Degree conditions for the existence of regular factors in 2-connected star-free graphs (2-連結スターフリーグラフにおける正則因子が存在するための次数条件)

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Contents

Chapter 1

Introduction

In this thesis, we focus on degree factor problems in graph theory, and investigate sufficient degree conditions for the existence of regular factors, in particular 3-factors and 6-factors, in graphs and best possibilities of the bounds of conditions. The factor of graph is one of the most important topics in graph theory, and it has been studied by many researchers and applied to a wide practical area of studies, for example, operations research, information theory, network theory and so on. It also serves as a mathematical tool for, for example, the edge-coloring problem of graphs. The name "degree factor" seem to be due to Akiyama and Kano in 1985 [1]. In 2007 [12], Plummer collected many results for problems of this vast area. Furthermore, many researchers have been still studied on degree factor problems. For more information on connected degree factors, see [8].

In Section 1.1, we shall show the history on factors in graph theory, which is related to our investigations, and the preceding studies. In Section 1.2, we propose our results.

We first give several definitions (for more comprehensive description of definitions of basic terms and symbols in graph theory, the reader is referred to Chapter 2). Our notation is standard, and is mostly taken from Diestel [4]. In this thesis, we consider only finite, simple, and undirected graphs with no loops and no multiple edges throughout. Let G be a graph. We let $V(G)$

and $E(G)$ denote the *vertex set* and the *edge set* of G, respectively. We also let $\delta(G)$ denote the *minimum degree*. For an integer $r \geq 1$, a subgraph F of G such that $V(F) = V(G)$ and $\deg_F(x) = r$ for all $x \in V(F)$ is called an *r-factor* of G. A complete bipartite graph $K_{1,t}$ with partite sets of cardinalities 1 and t is called a *t-star*. We say G is *$K_{1,t}$-free* or *t-star-free* if G has no $K_{1,t}$ as an induced subgraph.

1.1 History on factors and motivation

In this section, we first review the history on factors in graph theory briefly. The history dates back as far as the so-called Petersen's Theorem, which is one of the earliest results in the study of graph factors and can be stated as follows [11]:

Theorem 1.1.1 (Petersen [11]) *Every 2-edge-connected 3-regular graph contains a 1-factor (and hence also a 2-factor).*

For matchings in bipartite graphs, Hall [7] obtained the following theorem in 1935, which is the fundamental matching theorem on bipartite graphs and remains one of the most widly applied graph-theoretic results.

Theorem 1.1.2 (Hall [7], Hall's Marriage Theorem) *Let G be a bipartite graph with bipartite sets X and Y. Then G has a matching saturating all vertices of X if and only if for any $S \subseteq X$,*

$$|N(S)| \geq |S|.$$

In 1947, W. T. Tutte extended Theorem 1.1.2 to obtain the following theorem, which is a characterization for the existence of perfect matchings, i.e., 1-factors, in non-bipartite graphs [15].

Theorem 1.1.3 (Tutte [15], Tutte's 1-Factor Theorem) *Let G be a graph. Then, a graph G has a 1-factor if and only if for each subset $S \subseteq V(G)$,*

$$q(G - S) \leq |S|,$$

where $q(G-S)$ denotes the number of those components C of $G-S$ such that $|V(C)|$ is odd.

This result of Tutte was later generalized by himself in 1952 as follows [16]:

Theorem 1.1.4 (Tutte [16], Tutte's f-Factor Theorem) *Let G be a graph. Then, a graph G has an f-factor if and only if for each subsets S and $T \subseteq V(G)$ with $S \cap T = \emptyset$,*

$$\theta(S,T) := \sum_{x \in S} f(x) + \sum_{y \in T} \Big(\deg_{G-S}(y) - f(y) \Big) - h(S,T) \geq 0,$$

where $h(S,T)$ denotes the number of those components C of $G-S-T$ such that $\sum_{x \in V(C)} f(x) + |E(C,T)|$ is odd.

Many results concerning degree factor problems depend upon this theorem, where the function $\theta(S,T)$ is often called as "Tutte's evaluation function".

We next concentrate on the existence of regular factors in terms of star-free condition although there are many different conditions. Star-free condition has been popular to examine non-trivial sufficient conditions for a graph to have a certain factor. The most commonly expressed reason for this is that many graph problems become simpler by restricting to the class of star-free graphs. For example, a certain NP-hard problem in arbitrary graphs may possibly be polynomial under the condition. Another reason is that if forbidding a single subgraph for regular factors, then it must be a star. Arguing as Theorem 3 in [3], we obtain the following proposition.

Proposition 1.1.5 *Let k, d and r be possitive integers with $r \geq k$, $r \geq d$ and $r \geq 2$, and let H be a connected graph. If every k-connected H-free graph with minimum degree at least d has an r-factor, then H is a star.*

Proof. Set $p = \max\{k, d\}$. Let $G_1 = K_p + (p+1)K_1$ and $G_2 = K_{p,p+1}$, then both of G_1 and G_2 are k-connected graphs with minimum degree at least d, and neither G_1 nor G_2 has an r-factor. Since H is a connected induced subgraph of G_1, H is a star or H contains a triangle. Moreover since H is an induced subgraph of G_2, H is triangle-free, which implies H is a star. $\square$

As a first step, we now give a sufficient condition for a graph to have a 1-factor by using $K_{1,3}$-free (i.e., claw-free) condition. This result was obtained by Sumner [13] in 1976.

Theorem 1.1.6 (Sumner [13]) (i) *If G is a connected claw-free graph of even order, then G has a 1-factor.*

(ii) *If G is a k-connected $K_{1,k+1}$-free ($k \geq 2$) of even order, then G has a 1-factor.*

Now we present a minimum degree condition sufficient to guarantee an r-factor in a $K_{1,t}$-free graph, was proved by Ota and Tokuda in [10].

Theorem 1.1.7 (Ota and Tokuda [10]) *Let t and r be integers with $t \geq 3$ and $r \geq 2$. Let G be a connected $K_{1,t}$-free graph, and suppose that*

$$\delta(G) \geq \left(t + \frac{t-1}{r}\right)\left\lceil \frac{rt}{2(t-1)} \right\rceil - \frac{t-1}{r}\left\lceil \frac{rt}{2(t-1)} \right\rceil^2 + t - 3.$$

In the case where r is odd, suppose further that $t \leq r+1$ and $r|V(G)|$ is even. Then G has an r-factor.

Since the above degree condition forms complex, we examine restricted two cases where $r = 3$ and $r = 6$. In the case where $r = 2$, 3, 4 and 6, the minimum degree condition in Theorem 1.1.7 takes the following simple forms.

Corollary 1.1.8 *Let $t \geq 3$ be an integer. Let G be a connected $K_{1,t}$-free graph, and suppose that $\delta(G) \geq 2t - 2$. Then G has a 2-factor.*

Corollary 1.1.9 *Let $3 \leq t \leq 4$ be an integer. Let G be a connected $K_{1,t}$-free graph with $|V(G)|$ even, and suppose that $\delta(G) \geq 5$ or $\delta(G) \geq 7$ according as $t = 3$ or $t = 4$. Then G has a 3-factor.*

Corollary 1.1.10 *Let $t \geq 3$ be an integer. Let G be a connected $K_{1,t}$-free graph, and suppose that $\delta(G) \geq (5t-3)/2$. Then G has a 4-factor.*

Corollary 1.1.11 *Let $t \geq 3$ be an integer. Let G be a connected $K_{1,t}$-free graph, and suppose that $\delta(G) \geq 9$ or $\delta(G) \geq 3t-1$ according as $t = 3$ or $t \geq 4$. Then G has a 6-factor.*

The minimum degree condition in Theorem 1.1.7 is best possible, and hence so are those in Corollaries 1.1.8, 1.1.9, 1.1.10 and 1.1.11. On the other hand, as for Corollaries 1.1.8 and 1.1.10, if we add the assumption that G is 2-connected, then we can relax the minimum degree condition, as in shown in the following two results which were proved in [2] and [5], respectively.

Theorem 1.1.12 (Aldred et al. [2]) *Let $t \geq 3$ be an integer. Let G be a 2-connected $K_{1,t}$-free graph, and suppose that $\delta(G) \geq t$. Then G has a 2-factor.*

Theorem 1.1.13 (Egawa and Kotani [5]) *Let $t \geq 3$ be an integer. Let G be a 2-connected $K_{1,t}$-free graph, and suppose that $\delta(G) \geq (3t+1)/2$. Then G has a 4-factor.*

Our investigations and results are similarly to weaken the minimum degree condition in Corollaries 1.1.9 and 1.1.11 by assuming that G is 2-connected.

1.2 The results

Theorem 1.2.1 *Let $t \geq 3$ be an integer. Let G be a 2-connected $K_{1,t}$-free graph. Then the followings hold.*

(i) *If $t \geq 4$ and $\delta(G) \geq 2t+1$, then G has a 6-factor.*

(ii) *If $t = 3$ and $\delta(G) \geq 8$, then G has a 6-factor.*

Theorem 1.2.2 *Let $3 \leq t \leq 4$ be an integer. Let G be a 2-connected $K_{1,t}$-free graph with $|V(G)|$ even and suppose that $\delta(G) \geq t+1$. Then G has a 3-factor.*

Theorem 1.2.3 *Let $5 \leq t \leq 7$ be an integer. Let G be a 2-connected $K_{1,t}$-free graph with $|V(G)|$ even and suppose that $\delta(G) \geq t+2$. Then G has a 3-factor.*

Theorem 1.2.1 in [17] is our result, and so are Theorems 1.2.2 and 1.2.3 in [6]. The proof of them depends on Theorem 1.1.4.

Each of the degree conditions in the above theorems is best possible. To see this, we later mention about sharpness examples in Sections 3.4 of Chapter 3 and 4.6 of Chapter 4.

We shall close this section with the following conjecture, which implies Theorems 1.1.12, 1.1.13 and 1.2.1.

Conjecture 1.2.4 *Let t and r be integers with $t \geq 3$ and $r \geq 1$. Let G be a 2-connected $K_{1,t}$-free graph, and suppose that*

$$\delta(G) \geq \left(t + \frac{t-1}{2r}\right)\left\lceil \frac{rt}{t-1} \right\rceil - \frac{t-1}{2r}\left\lceil \frac{rt}{t-1} \right\rceil^2 - 1.$$

Then G has a $2r$-factor.

Chapter 2

Definitions and terminologies

We begin our study of graphs by introducing many of the basic concepts that we shall encounter throughout our investigations. This chapter gives definitions of basic terms and symbols in graph theory.

2.1 Graphs

A *graph* G is an ordered pair (V, E) of two disjoint sets V and E together with a mapping φ which associates with each element of E an unordered pair of two distinct elements of V (see Figure 2.1). The elements of V are called the *vertices* (the singular is *vertex*) of the graph G, and the elements of E are called its *edges*, which is a (possibly empty) set of unordered pairs of distinct its vertices. The vertex set of a graph G is denoted by $V(G)$, and its edge set by $E(G)$. The cardinality of $V(G)$ is called the *order* of G and is commonly denoted by $n(G)$, or more simply by n when the graph under consideration is clear; while the cardinality of $E(G)$ is called *size* of G and is often denoted by $m(G)$ or m. If both of $V(G)$ and $E(G)$ are finite sets, G is called a *finite* graph. If $V(G) = \emptyset$, G is called an *empty graph*.

If $e \in E(G)$ and u and v are the vertices which constitute the unordered pair associated with e by φ_G, then we call u and v the *endvertices* of e, and say that e *joins* u and v and write $e = uv$ (or $e = vu$). When a vertex u is

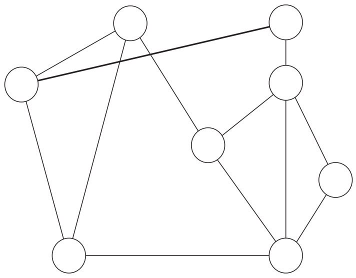

Figure 2.1: A graph

an endvertex of an edge e of G, we also say that u is *incident* with e and e is incident with u. For $u, v \in V(G)$, u and v are said to be *adjacent* if they are joined by an edge. A set $U \subseteq V(G)$ is *independent* if no two vertices in U are adjacent in G. For $e, f \in E(G)$ with $e \neq f$, e and f are said to be *adjacent* if they have an endvertex in common.

Two graphs are *isomorphic* if there exists a one-to-one correspondence between their vertex sets which preserves adjacency. When two graphs G and G' are isomorphic, we write $G \simeq G'$ (or $G = G'$ if there is no danger of confusion). A graph isomorphic to a given graph G is often referred to as a *copy* of G.

2.2 Subgraphs and operations on graphs

Let G be a graph. A graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ is called a *subgraph* of G. A subgraph H of G is *spanning* if the vertex set of H is equal to the vertex set of G. Let $X \subseteq V(G)$. The subgraph *induced* by X in G, denoted by $G[X]$, is the graph defined by $V(G[X]) = X$ and $E(G[X]) = \{e \in E(G) \mid \text{both endvertices of } e \text{ belong to } X\}$. We define the graph $G - X$ as $G[V(G) - X]$. If $X = \{x\}$ is a singleton, we simply write $G - x$ instead of $G - \{x\}$ (see Figure 2.2).

Let now G and H be two graphs with $V(G) \cap V(H) = \emptyset$. We let $G \cup H$ denote the graph defined by $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) =$

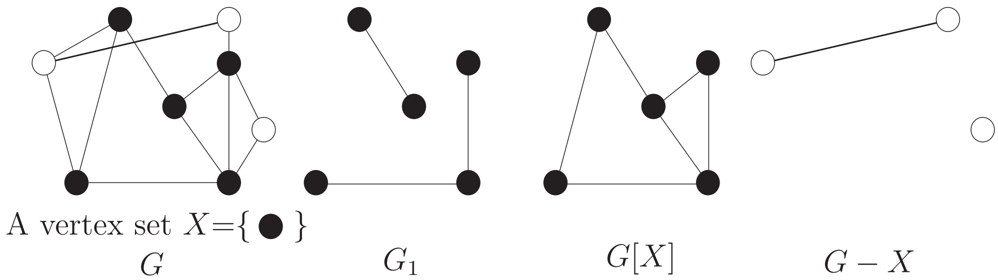

Figure 2.2: A subgraph G_1 of G with $V(G_1) = X$, $G[X]$ and $G - X$

$E(G) \cup E(H)$, and we call $G \cup H$ the *union* of G and H. For $s \geq 1$, sG denotes the union of s vertex-disjoint copies of G. The *join* of G and H, denoted by $G + H$, is the graph defined by $V(G + H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{uv \mid u \in V(G), v \in V(H)\}$. The *complement* of G, denoted by $\overline{G}$, is the graph defined by $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{uv \mid u, v \in V(G),\ u \neq v, uv \notin E(G)\}$. Note that $\overline{\overline{G}} = G$ (see Figure 2.3).

