal operator defined for $f \in \mathcal{S}\left(\mathbf{R}^d\right)$ by

$$T_{\Gamma,h}f(x) = \text{p.v.} \int_{\mathbf{R}^n} f\left(x - \Gamma\left(y\right)\right) K\left(y\right) dy \tag{1.2}$$

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and

$$\mathcal{M}_{\Gamma,h}f(x)=\sup_{r>0}\frac{1}{r^n}\int\limits_{|y|\leq r}|f(x-\Gamma(y)|\,|\Omega\,(y)|\,|h(|y|)|\,dy \tag{1.3}$$

where

$$K(y)=\frac{\Omega(y)}{|y|^n}h(|y|). \tag{1.4}$$

For $\gamma > 1$, let $\Delta_\gamma\,(\mathbf{R}^+)$ denote the set of measurable functions h on $\mathbf{R}^+$ such that

$$\sup_{R>0}\frac{1}{R}\int\limits_0^R |h\,(t)|^\gamma\,dt<\infty.$$

The L^p mapping properties of singular integral operators along curves and surfaces has been extensively studied by a number of authors (see [St2], [CNSW], among others). For example, it is known that if Γ satisfies a "finite type" condition at 0 then $T_{\Gamma,1}$ and $\mathcal{M}_{\Gamma,1}$ are bounded on L^p for $1 < p < \infty$ (see [St1] and [FGP2]).

In [KWWZ], Kim, Wainger, Wright and Ziesler studied singular integrals and maximal functions along surfaces of revolution without imposing the "finite type" condition on Γ. More precisely, they proved the following two L^p boundedness results of the operators $T_{\phi,h} = T_{\Gamma,h}$ and $\mathcal{M}_{\phi,h} = \mathcal{M}_{\Gamma,h}$ for $\Gamma(x) = (x, \phi(|x|))$ and a suitable function ϕ. Here, for $f \in \mathcal{S}\left(\mathbf{R}^{n+1}\right)$ and $(x, x_{n+1}) \in \mathbf{R}^n \times \mathbf{R}^1$, the singular integral $T_{\phi,h}$ is defined by

$$T_{\phi,h}f(x,x_{n+1})=\text{p.v.}\int\limits_{\mathbf{R}^n} f\,(x-y,x_{n+1}-\phi\,(|y|))\,K\,(y)\,dy \tag{1.5}$$

and the related maximal function $\mathcal{M}_{\phi,h}$ is defined as

(1.6)

$$\mathcal{M}_{\phi,h}f(x,x_{n+1})=\sup_{r>0}\frac{1}{r^n}\int\limits_{|y|\leq r}|f\,(x-y,x_{n+1}-\phi\,(|y|))|\,|\Omega\,(y)|\,|h(|y|)|\,dy.$$

The gist of their first result is that as far as the L^2 boundedness is concerned, the results for the operators $T_{\phi,1}$ and $\mathcal{M}_{\phi,1}$ are quite different. In fact, they proved the following:

Theorem 1. *Let K, $T_{\phi,h}$ and $\mathcal{M}_{\phi,h}$ be given as in* (1.1), (1.4)-(1.6). *Assume that $\Omega \in C^\infty(\mathbf{S}^{n-1})$, $h \equiv 1$ and $\phi(\cdot)$ is in C^1 of $[0,\infty)$. Then there exists a* *positive constant* C such that

$$\|T_{\phi,1}(f)\|_{L^2(\mathbf{R}^{n+1})} \le C\,\|f\|_{L^2(\mathbf{R}^{n+1})} \text{ for every } f \in L^2(\mathbf{R}^{n+1}),$$

whereas the analogous statement for $\mathcal{M}_{\phi,1}$ *is false.*

For other values of p, the authors imposed a convexity condition on ϕ and they proved the following:

Theorem 2. *Let* K, $T_{\phi,h}$ *and* $\mathcal{M}_{\phi,h}$ *be given as in* (1.1), (1.4)-(1.6)*. Assume that* $\Omega \in C^\infty(\mathbf{S}^{n-1})$, $h \equiv 1$ *and* $\phi(\cdot)$ *is in* C^2 *of* $[0,\infty)$, *that* ϕ *is convex, and increasing. Then there exists a positive constant* C_p *such that*

$$\|T_{\phi,1}(f)\|_{L^p(\mathbf{R}^{n+1})} \le C_p\,\|f\|_{L^p(\mathbf{R}^{n+1})} \text{ for } 1 < p < \infty$$

and

$$\|\mathcal{M}_{\phi,1}(f)\|_{L^p(\mathbf{R}^{n+1})} \le C_p\,\|f\|_{L^p(\mathbf{R}^{n+1})} \text{ for } 1 < p \le \infty.$$

Even though the authors of [KWWZ] imposed the condition $\Omega \in C^\infty(\mathbf{S}^{n-1})$ in Theorems 1 and 2, by a minor modification of their proofs, one can show that the L^p boundedness of the operators $T_{\Gamma,1}$ and $\mathcal{M}_{\Gamma,1}$ continues to hold under the condition $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$. A question that arises naturally is whether the operators $T_{\Gamma,1}$ and $\mathcal{M}_{\Gamma,1}$ are bounded under a weaker condition on Ω?

