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Remarks on linear Schrödinger evolution equations with Coulomb potential with moving center

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Abstract. This paper is concerned with Cauchy problems for the linear Schrödinger evolution equation:

$$i\,\partial_t u(x,t) + \Delta u(x,t) + |x-a(t)|^{-1}u(x,t) + V_1(x,t)u(x,t) = f(x,t)$$

in $\mathbb{R}^N \times [0,T]$, subject to initial condition: $u(x,0) = u_0(x) \in H^2(\mathbb{R}^N) \cap H_2(\mathbb{R}^N)$, where $i := \sqrt{-1}$, $N \geq 3$, $a : [0,T] \to \mathbb{R}^N$ expresses the center of the Coulomb potential, V_1 and $f : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ are another potential and an inhomogeneous term while

$$H_2(\mathbb{R}^N) := \bigl\{v \in L^2(\mathbb{R}^N); |x|^2 v \in L^2(\mathbb{R}^N)\bigr\}.$$

The strong formulation of this problem (with $f \equiv 0$ and $N = 3$) has been solved by Baudouin-Kavian-Puel (2005) partly with formal computation. In this paper we reconstruct their argument with rigorous proofs. Moreover, we show that the solution u satisfies the energy estimate

$$\|\partial_t u(t)\| + \|u(t)\|_{H^2\cap H_2} \leq C_0(\|u_0\|_{H^2\cap H_2} + \|f\|_F),$$

where $C_0 > 0$ is a constant depending on a, V_1 and T, while $\|f\|_F$ is some norm of f.

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§1. Introduction

In this paper we consider Cauchy problems for the Schrödinger equation:

$$\text{(SE)}\quad \begin{cases} i\,\partial_t u(x,t) + \Delta u(x,t) + \dfrac{u(x,t)}{|x-a(t)|} + V_1(x,t)u(x,t) = f(x,t), \\ \hfill (x,t) \in \mathbb{R}^N \times [0,T], \\ u(x,0) = u_0(x), \hfill x \in \mathbb{R}^N \end{cases}$$

in $L^2(\mathbb{R}^N)$, $N \ge 3$, under the assumption (which is the same as in Baudouin, Kavian and Puel [**1**]) that $a : [0,T] \to \mathbb{R}^N$ and the potential $V_1 : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ satisfy

(a) $a \in W^{2,1}(0,T)^N = W^{2,1}(0,T;\mathbb{R}^N)$,

(V1) $\langle x\rangle^{-2}V_1 \in W^{1,1}(0,T;L^\infty(\mathbb{R}^N))$,

(V2) $\langle x\rangle^{-2}\nabla V_1 \in L^1(0,T;L^\infty(\mathbb{R}^N))^N$.

Here we employ the usual notations of function spaces. Namely, we denote the Lebesgue and L^2-type Sobolev spaces by

$$L^p = L^p(\mathbb{R}^N),\ \ p \in [1,\infty], \quad H^s = H^s(\mathbb{R}^N),\ \ s = 1,2,$$

with norm $\|\cdot\|_{L^p}$ and

$$\|v\|_{H^1} := (\|v\|^2 + \|\nabla v\|^2)^{1/2} = \|(1-\Delta)^{1/2}v\|, \quad \|v\|_{H^2} := \|(1-\Delta)v\|.$$

We define the vector valued Lebesgue and Sobolev spaces. Let X be a Banach space with norm $\|\cdot\|_X$. Then $L^1(I;X)$ is the class of measurable functions $u : I \to X$ such that

$$\|u\|_{L^1(0,T;\,X)} := \int_I \|u(t)\|_X\,dt < \infty,$$

while $W^{m,1}(I;X)$ is the class of u such that $(\partial/\partial t)^j u \in L^1(I;X)$ for every $0 \le j \le m$. Also we use the abbreviation for L^2-norm and inner product:

$$\|\cdot\| := \|\cdot\|_{L^2}, \quad (\cdot,\cdot) := (\cdot,\cdot)_{L^2}.$$

Setting $\langle x\rangle := (1+|x|^2)^{1/2}$, we define as

$$H_s = H_s(\mathbb{R}^N) := \big\{v \in L^2(\mathbb{R}^N); \langle x\rangle^s v \in L^2(\mathbb{R}^N)\big\},\ \ s > 0.$$

It is easy to see that H_s is the image of H^s under the Fourier transform, with norm $\|v\|_{H_s} := \|\langle x\rangle^s v\|$. In this connection, it is useful to introduce

$$\|v\|_{H^m\cap H_s} := (\|v\|^2_{H^m} + \|v\|^2_{H_s})^{1/2} \quad (m,s = 1,2).$$

Before stating our result we review the main theorem in [**1**] and its proof. In [**1**] they established the case where $N = 3$ in the following theorem:

Theorem 1.1 ([**1**, Theorems 1 and 2]). *Let $f \equiv 0$ and assume that a and V_1 satisfy conditions* (a)*,* (V1) *and* (V2)*. Then the Cauchy problem* (SE) *with initial value $u_0 \in H^2(\mathbb{R}^N) \cap H_2(\mathbb{R}^N)$ has a unique solution u such that*

$$u \in W^{1,\infty}(0,T;L^2(\mathbb{R}^N)) \cap C_{\mathrm{w}}([0,T];H^2(\mathbb{R}^N) \cap H_2(\mathbb{R}^N)), \tag{1.1}$$

$$u \in L^{\infty}(0,T;H^2(\mathbb{R}^N) \cap H_2(\mathbb{R}^N)) \cap C([0,T];H^1(\mathbb{R}^N) \cap H_1(\mathbb{R}^N)), \tag{1.2}$$

where $C_{\mathrm{w}}(I;H)$ is the space of all weakly continuous functions on I into H.

Here we sketch their proof in [**1**]. For $\varepsilon > 0$ set

$$\begin{aligned} V_0^{\varepsilon}(x,t) &:= (\varepsilon^2 + |x - a(t)|^2)^{-1/2}, \\ V_1^{\varepsilon}(x,t) &:= ((T_\varepsilon \circ V_1) * \zeta_\varepsilon)(x,t) \\ &= \int_{\mathbb{R}^N \times \mathbb{R}} T_\varepsilon(V_1(x - \varepsilon y, t - \varepsilon s))\chi(s)\rho(y)\,dsdy, \end{aligned} \tag{1.3}$$

where $T_\varepsilon(r) := |r|^{-1} r \min\{|r|, \varepsilon^{-1}\}$ and χ, ρ are the mollifiers on $\mathbb{R}$ and $\mathbb{R}^N$, respectively, and hence $\zeta_\varepsilon(x,t) := \varepsilon^{-(1+N)}\chi(t/\varepsilon)\rho(x/\varepsilon)$. Then they consider the approximate problem

$$\begin{cases} i\,\partial_t u_\varepsilon(x,t) + \Delta u_\varepsilon(x,t) + V_0^{\varepsilon}(x,t)u_\varepsilon(x,t) + V_1^{\varepsilon}(x,t)u_\varepsilon(x,t) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad (x,t) \in \mathbb{R}^N \times [0,T], \\ u_\varepsilon(x,0) = u_0(x), \qquad\qquad\qquad\qquad x \in \mathbb{R}^N \end{cases} \tag{1.4}$$

with $u_0 \in H^2(\mathbb{R}^N) \cap H_2(\mathbb{R}^N)$ and obtain its solution

$$u_\varepsilon \in C^1([0,T];L^2(\mathbb{R}^N)) \cap C([0,T];H^2(\mathbb{R}^N) \cap H_2(\mathbb{R}^N)), \tag{1.5}$$

satisfying the energy estimates:

$$\|\partial_t u_\varepsilon(t)\| + \|u_\varepsilon(t)\|_{H^2 \cap H_2} \le C\|u_0\|_{H^2 \cap H_2} \quad \forall\, t \in [0,T].$$

Since $C > 0$ is independent of ε, they can extract a subfamily $(u_{\varepsilon'})$ which converges weakly* to a solution to (SE) satisfying (1.1) and (1.2).

Now we are in a position to point out that two parts of their argument should be modified (though the conclusion of Theorem 1.1 remains true).

On the one hand, to approximate V_1 they employ $V_1^{\varepsilon} \in C([0,T];C_b^2(\mathbb{R}^N))$ defined by (1.3), where C_b^2 denotes the space of all bounded C^2-functions with bounded first and second derivatives. They assert that "the norm of V_1^{ε} is bounded by the norm of V_1 in the space where it is defined". This kind of boundedness is essential in [**1**, Sections 3 and 4.2]. However, it seems impossible to derive such an estimate even if $V_1(t)(x) = V_1(x,t)$ is replaced with its "extension by 0":

$$\bar{V_1}(t) := \begin{cases} V_1(t), & t \in [0,T], \\ 0, & t \in \mathbb{R} \setminus [0,T]. \end{cases}$$

This means that the definition of V_1^ε should be modified (as is done in (2.2) below).