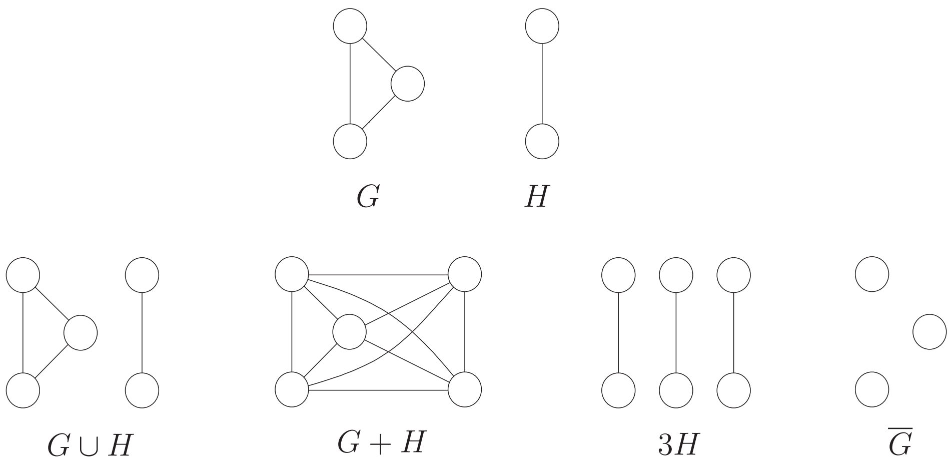

Figure 2.3: Graphs $G \cup H$, $G + H$, $3H$ and $\overline{G}$

2.3 Neighborhood and degree

Let G be a graph, and let v be a vertex of G. The *neighborhood* of v, denoted by $N_G(v)$, is the set of all vertices which are adjacent to v. For $X \subseteq V(G)$, let $N_G(X) = \bigcup_{v \in X} N_G(v)$. If there is no ambiguity, we write $N(v)$ and $N(X)$ for $N_G(v)$ and $N_G(X)$, respectively. The *degree* of v, denoted by $\deg_G(v)$, is the number of edges of G incident with v. Note that if G is a graph, then $\deg_G(v) = |N_G(v)|$. The *minimum degree* of G is the minimum of $\deg_G(x)$ as x ranges over $V(G)$, and is denoted by $\delta(G)$. If all vertices of G have the same degree k, then G is *k-regular*, or simply *regular*. Note that the edge set of a 1-regular subgraph of G is a perfect matching of G.

For subsets $S, T \subseteq V(G)$ with $S \cap T = \emptyset$, we let $E_G(S,T)$ denote the set of edges of G joining a vertex in S and a vertex in T. If there is no ambiguity, we write $E(S,T)$ for $E_G(S,T)$. When S or T consists of a single vertex, say $S = \{u\}$ or $T = \{v\}$, we write $E_G(u,T)$ or $E_G(S,v)$ for $E_G(S,T)$. Note that if $S = \{u\}$ and G is a graph, then $|E_G(u,T)| = |N_G(u) \cap T|$.

2.4 Complete graphs and bipartite graphs

A graph is *complete* if all of its vertices are pairwise adjacent. We denote by K_n the complete graph of order n (see Figure 2.4). In particular, K_3 is called a *triangle*. A graph is *bipartite* if its vertex set can be partitioned into two subsets X and Y so that every edge has one endvertex in X and one endvertex in Y, and we call X and Y its *partite sets*. A bipartite graph G with partite sets X and Y is a *complete bipartite graph* if $E(G) = \{uv \mid u \in X, v \in Y\}$. The complete bipartite graph with partite sets of cardinalities of n and m is denoted by $K_{n,m}$ (see Figure 2.5). In particular, the $K_{1,t}$ is often called as a *t-star* and the $K_{1,3}$ as a *claw*.

2.5 Paths and cycles

A graph P of order m of the form $V(P) = \{u_i \mid 1 \leq i \leq m\}$ and $E(P) = \{u_i u_{i+1} \mid 1 \leq i \leq m-1\}$ is called a *path*. We say that u_1 and u_m are *connected* by P. The vertices u_1 and u_m are called the *endvertices* of P, and the vertices u_i $(2 \leq i \leq m-1)$ are called the *internal vertices* of P. For a graph G and

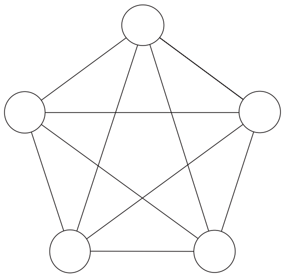

Figure 2.4: Complete graph K_5

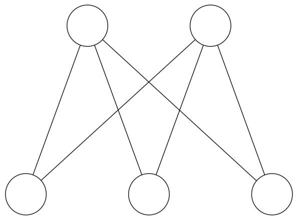

Figure 2.5: Complete bipartite graph $K_{2,3}$

two vertices $u, u' \in V(G)$, we call a path P of G a *u-u' path* if u and u' are connected by P. We define the *length* of a path as the number of its edges. We let P_l denote the path of order l (see Figure 2.6). Note that P_l has length $l-1$. A graph C of order m ($m \geq 3$) the form $V(C) = \{v_i \mid 1 \leq i \leq m\}$ and $E(C) = \{v_i v_{i+1} \mid 1 \leq i \leq m-1\} \cup \{v_m v_1\}$ is called a *cycle*. We define the *length* of a cycle as the number of its edges. Note that for a cycle, the length is the same as the order. We let C_l denote the cycle of order l (see Figure 2.7). Paths and cycles are often denoted by a sequence of vertices along edges. For example, a path P with $V(P) = \{u_i \mid 1 \leq i \leq m\}$ and $E(P) = \{u_i u_{i+1} \mid 1 \leq i \leq m-1\}$ is denoted by $u_1 u_2 \cdots u_m$, and a cycle C with $V(C) = \{v_i \mid 1 \leq i \leq m\}$ and $E(C) = \{v_i v_{i+1} \mid 1 \leq i \leq m-1\} \cup \{v_m v_1\}$ is denoted by $v_1 v_2 \cdots v_m v_1$. For a graph G, a *Hamiltonian path* of G is a spanning path of G and, similarly, a *Hamiltonian cycle* of G is a spanning cycle of G.

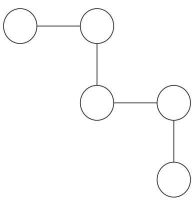

Figure 2.6: Path P_5

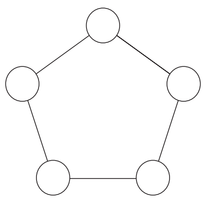

Figure 2.7: Cycle C_5

2.6 Component and (edge-)connectivity

A graph G is *connected* if for any two vertices $u, v \in V(G)$, G has an u-v path; otherwise, G is *disconnected*. A maximal connected subgraph of G is called a *component* of G. For an integer l, a graph G is called *l-connected* if $|V(G)| \geq l+1$ and $G - X$ is connected for any subset $X \subseteq V(G)$ with $|X| \leq l-1$. Note that every nonempty graph is 0-connected. The largest integer l such that G is l-connected is the *connectivity* of G, and is denoted by $\kappa(G)$. For an integer l, a graph G is called *l-edge-connected* if $|V(G)| \geq 2$ and $G - F$ is connected for any subset $F \subseteq E(G)$ with $|F| \leq l-1$. Note that every graph of order at least two is 0-edge-connected. The largest integer l such that G is l-edge-connected is the *edge-connectivity* of G, and is denoted by $\lambda(G)$.

2.7 $K_{1,t}$-freeness

Let G be a graph, and H a connected graph. A graph G is said to be *H-free* if G does not contain H as an induced subgraph. We also say that H is *forbidden* in G. For a set of connected graphs $\mathcal{F}$, G is said to be *$\mathcal{F}$-free* if G is F-free for every $F \in \mathcal{F}$. In this thesis, we deal with a single forbidden subgraph $K_{1,t}$.

2.8 f-factors

Let G be a graph, and let f be a non-negative integer-valued function defined on $V(G)$, denoted by $f : V(G) \to \mathbb{Z}^+ \cup \{0\}$. Then G has a spanning subgraph F satisfying $\deg_F(x) = f(x)$ for all $x \in V(G)$ if and only if the Tutte's evaluation function $\theta(S, T) \geq 0$ for any disjoint $S, T \subseteq V(G)$ (see Theorem 1.1.4). This spanning subgraph is called as an *f-factor*. If $f(x) = r$ for all $v \in V(G)$, then an f-factor is called an *r-factor*.

2.9 Other notations of graphs

In the rest of this chapter, we shall show another results on degree factors problems. For even $b \geq 2$, an even $[2, b]$-factor is a spanning subgraph each of whose degree is even between 2 and b. The following theorem is a special case of the parity (g, f)-factor theorem of Lovász.

Theorem 2.9.1 *Let G be a graph and $b \geq 2$ an even integer. A graph G has an even $[2, b]$-factor if and only if*

$$\eta(S,T) = b|S| + \sum_{y \in T} \deg_{G-S}(y) - 2|T| - h(S,T) \geq 0$$

for any subsets S and T of $V(G)$ with $S \cap T = \emptyset$, where $h(S,T)$ denotes the number of components C of $G - S - T$ such that $|E(T, V(C))|$ is odd.

By using the above theorem, we obtain a result on even $[2, b]$-factor as follows [14].

Theorem 2.9.2 *Let $b \geq 2$ be an even integer and let G be a 2-edge-connected graph. If*

$$\max\{\deg_G(x), \deg_G(y)\} \geq \max\left\{\frac{2n}{2+b}, 3\right\}$$

for any two nonadjacent vertices x and y of G, then G has an even [2, b]-factor.

In order to prove Theorem 2.9.2, we consider two cases depending on the order of a graph. In fact, we prove the following theorems.

Theorem 2.9.3 *Let $b \geq 2$ be an even integer and let G be a 2-edge-connected graph of order $n \geq b + 3$. If*

$$\max\{\deg_G(x), \deg_G(y)\} \geq \frac{2n}{2+b} \tag{2.9.1}$$

for any two nonadjacent vertices x and y of G, then G has an even $[2, b]$-factor.

For $n \leq b+2$, we need another condition for the existence of an even $[2,b]$-factor as follows.

Theorem 2.9.4 *Let $b \geq 2$ be an even integer and let G be a 2-edge-connected graph of order $n \leq b+2$. If*

$$\max\{\deg_G(x), \deg_G(y)\} \geq 3 \tag{2.9.2}$$

for any two nonadjacent vertices x and y of G, then G has an even $[2,b]$-factor.

Combining these, we can obtain Theorem 2.9.2. This result is sharp in three senses. The above lower bounds of the degree conditions (2.9.1) and (2.9.2) are best possible. Moreover, the hypothesis "2-edge-connected" cannot be dropped. For Theorem 2.9.4, the lower bound $n \geq b+3$ is also sharp.

Chapter 3

6-factors in 2-connected star-free graphs

In Section 3.1, we show preliminary results to prove Theorem 1.2.1. In Section 3.2, we prove Theorem 1.2.1 (i). In Section 3.3, we prove Theorem 1.2.1 (ii). In Section 3.4, we give a sharpness example of Theorem 1.2.1.

3.1 Preliminary results

In this section, we state preliminary lemmas, which we use in the proof of Theorem 1.2.1.

Let G be a graph. For S, $T \subseteq V(G)$ with $S \cap T = \emptyset$, denote $\theta_6(S,T)$ by

$$\theta_6(S,T) = 6|S| + \sum_{y \in T} \big(\deg_{G-S}(y) - 6\big) - h_6(S,T),$$

where $h_6(S,T)$ denotes the number of components C of $G - S - T$ such that $|E(T,C)|$ is odd. The following lemma is a special case of Theorem 1.1.4.

Lemma 3.1.1 (i) *G has a 6-factor if and only if $\theta_6(S,T) \geq 0$ for every $S, T \subseteq V(G)$ with $S \cap T = \emptyset$.*

(ii) *$\theta_6(S,T)$ is even for every $S, T \subseteq V(G)$ with $S \cap T = \emptyset$.* □

The following lemma is proved in [9], but we include its proof for the convenience of the reader.

Lemma 3.1.2 *Suppose that G has no 6-factors, and choose $S, T \subseteq V(G)$ with $S \cap T = \emptyset$ and $\theta_6(S,T) < 0$ so that $|S \cup T|$ is as large as possible. Then $|V(C)| \geq 3$ for every component C of $G - S - T$.*

Proof. Note that we have $\theta_6(S,T) \leq -2$ by Lemma 3.1.1 (ii). Suppose that there exists a component C of $G-S-T$ with $|V(C)| \leq 2$, and take $v \in V(C)$. If $|E(v,T)| \leq 5$, then $\sum_{y \in T \cup \{v\}}(\deg_{G-S}(y) - 6) = \sum_{y \in T}(\deg_{G-S}(y) - 6) + \deg_{G-S}(v) - 6 \leq \sum_{y \in T}(\deg_{G-S}(y) - 6) + (|E(v,T)|+1) - 6 \leq \sum_{y \in T}(\deg_{G-S}(y) - 6)$ and $h_6(S, T \cup \{v\}) \geq h_6(S,T) - 1$, and hence $\theta_6(S, T \cup \{v\}) \leq \theta_6(S,T) + 1 \leq -1$. If $|E(v,T)| \geq 6$, then $\sum_{y \in T}(\deg_{G-(S \cup \{v\})}(y) - 6) = \sum_{y \in T}(\deg_{G-S}(y) - 6) - |E(v,T)| \leq \sum_{y \in T}(\deg_{G-S}(y) - 6) - 6$ and $h_6(S \cup \{v\}, T) \geq h_6(S,T) - 1$, and hence $\theta_6(S \cup \{v\}, T) \leq \theta_6(S,T) + 1 \leq -1$. In either case, we get a contradiction to the maximality of $|S \cup T|$. $\square$

The purpose of the rest of this section is to prove inequalities (Lemmas 3.1.5 and 3.1.7), which play an important role in the proof of Theorem 1.2.1. Throughout the rest of this section, we let f, f' and g' denote the functions defined by

$$f(t,d;\alpha,\beta) = \frac{6(2t-d+1)}{t-1} + \left(d - \frac{t-7}{t-1}\alpha - \frac{\beta}{3} - 6\right)(d - \alpha - \beta + 1),$$

$$f'(d;\alpha,\beta) = 24 - 3d + \left(d + 2\alpha - \frac{\beta}{3} - 6\right)(d - \alpha - \beta + 1), \text{and}$$

$$g'(d;\alpha,\beta,\gamma) = 24 - 3d + \left(d + 2\alpha - \frac{\beta}{3} - \frac{\gamma}{5} - 6\right)(d - \alpha - \beta - \gamma + 1)$$

(see Sections 3.2 and 3.3 for the role of the functions f, f' and g' in the proof of Theorem 1.2.1).