Our main purpose in this paper is to give a positive answer to the above question by considering Ω in the block space $B_q^{0,0}(\mathbf{S}^{n-1})$ (the definition of these spaces will be recalled in Section 2). It should be noted that $B_q^{0,0}\left(\mathbf{S}^{n-1}\right)$ contains $L^q\left(\mathbf{S}^{n-1}\right)$ as a proper subspace for each $q > 1$ and

$$\bigcup_{q>1} L^q(\mathbf{S}^{n-1}) \subsetneqq \bigcup_{q>1} B_q^{0,0}(\mathbf{S}^{n-1}).$$

We shall now give precise statements of our results, beginning with an L^2 theorem:

Theorem 3. *Let* K *and* $T_{\phi,h}$ *be given as in* (1.1), (1.4)-(1.5)*. Assume that* $\Omega \in B_q^{0,0}(\mathbf{S}^{n-1})$, $h \in \Delta_\gamma(\mathbf{R}^+)$ *for some* $\gamma > 1$ *and* $\phi \in C^1([0,\infty))$*. Then there exists a positive constant* C_p

$$\|T_{\phi,h}(f)\|_{L^2(\mathbf{R}^{n+1})} \le C_p\,\|f\|_{L^2(\mathbf{R}^{n+1})} \text{ for every } f \in L^2\left(\mathbf{R}^{n+1}\right). \tag{1.7}$$

The following is our main result regarding the operators $T_{\phi,h}$ and $\mathcal{M}_{\phi,h}$.

Theorem 4. *Let* K, $T_{\phi,h}$ *and* $\mathcal{M}_{\phi,h}$ *be given as in* (1.1), (1.4)-(1.6) *and let* γ' *be the conjugate index to* γ*. Assume that* $\Omega \in B_q^{0,0}(\mathbf{S}^{n-1})$, $h \in \Delta_\gamma(\mathbf{R}^+)$ *for* *some* $\gamma > 1$ *and* ϕ *is in* $C^2([0,\infty))$*, convex, and increasing. Then*

(i) *for every* p *satisfying* $\left|\frac{1}{p}-\frac{1}{2}\right| < \min\left\{\frac{1}{2},\frac{1}{\gamma'}\right\}$*, there exists a positive constant* C_p *such that*

$$\|T_{\phi,h}(f)\|_{L^p(\mathbf{R}^{n+1})} \le C_p \|f\|_{L^p(\mathbf{R}^{n+1})} \tag{1.8}$$

for every $f \in L^p\left(\mathbf{R}^{n+1}\right)$.

(ii) *for all* $\gamma' < p \le \infty$*, there exists a positive constant* C_p *such that*

$$\|\mathcal{M}_{\phi,h}(f)\|_{L^p(\mathbf{R}^{n+1})} \le C_p \|f\|_{L^p(\mathbf{R}^{n+1})} \tag{1.9}$$

for all $f \in L^p\left(\mathbf{R}^{n+1}\right)$.

We point out that the appearance of the function h in the kernel was first allowed by R. Fefferman in [Fe] ($\gamma = \infty$), and then in [DR] ($\gamma \ge 2$) in their study of the singular integral operator $T_{\Gamma,h}$ for $d = n+1$ and $\Gamma(y) = (y,0)$.

We are also interested in studying the L^p boundedness of the maximal truncated singular integral operators $T^*_{\phi,h}$. The operators $T^*_{\phi,h}$ defined by

$$T^*_{\phi,h}f(x,x_{n+1}) = \sup_{\varepsilon>0}\left|\int\limits_{|y|>\varepsilon} f\left(x-y, x_{n+1}-\phi\left(|y|\right)\right)K(y)\,dy\right|. \tag{1.10}$$

Our result regarding $T^*_{\phi,h}$ is the following:

Theorem 5. *Let* K *and* $T^*_{\phi,h}$ *be given as in* (1.1), (1.4)-(1.6) *and* (1.10). *Assume that* $\Omega \in B_q^{0,0}(\mathbf{S}^{n-1})$. *Then there exists a positive constant* C_p *such that*

(i) *if* $h \in \Delta_\gamma(\mathbf{R}^+)$ *for some* $\gamma > 1$ *and* $\phi \in C^1([0,\infty))$,

$$\left\|T^*_{\phi,h}(f)\right\|_{L^2(\mathbf{R}^{n+1})} \le C_p \|f\|_{L^2(\mathbf{R}^{n+1})} \text{ for every } f \in L^2\left(\mathbf{R}^{n+1}\right), \tag{1.11}$$

and

(ii) *if* $h \in L^\infty(\mathbf{R}^+)$ *and* $\phi \in C^2([0,\infty))$*, convex, and increasing,*

$$\left\|T^*_{\phi,h}(f)\right\|_{L^p(\mathbf{R}^{n+1})} \le C_p \|f\|_{L^p(\mathbf{R}^{n+1})} \tag{1.12}$$

for all $1 < p < \infty$ *and* $f \in L^p\left(\mathbf{R}^{n+1}\right)$.

We remark that as in the classical theory of singular integrals, our result in Theorem 5 implies that the truncated integrals

$$\int\limits_{\varepsilon\le|u|\le N} f\left(x-u, x_{n+1}-\phi\left(|u|\right)\right)K(u)\,du \tag{1.13}$$

converge almost everywhere as $\varepsilon \to 0^+$ and $N \to \infty$ for every $f \in L^p(\mathbf{R}^{n+1})$ with $1 < p < \infty$.

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§2. Certain block spaces on $\mathbf{S}^{n-1}$

The notion of block spaces was introduced in [TW] by M. H. Taibleson and G. Weiss in their studies of a.e. convergence of the Fourier series. For more information about the applications of block spaces in harmonic analysis one may consult the book [LTW]. Let us now begin by recalling the definition of a block function on $\mathbf{S}^{n-1}$.

