Integral sections of elliptic surfaces and degenerated (2, 3) torus decompositions of a 3-cuspidal quartic

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Abstract. In this note, we consider when a plane curve given by a polynomial of the form

\[ x^3 + a_1(t)x^2 + a_2(t)x + a_3(t) = 0, \]

where \( \deg_t a_i(t) \leq id \) (\( d \): even), has degenerated (2, 3) torus decompositions by using arithmetic properties of elliptic surfaces and show that a 3-cuspidal quartic has infinitely many degenerated (2, 3) torus decompositions.

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§1. Introduction

In this note, all varieties are defined over the field of complex numbers \( \mathbb{C} \). Let \( d \) be an even positive integer and let \( p(t, x) \in \mathbb{C}[t, x] \) be a polynomial of the form

\[ x^3 + a_1(t)x^2 + a_2(t)x + a_3(t) = 0, \]

where \( \deg_t a_i(t) \leq id \). Our aim of this note is to consider when \( p(t, x) \) has a decomposition of the form

\[ (*) \quad p(t, x) = (x - x_0(t))^3 + (c_0(t)x + c_1(t))^2, \quad x_0(t), c_0(t), c_1(t) \in \mathbb{C}[t]. \]

The right hand side of (*) is called a (2, 3) torus decomposition of the affine curve given by \( p(t, x) = 0 \). Such decompositions have been considered in, for example, [13, 14, 5, 3] from viewpoint of the topology of the complements to \( \{ p(t, x) = 0 \} \). In this note we add another remark to this problem. In order to state our criterion, we need to introduce some notation.
Let $E$ be an elliptic curve defined over the rational function field of one variable $\mathbb{C}(t)$ given by

$$E : y^2 = p(t, x),$$

and we denote the set of $\mathbb{C}(t)$-rational points and the point at infinity $O$ by $E(\mathbb{C}(t))$. It is well-known that $E(\mathbb{C}(t))$ becomes an abelian group, $O$ being the zero element. Now our first statement is as follows:

**Proposition 1.** Assume that both of plane curves given by

$$p(t, x) = 0 \quad \text{and} \quad s^3d^4p(1/s, x'/s^d) = 0$$

have at worst simple singularities (see [2] for simple singularities) in both of $(t, x)$ and $(s, x')$ planes. Then $p(t, x)$ has a decomposition as in $(\ast)$ if and only if $E(\mathbb{C}(t))$ has a point $P$ of order $3$. The polynomial $x_0(t)$ is given by the $x$-coordinate of $P$.

As an application of Proposition 1, we have the following theorem:

**Theorem 1.** Let $Q$ be a quartic with $3$ cusps and choose a smooth point $z_o$ on $Q$. There exists a unique irreducible conic $C$ as follows:

(i) $C$ is tangent to $Q$ at $z_o$ and passes through three cusps of $Q$.

(ii) Let $F_Q$, $F_C$, and $L_{z_o}$ be defining equations of $Q$, $C$ and the tangent line $L_{z_o}$ of $Q$ at $z_o$, respectively. Then there exists a homogeneous polynomial $G$ of degree $3$ such that

$$(\ast\ast) \quad L_{z_o}^2 F_Q = F_C^3 + G^2.$$

**Remark 1.**

- Following [10], we call the decomposition of $F_Q$ as in $(\ast\ast)$ a **degenerated** $(2, 3)$ torus decomposition of projective plane curves. The statement of Theorem 1 can be found in [10, 5.3.2]. We, however, consider that our point of view explains geometry behind the statement, and hope that it is worthwhile mentioning.

- The $5$ $(2, 3)$ torus decompositions given in [5] can also be found by Proposition 1. In the terminology of [5], our statement can be rephrased: $Q$ has infinitely many invisible $(2, 3)$ torus decompositions.

- Let $z_o$ be one of $3$ cusps of $Q$ and $L_{\max,z_o}$ is the tangent line at $z_o$. Then we also have a degenerated $(2, 3)$ decomposition by using $L_{\max,z_o}$. This is informed the authors by M. Kawashima. In fact, it is enough to
check the statement for one explicit example as any 3-cuspidal quartic is projectively equivalent to each other. For example, we have:

\[ Z^2(3T^4 - 2T^3X - 3T^2Z^2 + X^2Z^2 + Z^4) = (XZ^2 - T^3Z^2 - (T^2 - Z^2)^3, \]

where \([T, X, Z]\) denote homogeneous coordinates. Note that \([0, 1, 0]\) is a cusp and \(Z = 0\) is the maximal tangent line, \(L_{\text{max}, z_0}\). This statement can also be found in [10, 5.3.2]

§2. Preliminaries

2.1. Existence of \(C\)

We first show that the conic \(C\) in Theorem 1 exists. Let \([T, X, Z]\) be homogeneous coordinates of \(\mathbb{P}^2\).