Lemma 3.1.3 *Let t, d and α be integers with $t \geq 3$ and $0 \leq \alpha \leq d < \frac{t-7}{t-1}\alpha + 6$. Then the followings hold.*

(i) *$f(t,d;\alpha,0) \geq 0$ unless $t=3, d=4$ and $\alpha=0$.*

(ii) *If $t=3, d=4$ and $\alpha=0$, then $f(t,d;\alpha,0)=-1$.*

Proof. From the assumption that $0 \leq \alpha \leq d < (t-7)\alpha/(t-1)+6$, it follows that $d-6<\alpha \leq d$, i.e.,

$$d-5 \leq \alpha \leq d. \tag{3.1.1}$$

Since (3.1.1) implies $\alpha-d+6 \geq 1$ and $d-\alpha+1 \geq 1$ and since $(\alpha-d+6)+(d-\alpha+1)=7$, we get

$$(\alpha-d+6)(d-\alpha+1) \leq 12. \tag{3.1.2}$$

If $d \leq 3$, then $6(2t-d+1)/(t-1) \geq 12$, and hence it follows from (3.1.2) that

$$\begin{aligned} f(t,d;\alpha,0) &= \frac{6(2t-d+1)}{t-1} - \left(\frac{t-7}{t-1}\alpha - d + 6\right)(d-\alpha+1) \\ &\geq 12-(\alpha-d+6)(d-\alpha+1) \geq 0. \end{aligned}$$

Thus we may assume $d \geq 4$. If $\alpha=0$, then it follows from (3.1.1) that $d=4$ or 5. If $\alpha=0$ and $d=4$, then $f(t,4;0,0)=2-6/(t-1)$. Hence we see that if $t=3$, then $f(3,4;0,0)=-1$; if $t \geq 4$, then $f(t,4;0,0) \geq 0$. We also see that if $\alpha=0$ and $d=5$, then $f(t,5;0,0)=6-12/(t-1) \geq 0$.

Thus we may assume $\alpha \geq 1$. Then it follows from (3.1.1) that $\alpha(d-\alpha+1) \geq d$. Hence

$$\begin{aligned} \left(\frac{t-7}{t-1}\alpha - d + 6\right)(d-\alpha+1) &= (\alpha-d+6)(d-\alpha+1) - \frac{6}{t-1}\alpha(d-\alpha+1) \\ &\leq 12 - \frac{6d}{t-1} \end{aligned}$$

by (3.1.2). Consequently

$$\begin{aligned} f(t,d;\alpha,0) &= 12 - \frac{6(d-3)}{t-1} - \left(\frac{t-7}{t-1}\alpha - d + 6\right)(d-\alpha+1) \\ &\geq \frac{18}{t-1} > 0, \end{aligned}$$

as desired. □

Lemma 3.1.4 *Let t, d and β be integers with $t \geq 4$ and $0 \leq \beta \leq d < \frac{\beta}{3} + 6$. Then $f(t, d; 0, \beta) \geq 0$.*

Proof. From $0 \leq \beta \leq d < \beta/3 + 6$, we get $\beta < 9$, and hence

$$0 \leq \beta \leq d \leq 8. \tag{3.1.3}$$

In the case where $\beta \geq 1$, $f(t, d; 0, \beta) \geq f(4, d; 0, \beta) \geq f(4, 2\beta/3 + 7/2; 0, \beta) = -\beta^2/9 + \beta - 1/4 > 0$ by (3.1.3). In the case where $\beta = 0$, since d is an integer, $f(t, d; 0, 0) \geq f(4, d; 0, 0) \geq f(4, 4; 0, 0) = 0$. Hence in either case, we obtain $f(t, d; 0, \beta) \geq 0$. □

Lemma 3.1.5 *Let t, d, α and β be nonnegative integers with $t \geq 4$ and $\alpha + \beta \leq d < \frac{t-7}{t-1}\alpha + \frac{\beta}{3} + 6$. Then $f(t, d; \alpha, \beta) \geq 0$.*

Proof. First assume $t \geq 10$. Then $(t - 7)/(t - 1) \geq 1/3$, which implies $d < (t - 7)(\alpha + \beta)/(t - 1) + 6$ and $f(t, d; \alpha, \beta) \geq f(t, d; \alpha + \beta, 0)$. Hence $f(t, d; \alpha, \beta) \geq 0$ by Lemma 3.1.3.

Next assume $4 \leq t \leq 9$. Then $1/3 > (t - 7)/(t - 1)$, which implies $f(t, d; \alpha, \beta) > f(t, d; 0, \alpha + \beta) \geq 0$. Hence $f(t, d; \alpha, \beta) \geq 0$ by Lemma 3.1.4.
□

Lemma 3.1.6 *Let d, α and β be nonnegative integers with $\alpha + \beta \leq d < -2\alpha + \frac{\beta}{3} + 6$. Then $f'(d; \alpha, \beta) \geq 0$ unless $d = 8$, $\alpha = 0$ and $\beta = 7$ or 8.*

Proof. Since $0 \leq \alpha \leq d$, $f'(d; \alpha, 0) \geq \min\{f'(d; 0, 0), f'(d; d, 0)\} = \min\{d^2 - 8d + 18, 18\} > 0$. Hence

$$f'(d; \alpha, 0) \geq 0.$$

Since $0 \leq \beta \leq d < \beta/3 + 6$, we get $\beta < 9$, and hence $0 \leq \beta \leq d \leq 8$. If $d \leq 4$, then $24 - 3d \geq 12 \geq (\beta/3 - d + 6)(d - \beta + 1)$; if $5 \leq d \leq 7$, then $f'(d; 0, \beta) \geq f'(d; 0, 2d - 17/2) = -d^2/3 + 10d/3 - 73/12 > 0$. Now we may assume $d = 8$. If $\beta \leq 6$, then $f'(8; 0, \beta) = \beta^2/3 - 5\beta + 18 \geq 0$; if $\beta = 7$ or 8,

$f'(8; 0, \beta) = -2/3$. Consequently

$$f'(d; 0, \beta) \geq 0 \text{ unless } d = 8 \text{ and } \beta = 7 \text{ or } 8, \tag{3.1.4}$$

$$\text{If } d = 8 \text{ and } \beta = 7 \text{ or } 8, \text{then } f'(d; 0, \beta) = -2/3. \tag{3.1.5}$$

If $\alpha = 0$, the desired conclusion immediately follows from (3.1.4). Thus we may assume $\alpha \geq 1$. Then $f'(d; \alpha, \beta) = f'(d; 0, \alpha + \beta) + 7\alpha(d - \alpha - \beta + 1)/3$. Hence $f'(d; \alpha, \beta) \geq -2/3 + 7/3 > 0$ by (3.1.4) and (3.1.5). □

Lemma 3.1.7 *Let d, α, β and γ be nonnegative integers with $\alpha + \beta + \gamma \leq d < -2\alpha + \frac{\beta}{3} + \frac{\gamma}{5} + 6$. Then $g'(d; \alpha, \beta, \gamma) \geq 0$ unless $d = 8$, $\alpha = 0$ and $\beta + \gamma = 7$ or 8.*

Proof. Since $g'(d; \alpha, \beta, \gamma) \geq f'(d; \alpha, \beta + \gamma)$, Lemma 3.1.6 follows the desired conclusion immediately. □

3.2 Proof of Theorem 1.2.1 (i)

Let t, G be as in Theorem 1.2.1 (i); thus $t \geq 4$, and G is a 2-connected $K_{1,t}$-free graph with $\delta(G) \geq 2t + 1$.

By way of contradiction, suppose that G does not have a 6-factor. Then by Lemma 3.1.1, there exist $S, T \subseteq V(G)$ such that $S \cap T = \emptyset$ and $\theta_6(S, T) < 0$. If $T = \emptyset$, then $h_6(S, T) = 0$, and hence $\theta_6(S, T) = 6|S| \geq 0$, a contradiction. Thus $T \neq \emptyset$. Suppose that $|S \cup T| \leq 1$. Then $S = \emptyset$ and $|T| = 1$. Since $\delta(G) \geq 2t+1 \geq 9$, we have $\sum_{y \in T}(\deg_{G-S}(y) - 6) \geq 3$. Since G is 2-connected and $|S \cup T| \leq 1$, $G - S - T$ is connected, which implies $h_6(S, T) \leq 1$. Hence $\theta_6(S, T) \geq 3 - 1 > 0$, a contradiction. Thus $|S \cup T| \geq 2$.

Now we may assume that we have chosen S and T so that $|S \cup T|$ is as large as possible. Then by Lemma 3.1.2, $|V(C)| \geq 3$ for every component C of $G - S - T$. We call a component C of $G - S - T$ an odd component or an even component according as $|E(T, C)|$ is odd or even. We proceed to estimate the cardinarity of S from below by using the assumption that G is 2-connected and $K_{1,t}$-free (Claim 3.2.4).

Let $C_1, ..., C_a, ..., C_{a+b}, ..., C_k$ be the components of $G - S - T$ such that $|E(T, C_i)| = 1$ for each $i = 1, ..., a$, $|E(T, C_j)| \geq 3$ is odd for each $a < j \leq a+b$ and $|E(T, C_k)|$ is even for each $k > a + b$. Then $h_6(S, T) = a + b$. For each $y \in T$, set $\alpha(y) = \sum_{1 \leq i \leq a} |E(y, C_i)|$ and $\beta(y) = \sum_{a+1 \leq i \leq k} |E(y, C_i)|$. Then

$$\alpha(y) + \beta(y) = |E(y, V(G) - S - T)|; \qquad (3.2.1)$$

in particular,

$$\alpha(y) + \beta(y) \leq \deg_{G-S}(y). \qquad (3.2.2)$$

Claim 3.2.1 (i) $a = \sum_{y \in T} \alpha(y)$,

(ii) *For each i with $a + 1 \leq i \leq a + b$, $\sum_{y \in T} |E(y, C_i)| \geq 3$,*

(iii) $b \leq (\sum_{y \in T} \beta(y))/3$.

Proof. We have $a = \sum_{1 \leq i \leq a} |E(T, C_i)| = \sum_{1 \leq i \leq a} \sum_{y \in T} |E(y, C_i)| = \sum_{y \in T} \alpha(y)$, which proves (i). Let $a + 1 \leq i \leq a + b$. Then $|E(T, C_i)| \neq 1$ and $|E(T, C_i)|$ is odd. Hence $\sum_{y \in T} |E(y, C_i)| = |E(T, C_i)| \geq 3$. Thus (ii) is proved. By (ii), $b \leq (\sum_{a+1 \leq i \leq a+b} \sum_{y \in T} |E(y, C_i)|)/3 \leq (\sum_{a+1 \leq i \leq k} \sum_{y \in T} |E(y, C_i)|)/3 = (\sum_{y \in T} \beta(y))/3$, which proves (iii). $\square$

Recall that $T \neq \emptyset$. We choose vertices $z_1, ..., z_m$ of T and define subsets $N_1, ..., N_m$ of T inductively by the following procedure. First let $z_1 \in T$ be a vertex such that $\deg_{G-S}(z_1) - \frac{(t-7)\alpha(z_1)}{t-1} - \frac{\beta(z_1)}{3} \leq \deg_{G-S}(y) - \frac{(t-7)\alpha(y)}{t-1} - \frac{\beta(y)}{3}$ for all $y \in T$, and set $N_1 = (N(z_1) \cap T) \cup \{z_1\}$. Now let $j \geq 2$, and assume that $z_1, ..., z_{j-1}$ and $N_1, ..., N_{j-1}$ have been defined. If $T - \left(\bigcup_{1 \leq i \leq j-1} N_i\right) \neq \emptyset$, then let $z_j \in T - \left(\bigcup_{1 \leq i \leq j-1} N_i\right)$ be a vertex such that $\deg_{G-S}(z_j) - \frac{(t-7)\alpha(z_j)}{t-1} - \frac{\beta(z_j)}{3} \leq \deg_{G-S}(y) - \frac{(t-7)\alpha(y)}{t-1} - \frac{\beta(y)}{3}$ for all $y \in T - \left(\bigcup_{1 \leq i \leq j-1} N_i\right)$, and set $N_j = \left(N(z_j) \cap \left(T - \left(\bigcup_{1 \leq i \leq j-1} N_i\right)\right)\right) \cup \{z_j\}$; if $T - \left(\bigcup_{1 \leq i \leq j-1} N_i\right) = \emptyset$, then let $m = j - 1$ and terminate the procedure.

Claim 3.2.2 (i) $\{z_1, ..., z_m\}$ *is independent.*

(ii) $T = \bigcup_{1 \le j \le m} N_j$ *(disjoint union).*

(iii) *For each* $1 \le j \le m$, $\deg_{G-S}(z_j) - \frac{(t-7)\alpha(z_j)}{t-1} - \frac{\beta(z_j)}{3} \le \deg_{G-S}(y) - \frac{(t-7)\alpha(y)}{t-1} - \frac{\beta(y)}{3}$ *for all* $y \in N_j$.

(iv) *For each* $1 \le j \le m$, $|N_j| \le \deg_{G-S}(z_j) - \alpha(z_j) - \beta(z_j) + 1$.