On the other hand, they set $v_\varepsilon(y,t) := u_\varepsilon(x,t)$, $y := x - a(t)$, to get the estimate of $\|\partial_t u_\varepsilon\|$ in the proof of [**1**, Lemma 8]. Then they employ the equation

$$\begin{aligned} &i\,\partial_t(\partial_t v_\varepsilon) + \Delta(\partial_t v_\varepsilon) + (|y|^2+\varepsilon^2)^{-1/2}\partial_t v_\varepsilon + V_1^\varepsilon(y+a(t),t)(\partial_t v_\varepsilon) \\ &= i\,\frac{d^2a}{dt^2}(t)\cdot\nabla v_\varepsilon + i\,\frac{da}{dt}(t)\cdot\nabla(\partial_t v_\varepsilon) - \frac{da}{dt}(t)\cdot\nabla V_1^\varepsilon(y+a(t),t) \\ &- (\partial_t V_1^\varepsilon)(y+a(t),t)v_\varepsilon \end{aligned}$$

(in which, actually, $|y|^{-1}$ and V_1 are used instead of $(|y|^2+\varepsilon^2)^{-1/2}$ and V_1^ε, respectively). However, $\partial_t(\partial_t v_\varepsilon)$ does not make sense in view of (1.5) and we believe that $\partial_t(\partial_t v_\varepsilon)$ should be replaced with its difference quotient:

$$(D_h(\partial_t v_\varepsilon))(y,t) = (\partial_t(D_h v_\varepsilon))(y,t) = \frac{1}{h}[(\partial_t v_\varepsilon)(y,t+h) - (\partial_t v_\varepsilon)(y,t)]$$

for $h > 0$ (as is done in Lemma 3.4 below).

In this context the purpose of this paper is to rewrite the original proof in [**1**] correctly and to establish Theorem 1.1 with an inhomogeneous term.

Theorem 1.2. *In addition to* (a), (V1) *and* (V2) *assume that f satisfies*

$$f \in W^{1,1}(0,T;L^2(\mathbb{R}^N)) \cap L^1(0,T;H^1(\mathbb{R}^N)\cap H_2(\mathbb{R}^N)). \tag{1.6}$$

Then (SE) *with initial value $u_0 \in H^2(\mathbb{R}^N)\cap H_2(\mathbb{R}^N)$ has a unique solution satisfying* (1.1), (1.2) *and the energy estimate*:

$$\|\partial_t u(t)\| + \|u(t)\|_{H^2\cap H_2} \le C_0(\|u_0\|_{H^2\cap H_2} + \|f\|_F), \tag{1.7}$$

where $C_0 > 0$ is a constant depending on a, V_1 and T, while $\|f\|_F$ is given as follows:

$$\|f\|_F := \|f\|_{L^\infty(0,T;\,L^2)} + \int_0^T (\|\partial_t f(t)\| + \alpha_1\|f(t)\|_{H^1} + \|f(t)\|_{H_2})\,dt;$$

$\alpha_1 > 0$ is a constant depending on a.

This type of result has already been obtained by Wüller [**11**] under the conditions different from those in [**1**] and ours. Of course, he dealt with the equation with time-dependent potential

$$i\,\partial_t u(x,t) + \Delta u(x,t) + V(x,t)u(x,t) = 0. \tag{1.8}$$

However, in the simplest case he assumes that

$$V(x,t) := |x-a(t)|^{-1} + v_2(x-a(t)),$$

where v_2 is a suitable bounded function. That is, comparing with the above-mentioned potential V_1 satisfying conditions (V1) and (V2), this is a strong restriction. By virtue of this restriction, setting $w(y,t) = u(x,t)$, $y = x - a(t)$, one can get a new equation with time-independent potentials

$$i\,\partial_t w + \Delta w - i\Big(\frac{da}{dt}(t)\cdot\nabla\Big)w + \frac{w}{|y|} + v_2(y)w = 0. \tag{1.9}$$

Then it is possible to prove the unique existence of solutions of (1.9) (and hence of (1.8)) with initial value $u_0 \in H^2(\mathbb{R}^N)\cap H_2(\mathbb{R}^N)$ according to a general theory of evolution equations developed by Kato [**6**], [**7**].

In a series of papers [**12**]–[**14**] Yajima has been considering the Schrödinger evolution equation containing time-dependent (scalar and vector) potentials. In [**13**] he discusses three methods such as energy method (Section 3.1), method via semi-group theory (Section 3.2) and method by integral equation (Section 3.3). At the end of Section 3.2 he comments that the main theorem does not accommodate the Coulomb potential $|x-a(t)|^{-1}$ in $\mathbb{R}^3$, where $a(t) \in \mathbb{R}^3$ is a smooth function. At the beginning of Section 3.3 he mentions that the third method can handle more singular potentials than those treated in Sections 3.1 and 3.2. In fact, he treated (1.8) with

$$V(x,t) = W_0(x,t) + |x-a(t)|^{-1}$$

as a typical case in which $N = 3$. Here $W_0(\cdot,t) \in C^\infty(\mathbb{R}^3)$ satisfies

$$|D^\alpha W_0(x,t)| \le C_\alpha \quad \forall\, \alpha \in \mathbb{Z}_+^3 \ \ (|\alpha| \ge 2),$$

while $|x-a(t)|^{-1}$ is decomposed as

$$|x-a(t)|^{-1} = W_1(x,t) + W_2(x,t), \tag{1.10}$$

where $W_1 \in L^4(0,T;L^2(\mathbb{R}^3))$ and $W_2 \in L^1(0,T;L^\infty(\mathbb{R}^3))$ are given by

$$W_1(x,t) := \begin{cases} |x-a(t)|^{-1}, & |x-a(t)| < 1,\\ 0, & \text{otherwise} \end{cases}$$

and $W_2(x,t) := |x-a(t)|^{-1} - W_1(x,t)$ (for this sufficient condition we refer [**12**, Theorem 1.1] to avoid the use of Lorentz spaces in [**13**, Theorem 3.9]); note that the idea of the decomposition (1.10) goes back to Kato [**8**, Section V.5.3]. We feel that it is desirable to replace $W_0(\cdot,t) \in C^\infty(\mathbb{R}^3)$ with some weaker condition.

Incidentally, we shall use a mixture of energy method and method via semi-group theory in this paper. In fact, we use semi-group method to solve the approximate problem, while energy method is used for the convergence of approximate solutions. We note further that only energy method has been used in [**1**].

Remark 1. If a, V_1 and f are defined on $[-T,T]$, then we can obtain a unique solution satisfying

$$u \in W^{1,\infty}(-T,T;L^2(\mathbb{R}^N)) \cap C_{\mathrm{w}}([-T,T];H^2(\mathbb{R}^N)\cap H_2(\mathbb{R}^N)),$$
$$u \in L^{\infty}(-T,T;H^2(\mathbb{R}^N)\cap H_2(\mathbb{R}^N)) \cap C([-T,T];H^1(\mathbb{R}^N)\cap H_1(\mathbb{R}^N))$$

(see, e.g., [**10**, Remark 1.3]).

Remark 2. Theorem 1.2 is rather unsatisfactory. In fact, strong solutions u obtained in Theorem 1.2 or Remark 1 are expected to be C^1-solutions:

$$u \in C^1([0,T];L^2(\mathbb{R}^N)) \cap C([0,T];H^2(\mathbb{R}^N)\cap H_2(\mathbb{R}^N))$$

or

$$u \in C^1([-T,T];L^2(\mathbb{R}^N)) \cap C([-T,T];H^2(\mathbb{R}^N)\cap H_2(\mathbb{R}^N)).$$

Roughly speaking, both Wüller [**11**, Theorem in Section 5] and Yajima [**13**, Theorem 3.10] have already established this assertion under stronger assumptions. We are planning to discuss this problem in a forthcoming paper.

In Section 2 we define $V_0^\varepsilon, V_1^\varepsilon$ and f_ε more carefully and prepare some lemmas to consider our approximate problem. Here, not only V_1^ε but also V_0^ε are different from those in [**1**], while f_ε is new. In Section 3 we prove that the family $\{u_\varepsilon\}$ of approximate solutions satisfies the energy estimate

$$\|\partial_t u_\varepsilon(t)\| + \|u_\varepsilon(t)\|_{H^2\cap H_2} \le C_0(\|u_0\|_{H^2\cap H_2} + \|f\|_F).$$

In the proof we have to show that

$$\begin{aligned}\|(D_h v_\varepsilon)(t)\| - \|(D_h v_\varepsilon)(0)\| \le &\int_0^t \Big\|\Big(D_h \frac{da_\varepsilon}{ds}\Big)(s)\cdot \nabla v_\varepsilon(s)\Big\|\, ds \\ &+ \int_0^t \big\|(D_h V_1^\varepsilon)(\cdot + a_\varepsilon(s), s) v_\varepsilon(s)\big\|\, ds \\ &+ \int_0^t \big\|(D_h f_\varepsilon)(\cdot + a_\varepsilon(s), s)\big\|\, ds,\end{aligned}$$

where a_ε is an approximation of a. By virtue of the estimate we can extract a subsequence of $\{u_\varepsilon\}$ which converges weakly* in $L^\infty(0,T;L^2(\mathbb{R}^N))$. In this way we can prove the existence and uniqueness of strong solutions to (SE) satisfying (1.1) and (1.2).

§2. Preliminaries

For a Banach space X let $\varphi \in W^{1,1}(0,T;X)$. Then, as in Brézis [**3**, Théorème VIII.5], we define the extension operator $P : W^{1,1}(0,T;X) \to W^{1,1}(\mathbb{R};X)$ by

$$(P\varphi)(t) := \begin{cases} \varphi(t), & t \in [0,T], \\ \Big(2 - \dfrac{t}{T}\Big)\varphi(2T-t), & t \in (T, 2T], \\ \Big(1 + \dfrac{t}{T}\Big)\varphi(-t), & t \in [-T, 0), \\ 0, & \text{otherwise.} \end{cases}$$

In fact, we can prove

Lemma 2.1. *Let $\varphi \in W^{1,1}(0,T;X)$. Then $P\varphi \in W^{1,1}(\mathbb{R};X)$, with*

(a) $\|P\varphi\|_{L^\infty(\mathbb{R};\,X)} = \|\varphi\|_{L^\infty(0,T;\,X)}$.