Lemma 2.1. (i) Let \(C\) be a conic tangent to \(\{T = 0\}, \{X = 0\}\) and \(\{Z = 0\}\) in \(\mathbb{P}^2\). Let \(Q\) be the standard quadratic transformation (or the standard Cremona transformation) with respect to \(\{T = 0\}, \{X = 0\}\) and \(\{Z = 0\}\). Then \(Q(C)\) is a quartic whose singularities are only 3 cusps at \([0, 0, 1], [0, 1, 0]\) and \([1, 0, 0]\).

(ii) Let \(L\) be the line tangent to \(C\) at a point \(P = [T_0, X_0, Z_0] \in C\). If \(L\) is different from \(\{T = 0\}, \{X = 0\}\) and \(\{Z = 0\}\), then \(Q(L)\) is a conic tangent to \(Q(C)\) at \(Q(P) = [X_0Z_0, T_0Z_0, T_0X_0]\) and passes through \([0, 0, 1], [0, 1, 0]\) and \([1, 0, 0]\).

(iii) Conversely any conic such that it is tangent to a smooth point of a 3-cuspidal quartic \(Q\) and passes through the 3 cusps of \(Q\) can be obtained as above.

Since both of these statements are well-known, we omit their proofs. Let \(L_{Q(P)}\) be the tangent line to \(Q(C)\) at \(Q(P)\) and let \(\Phi\) be a coordinate change such that \(L_{Q(P)}\) is transformed into the line \(Z = 0\) and \(Q(P)\) is mapped to \([0, 1, 0]\).

Then \(\Phi(Q(C))\) has an affine equation of the form \(x^3 + b_1(t)x^2 + b_2(t)x + b_3(t) = 0\), where \(t = T/Z, x = X/Z, b_i(t) \in \mathbb{C}[t]\) and \(\deg b_i(t) \leq i + 1\). Also \(\Phi(Q(L))\) is given by an equation of the form \(x - x_o(t) = 0\), where \(x_o(t) \in \mathbb{C}[t]\) and \(\deg x_o(t) = 2\).

2.2. Elliptic Surfaces

As for details on the results in this subsection, we refer to [6], [7], [8], [12], [16] and [1].
2.2.1. Some terminologies

Throughout this article, an elliptic surface always means a smooth projective surface $S$ with a fibration $\varphi : S \to C$ over a smooth projective curve, $C$, as follows:

(i) There exists non empty finite subset $\text{Sing}(\varphi) \subset C$ such that $\varphi^{-1}(v)$ is a smooth curve of genus 1 for $v \in C \setminus \text{Sing}(\varphi)$, while $\varphi^{-1}(v)$ is not a smooth curve of genus 1 for $v \in \text{Sing}(\varphi)$.

(ii) There exists a section $O : C \to S$ (we identify $O$ with its image in $S$).

(iii) there is no exceptional curve of the first kind in any fiber.

In this note, we only consider an elliptic surface over $\mathbb{P}^1$, $\varphi : S \to \mathbb{P}^1$.

We call $F_v = \varphi^{-1}(v)(v \in \text{Sing}(\varphi))$ a singular fiber over $v$. In order to describe the type of singular fibers, we use notation given in Kodaira ([6]). We denote the irreducible decomposition of $F_v$ by

$$F_v = \Theta_{v,0} + \sum_{i=1}^{m_v-1} a_{v,i}\Theta_{v,i},$$

where $m_v$ is the number of irreducible components of $F_v$ and $\Theta_{v,0}$ is the irreducible component with $\Theta_{v,0}O = 1$. We call $\Theta_{v,0}$ the identity component. We also define a subset $\text{Red}(\varphi)$ of $\text{Sing}(\varphi)$ to be $\text{Red}(\varphi) := \{ v \in \text{Sing}(\varphi) \mid F_v \text{ is reducible} \}$. For $s \in \text{MW}(S)$, $s$ is said to be integral if $sO = 0$. It is known that any torsion element in $\text{MW}(S)$ is integral (cf. [7]).

Let $\text{MW}(S)$ be the set of sections of $\varphi : S \to \mathbb{P}^1$. By our assumption, $\text{MW}(S) \neq \emptyset$. On a smooth fiber $F$ of $\varphi$, by regarding $F \cap O$ as the zero element, we can consider the abelian group structure on $F$. Hence for $s_1, s_2 \in \text{MW}(S)$, one can define the addition $s_1 + s_2$ or the multiplication-by-$m$ map $[m]s_1$ on $\mathbb{P}^1 \setminus \text{Sing}(\varphi)$. By [6, Theorem 9.1], $s_1 + s_2$ and $[m]s_1$ can be extended over $\mathbb{P}^1$, and we can consider $\text{MW}(S)$ as an abelian group. On the other hand, we can regard the generic fiber $E := S_q$ of $S$ as a curve of genus 1 over $\mathbb{C}(\mathbb{P}^1)$, the rational function field of $\mathbb{P}^1$. The restriction of $O$ to $E$ gives rise to a $\mathbb{C}(\mathbb{P}^1)$-rational point of $E$, and one can regard $E$ as an elliptic curve over $\mathbb{C}(\mathbb{P}^1) \cong \mathbb{C}(t)$, $O$ being the zero element. By considering the restriction to the generic fiber for each section, $\text{MW}(S)$ can be identified with the set of $\mathbb{C}(t)$-rational points $E(\mathbb{C}(t))$. Conversely, any element $P \in E(\mathbb{C}(t))$ gives rise to a section determined by $P$, which we denote by $s_P$. We also denote the addition and the multiplication-by-$m$ map on $E(\mathbb{C}(t))$ by $+$ and $[m]$, respectively.