Proof. Statements (i) through (iii) follow from the definition of z_j and N_j. Let $1 \le j \le m$. Then $|N_j| \le |N(z_j) \cap T| + 1 = \deg_{G-S}(z_j) - |E(z_j, V(G) - S - T)| + 1 = \deg_{G-S}(z_j) - \alpha(z_j) - \beta(z_j) + 1$ by (3.2.1). □

Let $1 \le i \le a$. Since $|V(C_i)| \ge 3$ and G is 2-connected, there exists an edge joining S and $V(C_i) - N(T)$. Let $x_i u_i$ be such an edge ($x_i \in S$, $u_i \in V(C_i) - N(T)$). Set $L = \{u_i | 1 \le i \le a\}$ and thus $|L| = a$. For each $x \in S$, let $L(x) = \{u_i | 1 \le i \le a, x_i = x\}$. Clearly

$$L(x) \subseteq N(x). \tag{3.2.3}$$

Also

$$L = \bigcup_{x \in S} L(x) \text{ (disjoint union),}$$

and hence

$$\sum_{x \in S} |L(x)| = a. \tag{3.2.4}$$

Claim 3.2.3 (i) *L is independent.*

(ii) $\{z_1, ..., z_m\} \cup L$ *is independent.*

(iii) *For each* $x \in S$, $(N(x) \cap \{z_1, ..., z_m\}) \cup L(x)$ *is independent.*

Proof. For every i, $i' \in \{1, ..., a\}$ with $i \ne i'$, u_i and $u_{i'}$ belong to distinct components of $G - S - T$. Hence (i) holds. Further for every $1 \le i \le a$, $u_i \notin N(T)$. Consequently (ii) follows from Claim 3.2.2 (i). Also (iii) follows from (ii). □

Claim 3.2.4 $|S| \geq \dfrac{1}{t-1}\Bigg(\sum_{1\leq j\leq m}\Big(|E(z_j,S)| + \sum_{y\in N_j}\alpha(y)\Big)\Bigg)$.

Proof. Since G is $K_{1,t}$-free, it follows from (3.2.3) and Claim 3.2.3 (iii) that $|E(x,\{z_1,...,z_m\})| + |L(x)| \leq t-1$ for every $x \in S$. Note that by (3.2.4), Claims 3.2.1 (i) and 3.2.2 (ii), $\sum_{x\in S}|E(x,\{z_1,...,z_m\})| = |E(S,\{z_1,...,z_m\})| = \sum_{1\leq j\leq m}|E(z_j,S)|$, $\sum_{x\in S}|L(x)| = a = \sum_{y\in T}\alpha(y) = \sum_{1\leq j\leq m}\sum_{y\in N_j}\alpha(y)$. Consequently we obtain $(t-1)|S| \geq \sum_{x\in S}\Big(|E(x,\{z_1,...,z_m\})| + |L(x)|\Big) = \sum_{1\leq j\leq m}\Big(|E(z_j,S)| + \sum_{y\in N_j}\alpha(y)\Big)$, as desired. □

We now estimate $\theta_6(S,T)$ from below by using the assumption that $\delta(G) \geq 2t+1$. For each $1 \leq j \leq m$, set

$$\begin{aligned} p_j &= \frac{6}{t-1}|E(z_j,S)| + \left(\deg_{G-S}(z_j) - \frac{t-7}{t-1}\alpha(z_j) - \frac{\beta(z_j)}{3} - 6\right)|N_j|, \\ r_j &= \frac{6(2t - \deg_{G-S}(z_j)+1)}{t-1} \\ &\quad + \left(\deg_{G-S}(z_j) - \frac{t-7}{t-1}\alpha(z_j) - \frac{\beta(z_j)}{3} - 6\right) \\ &\qquad\qquad (\deg_{G-S}(z_j) - \alpha(z_j) - \beta(z_j) + 1). \end{aligned}$$

Claim 3.2.5 $\theta_6(S,T) \geq \sum_{1\leq j\leq m} p_j$.

Proof. Note that $h_6(S,T) = a + b \leq \sum_{1\leq j\leq m}\sum_{y\in N_j}(\alpha(y) + \beta(y)/3)$ and $\sum_{y\in T}(\deg_{G-S}(y) - 6) = \sum_{1\leq j\leq m}\sum_{y\in N_j}(\deg_{G-S}(y) - 6)$ by Claims 3.2.1(i), (iii) and 3.2.2 (ii). Therefore it follows from Claims 3.2.4 and 3.2.2 (iii) that

$$\begin{aligned} \theta_6(S,T) &\geq \sum_{1\leq j\leq m}\Bigg(\frac{6}{t-1}|E(z_j,S)| \\ &\qquad + \sum_{y\in N_j}\Big(\deg_{G-S}(y) - \Big(1 - \frac{6}{t-1}\Big)\alpha(y) - \frac{\beta(y)}{3} - 6\Big)\Bigg) \\ &\geq \sum_{1\leq j\leq m}\Bigg(\frac{6}{t-1}|E(z_j,S)| \\ &\qquad + \Big(\deg_{G-S}(z_j) - \frac{t-7}{t-1}\alpha(z_j) - \frac{\beta(z_j)}{3} - 6\Big)|N_j|\Bigg), \end{aligned}$$

as desired. □

Claim 3.2.6 *Let $1 \leq j \leq m$, and suppose that* $\deg_{G-S}(z_j) - \frac{(t-7)\alpha(z_j)}{t-1} - \frac{\beta(z_j)}{3} - 6 < 0$. *Then* $p_j \geq r_j$.

Proof. Since $|N_j| \leq \deg_{G-S}(z_j) - \alpha(z_j) - \beta(z_j) + 1$ by Claim 3.2.2 (iv) and $|E(z_j, S)| \geq 2t + 1 - \deg_{G-S}(z_j)$ by the assumption that $\delta(G) \geq 2t + 1$, the desired inequality follows immediately. □

Claim 3.2.7 *For each $1 \leq j \leq m$, $p_j \geq 0$.*

Proof. We may assume $\deg_{G-S}(z_j) - \frac{(t-7)\alpha(z_j)}{t-1} - \frac{\beta(z_j)}{3} - 6 < 0$. Then $p_j \geq r_j$ by Claim 3.2.6. In view of (3.2.2), it follows from Lemma 3.1.5 that $r_j \geq 0$. Hence $p_j \geq 0$. □

Now it follows from Claims 3.2.5 and 3.2.7 that $\theta_6(S,T) \geq 0$, which contradicts the assumption that $\theta_6(S,T) < 0$. Thus Theorem 1.2.1 (i) is proved.

3.3 Proof of Theorem 1.2.1 (ii)

Now let t and G be as in Theorem 1.2.1 (ii); thus $t = 3$ and G is a 2-connected $K_{1,3}$-free graph with $\delta(G) \geq 8$. Here we make a similar definition as in Section 3.2.

Let $C_1, ..., C_k$ be the components of $G-S-T$. We may assume that there exist a and b with $0 \leq a+b \leq k$ such that $|E(T, C_i)| = 1$ for each $1 \leq i \leq a$, $|E(T, C_i)| = 3$ for each $a+1 \leq i \leq a+b$, and $|E(T, C_i)| = 2$ or $|E(T, C_i)| \geq 4$ for each $a+b+1 \leq i \leq k$. Then the components $C_1, ..., C_{a+b}$ are odd components. We may further assume that there exists c with $0 \leq c \leq k-a-b$ such that C_i is an odd component for each $a+b+1 \leq i \leq a+b+c$ and C_i is an even component for each $a+b+c+1 \leq i \leq k$. Then $h_6(S,T) = a+b+c$. For each $y \in T$, set $\alpha(y) = \sum_{1 \leq i \leq a} |E(y, C_i)|$, $\beta(y) = \sum_{a+1 \leq i \leq a+b} |E(y, C_i)|$ and $\gamma(y) = \sum_{a+b+1 \leq i \leq k} |E(y, C_i)|$. Then $\alpha(y)+\beta(y)+\gamma(y) = |E(y, V(G)-S-T)|$; in particular,

$$\alpha(y) + \beta(y) + \gamma(y) \leq \deg_{G-S}(y). \tag{3.3.1}$$

Moreover we obtain the following similar claim to Claim 3.2.1.

Claim 3.3.1 (i) $a = \sum_{y \in T} \alpha(y)$.

(ii) $b = (\sum_{y \in T} \beta(y))/3$.

(iii) $c \leq (\sum_{y \in T} \gamma(y))/5$. □

Recall that $T \neq \emptyset$. We choose vertices $z_1, ..., z_m$ of T and define subsets $N_1, ..., N_m$ of T inductively by the following procedure in the similar way to the paragraph preceding the statement of Claim 3.2.2. First let $z_1 \in T$ be a vertex such that $\deg_{G-S}(z_1)+2\alpha(z_1)-\frac{\beta(z_1)}{3}-\frac{\gamma(z_1)}{5} \leq \deg_{G-S}(y)+2\alpha(y)-\frac{\beta(y)}{3}-\frac{\gamma(y)}{5}$ for all $y \in T$, and set $N_1 = (N(z_1) \cap T) \cup \{z_1\}$. Now let $j \geq 2$, and assume that $z_1, ..., z_{j-1}$ and $N_1, ..., N_{j-1}$ have been defined. If $T - \bigl(\bigcup_{1 \leq i \leq j-1} N_i\bigr) \neq \emptyset$, then let $z_j \in T - \bigl(\bigcup_{1 \leq i \leq j-1} N_i\bigr)$ be a vertex such that $\deg_{G-S}(z_j) + 2\alpha(z_j) - \frac{\beta(z_j)}{3} - \frac{\gamma(z_j)}{5} \leq \deg_{G-S}(y)+2\alpha(y)-\frac{\beta(y)}{3}-\frac{\gamma(y)}{5}$ for all $y \in T - \bigl(\bigcup_{1 \leq i \leq j-1} N_i\bigr)$, and set $N_j = \Bigl(N(z_j) \cap \Bigl(T - \bigl(\bigcup_{1 \leq i \leq j-1} N_i\bigr)\Bigr)\Bigr) \cup \{z_j\}$; if $T - \bigl(\bigcup_{1 \leq i \leq j-1} N_i\bigr) = \emptyset$, then let $m = j-1$ and terminate the procedure. Then, arguing as in the proof of Claims 3.2.2 and 3.2.4, we obtain the following two claims.

Claim 3.3.2 (i) *For each* $1 \leq j \leq m, \deg_{G-S}(z_j) + 2\alpha(z_j) - \frac{\beta(z_j)}{3} - \frac{\gamma(z_j)}{5} \leq \deg_{G-S}(y) + 2\alpha(y) - \frac{\beta(y)}{3} - \frac{\gamma(y)}{5}$ *for all* $y \in N_j$.

(ii) *For each* $1 \leq j \leq m, |N_j| \leq \deg_{G-S}(z_j) - \alpha(z_j) - \beta(z_j) - \gamma(z_j) + 1$. □

Claim 3.3.3 $|S| \geq \dfrac{1}{t-1}\Bigl(\sum_{1 \leq j \leq m}\Bigl(|E(z_j, S)| + \sum_{y \in N_j} \alpha(y)\Bigr)\Bigr)$. □

We now estimate $\theta_6(S, T)$ from below by using the assumption that $\delta(G) \geq 8$. For each $1 \leq j \leq m$, set

$$\begin{aligned} p'_j &= 3|E(z_j, S)| + \Bigl(\deg_{G-S}(z_j) + 2\alpha(z_j) - \frac{\beta(z_j)}{3} - \frac{\gamma(z_j)}{5} - 6\Bigr)|N_j|, \\ r'_j &= 24 - 3\deg_{G-S}(z_j) \\ &\quad + \Bigl(\deg_{G-S}(z_j) + 2\alpha(z_j) - \frac{\beta(z_j)}{3} - \frac{\gamma(z_j)}{5} - 6\Bigr) \\ &\qquad\qquad (\deg_{G-S}(z_j) - \alpha(z_j) - \beta(z_j) - \gamma(z_j) + 1). \end{aligned}$$

Claim 3.3.4 $\theta_6(S,T) \geq \sum_{1\leq j\leq m} p'_j$.

Proof. Note that $h_6(S,T) = a+b+c \leq \sum_{1\leq j\leq m}\sum_{y\in N_j}(\alpha(y)+\beta(y)/3+\gamma(y)/5)$ by Claim 3.3.1. Then, by the similar argument in the proof of Claim3.2.5, the desired inequality follows from Claim3.3.2 (ii) and Claim 3.3.3. □

Claim 3.3.5 *Let* $1 \leq j \leq m$, *and suppose that* $\deg_{G-S}(z_j)+2\alpha(z_j)-\frac{\beta(z_j)}{3}-\frac{\gamma(z_j)}{5}-6<0$. *Then* $p'_j \geq r'_j$.

Proof. Since $|N_j| \leq \deg_{G-S}(z_j)-\alpha(z_j)-\beta(z_j)-\gamma(z_j)+1$ by Claim 3.3.2 (ii) and $|E(z_j,S)| \geq 8-\deg_{G-S}(z_j)$ by the assumption that $\delta(G)\geq 8$, the desired inequality follows immediately. □

Claim 3.3.6 *For each* $1 \leq j \leq m$, $p'_j \geq 0$.

Proof. We may assume that $\deg_{G-S}(z_j)+2\alpha(z_j)-\beta(z_j)/3-\gamma(z_j)/5-6<0$. Then $p'_j \geq r'_j$ by Claim 3.3.5. Thus, we will show that $r'_j \geq 0$. In view of (3.3.1), it follows from Lemma 3.1.7 that $r'_j \geq 0$ unless $\deg_{G-S}(z_j)=8$, and $\beta(z_j)+\gamma(z_j)=7$ or $\beta(z_j)+\gamma(z_j)=8$. Thus we may assume that $\deg_{G-S}(z_j)=8$, $\alpha(z_j)=0$ and $\beta(z_j)+\gamma(z_j)=7$ or 8. Note that if $\deg_{G-S}(z_j)=8$ and $\alpha(z_j)=0$, then $r'_j=(2-\beta(z_j)/3-\gamma(z_j)/5)(9-\beta(z_j)-\gamma(z_j))$.

Since G is $K_{1,3}$-free, the number of components C_i $(a+1\leq i\leq k)$ with $E(z_j,C_i)\neq\emptyset$ is at most 2. Thus from the definition of β, we have $\beta(z_j)\leq 6$, and hence $\gamma(z_j)\neq 0$. Once more taking into consideration the assumption that G is $K_{1,3}$-free, we have $\beta(z_j)\leq 3$. Therefore it suffices to examine separately the following eight cases;

(a) $\beta(z_j) = 0$ and $\gamma(z_j) = 7$,
(b) $\beta(z_j) = 0$ and $\gamma(z_j) = 8$,
(c) $\beta(z_j) = 1$ and $\gamma(z_j) = 6$,
(d) $\beta(z_j) = 1$ and $\gamma(z_j) = 7$,
(e) $\beta(z_j) = 2$ and $\gamma(z_j) = 5$,
(f) $\beta(z_j) = 2$ and $\gamma(z_j) = 6$,
(g) $\beta(z_j) = 3$ and $\gamma(z_j) = 4$,
(h) $\beta(z_j) = 3$ and $\gamma(z_j) = 5$.