Definition 6. *For $1 < q \le \infty$ we say that a measurable function $b(\cdot)$ on $\mathbf{S}^{n-1}$ is a $q-$block if it satisfies the following:*

(i) *supp $(b) \subseteq \mathcal{I}$ and*

(ii) $\|b\|_{L^q} \le |\mathcal{I}|^{-\frac{1}{q'}}$ *, where $\mathcal{I}$ is an interval on $\mathbf{S}^{n-1}$; i.e.,*

$$\mathcal{I} = \{x' \in \mathbf{S}^{n-1} : \left|x' - x_0'\right| < \alpha \text{ for some } x_0' \in \mathbf{S}^{n-1}\} \text{ and } |\mathcal{I}| = \sigma(\mathcal{I}).$$

The block spaces $B_q^{0,0}$ on $\mathbf{S}^{n-1}$ are defined as follows:

Definition 7. *The function space $B_q^{0,0}\left(\mathbf{S}^{n-1}\right)$, $1 < q \le \infty$, consists of all functions $\Omega \in L^1\left(\mathbf{S}^{n-1}\right)$ of the form $\Omega = \sum_{\mu=1}^{\infty} c_\mu b_\mu$ where $c_\mu \in \mathbf{C}$; each b_μ is a $q-$block supported in an interval $\mathcal{I}_\mu$; and*

$$M_q^{0,0}\left(\{c_\mu\}\right) = \sum_{\mu=1}^{\infty} \left|c_\mu\right| \left(1 + \log \frac{1}{\left|\mathcal{I}_\mu\right|}\right) < \infty. \tag{2.1}$$

The special class of functions $B_q^{0,0}$ was first introduced by Jiang and Lu in their study of singular integral operators of Calderón-Zygmund type (see [LTW]). Every $q-$block $b(\cdot)$ described in Definition 6 has a companion function $\tilde{b}(\cdot)$ given by

$$\tilde{b}(x) = b(x) - \int_{\mathbf{S}^{n-1}} b(u) d\sigma(u). \tag{2.2}$$

Then $\tilde{b}$ has the following properties:

$$\int_{\mathbf{S}^{n-1}} \tilde{b}\,(u)\, d\sigma\,(u) = 0, \tag{2.3}$$

$$\left\|\tilde{b}\right\|_{L^q(\mathbf{S}^{n-1})} \le 2\,|\mathcal{I}|^{-\frac{1}{q'}} \tag{2.4}$$

and

$$\left\|\tilde{b}\right\|_{L^1(\mathbf{S}^{n-1})} \le 2. \tag{2.5}$$

The function $\tilde{b}$ is called the blocklike function corresponding to the block function b.

§3. Certain Fourier transform estimates related to blocklike functions

We shall begin by introducing a class of measures associated to a given blocklike function $\tilde{b}$.

Definition 8. *Let* $\tilde{b}(\cdot)$ *be a blocklike function defined as in* (2.2) *and* $\phi(\cdot)$ *be an arbitrary function on* $\mathbf{R}^+$. *Define the measures* $\left\{\sigma_{k,\tilde{b}} : k \in \mathbf{Z}\right\}$ *and the maximal operator* $\sigma^*_{\tilde{b}}$ *on* $\mathbf{R}^{n+1}$ *by*

$$\int\limits_{\mathbf{R}^{n+1}} f\, d\sigma_{k,\tilde{b}} = \int\limits_{2^k \le |u| < 2^{k+1},\ u\in\mathbf{R}^n} f\left(u, \phi\left(|u|\right)\right) h\left(|u|\right) \frac{\tilde{b}\left(u'\right)}{|u|^n} du, \tag{3.1}$$

and

$$\sigma^*_{\tilde{b}}\left(f\right) = \sup_{k\in\mathbf{Z}} \left|\left|\sigma_{k,\tilde{b}}\right| * f\right| \tag{3.2}$$

where $u' = u/\left|u\right| \in \mathbf{S}^{n-1}$.

However, these measures are only useful in the case $|\mathcal{I}| \ge e^{-2}$ where $\mathcal{I}$ is the support of b. For the case $|\mathcal{I}| < e^{-2}$ we need to define the following measures.

Definition 9. *Let* $\tilde{b}(\cdot)$ *be a* $q-$*blocklike function defined as in* (2.2) *and* $\phi(\cdot)$ *be an arbitrary function on* $\mathbf{R}^+$. *We define the measures* $\{\lambda_{k,\tilde{b}} : k \in \mathbf{Z}\}$ *and the maximal operators* $\lambda^*_{\tilde{b}}$ *on* $\mathbf{R}^{n+1}$ *by*

$$\int\limits_{\mathbf{R}^{n+1}} f\, d\lambda_{k,\tilde{b}} = \int\limits_{\omega^k \le |u| < \omega^{k+1}, u\in\mathbf{R}^n} f\left(u, \phi(|u|)\right) \frac{\tilde{b}\left(u'\right)}{|u|^n} h\left(|u|\right) du \tag{3.3}$$

and

$$\lambda^*_{\tilde{b}} f\left(x\right) = \sup_{k\in\mathbf{Z}} \left|\left|\lambda_{k,\tilde{b}}\right| * f(x)\right| \tag{3.4}$$

where $\omega = 2^{\log\left(\frac{1}{|\mathcal{I}|}\right)}$ *and* $|\mathcal{I}| < e^{-2}$.

Now let us establish the following result which will provide us with the necessary Fourier transform estimates related to $\tilde{b}$. One of the key points in these Fourier transform estimates is that the radial nature of the hypersurface $\Gamma(x) = (x, \phi(|x|))$ allows one to obtain these estimates without any condition on ϕ.