(b) $\|P\varphi\|_{L^1(\mathbb{R};\,X)} = 2\|\varphi\|_{L^1(0,T;\,X)} \le 2T\|\varphi\|_{L^\infty(0,T;\,X)}$.

(c) $\Big\|\dfrac{d}{dt}(P\varphi)\Big\|_{L^1(\mathbb{R};\,X)} \le 2\|\varphi\|_{L^\infty(0,T;\,X)} + 2\Big\|\dfrac{d}{dt}\varphi\Big\|_{L^1(0,T;\,X)}$.

Now put $V_0 = V_0(x,t) := |x - a(t)|^{-1}$. Then we consider the approximations of potentials V_0, V_1 and inhomogeneous term f.

Let $0 \le \chi \in C_0^\infty(\mathbb{R})$ and $0 \le \rho \in C_0^\infty(\mathbb{R}^N)$ such that $\|\chi\|_{L^1} = \|\rho\|_{L^1} = 1$ and $\operatorname{supp}\chi \subset [-1,1]$, $\operatorname{supp}\rho \subset \overline{B(0;1)} := \{x \in \mathbb{R}^N; |x| \le 1\}$, respectively. Let $0 \le \eta \in W^{1,\infty}(0,\infty)$ be defined as

$$\eta(r) := \begin{cases} 1, & r \in [0,1), \\ 2 - r, & r \in [1,2), \\ 0, & r \in [2,\infty). \end{cases}$$

For $\varepsilon > 0$ let $\chi_\varepsilon(t) := \varepsilon^{-1}\chi(t/\varepsilon)$, $\zeta_\varepsilon(x,t) := \varepsilon^{-(1+N)}\chi(t/\varepsilon)\rho(x/\varepsilon)$ and $\eta_\varepsilon(x) := \eta(\varepsilon|x|)$. Then by using the extension operator P we can define as

$$V_0^\varepsilon(x,t) := (\varepsilon^2 + |x - a_\varepsilon(t)|^2)^{-1/2}, \tag{2.1}$$

$$\begin{aligned} V_1^\varepsilon(x,t) &:= \big((\eta_\varepsilon(PV_1)) * \zeta_\varepsilon\big)(x,t) \\ &= \int_{B(0;1)} \Big[\int_{-1}^{1} \eta_\varepsilon(x - \varepsilon y)(PV_1)(x - \varepsilon y, t - \varepsilon s)\chi(s)\rho(y)\,ds\Big]dy, \end{aligned} \tag{2.2}$$

$$\begin{aligned} f_\varepsilon(x,t) &:= \big((Pf) * \zeta_\varepsilon\big)(x,t) \\ &= \int_{B(0;1)} \Big[\int_{-1}^{1} (Pf)(x - \varepsilon y, t - \varepsilon s)\chi(s)\rho(y)\,ds\Big]dy. \end{aligned} \tag{2.3}$$

In (2.1) a_ε is defined as

$$a_\varepsilon(t) := a(0) + \int_0^t \Big(\Big(P\frac{da}{ds}\Big) * \chi_\varepsilon\Big)(s)\,ds.$$

As is well-known, $\{V_1^\varepsilon\}$ and $\{f_\varepsilon\}$ are families in $C_0^\infty(\mathbb{R}^N \times \mathbb{R})$, while $\{a_\varepsilon\}$ is a family in $C_0^\infty(\mathbb{R})$. We shall see further that the properties of a_ε, V_1^ε and f_ε reflect those of a, V_1 and f, respectively.

Lemma 2.2. *Assume that a satisfies condition* (a)*. Put*

$$\alpha_1 := \Big\|\frac{da}{dt}\Big\|_{L^\infty(0,T)}. \tag{2.4}$$

Then a_ε has the following properties:

(a) $\Big\|\dfrac{da_\varepsilon}{dt}\Big\|_{L^\infty(\mathbb{R})} \le \alpha_1$.

(b) $\Big\|\dfrac{d^2a_\varepsilon}{dt^2}\Big\|_{L^1(\mathbb{R})} \le 2\alpha_1 + 2\Big\|\dfrac{d^2a}{dt^2}\Big\|_{L^1(0,T)}$.

(b)′ $\Big\|\dfrac{d^2a_\varepsilon}{dt^2}\Big\|_{L^\infty(\mathbb{R})} \le \varepsilon^{-1}\alpha_1\Big\|\dfrac{d\chi}{dt}\Big\|_{L^1(-1,1)}$.

Proof. (a) Since $\dfrac{da_\varepsilon}{dt}(t) = \Big(\Big(P\dfrac{da}{dt}\Big) * \chi_\varepsilon\Big)(t)$, we see from Lemma 2.1 (a) that

$$\Big\|\frac{da_\varepsilon}{dt}\Big\|_{L^\infty(\mathbb{R})} \le \Big\|P\frac{da}{dt}\Big\|_{L^\infty(\mathbb{R})} = \Big\|\frac{da}{dt}\Big\|_{L^\infty(0,T)} = \alpha_1.$$

(b) and (b)′ are proved similarly. □

Lemma 2.3. *Put $X := L^\infty(\mathbb{R}^N)$. Assume that V_1 satisfies conditions* (V1) *and* (V2)*. Then the useful properties of V_1^ε are summarized as follows*:

(a) $\Big\|\dfrac{V_1^\varepsilon}{\langle x\rangle^2}\Big\|_{L^\infty(\mathbb{R};\,X)} \le (1+\varepsilon)^2\Big\|\dfrac{V_1}{\langle x\rangle^2}\Big\|_{L^\infty(0,T;\,X)}$.

(b) $\Big\|\dfrac{\partial_t V_1^\varepsilon}{\langle x\rangle^2}\Big\|_{L^1(\mathbb{R};\,X)} \le 2(1+\varepsilon)^2\Big[\Big\|\dfrac{\partial_t V_1}{\langle x\rangle^2}\Big\|_{L^1(0,T;\,X)} + \Big\|\dfrac{V_1}{\langle x\rangle^2}\Big\|_{L^\infty(0,T;\,X)}\Big]$.

(c) $\Big\|\dfrac{\nabla V_1^\varepsilon}{\langle x\rangle^2}\Big\|_{L^1(\mathbb{R};\,X)} \le 2(1+\varepsilon)^2\Big[\Big\|\dfrac{\nabla V_1}{\langle x\rangle^2}\Big\|_{L^1(0,T;\,X)} + \varepsilon T\Big\|\dfrac{V_1}{\langle x\rangle^2}\Big\|_{L^\infty(0,T;\,X)}\Big]$.

This means that V_1^ε also satisfies conditions (V1) *and* (V2)*.*

Proof. Let $x \in \mathbb{R}^N$, $y \in B(0;1)$ and $\varepsilon > 0$. Then we see that

$$\langle x - \varepsilon y\rangle \le (1+\varepsilon)\langle x\rangle. \tag{2.5}$$

In fact, we can compute as

$$\langle x-\varepsilon y\rangle^2 \le 1+(|x|+\varepsilon|y|)^2 = \langle x\rangle^2 + 2\varepsilon|x| + \varepsilon^2 \le (1+\varepsilon)^2\langle x\rangle^2.$$

(a) Since $0 \le \eta_\varepsilon(x) \le 1$ on $\mathbb{R}^N$, we see from (2.5) that

$$\begin{aligned}\Big|\frac{V_1^\varepsilon(x,t)}{\langle x\rangle^2}\Big| &\le \frac{1}{\langle x\rangle^2}\int_{B(0;1)}\Big[\int_{-1}^1 \eta_\varepsilon(x-\varepsilon y)|(PV_1)(x-\varepsilon y, t-\varepsilon s)|\chi(s)\rho(y)\,ds\Big]\,dy \\ &\le (1+\varepsilon)^2\int_{B(0;1)}\Big[\int_{-1}^1 \frac{\big|(PV_1)(x-\varepsilon y,t-\varepsilon s)\big|}{\langle x-\varepsilon y\rangle^2}\chi(s)\rho(y)\,ds\Big]\,dy \\ &\le (1+\varepsilon)^2\Big\|\frac{PV_1}{\langle x\rangle^2}\Big\|_{L^\infty(\mathbb{R};\,X)} = (1+\varepsilon)^2\Big\|P\Big(\frac{V_1}{\langle x\rangle^2}\Big)\Big\|_{L^\infty(\mathbb{R};\,X)}.\end{aligned}$$

By virtue of Lemma 2.1 (a) we obtain the assertion.

(b) and (c) are proved in the same way as in (a). □

In the same way as in the proof of Lemma 2.3 we can obtain

Lemma 2.4. *Let f_ε be as defined in* (2.3) *and let f satisfy condition* (1.6). *Then*

(a) $\|f_\varepsilon\|_{L^1(\mathbb{R};\,L^2)} \le 2\|f\|_{L^1(0,T;\,L^2)}$.