In either case, it is immediate that $r'_j \geq 0$. Consequently we obtain the desired inequality. $\square$

By Claims 3.3.4 and 3.3.6, $\theta_6(S,T) \geq 0$. This contradicts the assumption that $\theta_6(S,T) < 0$. Therefore Theorem 1.2.1 (ii) is proved.

3.4 Sharpness

In this section, we first construct examples which show that in Theorem 1.2.1 (i), the lower bound $2t+1$ on $\delta(G)$ is best possible. Let $t \geq 4$ and set $r = 2t-3$. We first define a graph I of order $r + 4(t-1)$ by

$$V(I) = \{v_i | 1 \leq i \leq r\} \cup \{w_j, x_j, y_j, z_j | 1 \leq j \leq t-1\},$$

$$E(I) = \{v_i v_j | 1 \leq i < j \leq \lfloor r/2 \rfloor\} \cup \{v_i v_j | \lfloor r/2 \rfloor + 1 \leq i < j \leq r\}$$
$$\cup \{w_j x_j, w_j y_j, w_j z_j, x_j y_j, x_j z_j, y_j z_j | 1 \leq j \leq t-1\}.$$

Thus I is the union of a complete graph of order $\lfloor r/2 \rfloor$, a complete graph of order $\lceil r/2 \rceil$, and $t-1$ complete graphs of order 4. Set $S_1 = \{v_i | 1 \leq i \leq \lfloor r/2 \rfloor\}$, $S_2 = \{v_i | \lfloor r/2 \rfloor + 1 \leq i \leq r\}$, $T_1 = \{w_j, x_j, y_j, z_j | 1 \leq j \leq \lfloor (t-1)/2 \rfloor\}$, $T_2 = \{w_j, x_j, y_j, z_j | \lfloor (t-1)/2 \rfloor + 1 \leq j \leq t-1\}$. Let H be the graph obtained from I by joining each vertex in S_1 to all vertices in $T_1 \cup T_2$ and joining each vertex in S_2 to all vertices in T_2. Let $n \geq 1$, and let $H_1, ..., H_n$ be disjoint copies of H. For each k ($1 \leq k \leq n$), let $S_{k,1}$, $S_{k,2}$, $T_{k,1}$ and $T_{k,2}$ denote the subsets of $V(H_k)$ which correspond to S_1, S_2, T_1 and T_2, respectively.

Now let G be the graph obtained from the union of $H_1, ..., H_n$ by joining each vertex in $S_{k,2}$ to all vertices in $T_{k+1,1}$ for each $1 \leq k \leq n$ (we take $T_{n+1,1} = T_{1,1}$). Then G is 2-connected and $K_{1,t}$-free, and $\delta(G) = 2t$. However, we easily see that G has no 6-factor (for example, if we apply Lemma 3.1.1 in Section 3.1 with $S = \bigcup_{1 \leq k \leq n}(S_{k,1} \cup S_{k,2})$ and $T = \bigcup_{1 \leq k \leq n}(T_{k,1} \cup T_{k,2})$,then we get $\theta_6(S,T) = -6n$).

Next, we construct examples which show that in Theorem 1.2.1 (ii), the lower bound 8 on $\delta(G)$ is best possible. We first define I of order 13 by

$$V(I) = \{v_1, v_2, v_3\} \cup \{x_1, x_2, x_3, x_4, x_5\} \cup \{y_1, y_2, y_3, y_4, y_5\},$$

$$E(I) = \{v_2v_3\} \cup \{x_ix_j | 1 \leq i < j \leq 5\} \cup \{y_iy_j | 1 \leq i < j \leq 5\}.$$

Thus I is the union of a complete graph of order 1, a complete graph of order 2, and 2 complete graphs of order 5. Set $S_1 = \{v_1\}$, $S_2 = \{v_2, v_3\}$, $T_1 = \{x_1, x_2, x_3, x_4, x_5\}$, $T_2 = \{y_1, y_2, y_3, y_4, y_5\}$. Let H be the graph obtained from I by joining each vertex in S_1 to all vertices in $T_1 \cup T_2$ and joining each vertex in S_2 to all vertices in T_2. Let $n \geq 1$, and let $H_1, ..., H_n$ be disjoint copies of H. For each k ($1 \leq k \leq n$), let $S_{k,1}$, $S_{k,2}$, $T_{k,1}$ and $T_{k,2}$ denote the subsets of $V(H_k)$ which correspond to S_1, S_2, T_1 and T_2, respectively. Now let G be the graph obtained from the union of $H_1, ..., H_n$ by joining each vertex in $S_{k,2}$ to all vertices in $T_{k+1,1}$ for each $1 \leq k \leq n$ (we take $T_{n+1,1} = T_{1,1}$). Then G is 2-connected and $K_{1,3}$-free, and $\delta(G) = 7$. However, arguing as in the former examples, we easily see that G has no 6-factor.

Chapter 4

3-factors in 2-connected star-free graphs

In Section 4.1, we show preliminary results to prove Theorems 1.2.2 and 1.2.3. In Section 4.2, we fix notations for the proof of Theorems 1.2.2 and 1.2.3. In Section 4.3, we prove Theorem 1.2.3. In Section 4.4, we prove Theorem 1.2.2 for $t = 4$. In Section 4.5, we prove Theorem 1.2.3 for $t = 3$. In Section 4.6, we give sharpness examples of Theorems 1.2.2 and 1.2.3.

4.1 Preliminary results

In this section, we state preliminary lemmas, which we use in the proof of Theorems 1.2.2 and 1.2.3.

Let G be a graph. For S, $T \subseteq V(G)$ with $S \cap T = \emptyset$, define $\theta_3(S,T)$ by

$$\theta_3(S,T) = 3|S| + \sum_{y \in T}(\deg_{G-S}(y) - 3) - h_3(S,T),$$

where $h_3(S,T)$ denotes the number of those components C of $G - S - T$ such that $|E(T,C)| + |V(C)|$ is odd. The following lemma is also a special case of Theorem 1.1.4.

Lemma 4.1.1 (i) *The graph G has a 3-factor if and only if $\theta_3(S,T) \geq 0$ for all S, $T \subseteq V(G)$ with $S \cap T = \emptyset$.*

(ii) *If* $|V(G)|$ *is even, then whether* G *has a 3-factor or not,* $\theta_3(S,T)$ *is even for all* S, $T \subseteq V(G)$ *with* $S \cap T = \emptyset$.

The following lemma seems to be well-known, but we include its proof for the convenience of the reader.

Lemma 4.1.2 *Let* S, $T \subseteq V(G)$ *be subsets of* $V(G)$ *with* $S \cap T = \emptyset$ *for which* $\theta_3(S,T)$ *becomes smallest. Then the followings hold:*

(i) *Let* C *be a component of* $G - S - T$ *such that* $|E(T,C)| \leq 1$. *Then* $|V(C)| \geq 2$.

(ii) *Suppose that* S *and* T *are chosen so that* $|T|$ *is as small as possible, subject to the condition that* $\theta_3(S,T)$ *is smallest. Then* $\deg_{G[T]}(y) \leq 1$ *for every* $y \in T$.

Proof. Suppose that there exists a component C of $G - S - T$ such that $|E(T,C)| \leq 1$ and $|V(C)| = 1$. Write $V(C) = \{v\}$, and set $T' = T \cup \{v\}$. Then $\deg_{G-S}(v) \leq 1$ and $h_3(S,T') \geq h_3(S,T) - 1$ by the definition of $h_3(S,T)$, and hence

$$\begin{aligned}\theta_3(S,T') &= 3|S| + \sum_{y\in T'}(\deg_{G-S}(y) - 3) - h_3(S,T') \\ &\leq 3|S| + \sum_{y\in T}(\deg_{G-S}(y) - 3) + \deg_{G-S}(v) - 3 - h_3(S,T) + 1 \\ &\leq \theta_3(S,T) - 1,\end{aligned}$$

which contradicts the minimality of $\theta_3(S,T)$. Thus, (i) is proved.

Now let $y \in T$, and set $T' = T - \{y\}$. Then $h_3(S,T') \geq h_3(S,T) - |E(y, V(G) - S - T)|$, and hence $\theta_3(S,T') \leq \theta_3(S,T) - \deg_{G-S}(y) + 3 + |E(y, V(G) - S - T)| = \theta_3(S,T) - \deg_{G[T]}(y) + 3$. On the other hand, by the minimality of $|T|$ and Lemma 4.1.1 (ii), $\theta_3(S,T') \geq \theta_3(S,T) + 2$. Hence $\theta_3(S,T) + 2 \leq \theta_3(S,T) - \deg_{G[T]}(y) + 3$, which implies (ii). $\square$

4.2 Notation for arguing in this chapter

Let $3 \leq t \leq 7$, and let G be a 2-connected $K_{1,t}$-free graph of even order. In this section, we fix notation for the proof of Theorems 1.2.2 and 1.2.3.

Let $S, T \subseteq V(G)$ with $S \cap T = \emptyset$ for which $\theta_3(S,T)$ becomes the smallest. We choose S, $T \subseteq V(G)$ so that $|T|$ is as small as possible, subject to the condition that $\theta_3(S,T)$ is smallest. If $S \cup T = \emptyset$, then since G is connected and has even order, we get $h_3(S,T) = 0$, and hence $\theta_3(S,T) = 0$. Thus, we may assume that $S \cup T \neq \emptyset$.

Let $C_1, ..., C_k$ be the components of $G - S - T$. We may assume that there exists a with $0 \leq a \leq k$ such that $|E(T, C_i)| = 0$ for each $1 \leq i \leq a$ and $|E(T, C_i)| \geq 1$ for each $a+1 \leq i \leq k$. Note that if $S \neq \emptyset$ and $|T| + k \leq 1$, then $\sum_{y \in T}(3 - \deg_{G-S}(y)) + h_3(S,T) \leq 3$, and hence $\theta_3(S,T) \geq 3|S| - 3 \geq 0$. Thus, we may assume that if $S \neq \emptyset$, then we have $|T| + k \geq 2$.

Assume for the moment that $a \geq 1$, and let $1 \leq i \leq a$. By Lemma 4.1.2 (i), $|V(C_i)| \geq 2$. Recall that we have $S \cup T \neq \emptyset$ by the assumption made in the second paragraph. Since G is connected, it follows that $\emptyset \neq N(C_i) \cap (S \cup T) = N(C_i) \cap S$; in particular, $S \neq \emptyset$. By the assumption made at the end of the third paragraph, this implies $|T| + k \geq 2$, and hence $G - S \neq C_i$. Since G is 2-connected, we now see that there exist two independent edges joining S and $V(C_i)$. Let $x_i u_i$, $x'_i u'_i$ be such two edges (x_i, $x'_i \in S, u_i, u'_i \in V(C_i)$). Set $L = \{u_i, u'_i \mid 1 \leq i \leq a\}$. Then

$$|L| = 2a. \tag{4.2.1}$$

For each $x \in S$, let $L(x) = \{u_i \mid 1 \leq i \leq a, x_i = x\} \cup \{u'_i \mid 1 \leq i \leq a, x'_i = x\}$. Clearly,

$$L(x) \subseteq N(x) \text{ and } L(x) \text{ is independent.} \tag{4.2.2}$$

Also, $L = \bigcup_{x \in S} L(x)$ (disjoint union); and hence

$$\sum_{x \in S} |L(x)| = 2a \tag{4.2.3}$$

by (4.2.1). If $a = 0$, then we let $L(x) = \emptyset$ for each $x \in S$; thus, (4.2.2) and (4.2.3) hold in this case as well.

We now look at components of $G[T]$. Let $H_1, H_2, ..., H_m$ be the components of $G[T]$. Then

$$T = \bigcup_{1 \le \mu \le m} V(H_\mu) \text{ (disjoint union).} \tag{4.2.4}$$

In the remainder of this section, we assign a real number θ_μ to each H_μ, and show that $\theta_3(S, T) \ge \sum_{1 \le \mu \le m} \theta_\mu$. We first prove several claims concerning H_μ. Note that H_μ is a path of order 1 or 2 by Lemma 4.1.2 (ii). For each $1 \le \mu \le m$, set

$$\begin{aligned} I_\mu &= \{i \mid a+1 \le i \le k, E(H_\mu, C_i) \ne \emptyset\}, \\ I'_\mu &= \{i \in I_\mu \mid |E(H_\mu, C_i)| = 1\}, \text{and} \\ q_\mu &= \sum_{y \in V(H_\mu)} \deg_{G-S}(y). \end{aligned}$$

Claim 4.2.1 *Let $1 \le \mu \le m$.*

(i) *If $|H_\mu| = 1$, then $q_\mu \ge 2|I_\mu| - |I'_\mu|$ and $|N(H_\mu) \cap S| \ge \delta(G) - q_\mu$.*

(ii) *If $|H_\mu| = 2$, then $q_\mu \ge 2|I_\mu| - |I'_\mu| + 2$ and $|N(H_\mu) \cap S| \ge \delta(G) - \lfloor q_\mu/2 \rfloor$.*

Proof. This immediately follows from the definition of I_μ, I'_μ and q_μ. □

Set

$$\begin{aligned} J &= \{i \mid a+1 \le i \le k, |V(C_i)| \ge 2, \text{ there exists } \mu \text{ with } 1 \le \mu \le m \\ &\qquad \text{such that } N(C_i) \cap T \subseteq V(H_\mu) \text{ and } |N(H_\mu) \cap V(C_i)| = 1\}, \text{ and} \\ K &= \{i \mid a+1 \le i \le k, i \notin J, \text{ there exists } \mu \text{ with } 1 \le \mu \le m \text{ such that} \\ &\qquad N(C_i) \cap T \subseteq V(H_\mu) \text{ and } (N(C_i) \cap S) - N(H_\mu) \ne \emptyset\}. \end{aligned}$$

Let $j \in J$. Since G is 2-connected, it follows that there exists an edge joining S and $V(C_j) - N(T)$. Let $x_j u_j$ be such an edge ($x_j \in S$, $u_j \in V(C_j) - N(T)$). Let now $i \in K$. By the definition of K, there exists an edge joining $S - N(H_\mu)$

and $V(C_i)$, where H_μ is as in the definition of K. Let $x_i u_i$ be such an edge ($x_i \in S - N(H_\mu)$, $u_i \in V(C_i)$). For each $x \in S$, let

$$J(x) = \{u_j \mid j \in J, x_j = x\} \text{ and } K(x) = \{u_i \mid i \in K, x_i = x\}.$$

Clearly, $J(x) \cup K(x) \subseteq N(x)$. Since u and v belong to distinct components of $G - S - T$ for any u, $v \in L(x) \cup J(x) \cup K(x)$ with $u \neq v$, this together with (4.2.2) implies

$$L(x) \cup J(x) \cup K(x) \subseteq N(x) \text{ and } L(x) \cup J(x) \cup K(x) \text{ is independent.} \quad (4.2.5)$$

Also,

$$J = \bigcup_{x \in S} J(x) \text{ (disjoint union) and } K = \bigcup_{x \in S} K(x) \text{ (disjoint union).} \quad (4.2.6)$$

For each $x \in S$, let $\mathcal{N}(x) = \{H_\mu \mid 1 \leq \mu \leq m, x \in N(H_\mu)\}$.