Proposition 10. *Let* $h \in \Delta_\gamma\left(\mathbf{R}^+\right)$ *for some* $\gamma, 1 < \gamma \le 2$ *and let* $\tilde{b}$ *be a* $q-$*blocklike function defined as in* (2.2). *If* $|\mathcal{I}| < e^{-2}$, *then there exist constants* C *and* $0 < \beta < \frac{1}{q'}$ *such that*

$$\left|\hat{\lambda}_{k,\tilde{b}}(\xi, \tau)\right| \le C \log\left(\frac{1}{|\mathcal{I}|}\right) \left|\omega^k\ \xi\right|^{\pm\frac{\beta}{\gamma' \log|\mathcal{I}|}} \tag{3.5}$$

holds for all $k \in \mathbf{Z}$ where $\xi \in \mathbf{R}^n$, $\tau \in \mathbf{R}$ and $t^{\pm\alpha} = \inf\{t^{\alpha}, t^{-\alpha}\}$. The constant C is independent of k, $\tilde{b}$, ξ, τ and $\phi(\cdot)$.

Proof. By Hölder's inequality we have

$$\left|\hat{\lambda}_{k,\tilde{b}}(\xi,\tau)\right| \leq \left(\int_{\omega^k}^{\omega^{k+1}} |h(t)|^{\gamma} \frac{dt}{t}\right)^{\frac{1}{\gamma}} \left(\int_1^{\omega} |S_k(t)|^{\gamma'} \frac{dt}{t}\right)^{\frac{1}{\gamma'}}$$

where

$$S_k(t) = \int_{\mathbf{S}^{n-1}} e^{i(\omega^k t\xi\cdot x + \tau\phi(\omega^k t))} \tilde{b}(x) d\sigma(x).$$

Since $|S_k(t)| \leq 2$ we get immediately

$$\left|\hat{\lambda}_{k,\tilde{b}}(\xi,\tau)\right| \leq C \left(\sum_{s=1}^{[\log\frac{1}{|\mathcal{I}|}]+1} \int_{\omega^k 2^{s-1}}^{\omega^k 2^s} |h(t)|^{\gamma} \frac{dt}{t}\right)^{\frac{1}{\gamma}} \left(\int_1^{\omega} |S_k(t)|^2 \frac{dt}{t}\right)^{\frac{1}{\gamma'}}.$$

However, by integration by parts

$$\left|\int_1^{\omega} e^{i\omega^k t\xi\cdot(x-y)} \frac{dt}{t}\right| \leq C \min\left\{\log\left(\frac{1}{|\mathcal{I}|}\right), \left|\omega^k \xi\cdot(x-y)\right|^{-1}\right\}$$

$$\leq C \log\left(\frac{1}{|\mathcal{I}|}\right) \left|\omega^k\xi\right|^{-\beta} \left|\xi'\cdot(x-y)\right|^{-\beta}$$

and

$$|S_k(t)|^2 = \int_{\mathbf{S}^{n-1}}\int_{\mathbf{S}^{n-1}} \tilde{b}(x)\, \overline{\tilde{b}(y)} e^{i\omega^k t(x-y)\cdot\xi}\, d\sigma(x)\, d\sigma(y)$$

where $\xi' = \frac{\xi}{|\xi|}$, and $\beta > 0$ with $0 < \beta q' < 1$. Therefore, by Hölder's inequality

$$\left|\hat{\lambda}_{k,\tilde{b}}(\xi,\tau)\right| \leq$$

$$C \log\left(\frac{1}{|\mathcal{I}|}\right) \left|\omega^k\xi\right|^{-\frac{\beta}{\gamma'}} \left|\int_{\mathbf{S}^{n-1}}\int_{\mathbf{S}^{n-1}} \left|\tilde{b}(x)\tilde{b}(y)\right| \left|\xi'\cdot(x-y)\right|^{-\beta} d\sigma(x)\, d\sigma(y)\right|^{\frac{1}{\gamma'}}$$

$$\leq C\log\left(\frac{1}{|\mathcal{I}|}\right)\left|\omega^k\xi\right|^{-\frac{\beta}{\gamma'}}\left\|\tilde{b}\right\|_{L^q(\mathbf{S}^{n-1})}^{\frac{2}{\gamma'}}\left\{\int\limits_{\mathbf{S}^{n-1}}\int\limits_{\mathbf{S}^{n-1}}|x_1-y_1|^{-\beta q'}\,d\sigma(x)\,d\sigma(y)\right\}^{\frac{1}{\gamma q'}}.$$

where $x=(x_{1,\cdots},x_n)$ and $y=(y_{1,\cdots},y_n)$. Since the last integral is finite, by (2.4) we obtain

$$\left|\hat{\lambda}_{k,\tilde{b}}(\xi,\tau)\right|\leq C\log\left(\frac{1}{|\mathcal{I}|}\right)|\mathcal{I}|^{-\frac{2}{q'\gamma'}}\left|\omega^k\xi\right|^{-\frac{\beta}{\gamma'}}.$$

By interpolation between this estimate and the trivial estimate

$$\left|\hat{\lambda}_{k,\tilde{b}}(\xi,\tau)\right|\leq C\log\left(\frac{1}{|\mathcal{I}|}\right)$$

we get the estimate in (3.5) with a plus sign in the exponent. To get the second estimate, we use the mean zero property (2.3) of $\tilde{b}$ to get

$$\left|\hat{\lambda}_{k,\tilde{b}}(\xi,\tau)\right|\leq\int\limits_{\mathbf{S}^{n-1}}\int\limits_{1}^{\omega}\left|e^{-i\{\omega^k t\xi\cdot x+\tau\phi(\omega^k t)\}}-e^{-i\tau\phi(\omega^k t)}\right|\left|h(\omega^k t|\right)\left|\tilde{b}(x)\right|\frac{dt}{t}d\sigma(x).$$

Hence, $\left|\hat{\lambda}_{k,\tilde{b}}(\xi,\tau)\right|\leq C\omega\left|\omega^k\xi\right|$ which, when combined with the trivial estimate $\left|\hat{\lambda}_{k,\tilde{b}}(\xi,\tau)\right|\leq C\log\left(\frac{1}{|\mathcal{I}|}\right)$, yields the second estimate in (3.5). This completes the proof of our proposition.