(a)′ $\|f_\varepsilon\|_{L^\infty(\mathbb{R};\,L^2)} \le \|f\|_{L^\infty(0,T;\,L^2)}$.

(b) $\|\partial_t f_\varepsilon\|_{L^1(\mathbb{R};\,L^2)} \le 2\|\partial_t f\|_{L^1(0,T;\,L^2)} + 2\|f\|_{L^\infty(0,T;\,L^2)}$.

(c) $\|\nabla f_\varepsilon\|_{L^1(\mathbb{R};\,L^2)} \le 2\|\nabla f\|_{L^1(0,T;\,L^2)}$.

(d) $\|f_\varepsilon\|_{L^1(\mathbb{R};\,H_2)} \le 2(1+\varepsilon)^2\|f\|_{L^1(0,T;\,H_2)}$.

The following proposition has been established by Fujiwara [**4**]:

Proposition 2.5. *For $\varepsilon > 0$ let V_0^ε, V_1^ε, f_ε be as above. Put*

$$V^\varepsilon := V_0^\varepsilon + V_1^\varepsilon.$$

Then the approximate problem:

$$(\mathrm{SE})_\varepsilon \qquad \begin{cases} i\,\partial_t u_\varepsilon(x,t) + \Delta u_\varepsilon(x,t) + V^\varepsilon(x,t)u_\varepsilon(x,t) = f_\varepsilon(x,t), \\ \hfill (x,t)\in\mathbb{R}^N\times[0,T], \\ u(x,0) = u_0(x), \hfill x\in\mathbb{R}^N \end{cases}$$

with $u_0 \in H^2(\mathbb{R}^N)\cap H_2(\mathbb{R}^N)$ has a unique solution

$$u_\varepsilon \in C^1([0,T];L^2(\mathbb{R}^N)) \cap C([0,T];H^2(\mathbb{R}^N)\cap H_2(\mathbb{R}^N)).$$

Here we verify this proposition from the view point of the abstract theory.

Proof. We apply [**10**, Theorems 1.2 and 1.4] by setting $X := L^2(\mathbb{R}^N)$ and

$$\begin{aligned} A_\varepsilon(t) &:= i^{-1}[\Delta + V^\varepsilon(t)], \quad g(t)(x) := i^{-1} f_\varepsilon(x,t), \\ S &:= 1 + \Delta^2 + |x|^4, \quad D(S) := H^4(\mathbb{R}^N) \cap H_4(\mathbb{R}^N). \end{aligned}$$

We have to verify several conditions of [**10**, Theorem 1.2]. Here we show only the key inequality:

$$|\operatorname{Re}(A_\varepsilon(t)u, Su)_X| \le \beta \|S^{1/2}u\|_X^2, \quad u \in D(S),\ 0 \le t \le T. \tag{2.6}$$

In fact, we see from integration by parts that

$$\begin{aligned} \operatorname{Re}(A_\varepsilon(t)u, Su)_X &= 4\operatorname{Im}(|x|^2u, (x\cdot\nabla)u) + \operatorname{Im}\big(\Delta V_0^\varepsilon(t)u + 2\nabla V_0^\varepsilon(t)\cdot\nabla u, \Delta u\big) \\ &\quad + \operatorname{Im}\big(\Delta V_1^\varepsilon(t)u + 2\nabla V_1^\varepsilon(t)\cdot\nabla u, \Delta u\big). \end{aligned}$$

Then it follows from the properties of mollifiers that

$$\begin{aligned} &|\operatorname{Re}(A_\varepsilon(t)u, Su)_X| \\ &\le 4\|u\|_{H_2}^{3/2}\|u\|_{H^2}^{1/2} + \frac{3N-2}{2\varepsilon^2(N-2)}\|u\|_{H^2}^2 \\ &+ (1+\varepsilon^{-1})^2\|\Delta\rho\|_{L^1}\left\|\frac{V_1}{\langle x\rangle^2}\right\|_{L^\infty(0,T;\,L^\infty)}\|u\|_{H^2}\|u\|_{H_2} \\ &+ (1+\varepsilon^{-1})^2\|\nabla\rho\|_{L^1}\left\|\frac{V_1}{\langle x\rangle^2}\right\|_{L^\infty(0,T;\,L^\infty)}\big(\|u\|_{H^2}^2 + 2\|u\|_{H^2}^{3/2}\|u\|_{H_2}^{1/2}\big). \end{aligned}$$

Putting

$$\beta = \beta_\varepsilon := 3 + \frac{3N-2}{2\varepsilon^2(N-2)} + \frac{1}{2}(1+\varepsilon^{-1})^2\big(\|\Delta\rho\|_{L^1} + 5\|\nabla\rho\|_{L^1}\big)\left\|\frac{V_1}{\langle x\rangle^2}\right\|_{L^\infty(0,T;\,L^\infty)}$$

and using Young's inequality, we obtain (2.6). □

Finally, we prepare a Gronwall type lemma.

Lemma 2.6 (Brézis [**2**, Lemma A.5])**.** *Let $m(\cdot) \in L^1(0,T)$ be a nonnegative function, α_0 a nonnegative constant. Let $\phi(\cdot) \in L^\infty(0,T)$ satisfy the integral inequality*:

$$|\phi(t)|^2 \le \alpha_0^2 + 2\int_0^t m(s)|\phi(s)|\,ds \quad \forall\, t \in [0,T].$$

Then one has

$$|\phi(t)| \le \alpha_0 + \int_0^t m(s)\,ds \quad \forall\, t \in [0,T].$$

§3. Strong solution of the Schrödinger equation

This section is a reconstruction of [**1**, Section 4]. First, we show some estimates for the family $\{u_\varepsilon\}$ of solutions to $(\mathrm{SE})_\varepsilon$ with initial value $u_0 \in H^2(\mathbb{R}^N) \cap H_2(\mathbb{R}^N)$. Next, we consider the convergence of $\{u_\varepsilon\}$ and show that (SE) has a unique strong solution satisfying (1.1) and (1.2).

3.1. Some estimates for approximate solutions

Let α_1 be as defined in (2.4) and put

$$N(V_1, \langle x\rangle^{-2}) := \Big\| \frac{V_1}{\langle x\rangle^2} \Big\|_{L^\infty(0,T;\,X)} + \Big\| \frac{\partial_t V_1}{\langle x\rangle^2} \Big\|_{L^1(0,T;\,X)} + \Big\| \frac{\nabla V_1}{\langle x\rangle^2} \Big\|_{L^1(0,T;\,X)}. \tag{3.1}$$

The purpose of this subsection is to prove that $\|\partial_t u_\varepsilon(t)\|$ and $\|u_\varepsilon(t)\|_{H^2\cap H_2}$ are bounded on $[0,T]$ as ε tends to zero. Actually, the boundedness of $\|\partial_t u_\varepsilon(t)\|$ is reduced to that of $\|u_\varepsilon(t)\|_{H^2\cap H_2}$. That is, we have

Lemma 3.1. *Let u_ε be a solution to $(\mathrm{SE})_\varepsilon$ with $\varepsilon \in (0,1]$. Then*
(a) $\|u_\varepsilon(t)\| \le \|u_0\| + 2\|f\|_{L^1(0,T;\,L^2)}$ *for* $t \in [0,T]$.
(b) *Put* $C_1 := 1 + (N-2)^{-1} + 4N(V_1, \langle x\rangle^{-2})$. *Then*

$$\|\partial_t u_\varepsilon(t)\| \le C_1 \|u_\varepsilon(t)\|_{H^2\cap H_2} + \|f\|_{L^\infty(0,T;\,L^2)} \quad \forall\, t \in [0,T]. \tag{3.2}$$

Proof. (a) We start with

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\|u_\varepsilon(s)\|^2 &= \mathrm{Re}\,\big(\partial_s u_\varepsilon(s), u_\varepsilon(s)\big) = \mathrm{Im}\,\big(f_\varepsilon(s), u_\varepsilon(s)\big)\\ &\le \|f_\varepsilon(s)\| \cdot \|u_\varepsilon(s)\|.\end{aligned}$$

Integrating this inequality on $[0,t]$, we have

$$\|u_\varepsilon(t)\|^2 \le \|u_0\|^2 + 2\int_0^t \|f_\varepsilon(s)\| \cdot \|u_\varepsilon(s)\|\, ds.$$

Thus the assertion is a consequence of Lemma 2.6 and Lemma 2.4 (a).

(b) The assertion is based on the following inequality:

$$\|\partial_t u_\varepsilon(t)\| \le \|\Delta u_\varepsilon(t)\| + \big\| |x - a_\varepsilon(t)|^{-1} u_\varepsilon(t) \big\| + \|V_1^\varepsilon(t) u_\varepsilon(t)\| + \|f_\varepsilon(t)\|.$$

In fact, it follows from Hardy's inequality that

$$\big\| |x - a_\varepsilon(t)|^{-1} u_\varepsilon(t) \big\| \le \frac{2}{N-2}\|\nabla u_\varepsilon(t)\| \le \frac{1}{N-2}\|u_\varepsilon(t)\|_{H^2}. \tag{3.3}$$

On the other hand, we see from Lemmas 2.3 (a) and 2.4 (a)′ that

$$\|V_1^\varepsilon(t) u_\varepsilon(t)\| \le 4N(V_1, \langle x\rangle^{-2}) \|u_\varepsilon(t)\|_{H_2} \tag{3.4}$$

and $\|f_\varepsilon(t)\| \le \|f\|_{L^\infty(0,T;\,L^2)}$, respectively. Therefore we obtain (3.2). □

The boundedness of $\|\partial_t u_\varepsilon(t)\|$ and $\|u_\varepsilon(t)\|_{H^2\cap H_2}$ is proved by using the energy estimates for the family $\{u_\varepsilon\}$.