Claim 4.2.2 (i) *If $\mu_1 \neq \mu_2$, then $E(H_{\mu_1}, H_{\mu_2}) = \emptyset$. In particular, for each $x \in S$, we have $E(H_{\mu_1}, H_{\mu_2}) = \emptyset$ for any H_{μ_1}, $H_{\mu_2} \in \mathcal{N}(x)$ with $\mu_1 \neq \mu_2$.*

(ii) *Let $x \in S$. Then $E(u, H_\mu) = \emptyset$ for any $u \in L(x) \cup J(x) \cup K(x)$ and for any $H_\mu \in \mathcal{N}(x)$.*

Proof. Statement (i) follows from the definition of H_μ. To prove (ii), suppose that there exists an edge uy such that $u \in L(x) \cup J(x) \cup K(x)$ and $y \in V(H_\mu)$ for some $H_\mu \in \mathcal{N}(x)$. Since $(L(x) \cup J(x)) \cap N(T) = \emptyset$ by the definition of $L(x)$ and $J(x)$, it follows that $u \in K(x)$. Then by the definition of $K(x)$, $x \notin N(H_\mu)$, which contradicts the assumption that $H_\mu \in \mathcal{N}(x)$. Thus, (ii) also holds. □

Claim 4.2.3 $(t-1)|S| \geq \sum_{1 \leq \mu \leq m} |N(H_\mu) \cap S| + 2a + |J \cup K|$.

Proof. Since G is $K_{1,t}$-free, it follows from (4.2.5) and Claim 4.2.2 (i), (ii) that $|\mathcal{N}(x)| + |L(x)| + |J(x)| + |K(x)| \leq t - 1$ for every $x \in S$. Since

$\sum_{x\in S}|\mathcal{N}(x)| = \sum_{1\le\mu\le m}|N(H_\mu)\cap S|$, this together with (4.2.3) and (4.2.6) implies

$$\begin{aligned}(t-1)|S| &\ge \sum_{x\in S}(|\mathcal{N}(x)|+|L(x)|+|J(x)|+|K(x)|)\\ &= \sum_{1\le\mu\le m}|N(H_\mu)\cap S|+\sum_{x\in S}|L(x)|+\sum_{x\in S}|J(x)|+\sum_{x\in S}|K(x)|\\ &= \sum_{1\le\mu\le m}|N(H_\mu)\cap S|+2a+|J\cup K|,\end{aligned}$$

as desired. □

Claim 4.2.4 *If $T=\emptyset$, then $\theta_3(S,T)\ge 0$.*

Proof. By Claim 4.2.3, $|S|\ge 2a/(t-1)$. Since $T=\emptyset$, we have $a=k$, which implies $h_3(S,T)\le k=a$. Hence $\theta_3(S,T)\ge 6a/(t-1)-a\ge 0$. □

In view of Claim 4.2.4, we may assume that $T\ne\emptyset$. For each μ ($1\le\mu\le m$) and for each i ($a+1\le i\le k$), we set

$$w(H_\mu,C_i)=\begin{cases}0 & (\text{if } N(C_i)\cap V(H_\mu)=\emptyset),\\ 1/2 & (\text{if } N(C_i)\cap V(H_\mu)\ne\emptyset \text{ and } N(C_i)\cap T\not\subseteq V(H_\mu)),\\ 1 & (\text{if } N(C_i)\cap V(H_\mu)\ne\emptyset \text{ and } N(C_i)\cap T\subseteq V(H_\mu)).\end{cases}$$

Then for each i ($a+1\le i\le k$), we have

$$\sum_{1\le\mu\le m}w(H_\mu,C_i)\ge 1, \tag{4.2.7}$$

and for each μ ($1\le\mu\le m$), we have

$$\sum_{i\in I_\mu}w(H_\mu,C_i)\le |I_\mu|. \tag{4.2.8}$$

Claim 4.2.5 *Let $1\le\mu\le m$. Then* $\displaystyle\sum_{i\in I_\mu}w(H_\mu,C_i)-\frac{3}{t-1}|I_\mu\cap J|\le |I_\mu|-\frac{|I'_\mu|}{2}$.

Proof. Let $i \in I'_\mu$. If $N(C_i) \cap T \not\subseteq V(H_\mu)$, then $w(H_\mu, C_i) = 1/2$. If $N(C_i) \cap T \subseteq V(H_\mu)$, then $|V(C_i)| \geq 2$ by Lemma 4.1.2 (i), and hence $i \in J$ by the definition of J, which implies $w(H_\mu, C_i) - 3|\{i\} \cap J|/(t-1) \leq 1 - 3/(t-1) \leq 1/2$ because of $t \leq 7$. Thus, if $i \in I'_\mu$, then $w(H_\mu, C_i) - 3|\{i\} \cap J|/(t-1) \leq 1/2$. Hence

$$\begin{aligned}\sum_{i \in I_\mu} w(H_\mu, C_i) - \frac{3}{t-1}|I_\mu \cap J| &\leq \sum_{i \in I_\mu - I'_\mu} w(H_\mu, C_i) + \sum_{i \in I'_\mu} w(H_\mu, C_i) \\ &\quad - \frac{3}{t-1}|I'_\mu \cap J| \\ &\leq |I_\mu - I'_\mu| + \frac{|I'_\mu|}{2} \\ &= |I_\mu| - \frac{|I'_\mu|}{2}.\end{aligned}$$

□

We now estimate $\theta_3(S,T)$ from below. For each $1 \leq \mu \leq m$, set

$$\begin{aligned}\theta_\mu &= \frac{3}{t-1}|N(H_\mu) \cap S| + q_\mu - 3|V(H_\mu)| \\ &\quad + \frac{3}{t-1}|I_\mu \cap (J \cup K)| - \sum_{i \in I_\mu} w(H_\mu, C_i).\end{aligned}$$

Claim 4.2.6 $\theta_3(S,T) \geq \sum_{1 \leq \mu \leq m} \theta_\mu$.

Proof. Note that

$$\begin{aligned}k - a &\leq \sum_{a+1 \leq i \leq k} \sum_{1 \leq \mu \leq m} w(H_\mu, C_i) \\ &= \sum_{1 \leq \mu \leq m} \sum_{a+1 \leq i \leq k} w(H_\mu, C_i) \\ &= \sum_{1 \leq \mu \leq m} \sum_{i \in I_\mu} w(H_\mu, C_i)\end{aligned}$$

by (4.2.7). Hence $h_3(S,T) \leq k \leq a + \sum_{1 \leq \mu \leq m} \sum_{i \in I_\mu} w(H_\mu, C_i)$. By (4.2.4), $\sum_{y \in T}(\deg_{G-S}(y) - 3) = \sum_{1 \leq \mu \leq m}\Big(\sum_{y \in V(H_\mu)} \deg_{G-S}(y) - 3|V(H_\mu)|\Big)$. Therefore, it follows from Claim 4.2.3 that

$$\begin{aligned}\theta_3(S,T) &= 3|S| + \sum_{y\in T}(\deg_{G-S}(y) - 3) - h_3(S,T)\\ &\geq \frac{3}{t-1}\Big(\sum_{1\leq\mu\leq m}|N(H_\mu)\cap S| + 2a + |J\cup K|\Big)\\ &\quad + \sum_{1\leq\mu\leq m}\Big(\sum_{y\in V(H_\mu)}\deg_{G-S}(y) - 3|V(H_\mu)|\Big)\\ &\quad -\Big(a + \sum_{1\leq\mu\leq m}\sum_{i\in I_\mu} w(H_\mu, C_i)\Big)\\ &\geq \sum_{1\leq\mu\leq m}\Big\{\frac{3}{t-1}\Big(|N(H_\mu)\cap S| + |I_\mu\cap(J\cup K)|\Big)\\ &\quad + \sum_{y\in V(H_\mu)}\deg_{G-S}(y) - 3|V(H_\mu)| - \sum_{i\in I_\mu} w(H_\mu, C_i)\Big\},\end{aligned}$$

as desired. □

4.3 Proof of Theorem 1.2.3

Let t and G be as in Theorem 1.2.3; thus, $5 \leq t \leq 7$, and G is a 2-connected $K_{1,t}$-free graph with $\delta(G) \geq t+2$. We continue with the notation of Section 4.2 with $5 \leq t \leq 7$. We here do not make use of K; that is to say, for each $1 \leq \mu \leq m$, set

$$\begin{aligned}\theta'_\mu &= \frac{3}{t-1}|N(H_\mu)\cap S| + q_\mu - 3|V(H_\mu)|\\ &\quad + \frac{3}{t-1}|I_\mu\cap J| - \sum_{i\in I_\mu} w(H_\mu, C_i).\end{aligned}$$

Then $\theta_3(S,T) \geq \sum_{1\leq\mu\leq m}\theta'_\mu$ by Claim 4.2.6. Thus, it suffices to show that $\theta'_\mu \geq 0$ for each $1 \leq \mu \leq m$. In view of Lemma 4.1.2 (ii), we divide the proof into the following two cases:

Case 1. $|V(H_\mu)| = 1$.

Since G is $K_{1,t}$-free, we have $|I_\mu| \le t-1$, and hence $(7-t)(2|I_\mu|-|I'_\mu|) \le 2(7-t)(t-1) \le 16$. Consequently, it follows from Claims 4.2.1 (i) and 4.2.5 that

$$\begin{aligned}
\theta'_\mu &\ge \frac{3}{t-1}(t+2-q_\mu) + q_\mu - 3 - |I_\mu| + \frac{|I'_\mu|}{2} \\
&= \frac{9}{t-1} + \frac{t-4}{t-1}q_\mu - |I_\mu| + \frac{|I'_\mu|}{2} \\
&\ge \frac{9}{t-1} + \frac{t-4}{t-1}(2|I_\mu| - |I'_\mu|) - |I_\mu| + \frac{|I'_\mu|}{2} \\
&= \frac{9}{t-1} - \frac{7-t}{2(t-1)}(2|I_\mu| - |I'_\mu|) > 0.
\end{aligned}$$

Case 2. $|V(H_\mu)| = 2$.

By Claims 4.2.1 (ii) and 4.2.5,

$$\begin{aligned}
\theta'_\mu &\ge \frac{3}{t-1}\left(t+2-\frac{q_\mu}{2}\right) + q_\mu - 6 - |I_\mu| + \frac{|I'_\mu|}{2} \\
&\ge \frac{3(t+2)}{t-1} + \frac{2t-5}{2(t-1)}q_\mu - 6 - |I_\mu| + \frac{|I'_\mu|}{2} \\
&\ge \frac{3(t+2)}{t-1} + \frac{2t-5}{2(t-1)}(2|I_\mu| - |I'_\mu| + 2) - 6 - |I_\mu| + \frac{|I'_\mu|}{2} \\
&= \frac{7-t}{t-1} + \frac{t-4}{2(t-1)}(2|I_\mu| - |I'_\mu|) \ge 0.
\end{aligned}$$

4.4 Proof of Theorem 1.2.2 for $t = 4$

Let G be as in Theorem 1.2.2 with $t = 4$, thus, G is a 2-connected $K_{1,4}$-free graph with $\delta(G) \ge 5$. We continue with the notation of Section 4.2 with $t = 4$; thus,

$$\theta_\mu = |N(H_\mu) \cap S| + q_\mu - 3|V(H_\mu)| + |I_\mu \cap (J \cup K)| - \sum_{i \in I_\mu} w(H_\mu, C_i).$$

In view of Claim 4.2.6, it suffices to show that $\theta_\mu \ge 0$ for each $1 \le \mu \le m$. We divide the proof into the following two cases:

Case 1. $|V(H_\mu)| = 1$.

Write $V(H_\mu) = \{y\}$. Note that $q_\mu = \deg_{G-S}(y)$ and $|N(H_\mu) \cap S| = \deg_G(y) - q_\mu$. Hence

$$\theta_\mu = \deg_G(y) - 3 + |I_\mu \cap (J \cup K)| - \sum_{i \in I_\mu} w(H_\mu, C_i).$$

Since G is $K_{1,4}$-free, $|I_\mu| \leq 3$. Hence if $\deg_G(y) \geq 6$ or $I_\mu \cap (J \cup K) \neq \emptyset$ or $|I_\mu| \leq 2$, then we get $\theta_\mu \geq \deg_G(y) - 3 + |I_\mu \cap (J \cup K)| - |I_\mu| \geq 0$ by (4.2.8). Thus, we may assume that

$$\deg_G(y) = 5, \tag{4.4.1}$$

$$I_\mu \cap (J \cup K) = \emptyset \text{ and} \tag{4.4.2}$$

$$|I_\mu| = 3. \tag{4.4.3}$$

By (4.4.1) and (4.4.2), $\theta_\mu = 2 - \sum_{i \in I_\mu} w(H_\mu, C_i)$.