By arguments similar to the proof of Proposition 10 we get the following result which is needed for the case $|\mathcal{I}|\geq e^{-2}$.

Proposition 11. *Let $h\in\Delta_\gamma(\mathbf{R}^+)$ for some $\gamma, 1<\gamma\leq 2$ and let $\tilde{b}$ be a $q-$blocklike function defined as in* (2.2)*. Let $\phi(\cdot)$ be an arbitrary function on $\mathbf{R}^+$. If $|\mathcal{I}|\geq e^{-2}$, then there exists a constant C and $0<\beta<\frac{1}{q'}$ such that*

$$\left|\hat{\sigma}_{k,\tilde{b}}(\xi,\tau)\right|\leq C\left|2^k\ \xi\right|^{\pm\frac{\beta}{\gamma'}}\tag{3.6}$$

holds for all $k\in\mathbf{Z}$. The constant C is independent of k, $\tilde{b},\xi,\tau$ and $\phi(\cdot)$.

§4. Maximal functions and singular integrals

We shall need the following result from [AqP] which is an extension of a result of Duoandikoetxea and Rubio de Francia in [DR] (see also [FP]).

Lemma 12. *Let $\{\sigma_k : k\in\mathbf{Z}\}$ be a sequence of Borel measures on $\mathbf{R}^n$ and $L:\mathbf{R}^n\to\mathbf{R}^d$ be a linear transformation. Suppose that for all $k\in\mathbf{Z}$, $\xi\in\mathbf{R}^n$, for some $\eta\in[2,\infty)$, α, $C>0$ and for some $B>1$ we have*

(i) $|\hat{\sigma}_k(\xi)| \le CB(\eta^{kB}\,|L(\xi)|\,)^{\pm\frac{\alpha}{B}}$;
(ii) *For some* $p_0 \in (2,\infty)$

$$\left\| (\sum_{k\in\mathbf{Z}} |\sigma_k * g_k|^2)^{\frac{1}{2}} \right\|_{p_0} \le CB \left\| (\sum_{k\in\mathbf{Z}} |g_k|^2)^{\frac{1}{2}} \right\|_{p_0} \tag{4.1}$$

for arbitrary functions $\{g_k\}$ *on* $\mathbf{R}^n$. *Then for* $p_0' < p < p_0$ *there exists a positive constant* C_p *such that*

$$\left\| \sum_{k\in\mathbf{Z}} \sigma_k * f \right\|_{L^p(\mathbf{R}^n)} \le C_p B \, \|f\|_{L^p(\mathbf{R}^n)} \tag{4.2}$$

and

$$\left\| (\sum_{k\in\mathbf{Z}} |\sigma_k * f|^2)^{\frac{1}{2}} \right\|_{L^p(\mathbf{R}^n)} \le C_p B \, \|f\|_{L^p(\mathbf{R}^n)} \tag{4.3}$$

hold for all f *in* $L^p(\mathbf{R}^n)$. *The constant* C_p *is independent of* B *and the linear transformation* L.

By tracking the constants in the proof of the lemma given in ([DR], page 544) we have the following:

Lemma 13. *Let* λ_k *be a sequence of Borel measures in* $\mathbf{R}^n$ *and let* $\lambda^*(f) = \sup_{k\in\mathbf{Z}} ||\lambda_k| * f|$. *Assume that*

$$\|\lambda^*(f)\|_q \le B \, \|f\|_q \text{ for some } q > 1 \text{ and } B > 0. \tag{4.4}$$

Then, for arbitrary functions $\{g_k\}$ *on* $\mathbf{R}^n$ *and* $\left|\frac{1}{p_0} - \frac{1}{2}\right| = \frac{1}{2q}$, *the following inequality holds*

$$\left\| (\sum_{k\in\mathbf{Z}} |\lambda_k * g_k|^2)^{\frac{1}{2}} \right\|_{p_0} \le (B \sup_{k\in\mathbf{Z}} \|\lambda_k\|)^{\frac{1}{2}} \left\| (\sum_{k\in\mathbf{Z}} |g_k|^2)^{\frac{1}{2}} \right\|_{p_0}. \tag{4.5}$$

Our aim now is to establish the following result.