Proposition 3.2. *Let u_ε be a solution to* (SE)$_\varepsilon$. *Then for $\varepsilon \in (0,1]$ there exists a constant $C_0 > 0$ independent of ε such that*

$$\|\partial_t u_\varepsilon(t)\| + \|u_\varepsilon(t)\|_{H^2\cap H_2} \le C_0\big(\|u_0\|_{H^2\cap H_2} + \|f\|_F\big), \tag{3.5}$$

where $\|f\|_F$ is given by

$$\|f\|_F := \|f\|_{L^\infty(0,T;\,L^2)} + \int_0^T (\|\partial_t f(t)\| + \alpha_1\|f(t)\|_{H^1} + \|f(t)\|_{H_2})\,dt. \tag{3.6}$$

To prove Proposition 3.2 we prepare three lemmas (Lemmas 3.3–3.5). The first (Lemma 3.3 yielding the estimate of $\|u_\varepsilon(t)\|_{H_2}$) simplifies the argument in [**1**, Lemma 7]. The second (Lemma 3.4 yielding the estimate of $\|\partial_t u_\varepsilon(t)\|$) is similar to [**1**, Lemma 8]. However, we will give a rigorous proof employing difference quotients as in Kato [**5**, Lemma 4.2]. The third (Lemma 3.5) yields the estimate of $\|u_\varepsilon(t)\|_{H^2}$ based on Lemmas 3.1 and 3.4. This leads us to (3.5).

Lemma 3.3. *Let u_ε be a solution to* (SE)$_\varepsilon$ *with $\varepsilon \in (0,1]$. Then u_ε satisfies*

$$\|u_\varepsilon(t)\|_{H_2}^{1/2} \le \|u_0\|_{H_2}^{1/2} + 2\int_0^t \|u_\varepsilon(s)\|_{H^2}^{1/2}\,ds + 2\sqrt{2}\,\|f\|_{L^1(0,T;\,H_2)}^{1/2}. \tag{3.7}$$

Proof. Put $B_n(x) := \langle x\rangle^2(1+n^{-1}\langle x\rangle^2)^{-1}$. Then $B_n(x) < \min\{n, \langle x\rangle^2\}$. Noting that V_0^ε and V_1^ε are real-valued, we see from (SE)$_\varepsilon$ that

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\|B_n u_\varepsilon(s)\|^2 &= \mathrm{Re}\,\big(\partial_s u_\varepsilon(s), B_n^2 u_\varepsilon(s)\big) = \mathrm{Im}\,\big(i\partial_s u_\varepsilon(s), B_n^2 u_\varepsilon(s)\big)\\ &= \mathrm{Im}\,\big(-\Delta u_\varepsilon(s) + f_\varepsilon(s), B_n^2 u_\varepsilon(s)\big)\\ &= \mathrm{Im}\,\big(4((1+n^{-1}\langle x\rangle^2)^{-2}x\cdot\nabla)u_\varepsilon(s) + B_n f_\varepsilon(s), B_n u_\varepsilon(s)\big).\end{aligned}$$

We can use the following inequality:

$$\big\||x|\nabla u\big\|^2 \le \|u-\Delta u\|\cdot\|\langle x\rangle^2 u\| = \|u\|_{H^2}\|u\|_{H_2}. \tag{3.8}$$

In fact, we have the equality $\mathrm{Re}(u-\Delta u, u+|x|^2u) = \|u\|^2 + \|\nabla u + xu\|^2 + \big\||x|\nabla u\big\|^2$. By virtue of (3.8) the Cauchy-Schwarz inequality applies to give

$$\frac{1}{2}\frac{d}{ds}\|B_n u_\varepsilon(s)\|^2 \le \big[4\,\|u_\varepsilon(s)\|_{H^2}^{1/2}\|u_\varepsilon(s)\|_{H_2}^{1/2} + \|f_\varepsilon(s)\|_{H_2}\big]\|u_\varepsilon(s)\|_{H_2}.$$

Integrating this inequality on $[0,t]$ and letting $n\to\infty$, we have

$$\|u_\varepsilon(t)\|_{H_2}^2 \le \|u_0\|_{H_2}^2 + 2\int_0^t \big[4\,\|u_\varepsilon(s)\|_{H^2}^{1/2}\|u_\varepsilon(s)\|_{H_2}^{1/2} + \|f_\varepsilon(s)\|_{H_2}\big]\|u_\varepsilon(s)\|_{H_2}\,ds.$$

It then follows from Lemma 2.6 and Lemma 2.4 (d) that

$$\begin{aligned}\|u_\varepsilon(t)\|_{H_2} &\le \|u_0\|_{H_2} + \int_0^t \big[4\,\|u_\varepsilon(s)\|_{H^2}^{1/2}\|u_\varepsilon(s)\|_{H_2}^{1/2} + \|f_\varepsilon(s)\|_{H_2}\big]\,ds \\ &\le \|u_0\|_{H_2} + 8\,\|f\|_{L^1(0,T;\,H_2)} + 4\int_0^t \|u_\varepsilon(s)\|_{H^2}^{1/2}\|u_\varepsilon(s)\|_{H_2}^{1/2}\,ds.\end{aligned}$$

To obtain (3.7) we can again apply Lemma 2.6. □

Lemma 3.4. *Let u_ε be a solution to* (SE)$_\varepsilon$ *with $\varepsilon \in (0,1]$. Then u_ε satisfies*

$$\begin{aligned}&\|\partial_t u_\varepsilon(t)\| - \alpha_1\|\nabla u_\varepsilon(t)\| \\ &\quad\le (\alpha_1 + C_1)\|u_0\|_{H^2\cap H_2} + \int_0^t \gamma_{1,\varepsilon}(s)\|u_\varepsilon(s)\|_{H^1\cap H_2}\,ds + 3\|f\|_F,\end{aligned} \tag{3.9}$$

where C_1 is the same as in Lemma 3.1 (b), *while $\gamma_{1,\varepsilon} \in L^1(0,T)$ is given by*

$$\gamma_{1,\varepsilon}(t) := \Big|\frac{d^2 a_\varepsilon}{dt^2}(t)\Big| + \Big\|\frac{\partial_t V_1^\varepsilon(t)}{\langle x\rangle^2}\Big\|_{L^\infty} + \alpha_1\Big\|\frac{\nabla V_1^\varepsilon(t)}{\langle x\rangle^2}\Big\|_{L^\infty}. \tag{3.10}$$

Remark 3. Here $\|\gamma_{1,\varepsilon}\|_{L^1(0,T)}$ is bounded as ε tends to zero. In fact, we see from Lemma 2.2 (b), Lemma 2.3 (b) and (c) that

$$\|\gamma_{1,\varepsilon}\|_{L^1(0,T)} \le M, \tag{3.11}$$

where $M := 2\alpha_1 + 2\Big\|\dfrac{d^2 a}{dt^2}\Big\|_{L^1(0,T)} + 8[1+\alpha_1(1+T)]N(V_1, \langle x\rangle^{-2})$.

We will give the proof of Lemma 3.4 in Section 3.2.

Lemma 3.5. *Let u_ε be a solution to* (SE)$_\varepsilon$ *with $\varepsilon \in (0,1]$. Then u_ε satisfies*

$$\begin{aligned}\|u_\varepsilon(t)\|_{H^2}^{1/2} &\le \big(\sqrt{\alpha_1 + C_1} + C_2\big)\|u_0\|_{H^2\cap H_2}^{1/2} + 2\sqrt{N(V_1, \langle x\rangle^{-2})}\|u_\varepsilon(t)\|_{H_2}^{1/2} \\ &\quad + \Big(\int_0^t \gamma_{1,\varepsilon}(s)\|u_\varepsilon(s)\|_{H^1\cap H_2}\,ds\Big)^{1/2} + 2(1+C_2)\|f\|_F^{1/2},\end{aligned} \tag{3.12}$$

where $C_1 > 0$ and $\gamma_{1,\varepsilon} \in L^1(0,T)$ are given in Lemma 3.1 (b) *and* (3.10), *respectively, while*

$$C_2 := 1 + \alpha_1 + \frac{2}{N-2}.$$

Proof. Since u_ε is a solution to (SE)$_\varepsilon$, it follows from (3.3) and (3.4) that

$$\begin{aligned}\|u_\varepsilon(t)\|_{H^2} &\le \|u_\varepsilon(t)\| + \|\Delta u_\varepsilon(t)\| \\ &\le \|u_\varepsilon(t)\| + \|\partial_t u_\varepsilon(t)\| + \|V_0^\varepsilon(t)u_\varepsilon(t)\| + \|V_1^\varepsilon(t)u_\varepsilon(t)\| + \|f_\varepsilon(t)\| \\ &\le \big\{\|\partial_t u_\varepsilon(t)\| - \alpha_1\|\nabla u_\varepsilon(t)\|\big\} + \|u_\varepsilon(t)\| + \Big(\alpha_1 + \frac{2}{N-2}\Big)\|\nabla u_\varepsilon(t)\| \\ &\quad + (1+\varepsilon)^2 N(V_1, \langle x\rangle^{-2})\|u_\varepsilon(t)\|_{H_2} + \|f\|_F.\end{aligned}$$