By (4.4.1), $q_\mu \leq 5$. First assume that $q_\mu \leq 4$. Then by (4.4.3), there exist i_1, $i_2 \in I_\mu$ with $i_1 \neq i_2$ such that $|E(y, C_{i_1})| = |E(y, C_{i_2})| = 1$. Suppose that $w(H_\mu, C_{i_1}) = 1$. Then $N(C_{i_1}) \cap T \subseteq V(H_\mu) = \{y\}$. Hence $|E(T, C_{i_1})| = |E(y, C_{i_1})| = 1$. By Lemma 4.1.2 (i) and the definition of J, this implies that $i_1 \in J$, which contradicts (4.4.2). Thus, $w(H_\mu, C_{i_1}) = 1/2$. Similarly, $w(H_\mu, C_{i_2}) = 1/2$. This implies that $\sum_{i \in I_\mu} w(H_\mu, C_i) \leq 2$, and hence

$$\theta_\mu = 2 - \sum_{i \in I_\mu} w(H_\mu, C_i) \geq 0.$$

Next assume that $q_\mu = 5$. Then $N(H_\mu) \cap S = \emptyset$. Take $i \in I_\mu$. Suppose that $w(H_\mu, C_i) = 1$. Then $N(C_i) \cap T \subseteq V(H_\mu) = \{y\}$. Since G is 2-connected, we get $N(C_i) \cap S \neq \emptyset$, which implied that $(N(C_i) \cap S) - N(H_\mu) \neq \emptyset$ because $N(H_\mu) \cap S = N(y) \cap S = \emptyset$. In view of the definition of K, it follows that if $i \notin J$, then $i \in K$; that is to say, $i \in J \cup K$, which contradicts (4.4.2). Thus, $w(H_\mu, C_i) = 1/2$. Since $i \in I_\mu$ is arbitrary, it follows that $\sum_{i \in I_\mu} w(H_\mu, C_i) = 3/2$. Consequently,

$$\theta_\mu = 2 - \sum_{i \in I_\mu} w(H_\mu, C_i) > 0.$$

Case 2. $|V(H_\mu)| = 2$.

By Claim 4.2.1 (ii) and 4.2.5,

$$\begin{aligned}\theta_\mu &\geq \left(5 - \frac{q_\mu}{2}\right) + q_\mu - 6 - |I_\mu| + \frac{|I'_\mu|}{2} \\ &\geq -1 + \frac{1}{2}(2|I_\mu| - |I'_\mu| + 2) - |I_\mu| + \frac{|I'_\mu|}{2} = 0.\end{aligned}$$

4.5 Proof of Theorem 1.2.2 for $t = 3$

Let G be as in Theorem 1.2.2 with $t = 3$; thus, G is a 2-connected $K_{1,3}$-free graph with $\delta(G) \geq 4$. We continue with the notation of Section 4.2 with $t = 3$. We first prove a technical claim.

Claim 4.5.1 *If there exists H_μ ($1 \leq \mu \leq m$) such that $|V(H_\mu)| = 2$, $|I_\mu| = 2$, $q_\mu \geq 5$, $N(H_\mu) \cap S \neq \emptyset$ and $V(G) = (N(H_\mu) \cap S) \cup V(H_\mu) \cup (\bigcup_{i \in I_\mu} V(C_i))$, then $\theta_3(S, T) \geq 0$.*

Proof. We have $|S| = |N(H_\mu) \cap S| \geq 1$, $T = V(H_\mu)$ and $h_3(S, T) \leq |I_\mu| = 2$, and hence $\theta_3(S, T) = 3|S| + q_\mu - 6 - h_3(S, T) \geq 3 + 5 - 6 - 2 \geq 0$. □

In view of Claim 4.5.1, we may assume that there is no H_μ such that $|V(H_\mu)| = 2$, $|I_\mu| = 2$, $q_\mu \geq 5$, $N(H_\mu) \cap S \neq \emptyset$ and $V(G) = (N(H_\mu) \cap S) \cup V(H_\mu) \cup (\bigcup_{i \in I_\mu} V(C_i))$. Recall that

$$\begin{aligned}\theta_\mu = &\frac{3}{2}(|N(H_\mu) \cap S| + |I_\mu \cap (J \cup K)|) \\ &+ q_\mu - 3|V(H_\mu)| - \sum_{i \in I_\mu} w(H_\mu, C_i),\end{aligned}$$

for each $1 \leq \mu \leq m$. In view of Claim 4.2.6, it suffices to show that $\theta_\mu \geq 0$. We divide the proof into the following two cases:

Case 1. $|V(H_\mu)| = 1$.

Write $V(H_\mu) = \{y\}$. Note that $q_\mu = \deg_{G-S}(y)$. Since G is $K_{1,3}$-free, $|I_\mu| \leq 2$. This implies $\theta_\mu \geq q_\mu - 3 - \sum_{i \in I_\mu} w(H_\mu, C_i) \geq q_\mu - 5$ by (4.2.8).

Thus, we may assume that

$$q_\mu \leq 4. \tag{4.5.1}$$

Since $\delta(G) \geq 4$,

$$|N(H_\mu) \cap S| \geq 4 - q_\mu \tag{4.5.2}$$

by Claim 4.2.1 (i), and hence

$$3|N(H_\mu) \cap S|/2 + q_\mu \geq 6 - \left\lfloor \frac{q_\mu}{2} \right\rfloor. \tag{4.5.3}$$

Subcase 1.1. $|I_\mu| \leq 1$.

By (4.2.8), (4.5.1) and (4.5.3), we have

$$\begin{aligned} \theta_\mu &\geq \frac{3}{2}|N(H_\mu) \cap S| + q_\mu - 3|V(H_\mu)| - \sum_{i \in I_\mu} w(H_\mu, C_i) \\ &\geq 6 - \frac{4}{2} - 3 - 1 \geq 0. \end{aligned}$$

Subcase 1.2. $|I_\mu| = 2$.

Write $I_\mu = \{i_1, i_2\}$. By (4.2.8), we have $\sum_{i \in I_\mu} w(H_\mu, C_i) \leq 2$. If $q_\mu \leq 2$, then $3|N(H_\mu) \cap S|/2 + q_\mu \geq 5$ by (4.5.3), and hence $\theta_\mu \geq 3|N(H_\mu) \cap S|/2 + q_\mu - 3 - \sum_{i \in I_\mu} w(H_\mu, C_i) \geq 5 - 3 - 2 = 0$. Thus, we may assume that $q_\mu = 3$ or 4. If $|N(H_\mu) \cap S| \geq 4 - q_\mu + 1$ or $I_\mu \cap (J \cup K) \neq \emptyset$, then $3(|N(H_\mu) \cap S| + |I_\mu \cap (J \cup K)|)/2 + q_\mu \geq 3(4 - q_\mu + 1)/2 + q_\mu \geq 11/2$ by (4.5.2) , and hence $\theta_\mu \geq 3(|N(H_\mu) \cap S| + |I_\mu \cap (J \cup K)|)/2 + q_\mu - 3 - \sum_{i \in I_\mu} w(H_\mu, C_i) \geq 11/2 - 3 - 2 > 0$. Thus, we may assume that

$$\begin{aligned} |N(H_\mu) \cap S| &= 4 - q_\mu \text{ and} & (4.5.4) \\ I_\mu \cap (J \cup K) &= \emptyset. & (4.5.5) \end{aligned}$$

First assume that $q_\mu = 3$. Then i_1 or i_2, say i_1, satisfies $|E(y, C_{i_1})| = 1$. By (4.5.4), $|N(H_\mu) \cap S| = 1$, and hence $3|N(H_\mu) \cap S|/2 + q_\mu = 9/2$. Suppose that $w(H_\mu, C_{i_1}) = 1$. Then $N(C_{i_1}) \cap T \subseteq V(H_\mu) = \{y\}$. Hence $|E(T, C_{i_1})| =$

$|E(y, C_{i_1})| = 1$. By Lemma 4.1.2 (i) and the definition of J, this implies $i_1 \in J$, which contradicts (4.5.5). Thus, $w(H_\mu, C_{i_1}) = 1/2$. Hence

$$\theta_\mu = \frac{3}{2}|N(H_\mu) \cap S| + q_\mu - 3 - \sum_{i \in I_\mu} w(H_\mu, C_i) \geq \frac{9}{2} - 3 - \frac{3}{2} = 0.$$

Next assume that $q_\mu = 4$. By (4.5.4), $N(H_\mu) \cap S = \emptyset$. Hence $3|N(H_\mu) \cap S|/2 + q_\mu = 4$. Suppose that $w(H_\mu, C_{i_1}) = 1$. Then $N(C_{i_1}) \cap T \subseteq V(H_\mu) = \{y\}$. Since G is 2-connected, we get $N(C_{i_1}) \cap S \neq \emptyset$, which implies $(N(C_{i_1}) \cap S) - N(H_\mu) \neq \emptyset$ because $N(C_{i_1}) \cap S = \emptyset$. In view of the definition of K, it follows that if $i_1 \notin J$, then $i_1 \in K$; that is to say, $i_1 \in J \cup K$, which contradicts (4.5.5). Thus, $w(H_\mu, C_{i_1}) = 1/2$. Similarly, $w(H_\mu, C_{i_2}) = 1/2$. Consequently,

$$\theta_\mu = \frac{3}{2}|N(H_\mu) \cap S| + q_\mu - 3 - \sum_{i \in I_\mu} w(H_\mu, C_i) = 4 - 3 - 1 = 0.$$

Case 2. $|V(H_\mu)| = 2$.

Write $V(H_\mu) = \{y_1, y_2\}$. We start with a claim.

Claim 4.5.2 *Set* $I = \{i \in I_\mu \mid N(y_1) \cap V(C_i) \neq \emptyset\}$ *and* $I' = \{i \in I_\mu \mid N(y_2) \cap V(C_i) \neq \emptyset\}$. *Then the followings hold:*

(i) *We have* $|I| \leq 2$ *and* $|I'| \leq 2$,

(ii) *If* $|I| = 2$ *or* $|I'| = 2$, *then* $I \cap I' \neq \emptyset$.

Proof. Statement (i) immediately follows from the assumption that G is $K_{1,3}$-free. Assume that I or I', say I, satisfies $|I| = 2$. Since G is $K_{1,3}$-free, y_2 is adjacent to at least one vertex in $N(y_1) \cap (\bigcup_{i \in I} V(C_i))$, which implies $I \cap I' \neq \emptyset$. This proves (ii). $\square$

Note that $q_\mu = \deg_{G-S}(y_1) + \deg_{G-S}(y_2)$, and we have

$$q_\mu \geq 2 + |I_\mu| \tag{4.5.6}$$

by Claim 4.2.1 (ii). By Claim 4.5.2, $|I_\mu| \leq 3$. This implies $\theta_\mu \geq q_\mu - 6 - \sum_{i \in I_{mu}} w(H_\mu, C_i) \geq q_\mu - 9$ by (4.2.8). Thus, we may assume that

$$q_\mu \leq 8. \tag{4.5.7}$$

Since $\delta(G) \geq 4$,

$$|N(H_\mu) \cap S| \geq 4 - \left\lfloor \frac{q_\mu}{2} \right\rfloor \tag{4.5.8}$$

by Claim 4.2.1 (ii), and hence

$$\frac{3}{2}|N(H_\mu) \cap S| + q_\mu \geq 6 + q_\mu - \frac{3}{2}\left\lfloor \frac{q_\mu}{2} \right\rfloor. \tag{4.5.9}$$

Subcase 2.1. $|I_\mu| = 0$.

By (4.2.8) and (4.5.9),

$$\theta_\mu \geq \frac{3}{2}|N(H_\mu) \cap S| + q_\mu - 6 \geq 6 + \frac{q_\mu}{4} - 6 \geq 0.$$

Subcase 2.2 $|I_\mu| = 1$.

By (4.5.6), $q_\mu \geq 3$. Hence, by (4.2.8) and (4.5.9),

$$\begin{aligned} \theta_\mu &\geq \frac{3}{2}|N(H_\mu) \cap S| + q_\mu - 6 - \sum_{i \in I_\mu} w(H_\mu, C_i) \\ &\geq 6 + q_\mu - \frac{3}{2}\left\lfloor \frac{q_\mu}{2} \right\rfloor - 6 - 1 \geq 0. \end{aligned}$$

Subcase 2.3. $|I_\mu| = 2$.

Write $I_\mu = \{i_1, i_2\}$. In view of Claim 4.5.2 (ii), we may assume that $E(y_1, C_{i_1}) \neq \emptyset$ and $E(y_2, C_{i_2}) \neq \emptyset$. By (4.5.6), $q_\mu \geq 4$.