Proposition 14. *Let* $h \in \Delta_\gamma(\mathbf{R}^+)$ *for some* $\gamma > 1$ *and let* $\tilde{b}$ *be defined as in* (2.2). *Assume* ϕ *is in* $C^2([0,\infty))$, *convex, and increasing. Then for* $\gamma' < p \le \infty$ *and* $f \in L^p(\mathbf{R}^{n+1})$ *there exists a positive constant* C_p *which is independent of* $\tilde{b}$ *such that*

$$\left\| \lambda_{\tilde{b}}^*(f) \right\|_{L^p(\mathbf{R}^{n+1})} \le C_p \log\left(\frac{1}{|\mathcal{I}|}\right) \|f\|_{L^p(\mathbf{R}^{n+1})} \quad \text{if } |\mathcal{I}| < e^{-2} \tag{4.6}$$

and

$$\left\|\sigma_{\tilde{b}}^{*}(f)\right\|_{L^{p}(\mathbf{R}^{n+1})} \leq C_{p}\|f\|_{L^{p}(\mathbf{R}^{n+1})} \quad \text{if } |\mathcal{I}| \geq e^{-2}. \tag{4.7}$$

Proof. Assume first that $|\mathcal{I}| < e^{-2}$. Without loss of generality we may assume that $\tilde{b} \geq 0$ and $h \geq 0$. By Hölder's inequality we have

$$\lambda_{\tilde{b}}^{*}(f) \leq \left(\int_{\omega^{k}}^{\omega^{k+1}} |h(t)|^{\gamma} \frac{dt}{t}\right)^{\frac{1}{\gamma}} \left(\Upsilon_{\tilde{b}}^{*}(|f|^{\gamma'})\right)^{\frac{1}{\gamma'}} \leq C(\log\left(\frac{1}{|\mathcal{I}|}\right))^{\frac{1}{\gamma}} \left(\Upsilon_{\tilde{b}}^{*}(|f|^{\gamma'})\right)^{\frac{1}{\gamma'}}$$

where

$$\int_{\mathbf{R}^{n+1}} f\, d\Upsilon_{k,\tilde{b}} = \int_{\omega^{k} \leq |u| < \omega^{k+1}} f\left(u, \phi(|u|)\right) \frac{\tilde{b}\left(u'\right)}{|u|^{n}} du$$

and

$$\Upsilon_{\tilde{b}}^{*}(f) = \sup_{k \in \mathbf{Z}} \left|\left|\Upsilon_{k,\tilde{b}}\right| * f\right|.$$

Therefore, in order to prove (4.6) it suffices to prove that

$$\left\|\Upsilon_{\tilde{b}}^{*}(f)\right\|_{L^{p}(\mathbf{R}^{n+1})} \leq C_{p} \log\left(\frac{1}{|\mathcal{I}|}\right) \|f\|_{L^{p}(\mathbf{R}^{n+1})}$$

for $1 < p \leq \infty$. By following the same arguments as in the proof of Proposition 10 we obtain for some α, $0 < \alpha < 1$,

$$\left|\hat{\Upsilon}_{k,\tilde{b}}(\xi, \eta)\right| \leq C \log\left(\frac{1}{|\mathcal{I}|}\right) \left|2^{k(\log \frac{1}{|\mathcal{I}|})} \xi\right|^{\frac{\alpha}{\gamma' \log|\mathcal{I}|}} \tag{4.8}$$

and

$$\left|\hat{\Upsilon}_{k,\tilde{b}}(\xi, \eta) - \hat{\Upsilon}_{k,\tilde{b}}(0, \eta)\right| \leq C \log\left(\frac{1}{|\mathcal{I}|}\right) \left|2^{k(\log \frac{1}{|\mathcal{I}|})} \xi\right|^{-\frac{\alpha}{\gamma' \log|\mathcal{I}|}}. \tag{4.9}$$

We now choose a $\psi \in \mathcal{S}(\mathbf{R}^{n})$ such that $\hat{\psi}(\xi) = 1$ for $|\zeta| \leq 1$ and $\hat{\psi}(\xi) = 0$ for $|\xi| \geq 2$. Let $(\psi_{k})\hat{}(\xi) = \hat{\psi}\left(2^{k(\log \frac{1}{|\mathcal{I}|})} \xi\right)$, $k \in \mathbf{Z}$, and define the sequence of measures $\{\tau_{k}\}$ by

$$\hat{\tau}_{k}(\xi, \eta) = \hat{\Upsilon}_{k,\tilde{b}}(\xi, \eta) - (\psi_{k})\hat{}(\xi)\, \hat{\Upsilon}_{k,\tilde{b}}(0, \eta). \tag{4.10}$$

By (4.8)-(4.9) we obtain

$$\left|\hat{\tau}_{k}(\xi, \eta)\right| \leq C \log\left(\frac{1}{|\mathcal{I}|}\right) \left|2^{k(\log \frac{1}{|\mathcal{I}|})} \xi\right|^{\pm \frac{\alpha}{\gamma' \log|\mathcal{I}|}}. \tag{4.11}$$

Let $B(f)$ denote the square function and $\tau^*(f)$ be the maximal function defined as follows: $B(f)(x, x_{n+1}) = (\sum_{k\in\mathbf{Z}} |\tau_k * f(x, x_{n+1})|^2)^{\frac{1}{2}}$ and $\tau^*(f)(x, x_{n+1}) = \sup_{k\in\mathbf{Z}} ||\tau_k| * f(x, x_{n+1})|$. Also, let $H_\phi(f)$ denote the maximal function defined by

$$H_\phi f(y, y_{n+1}) = \sup_{k\in\mathbf{Z}} \left| \int_{\omega^k}^{\omega^{k+1}} f(y, y_{n+1} - \phi(t)) \frac{dt}{t} \right|.$$

By (4.10) we have

$$\Upsilon^*_{\tilde{b}} f(x, x_{n+1}) \leq B(f)(x, x_{n+1}) + C((\mathcal{M}_{\mathbf{R}^n} \otimes id_{\mathbf{R}^1}) \circ H_\phi)(f(x, x_{n+1})) \tag{4.12}$$

and

$$\tau^* f(x, x_{n+1}) \leq B(f)(x, x_{n+1}) + 2C((\mathcal{M}_{\mathbf{R}^n} \otimes id_{\mathbf{R}^1}) \circ H_\phi)(f(x, x_{n+1})) \tag{4.13}$$

where $\mathcal{M}_{\mathbf{R}^s}$ is the classical Hardy-Littlewood maximal function on $\mathbf{R}^s$.