Noting that

$$\begin{aligned}\|u_\varepsilon(t)\| + \Big(\alpha_1 + \frac{2}{N-2}\Big)\|\nabla u_\varepsilon(t)\| &\le C_2\|u_\varepsilon(t)\|_{H^1}\\ &\le C_2\|u_\varepsilon(t)\|^{1/2}\|u_\varepsilon(t)\|_{H^2}^{1/2},\end{aligned}$$

we see from (3.9) that

$$\begin{aligned}\|u_\varepsilon(t)\|_{H^2} \le{}& (\alpha_1 + C_1)\|u_0\|_{H^2\cap H_2} + C_2\|u_\varepsilon(t)\|^{1/2}\|u_\varepsilon(t)\|_{H^2}^{1/2}\\ &+ (1+\varepsilon)^2 N(V_1, \langle x\rangle^{-2})\|u_\varepsilon(t)\|_{H_2}\\ &+ \int_0^t \gamma_{1,\varepsilon}(s)\|u_\varepsilon(s)\|_{H^1\cap H_2}\,ds + 4\|f\|_F.\end{aligned}$$

Thus, completing the square, we have

$$\begin{aligned}\|u_\varepsilon(t)\|_{H^2}^{1/2} \le{}& \sqrt{\alpha_1 + C_1}\|u_0\|_{H^2\cap H_2}^{1/2} + C_2\|u_\varepsilon(t)\|^{1/2}\\ &+ 2\sqrt{N(V_1, \langle x\rangle^{-2})}\|u_\varepsilon(t)\|_{H_2}^{1/2}\\ &+ \Big(\int_0^t \gamma_{1,\varepsilon}(s)\|u_\varepsilon(s)\|_{H^1\cap H_2}\,ds\Big)^{1/2} + 2\,\|f\|_F^{1/2}.\end{aligned}$$

Using Lemma 3.1 (a), we can obtain (3.12). □

Now we are in a position to prove (3.5).

Proof of Proposition 3.2. First we show that

$$\|u_\varepsilon(t)\|_{H^2\cap H_2} \le 4C_3^2 \exp\Big(\int_0^t \gamma_{0,\varepsilon}(s)\,ds\Big)(\|u_0\|_{H^2\cap H_2} + 8\|f(t)\|_F), \tag{3.13}$$

where $C_3 > 0$ and $\gamma_{0,\varepsilon} \in L^1(0,T)$ are given by

$$\begin{aligned}C_3 &:= 1 + C_2 + \sqrt{\alpha_1 + C_1} + 2\sqrt{N(V_1, \langle x\rangle^{-2})},\\ \gamma_{0,\varepsilon}(t) &:= 4\gamma_{1,\varepsilon}(t) + 32T\Big[1 + 2\sqrt{N(V_1, \langle x\rangle^{-2})}\Big]^2.\end{aligned}$$

It follows from (3.12) that

$$\begin{aligned}\|u_\varepsilon(t)\|_{H^2\cap H_2}^{1/2} &\le \|u_\varepsilon(t)\|_{H^2}^{1/2} + \|u_\varepsilon(t)\|_{H_2}^{1/2}\\ &\le \big(\sqrt{\alpha_1 + C_1} + C_2\big)\|u_0\|_{H^2\cap H_2}^{1/2}\\ &\quad + \big[1 + 2\sqrt{N(V_1, \langle x\rangle^{-2})}\big]\|u_\varepsilon(t)\|_{H_2}^{1/2}\\ &\quad + \Big(\int_0^t \gamma_{1,\varepsilon}(s)\|u_\varepsilon(s)\|_{H^1\cap H_2}\,ds\Big)^{1/2} + 2(1 + C_2)\|f\|_F^{1/2}.\end{aligned}$$

Here $\|u_\varepsilon(t)\|_{H_2}^{1/2}$ is estimated by (3.7). Thus we have

$$\begin{aligned}\|u_\varepsilon(t)\|_{H^2\cap H_2}^{1/2} \le\, & C_3(\|u_0\|_{H^2\cap H_2}^{1/2} + 2\sqrt{2}\|f\|_F^{1/2}) \\ & + 2\big[1 + 2\sqrt{N(V_1, \langle x\rangle^{-2})}\big]\int_0^t \|u_\varepsilon(s)\|_{H^2}^{1/2}\,ds \\ & + \Big(\int_0^t \gamma_{1,\varepsilon}(s)\|u_\varepsilon(s)\|_{H^1\cap H_2}\,ds\Big)^{1/2}.\end{aligned}$$

Applying the integral inequality $\displaystyle\int_0^t \|u_\varepsilon(s)\|_{H^2}^{1/2}\,ds \le \sqrt{t}\Big(\int_0^t \|u_\varepsilon(s)\|_{H^2}\,ds\Big)^{1/2}$, we obtain

$$\|u_\varepsilon(t)\|_{H^2\cap H_2} \le 4C_3^2(\|u_0\|_{H^2\cap H_2} + 8\|f\|_F) + \int_0^t \gamma_{0,\varepsilon}(s)\|u_\varepsilon(s)\|_{H^2\cap H_2}\,ds.$$

This yields (3.13). Now let M be as in (3.11). Then $4C_3^2\exp(\|\gamma_{0,\varepsilon}\|_{L^1(0,T)})$ is bounded by

$$C_4 := 4C_3^2\exp\Big(4M + 32T^2\Big[1 + 2\sqrt{N(V_1, \langle x\rangle^{-2})}\Big]^2\Big).$$

We see from Lemma 3.1 (b) that

$$\|\partial_t u_\varepsilon\| + \|u_\varepsilon(t)\|_{H^2\cap H_2} \le (1 + C_1)C_4(\|u_0\|_{H^2\cap H_2} + 8\|f\|_F) + \|f\|_F.$$

This completes the proof of (3.5) with $C_0 := 1 + 8(1 + C_1)C_4$. □

3.2. Proof of Lemma 3.4

The argument is completely different from that in [**1**]. The proof is divided into three steps.

First step. We use the new unknown function

$$v_\varepsilon(y, t) := u_\varepsilon(x, t) = u_\varepsilon(y + a_\varepsilon(t), t).$$

It follows from (2.4) that

$$\|\partial_t v_\varepsilon(t) - \partial_t u_\varepsilon(t)\| \le \alpha_1\|\nabla u_\varepsilon(t)\| = \alpha_1\|\nabla v_\varepsilon(t)\|. \tag{3.14}$$

Since u_ε is a solution to $(\mathrm{SE})_\varepsilon$, $v_\varepsilon \in C^1([0, T]; L^2(\mathbb{R}^N)) \cap C([0, T]; H^2(\mathbb{R}^N) \cap H_2(\mathbb{R}^N))$ satisfies

$$\begin{cases} i\,\partial_t v_\varepsilon + \Delta v_\varepsilon - i\Big(\dfrac{da_\varepsilon}{dt}(t)\cdot\nabla\Big)v_\varepsilon + \dfrac{v_\varepsilon}{(|y|^2 + \varepsilon^2)^{1/2}} & \\ \qquad + V_1^\varepsilon(y + a_\varepsilon(t), t)v_\varepsilon = f_\varepsilon(y + a_\varepsilon(t), t), & (y, t) \in \mathbb{R}^N \times [0, T], \\ v_\varepsilon(y, 0) = u_0(y + a_\varepsilon(0)), & y \in \mathbb{R}^N. \end{cases} \tag{3.15}$$

Let $0 < h < \min\{\varepsilon, T\}$. Then we define for $t \in [0, T-h]$,

$$(D_h\varphi)(y,t) := \frac{1}{h}[\varphi(y,t+h) - \varphi(y,t)].$$

In this step we show that

$$\|(D_h v_\varepsilon)(t)\| \le \|(D_h v_\varepsilon)(0)\| + I_1(h) + I_2(h) + I_3(h), \tag{3.16}$$

where

$$\begin{aligned} I_1(h) &:= \int_0^t \Big\| \Big(D_h \frac{da_\varepsilon}{ds}\Big)(s) \cdot \nabla v_\varepsilon(s) \Big\| \, ds, \\ I_2(h) &:= \int_0^t \big\| D_h(V_1^\varepsilon(\cdot + a_\varepsilon(s), s)) v_\varepsilon(s) \big\| \, ds, \\ I_3(h) &:= \int_0^t \big\| D_h(f_\varepsilon(\cdot + a_\varepsilon(s), s)) \big\| \, ds. \end{aligned}$$

Let $s \in [0, t]$. Then we have

$$\frac{1}{2}\frac{d}{ds}\big\|(D_h v_\varepsilon)(s)\big\|^2 = \operatorname{Re} \frac{1}{h} \int_{\mathbb{R}^N} \big(\partial_t v_\varepsilon(y, s+h) - \partial_t v_\varepsilon(y,s)\big)\overline{(D_h v_\varepsilon)(y,s)}\, dy.$$