If $q_\mu = 5$, 7 or 8, then $3|N(H_\mu) \cap S|/2 + q_\mu \geq 8$ by (4.5.9), and hence $\theta_\mu \geq 3|N(H_\mu) \cap S|/2 + q_\mu - 6 - \sum_{i \in I_\mu} w(H_\mu, C_i) \geq 8 - 6 - 2 = 0$. Thus, we may assume that $q_\mu = 4$ or 6. If $|N(H_\mu) \cap S| \geq 4 - \lfloor q_\mu/2 \rfloor + 1$ or $I_\mu \cap (J \cup K) \neq \emptyset$, then

$$\frac{3}{2}\Big(|N(H_\mu) \cap S| + |I_\mu \cap (J \cup K)|\Big) + q_\mu \geq \frac{3}{2}\Big(4 - \left\lfloor \frac{q_\mu}{2} \right\rfloor + 1\Big) + q_\mu \geq \frac{17}{2}$$

by (4.5.8), and hence

$$\begin{aligned}\theta_\mu &\geq \frac{3}{2}\Big(|N(H_\mu) \cap S| + |I_\mu \cap (J \cup K)|\Big) + q_\mu - 6 - \sum_{i \in I_\mu} w(H_\mu, C_i) \\ &\geq \frac{17}{2} - 6 - 2 > 0.\end{aligned}$$

Thus, we may assume that

$$\begin{aligned}|N(H_\mu) \cap S| &= 4 - \left\lfloor \frac{q_\mu}{2} \right\rfloor \text{ and} &\quad (4.5.10)\\ I_\mu \cap (J \cup K) &= \emptyset. &\quad (4.5.11)\end{aligned}$$

First assume that $q_\mu = 4$. Then $|E(H_\mu, C_{i_1})| = |E(y_1, C_{i_1})| = 1$ and $|E(H_\mu, C_{i_2})| = |E(y_2, C_{i_2})| = 1$. By (4.5.10), $|N(H_\mu) \cap S| = 2$, and hence

$$\frac{3}{2}|N(H_\mu) \cap S| + q_\mu = 7.$$

Suppose that $w(H_\mu, C_{i_1}) = 1$. Then $N(C_{i_1}) \cap T \subseteq V(H_\mu)$. Hence $|E(T, C_{i_1})| = |E(H_\mu, C_{i_1})| = 1$. By Lemma 4.1.2 (i) and the definition of J, we get $i_1 \in J$, which contradicts (4.5.11). Thus, $w(H_\mu, C_{i_1}) = 1/2$. Similarly, $w(H_\mu, C_{i_2}) = 1/2$. Hence

$$\begin{aligned}\theta_\mu &= \frac{3}{2}|N(H_\mu) \cap S| + q_\mu - 6 - \sum_{i \in I_\mu} w(H_\mu, C_i) \\ &= 7 - 6 - 1 = 0.\end{aligned}$$

Next assume that $q_\mu = 6$. By (4.5.10), $|N(H_\mu) \cap S| = 1$. Hence

$$\frac{3}{2}|N(H_\mu) \cap S| + q_\mu = \frac{15}{2}.$$

By the assumption made immediately after the proof of Claim 4.5.1, we have $V(G) \neq \big(N(H_\mu) \cap S\big) \cup V(H_\mu) \cup V(C_{i_1}) \cup V(C_{i_2})$. Suppose that $w(H_\mu, C_{i_1}) = w(H_\mu, C_{i_2}) = 1$. Then $N(C_{i_1}) \cap T \subseteq V(H_\mu)$ and $N(C_{i_2}) \cap T \subseteq V(H_\mu)$. Hence $N(H_\mu \cup C_{i_1} \cup C_{i_2}) \cap \big(S \cup \big(T - V(H_\mu)\big)\big) = N(H_\mu \cup C_{i_1} \cup C_{i_2}) \cap S$. Since G is 2-connected, $|N(H_\mu) \cap S| = 1$ and $\big(N(H_\mu) \cap S\big) \cup V(H_\mu) \cup V(C_{i_1}) \cup V(C_{i_2}) \neq V(G)$, we get $\big(N(C_{i_1} \cup C_{i_2}) \cap S\big) - N(H_\mu) \neq \emptyset$. Hence i_1 or i_2, say i_1, satisfies $\big(N(C_{i_1}) \cap S\big) - N(H_\mu) \neq \emptyset$. In view of K, it follows that if $i_1 \notin J$, then

$i_1 \in K$; that is to say, $i_1 \in J \cup K$, which contradicts (4.5.11). Thus, we have $w(H_\mu, C_{i_1}) = 1/2$ or $w(H_\mu, C_{i_2}) = 1/2$, and hence $\sum_{i \in I_\mu} w(H_\mu, C_i) \leq 3/2$. Consequently,

$$\begin{aligned} \theta_\mu &= \frac{3}{2}|N(H_\mu) \cap S| + q_\mu - 6 - \sum_{i \in I_\mu} w(H_\mu, C_i) \\ &\geq \frac{15}{2} - 6 - \frac{3}{2} = 0. \end{aligned}$$

Subcase 2.4. $|I_\mu| = 3$.

Write $I_\mu = \{i_1, i_2, i_3\}$. By Claim 4.5.2, we may assume that $E(H_\mu, C_{i_1}) = E(y_1, C_{i_1}) \neq \emptyset$, $E(H_\mu, C_{i_2}) = E(y_2, C_{i_2}) \neq \emptyset$, and $E(y_1, C_{i_3}) \neq \emptyset$ and $E(y_2, C_{i_3}) \neq \emptyset$. Since G is $K_{1,3}$-free, this implies $N(H_\mu) \cap V(C_{i_3}) = N(y_1) \cap V(C_{i_3}) = N(y_2) \cap V(C_{i_3})$. Hence

$$q_\mu = |E(H_\mu, C_{i_1})| + |E(H_\mu, C_{i_2})| + 2|N(H_\mu) \cap V(C_{i_3})| + 2 \tag{4.5.12}$$

which, in particular, implies

$$q_\mu \geq 6. \tag{4.5.13}$$

It follows from (4.5.7) and (4.5.12) that

$$1 \leq |N(H_\mu) \cap V(C_{i_3})| \leq 2. \tag{4.5.14}$$

If $|N(H_\mu) \cap S| \geq 4 - \lfloor q_\mu/2 \rfloor + 1$ or $I_\mu \cap (J \cup K) \neq \emptyset$, then

$$\frac{3}{2}\Big(|N(H_\mu) \cap S| + |I_\mu \cap (J \cup K)|\Big) + q_\mu \geq \frac{3}{2}\Big(4 - \Big\lfloor \frac{q_\mu}{2} \Big\rfloor + 1\Big) + q_\mu \geq 9$$

by (4.5.13), and hence

$$\begin{aligned} \theta_\mu &\geq \frac{3}{2}\Big(|N(H_\mu) \cap S| + |I_\mu \cap (J \cup K)|\Big) + q_\mu - 6 - \sum_{i \in I_\mu} w(H_\mu, C_i) \\ &\geq 9 - 6 - 3 = 0. \end{aligned}$$

Thus, we may assume that

$$|N(H_\mu) \cap S| = 4 - \Big\lfloor \frac{q_\mu}{2} \Big\rfloor \text{ and} \tag{4.5.15}$$

$$I_\mu \cap (J \cup K) = \emptyset. \tag{4.5.16}$$

Claim 4.5.3 (i) *If* $|E(y_1, C_{i_1})| = 1$*, then* $w(H_\mu, C_{i_1}) = 1/2$*.*

(ii) *If* $|E(y_2, C_{i_2})| = 1$*, then* $w(H_\mu, C_{i_2}) = 1/2$*.*

(iii) *If* $|E(y_1, C_{i_1})| \geq 2$ *and* $|E(y_2, C_{i_2})| \geq 2$*, then* $w(H_\mu, C_{i_1}) = w(H_\mu, C_{i_2}) = 1/2$*.*

Proof. (i) If $w(H_\mu, C_{i_1}) = 1$, then $|E(T, C_{i_1})| = |E(y_1, C_{i_1})| = 1$, and hence $i_1 \in J$ by Lemma 4.1.2 (i), which contradicts (4.5.16).

(ii) We get (ii) by arguing as in (i).

(iii) It follows from (4.5.7) and (4.5.12) that $q_\mu = 8$, and hence $N(H_\mu) \cap S = \emptyset$ by (4.5.15). Now suppose that $w(H_\mu, C_{i_1}) = 1$. Then $N(C_{i_1}) \cap T = N(C_{i_1}) \cap V(H_\mu) = \{y_1\}$. Since G is 2-connected, we get $N(C_{i_1}) \cap S \neq \emptyset$. Since $N(H_\mu) \cap S = \emptyset$ and $|N(y_1) \cap V(C_{i_1})| = |E(y_1, C_{i_1})| \geq 2$, it follows that $i_1 \in K$, which contradicts (4.5.16). Thus, $w(H_\mu, C_{i_1}) = 1/2$. We similarly obtain $w(H_\mu, C_{i_2}) = 1/2$. □

Assume for the moment that $|N(H_\mu) \cap V(C_{i_3})| = 2$. Then $q_\mu = 8$ and $|E(H_\mu, C_{i_1})| = |E(H_\mu, C_{i_2})| = 1$ by (4.5.12). Hence $w(H_\mu, C_{i_1}) = w(H_\mu, C_{i_2}) = 1/2$ by Claim 4.5.3. Consequently,

$$\theta_\mu \geq q_\mu - 6 - \sum_{i \in I_\mu} w(H_\mu, C_i) \geq 8 - 6 - 2 = 0.$$

Thus, in view of (4.5.14), we may assume that $|E(N(H_\mu), V(C_{i_3})| = 1$. Suppose that $w(H_\mu, C_{i_3}) = 1$. Since $i_3 \notin J$ by (4.5.16), it follows that $|V(C_{i_3})| = 1$. Write $V(C_{i_3}) = \{u\}$. Since $\deg_G(u) \geq 4$, $|N(u) \cap S| \geq 2$. Since $|N(H_\mu) \cap S| \leq 1$ by (4.5.15), we get $i_3 \in K$, which contradicts (4.5.16). Thus, $w(H_\mu, C_{i_3}) = 1/2$. By Claim 4.5.3, at least one of $w(H_\mu, C_{i_1}) = 1/2$ and $w(H_\mu, C_{i_2}) = 1/2$ holds. Hence $\sum_{i \in I_\mu} w(H_\mu, C_i) \geq 2$. If $q_\mu = 7$ or 8, then $3|N(H_\mu) \cap S|/2 + q_\mu \geq 8$ by (4.5.9), and hence

$$\begin{aligned} \theta_\mu &= \frac{3}{2}|N(H_\mu) \cap S| + q_\mu - 6 - \sum_{i \in I_\mu} w(H_\mu, C_i) \\ &\geq 8 - 6 - 2 = 0. \end{aligned}$$

Thus, we may assume that $q_\mu = 6$. Then $|E(H_\mu, C_{i_1})| = |E(H_\mu, C_{i_2})| = 1$ by (4.5.12), and hence $w(H_\mu, C_{i_1}) = w(H_\mu, C_{i_2}) = 1/2$ by Claim 4.5.3. Consequently, $\sum_{i \in I_\mu} w(H_\mu, C_i) = 3/2$. Therefore,

$$\begin{aligned}\theta_\mu &= \frac{3}{2}|N(H_\mu) \cap S| + q_\mu - 6 - \sum_{i \in I_\mu} w(H_\mu, C_i) \\ &\geq \frac{15}{2} - 6 - \frac{3}{2} = 0.\end{aligned}$$

4.6 Sharpness

In this section, we construct examples which show that in Theorems 1.2.2 and 1.2.3, the lower bound on $\delta(G)$ is best possible. Let $3 \leq t \leq 7$, and set $r = t$ or $r = t - 1$ according as $5 \leq t \leq 7$ or $3 \leq t \leq 4$. We first define a graph I of order $r + 2(t-1)$ by

$$\begin{aligned}V(I) &= \{x_i \mid 1 \leq i \leq r\} \cup \{y_j, z_j \mid 1 \leq j \leq t-1\}, \\ E(I) &= \{x_i x_j \mid 1 \leq i < j \leq \lfloor r/2 \rfloor\} \cup \{x_i x_j \mid \lfloor r/2 \rfloor + 1 \leq i < j \leq r\} \\ &\quad \cup \{y_j z_j \mid 1 \leq j \leq t-1\}.\end{aligned}$$

Thus, I is the union of a complete graph of order $\lfloor r/2 \rfloor$, a complete graph of order $\lceil r/2 \rceil$, and $t-1$ complete graphs of order 2. Set $S_1 = \{x_i \mid 1 \leq i \leq \lfloor r/2 \rfloor\}$, $S_2 = \{x_i \mid \lfloor r/2 \rfloor + 1 \leq i \leq r\}$, $T_1 = \{y_j, z_j \mid 1 \leq j \leq \lfloor (t-1)/2 \rfloor\}$, and $T_2\{y_j, z_j \mid \lfloor (t-1)/2 \rfloor + 1 \leq j \leq t-1\}$. Let H be the graph obtained from I by joining each vertex in S_1 to all vertices in $T_1 \cup T_2$ and joining each vertex in S_2 to all vertices in T_2. Let $n \geq 2$ be an even integer, and let $H_1, ..., H_n$ be disjoint copies of H. For each k ($1 \leq k \leq n$), let $S_{k,1}$, $S_{k,2}$, $T_{k,1}$ and $T_{k,2}$ denote the subsets of $V(H_k)$ which correspond to S_1, S_2, T_1 and T_2, respectively. Now let G be the graph obtained from the union of $H_1, ..., H_n$ by joining each vertex in $S_{k,2}$ to all vertices in $T_{k+1,1}$ for each $1 \leq k \leq n$ (we take $T_{n+1,1} = T_{1,1}$). Then G is 2-connected and $K_{1,t}$-free, $|V(G)|$ is even, and $\delta(G) = r + 1$. However, we easily see that G does not have a 3-factor (for example, if we apply Lemma 4.1.1 in Section 4.1

with $S = \bigcup_{1\le k\le n}(S_{k,1} \cup S_{k,2})$ and $T = \bigcup_{1\le k\le n}(T_{k,1} \cup T_{k,2})$, then we get $\theta_3(S,T) = -n$ if $t = 5$, $\theta_3(S,T) = -2n$ if $t = 3$ or 6, and $\theta_3(S,T) = -3n$ if $t = 4$ or 7).

In Theorem 1.2.3, the condition $t \le 7$ cannot be dropped; that is to say, for any possitive integer δ, there exists a 2-connected $K_{1,8}$-free graph G of even order with $\delta(G) \ge \delta$ such that G has no 3-factor. To see this, let l be an odd integer with $l \ge \delta$, and let $C_1, ..., C_7$ be disjoint copies of the complete graph of order l. We first define a graph I of order $2 + 7l$ by

$$\begin{aligned} V(I) &= \{x_1, x_2\} \cup \bigcup_{1\le i\le 7} V(C_i), \\ E(I) &= \bigcup_{1\le i\le 7} E(C_i). \end{aligned}$$

Set $T_1 = \bigcup_{1\le i\le 3} V(C_i)$ and $T_2 = \bigcup_{4\le i\le 7} V(C_i)$. Let H be the graph obtained from I by joining x_1 to all vertices in $T_1 \cup T_2$ and joining x_2 to all vertices in T_2. Let $n \ge 2$ be an even integer, and let $H_1, ..., H_n$ be disjoint copies of H. For each k $(1 \le k \le n)$, let $x_{k,1}$ and $x_{k,2}$ denote the vertices which correspond to x_1 and x_2, respectively, and let $T_{k,1}$ and $T_{k,2}$ denote the subsets of $V(H_k)$ which correspond to T_1 and T_2, respectively. Now let G be the graph obtained from the union of $H_1, ..., H_n$ by joining $x_{k,2}$ to all vertices in $T_{k+1,1}$ for each $1 \le k \le n$ (we take $T_{n+1,1} = T_{1,1}$). Then G is 2-connected and $K_{1,8}$-free, $|V(G)|$ is even, and $\delta(G) = l+1 \ge \delta$. However, we easily see that G does not have a 3-factor (for example, if we apply Lemma 4.1.1 in Section 4.1 with $S = \bigcup_{1\le k\le n}\{x_{k,1}, x_{k,2}\}$ and $T = \emptyset$, then we get $\theta_3(S,T) = -n$).

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