We need now to show the boundedness in $L^p(\mathbf{R}^{n+1})$ for all $p > 1$ of the maximal operator H_ϕ. For $f \geq 0$ and $(y, y_{n+1}) \in \mathbf{R}^n \times \mathbf{R}^1$, we have

$$H_\phi f(y, y_{n+1}) = \sup_{k\in\mathbf{Z}} \left(\int_{\phi(\omega^k)}^{\phi(\omega^{k+1})} f(y, y_{n+1} - t) \frac{dt}{\phi^{-1}(t)\phi'(\phi^{-1}(t))} \right).$$

Without loss of generality, we may assume that $\phi(t) > \phi(0)$ for all $t > 0$. Since the function $\frac{1}{\phi^{-1}(t)\phi'(\phi^{-1}(t))}$ is non-negative, decreasing and its integral over $[\phi(\omega^k),\ \phi(\omega^{k+1})]$ is equal to $\log\left(\frac{1}{|\mathcal{I}|}\right)$ we have

$$H_\phi f(y, y_{n+1}) \leq C \log\left(\frac{1}{|\mathcal{I}|}\right) \mathcal{M}_{\mathbf{R}^1} f(y, \cdot)(y_{n+1}). \tag{4.14}$$

By (4.11) and Plancherel's theorem we obtain

$$\|B(f)\|_{L^2} \leq C \log\left(\frac{1}{|\mathcal{I}|}\right) \|f\|_{L^2}. \tag{4.15}$$

By the L^p boundedness of the Hardy-Littlewood maximal function, (4.13) and (4.15) we get

$$\|\tau^*(f)\|_{L^2} \le C \log\left(\frac{1}{|\mathcal{I}|}\right) \|f\|_{L^2} \tag{4.16}$$

with C independent of $\tilde{b}$. By applying Lemma 13 (with $q = 2$) along with the fact $\|\tau_k\| \le C \log\left(\frac{1}{|\mathcal{I}|}\right)$ we get

$$\left\|(\sum_{k\in\mathbf{Z}}(|\tau_k * g_k|^2)^{\frac{1}{2}}\right\|_{p_0} \le C_{p_0} \log\left(\frac{1}{|\mathcal{I}|}\right) \left\|(\sum_{k\in\mathbf{Z}}|g_k|^2)^{\frac{1}{2}}\right\|_{p_0} \tag{4.17}$$

if $\frac{1}{4} = \left|\frac{1}{p_0} - \frac{1}{2}\right|$. Now, by (4.11), (4.17) and Lemma 12 we obtain

$$\|B(f)\|_{L^p} \le C_p \log\left(\frac{1}{|\mathcal{I}|}\right) \|f\|_{L^p} \tag{4.18}$$

for all p satisfying $\frac{4}{3} < p < 4$ which, when combined with the L^p boundedness of the Hardy-Littlewood maximal function and (4.13)-(4.14), implies

$$\|\tau^*(f)\|_{L^p} \le C \log\left(\frac{1}{|\mathcal{I}|}\right) \|f\|_{L^p} \tag{4.19}$$

for all p satisfying $\frac{4}{3} < p < 4$. Applying (4.11), (4.19), Lemma 12 and Lemma 13 again gives that

$$\|B(f)\|_{L^p} \le C_p \log\left(\frac{1}{|\mathcal{I}|}\right) \|f\|_{L^p} \tag{4.20}$$

for every p satisfying $\frac{8}{7} < p < 8$. By proceeding as above we ultimately get that

$$\|B(f)\|_{L^p} \le C_p \log\left(\frac{1}{|\mathcal{I}|}\right) \|f\|_{L^p} \tag{4.21}$$

for all $p \in (1, \infty)$. Therefore, by (4.12), (4.14) and (4.21) we get that

$$\left\|\Upsilon^*_{\tilde{b}}(f)\right\|_{L^p} \le C_p \log\left(\frac{1}{|\mathcal{I}|}\right) \|f\|_{L^p} \tag{4.22}$$

for $p \in (1, \infty)$. Since the inequality

$$\left\|\Upsilon^*_{\tilde{b}}(f)\right\|_{L^\infty} \le C \log\left(\frac{1}{|\mathcal{I}|}\right) \|f\|_{L^\infty}$$

holds trivially, the proof of (4.6) is complete. To prove the inequality (4.7) we argue more or less as in the proof of (4.6). We omit the details.

By Proposition 14 and following the same ideas as those used in the proof of Theorem 7.5 in [FP] we get the following:

Proposition 15. *Let $h \in \Delta_\gamma(\mathbf{R}^+)$ for some $\gamma, 1 < \gamma \leq 2$ and let $\tilde{b}$ be a $q-$blocklike function. Assume ϕ is in $C^2([0,\infty))$, convex, and increasing. Then for any p satisfying $\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{\gamma'}$, there exists a constant C_p which is independent of $\tilde{b}$ such that*

$$\left\| \left(\sum_{k\in\mathbf{Z}} \left|\nu_{k,\tilde{b}} * g_k\right|^2\right)^{\frac{1}{2}} \right\|_p \leq C_p A \left\| \left(\sum_{k\in\mathbf{Z}} |g_k|^2\right)^{\frac{1}{2}} \right\|_p \tag{4.23}$$

for any $f \in L^p(\mathbf{R}^{n+1})$ where $\nu_{k,\tilde{b}} = \lambda_{k,\tilde{b}}$ if $|\mathcal{I}| < e^{-2}$, $\nu_{k,\tilde{b}} = \sigma_{k,\tilde{b}}$ if $|\mathcal{I}| \geq e^{-2}$, $A = \log\left(\frac{1}{|\mathcal{I}|}\right)$ if $|\mathcal{I}| < e^{-2}$ and $A = 1$ if $|\mathcal{I}| \geq e^{-2}$.