Using the symmetry of $-\Delta$, $i\nabla$ and the real-valuedness of potentials, we see from (3.15) that

$$\begin{aligned} \frac{1}{2}\frac{d}{ds}\big\|(D_h v_\varepsilon)(s)\big\|^2 = \operatorname{Re} &\int_{\mathbb{R}^N} \Big(D_h \frac{da_\varepsilon}{ds}\Big)(s) \cdot \nabla v_\varepsilon(y,s) \overline{(D_h v_\varepsilon)(y,s)}\, dy \\ - \operatorname{Im} &\int_{\mathbb{R}^N} D_h(V_1^\varepsilon(y + a_\varepsilon(s), s)) v_\varepsilon(y,s) \overline{(D_h v_\varepsilon)(y,s)}\, dy \\ + \operatorname{Im} &\int_{\mathbb{R}^N} D_h(f_\varepsilon(y + a_\varepsilon(s), s)) \overline{(D_h v_\varepsilon)(y,s)}\, dy. \end{aligned}$$

It follows from the Cauchy-Schwarz inequality that

$$\begin{aligned} \frac{1}{2}\frac{d}{ds}\|(D_h v_\varepsilon)(s)\|^2 \le &\Big\| \Big(D_h \frac{da_\varepsilon}{ds}\Big)(s) \cdot \nabla v_\varepsilon(s) \Big\| \cdot \|(D_h v_\varepsilon)(s)\| \\ &+ \|D_h(V_1^\varepsilon(\cdot + a_\varepsilon(s), s)) v_\varepsilon(s)\| \cdot \|(D_h v_\varepsilon)(s)\| \\ &+ \|D_h(f_\varepsilon(\cdot + a_\varepsilon(s), s))\| \cdot \|(D_h v_\varepsilon)(s)\|. \end{aligned}$$

Integrating this inequality on $[0, t]$, we have

$$\begin{aligned} &\|(D_h v_\varepsilon)(t)\|^2 - \|(D_h v_\varepsilon)(0)\|^2 \\ &\le 2\int_0^t \Big[\Big\| \Big(D_h \frac{da_\varepsilon}{ds}\Big)(s) \cdot \nabla v_\varepsilon(s) \Big\| + \|D_h(V_1^\varepsilon(\cdot + a_\varepsilon(s), s)) v_\varepsilon(s)\| \\ &\qquad\qquad + \|D_h(f_\varepsilon(\cdot + a_\varepsilon(s), s))\| \Big] \cdot \|(D_h v_\varepsilon)(s)\|\, dr. \end{aligned}$$

Therefore (3.16) is a consequence of Lemma 2.6.

Second step. Letting $h \downarrow 0$ of (3.16), we shall obtain the L^2-estimate of $\partial_t v_\varepsilon$:

$$\|\partial_t v_\varepsilon(t)\| \le \|\partial_t v_\varepsilon(0)\| + \int_0^t \gamma_{1,\varepsilon}(s)(\|\nabla u_\varepsilon(s)\| + \|u_\varepsilon(s)\|_{H_2})\, ds + 2\|f\|_F, \tag{3.17}$$

where $\gamma_{1,\varepsilon} \in L^1(0,T)$ is defined as (3.10). First we note in (3.16) that

$$\|\partial_t v_\varepsilon(t) - (D_h v_\varepsilon)(t)\| \to 0 \quad (h \downarrow 0) \quad \forall\, t \in [0,T],$$

where we have set $v_\varepsilon(t) := v_\varepsilon(T) + (t-T)(\partial_t v_\varepsilon)(T)$, $T \le t \le T+\varepsilon$.

Now we consider the convergence of $I_1(h)$, $I_2(h)$ and $I_3(h)$.

$$\lim_{h\downarrow 0} I_1(h) = \int_0^t \Big\| \frac{d^2 a_\varepsilon}{ds^2}(s) \cdot \nabla v_\varepsilon(s) \Big\|\, ds \le \int_0^t \gamma_{1,\varepsilon}(s) \|\nabla u_\varepsilon(s)\|\, ds. \tag{3.18}$$

$$\lim_{h\downarrow 0} I_2(h) = \int_0^t \Big\| \Big[\frac{d}{ds} V_1^\varepsilon(\cdot + a_\varepsilon(s), s)\Big] v_\varepsilon(s) \Big\|\, ds \le \int_0^t \gamma_{1,\varepsilon}(s) \|u_\varepsilon(s)\|_{H_2}\, ds. \tag{3.19}$$

$$\lim_{h\downarrow 0} I_3(h) = \int_0^t \Big\| \frac{d}{ds} f_\varepsilon(\cdot + a_\varepsilon(s), s) \Big\|\, ds \le 2\, \|f\|_F. \tag{3.20}$$

Let us show (3.19). To this end we can proceed as follows:

$$\begin{aligned}
&\int_0^t \Big\| \Big[\frac{d}{ds} V_1^\varepsilon(\cdot + a_\varepsilon(s), s)\Big] v_\varepsilon(s) - D_h(V_1^\varepsilon(\cdot + a_\varepsilon(s), s)) v_\varepsilon(s) \Big\|\, ds \\
&\le \int_0^t G_h(s) \|\langle \cdot + a_\varepsilon(s)\rangle^2 v_\varepsilon(s)\|\, ds \\
&\le \max_{0\le r\le T} \|u_\varepsilon(r)\|_{H_2} \int_0^T G_h(s)\, ds,
\end{aligned}$$

where we set

$$\begin{aligned}
G_h(s) &:= \Big\| \frac{1}{\langle \cdot + a_\varepsilon(s)\rangle^2} \Big[\frac{d}{ds} V_1^\varepsilon(\cdot + a_\varepsilon(s), s) - D_h(V_1^\varepsilon(\cdot + a_\varepsilon(s), s))\Big] \Big\|_{L^\infty} \\
&= \Big\| \frac{1}{\langle \cdot + a_\varepsilon(s)\rangle^2} \Big[\frac{d}{ds} V_1^\varepsilon(\cdot + a_\varepsilon(s), s) - \frac{1}{h}\int_s^{s+h} \frac{d}{dr} V_1^\varepsilon(\cdot + a_\varepsilon(r), r)\, dr\Big] \Big\|_{L^\infty}.
\end{aligned}$$

Setting

$$\begin{aligned}
U_\varepsilon(s) &:= \frac{1}{\langle \cdot + a_\varepsilon(s)\rangle^2} \frac{d}{ds} V_1^\varepsilon(\cdot + a_\varepsilon(s), s) \\
&= \frac{1}{\langle \cdot + a_\varepsilon(s)\rangle^2} \Big[\partial_t V_1^\varepsilon(\cdot + a_\varepsilon(s), s) + \frac{da_\varepsilon}{ds}(s) \cdot \nabla V_1^\varepsilon(\cdot + a_\varepsilon(s), s)\Big],
\end{aligned}$$

we can write as

$$\int_0^T G_h(s)\,ds \le J_1(h) + J_2(h),$$

where

$$J_1(h) := \int_0^T \Big\| U_\varepsilon(s) - \frac{1}{h}\int_s^{s+h} U_\varepsilon(r)\,dr \Big\|_{L^\infty} ds,$$

$$J_2(h) := \int_0^T \Big\| \frac{1}{h}\int_s^{s+h} \frac{\langle \cdot + a_\varepsilon(r)\rangle^2 - \langle \cdot + a_\varepsilon(s)\rangle^2}{\langle \cdot + a_\varepsilon(s)\rangle^2} U_\varepsilon(r)\,dr \Big\|_{L^\infty} ds.$$

Since $U_\varepsilon \in L^\infty(0,T;L^\infty(\mathbb{R}^N))$, we can conclude that $J_1(h) + J_2(h) \to 0$ as $h \downarrow 0$. In fact, we have that for $y \in \mathbb{R}^N$ and $r \in [s, s+h]$,

$$\begin{aligned}\frac{|\langle y + a_\varepsilon(r)\rangle^2 - \langle y + a_\varepsilon(s)\rangle^2|}{\langle y + a_\varepsilon(s)\rangle^2} &\le |a_\varepsilon(r) - a_\varepsilon(s)| + |a_\varepsilon(r) - a_\varepsilon(s)|^2 \\ &\le \alpha_1 h(1 + \alpha_1 h).\end{aligned}$$

This proves the equality in (3.19). Now we show the remaining inequality in (3.19). Noting that $\|\,[(d/ds)V_1^\varepsilon(\cdot + a_\varepsilon(s), s)]\, v_\varepsilon(s)\| \le \|U_\varepsilon(s)\|_{L^\infty}\|u_\varepsilon(s)\|_{H_2}$ and

$$\|U_\varepsilon(s)\|_{L^\infty} \le \Big\|\frac{\partial_t V_1^\varepsilon(s)}{\langle x\rangle^2}\Big\|_{L^\infty} + \alpha_1 \Big\|\frac{\nabla V_1^\varepsilon(s)}{\langle x\rangle^2}\Big\|_{L^\infty} \le \gamma_{1,\varepsilon}(s),$$

we obtain (3.19).

In the same way as in the proof of (3.19) we can show (3.18) and (3.20) (use Lemma 2.2 (b)′ and Lemma 2.4 (b), (c)).

Therefore by virtue of (3.18)–(3.20) we obtain (3.17).