Proofs of main results.

By assumption, Ω can be written as $\Omega = \sum_{\mu=1}^{\infty} c_\mu b_\mu$ where $c_\mu \in \mathbf{C}$, b_μ is a q-block supported on an interval $\mathcal{I}_\mu$ on $\mathbf{S}^{n-1}$ and $M_q^{0,0}(\{c_\mu\}) < \infty$. Since $\Delta_\gamma(\mathbf{R}^+) \subseteq \Delta_2(\mathbf{R}^+)$ when $\gamma \geq 2$, we may assume without any loss of generality that $1 < \gamma \leq 2$ and p satisfies $\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{\gamma'}$. For each $\mu = 1, 2, \ldots$, let $\tilde{b}_\mu$ be the blocklike function corresponding to b_μ. From the mean-value zero property of Ω we infer easily that

$$\Omega = \sum_{\mu=1}^{\infty} c_\mu \tilde{b}_\mu. \tag{4.24}$$

Therefore, we have

$$\|T_{\phi,h} f\|_p \leq \sum_{\mu=1}^{\infty} |c_\mu| \, \|T_\mu f\|_p \tag{4.25}$$

where

$$T_\mu f(x, x_{n+1}) = \text{p.v.} \int_{\mathbf{R}^n} f(x-u, x_{n+1} - \phi(|u|)) \frac{\tilde{b}_\mu(u')}{|u|^n} h(|u|)\, du. \tag{4.26}$$

First, Theorem 3 now follows easily from (4.25), Propositions 10-11 and Plancherel's theorem.

Next, by Propositions 10-11, Proposition 15 and Lemma 12 we get

$$\|T_\mu f\|_p = \left\| \sum_{k\in\mathbf{Z}} \lambda_{k,\tilde{b}_\mu} * f \right\|_p \leq C_p \log\left(\frac{1}{|\mathcal{I}_\mu|}\right) \|f\|_p \text{ if } |\mathcal{I}_\mu| < e^{-2}, \tag{4.27}$$

and

$$\|T_\mu f\|_p = \left\| \sum_{k\in\mathbf{Z}} \sigma_{k,\tilde{b}_\mu} * f \right\|_p \le C_p \|f\|_p \text{ if } |\mathcal{I}_\mu| \ge e^{-2}, \tag{4.28}$$

for every $f \in L^p(\mathbf{R}^{n+1})$ and for p satisfying $\left|\frac{1}{p}-\frac{1}{2}\right| < \frac{1}{\gamma'}$. Therefore, by (2.1), (4.25) and (4.27)-(4.28) we get (1.8). On the other hand, by using (4.24) it is easy to see that

$$\begin{aligned}\mathcal{M}_{\phi,h} f(x,\ x_{n+1}) &\le 4\sum_{\mu=1}^{\infty} |c_\mu|\, \sigma^*_{k,\tilde{b}_\mu}(|f|)(x,\ x_{n+1}) \\ &\le 4 \sum_{\mu=1,|\mathcal{I}_\mu|\ge e^{-2}}^{\infty} |c_\mu|\, \sigma^*_{k,\tilde{b}_\mu}(|f|)(x, x_{n+1}) \\ &\quad + 8 \sum_{\mu=1,|\mathcal{I}_\mu|< e^{-2}}^{\infty} |c_\mu|\, \lambda^*_{k,\tilde{b}_\mu}(|f|)(x, x_{n+1}).\end{aligned} \tag{4.29}$$

Hence, inequality (1.9) follows by using (4.6)-(4.7). This concludes the proof of Theorem 4.

Finally, a proof of Theorem 5 can be obtained as follows. By (3.5)-(3.6) and Proposition 14, and the techniques in [AqP] we get (1.12). We omit the details. On the other hand, the inequality (1.11) follows by Plancherel's theorem and the decomposition used in the proof of Lemma 8 in [AqP]. Again we omit the details.

§5. A result on oscillatory singular integrals

By Theorem 4 one can easily obtain the following L^p boundedness result of the following oscillatory singular integral operator S_λ defined by

$$S_\lambda f(x) = \text{p.v.} \int_{\mathbf{R}^n} e^{i\lambda\phi(|x-y|)} K(x-y) f(y) dy$$

where the phase ϕ is a function on $\mathbf{R}$ and $\lambda \in \mathbf{R}$. In fact, we have the following:

Theorem 16. *Let $K(x) = \frac{\Omega(x)}{|x|^n} h(|x|)$ where Ω satisfies (1.1), $\Omega \in B_q^{0,0}(\mathbf{S}^{n-1})$ and $h \in \Delta_\gamma(\mathbf{R}^+)$ for some $\gamma > 1$, and $q > 1$. Assume ϕ is in $C^2([0,\infty))$, convex, and increasing. Then the operator S_λ is bounded from $L^p(\mathbf{R}^n)$ to itself for all p satisfying $\left|\frac{1}{p}-\frac{1}{2}\right| < \min\left\{\frac{1}{2}, \frac{1}{\gamma'}\right\}$. The bound for the operator norm is independent of λ.*

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