Third step. We obtain the L^2-estimate of $\partial_t u_\varepsilon$. Thus we see from (3.14) and (3.17) that

$$\begin{aligned}&\|\partial_t u_\varepsilon(t)\| - \alpha_1\|\nabla u_\varepsilon(t)\| \\ &\le \|\partial_t v_\varepsilon(t)\| \le \|\partial_t v_\varepsilon(0)\| + \int_0^t \gamma_{1,\varepsilon}(s)\|u_\varepsilon(s)\|_{H^1\cap H_2}\,ds + 2\,\|f\|_F.\end{aligned}$$

Using again (3.14) with $t = 0$, we have

$$\begin{aligned}&\|\partial_t u_\varepsilon(t)\| - \alpha_1\|\nabla u_\varepsilon(t)\| \\ &\le \|\partial_t u_\varepsilon(0)\| + \alpha_1\|\nabla u_0\| + \int_0^t \gamma_{1,\varepsilon}(s)\|u_\varepsilon(s)\|_{H^1\cap H_2}\,ds + 2\,\|f\|_F.\end{aligned}$$

Therefore by virtue of (3.2) with $t = 0$ we obtain (3.9); note that $\|\nabla u_0\| \le \|u_0\|_{H^2\cap H_2}$.

3.3. Convergence and existence of strong solution

For $\varepsilon > 0$ let u_ε be a unique solution as in Proposition 2.5. Now let $\{\varepsilon_n\}$ be a null sequence: $\varepsilon_n > 0$ and $\varepsilon_n \to 0$ $(n \to \infty)$. Then we denote u_{ε_n} by u_n. Accordingly, we have

$$\begin{cases} i\,\partial_t u_n(x,t) + \Delta u_n(x,t) + V^n(x,t)u_n(x,t) = f_n(x,t), \\ \qquad\qquad\qquad\qquad\qquad\qquad (x,t) \in \mathbb{R}^N \times [0,T], \\ u_n(x,0) = u_0(x), \qquad\qquad\qquad x \in \mathbb{R}^N, \end{cases} \tag{3.21}$$

where $V^n := V_0^n + V_1^n$. In Section 3.1 it is proved that $\{u_n\}$ is bounded in $W^{1,\infty}(0,T;L^2(\mathbb{R}^N)) \cap L^\infty(0,T;H^2(\mathbb{R}^N) \cap H_2(\mathbb{R}^N))$. Since $L^\infty(0,T;L^2(\mathbb{R}^N))$ is the dual space of $L^1(0,T;L^2(\mathbb{R}^N))$ and $L^1(0,T;L^2(\mathbb{R}^N))$ is separable, there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ and $u \in L^\infty(0,T;L^2(\mathbb{R}^N))$ such that

$$u = \operatorname*{w^*\text{-}lim}_{k\to\infty} u_{n_k} \quad \text{in } L^\infty(0,T;L^2(\mathbb{R}^N)).$$

Therefore we conclude that

$$u \in W^{1,\infty}(0,T;L^2(\mathbb{R}^N)) \cap L^\infty(0,T;H^2(\mathbb{R}^N) \cap H_2(\mathbb{R}^N)) \tag{3.22}$$

(see Lions [**9**, Section 1.4]). Next we want to show that u is a solution to problem (SE). To do so we need the following convergences:

Lemma 3.6. *Let a_n be a_ε with ε replaced with ε_n. Let V_0^n, V_1^n and f_n be as in* (3.21)*. Then*

(a) *$a_n \to a$ in $W^{2,1}(0,T)$, with*

$$|a(t) - a_n(t)| \le 4\varepsilon_n\Big(\alpha_1 + \Big\|\frac{d^2a}{dt^2}\Big\|_{L^1(0,T)}\Big). \tag{3.23}$$

(b) *$V_0^n u \to |x - a(t)|^{-1}u$ in $L^\infty(0,T;L^2(\mathbb{R}^N)) \quad \forall\, u \in H^1(\mathbb{R}^N)$.*

(c) *$V_1^n u \to V_1 u$ in $L^\infty(0,T;L^2(\mathbb{R}^N)) \quad \forall\, u \in H_2(\mathbb{R}^N)$.*

(d) *$f_n \to f$ in $L^\infty(0,T;L^2(\mathbb{R}^N))$.*

Proof. (a) is a consequence of the properties of mollifier χ_ε. In fact, for $t \in [0,T]$ we have

$$\begin{aligned} |a(t) - a_n(t)| &\le \int_0^t \Big[\int_{-1}^1 \Big|\int_{s-\varepsilon_n r}^s \Big|\frac{d}{d\tau}\Big(P\frac{da}{d\tau}\Big)(\tau)\Big| d\tau\Big| \chi(r)\,dr\Big]\,ds \\ &\le \int_0^t \Big[\int_{s-\varepsilon_n}^{s+\varepsilon_n} \Big|\frac{d}{d\tau}\Big(P\frac{da}{d\tau}\Big)(\tau)\Big| d\tau\Big]\,ds \\ &\le \int_{-\varepsilon_n}^{t+\varepsilon_n} \Big(\int_{\tau-\varepsilon_n}^{\tau+\varepsilon_n} ds\Big)\Big|\frac{d}{d\tau}\Big(P\frac{da}{d\tau}\Big)(\tau)\Big|\,d\tau \\ &\le 2\varepsilon_n \Big\|\frac{d}{dt}\Big(P\frac{da}{dt}\Big)\Big\|_{L^1(\mathbb{R})}. \end{aligned}$$

By the property of the extension operator P we can obtain (3.23).

(b) Set $\varphi := |x-a(t)|^{-1}u \in L^2(\mathbb{R}^N)$ for $u \in H^1(\mathbb{R}^N)$ and t fixed. Then $\varphi \in L^2(\mathbb{R}^N)$ and

$$\big\||x-a(t)|^{-1}u - V_0^n(x,t)u\big\| = \big\|(1-|x-a(t)|\,V_0^n(x,t))\varphi\big\|.$$

Here we want to show that $|x-a(t)|V_0^n(x,t)$ is bounded with respect to n. In fact, since $|x-a(t)| \le |x-a_n(t)| + |a_n(t)-a(t)|$, we see from (3.23) that

$$|x-a(t)|V_0^n(x,t) = \frac{|x-a(t)|}{\sqrt{|x-a_n(t)|^2+\varepsilon_n^2}} \le 1+4\Big(\alpha_1+\Big\|\frac{d^2a}{dt^2}\Big\|_{L^1(0,T)}\Big).$$

Since $H^1(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N)$, it suffices to show that

$$\lim_{n\to\infty}\big\|(1-|x-a(t)|V_0^n(x,t))\psi\big\| = 0 \tag{3.24}$$

for each $\psi \in H^1(\mathbb{R}^N)$. By virtue of (3.23) we can compute as follows:

$$\begin{aligned}
&\big|1-|x-a(t)|V_0^n(x,t)\big| \\
&= \frac{\big||x-a_n(t)|^2+\varepsilon_n^2-|x-a(t)|^2\big|}{\sqrt{|x-a_n(t)|^2+\varepsilon_n^2}+|x-a(t)|}V_0^n(x,t) \\
&\le \frac{\varepsilon_n}{\sqrt{|x-a_n(t)|^2+\varepsilon_n^2}+|x-a(t)|}\cdot \varepsilon_n V_0^n(x,t) \\
&\quad + \frac{|x-a_n(t)|+|x-a(t)|}{\sqrt{|x-a_n(t)|^2+\varepsilon_n^2}+|x-a(t)|}\cdot |a_n(t)-a(t)|V_0^n(x,t) \\
&\le \varepsilon_n\Big(1+4\alpha_1+4\Big\|\frac{d^2a}{dt^2}\Big\|\Big)|x-a_n(t)|^{-1}.
\end{aligned}$$

Therefore (3.24) is a consequence of the Hardy inequality:

$$\begin{aligned}
&\big\|\big(1-|x-a(t)|V_0^n(x,t)\big)\psi\big\| \\
&\le \frac{2\varepsilon_n}{N-2}\Big(1+4\alpha_1+4\Big\|\frac{d^2a}{dt^2}\Big\|\Big)\|\nabla\psi\| \to 0 \quad \text{as } n\to\infty.
\end{aligned}$$

(c) and (d) are a consequence of the properties of mollifier ζ_ε and cut-off function η_ε. □

Thus u is a solution to problem (SE) in the sense of distribution satisfying (3.22). The energy estimate (1.7) is a consequence of (3.5) and the weak*-convergence of $\{u_n\}$. Now (1.7) guarantees the uniqueness. In fact, let u and v be strong solutions to (SE) with respective initial values u_0 and v_0. Then (1.7) yields that $\|u(t)-v(t)\|_{H^2\cap H_2} \le C_0\|u_0-v_0\|_{H^2\cap H_2}$.

It remains to derive the continuity of u as in (1.1) and (1.2). We see from $u \in W^{1,\infty}(0,T;L^2(\mathbb{R}^N))$ that

$$u \in C([0,T];L^2(\mathbb{R}^N)); \tag{3.25}$$

more precisely, u is Lipschitz continuous on $[0,T]$. Since $\|\Delta u\|$, $\|\langle x\rangle^2 u\|$ are bounded on $[0,T]$ and $H^2(\mathbb{R}^N) \cap H_2(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N)$, we can show that $-\Delta u$, $\langle x\rangle^2 u \in C_{\mathrm{w}}([0,T];L^2(\mathbb{R}^N))$. It turns out that

$$u \in C_{\mathrm{w}}([0,T];H^2(\mathbb{R}^N) \cap H_2(\mathbb{R}^N)).$$

Finally, (3.25) implies together with $u \in L^\infty(0,T;H^2(\mathbb{R}^N) \cap H_2(\mathbb{R}^N))$ that

$$u \in C([0,T];H^1(\mathbb{R}^N) \cap H_1(\mathbb{R}^N)).$$

This completes the proof of Theorem 1.2.

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