In [12], Shioda introduced a $\mathbb{Q}$-valued bilinear form on $E(\mathbb{C}(t))$ called the height pairing. We denote it by $\langle \cdot, \cdot \rangle$. For our later use, we give two basic properties of $\langle \cdot, \cdot \rangle$:
\begin{itemize}
  \item \(\langle P, P \rangle \geq 0\) for \(\forall P \in E(\mathbb{C}(t))\) and the equality holds if and only if \(P\) is an element of finite order in \(E(\mathbb{C}(t))\).
  
  \item An explicit formula for \(\langle P_1, P_2 \rangle (P_1, P_2 \in E(\mathbb{C}(t)))\) is given as follows:
  \[
  \langle P_1, P_2 \rangle = \chi(O_S) + s_{P_1}O + s_{P_2}O - s_{P_1}s_{P_2} - \sum_{v \in \text{Red}(c)} \text{Contr}_v(s_{P_1}, s_{P_2}),
  \]
  where \(s_{P_i} (i = 1, 2)\) denote the sections in \(\text{MW}(S)\) determined by \(P_i\) \((i = 1, 2)\), and \(\text{Contr}_v(s_{P_1}, s_{P_2})\) is determined at which component \(s_{P_1}\) and \(s_{P_2}\) meet at \(F_v\). As for explicit values of \(\text{Contr}_v(s_{P_1}, s_{P_2})\), we refer to [12, (8.16)]. Note that since \(s^2_{P_i} = -\chi(O_S)\), we have
  \[
  \langle P_1, P_1 \rangle = 2\chi(O_S) + 2s_{P_1}O - \sum_{v \in \text{Red}(c)} \text{Contr}_v(s_{P_1}, s_{P_1}),
  \]
\end{itemize}

2.2.2. Double cover construction of elliptic surfaces and their Weierstrass equations

Let \(\Sigma_d\) (\(d:\) even) be the Hirzebruch surface of degree \(d\). We first give a method in constructing elliptic surfaces over \(\mathbb{P}^1\) as double covers of \(\Sigma_d\) as follows:

Let \(\Delta_0\) and \(\Delta\) denotes sections of \(\Sigma_d\) with \(\Delta_0^2 = -d, \Delta^2 = d\) and \(\Delta_0 \cap \Delta = \emptyset\). Note that \(\Delta \sim \Delta_0 + df\), where \(f\) denotes a fiber of \(\Sigma_d \to \mathbb{P}^1\) and \(\sim\) means the linear equivalence of divisors. Let \(\mathcal{T}\) be a reduced divisor on \(\Sigma_d\) such that

(i) \(\mathcal{T} \sim 3\Delta (\sim 3(\Delta_0 + df))\), and

(ii) \(\mathcal{T}\) has at worst simple singularities (see [2] for simple singularities).

Let \(f' : S' \to \Sigma_d\) be the double cover with branch locus \(\Delta_f = \Delta_0 + \mathcal{T}\) (cf. [2, III, §7]). We denote the diagram of the canonical resolution by

\[
\begin{array}{ccc}
S' & \xleftarrow{\mu} & S \\
\downarrow{f'} & & \downarrow{f} \\
\Sigma_d & \xleftarrow{q} & \tilde{\Sigma}_d.
\end{array}
\]

(see [4]). Namely, \(\mu\) is the minimal resolution of singularities and \(q\) is a composition of blowing-ups so that the branch locus of \(f\) becomes smooth. Then the induced morphism \(\varphi : S \to \Sigma_d \to \mathbb{P}^1\) gives rise to an elliptic fibration over \(\mathbb{P}^1\).

Conversely it is known that any elliptic surface \(\varphi : S \to \mathbb{P}^1\) is obtained in this way.
We next consider a Weierstrass equation of the generic fiber of $S$. Choose affine open sets $U_1$ and $U_2$ of $\Sigma_d$ as in [1, 2.2.3]. Namely, $U_i \cong \mathbb{C}^2 (i = 1, 2)$ with coordinates $(t, x)$ (resp. $(s, x')$) on $U_1$ (resp. $U_2$) with relations $t = 1/s, x = x'/s^d$. With these coordinates, $T$ is given by equations of the form

$$p_T(t, x) = x^3 + a_1(t)x^2 + a_2(t)x + a_3(t), \quad a_i \in \mathbb{C}[t], \deg a_i \leq \text{id}.$$ 

on $U_1$ and $s^{3d}p_T(1/s, x'/s^d) = 0$ on $U_2$. Over $U_1$, $S'|_{\mu(U_1)}$ is given by

$$y^2 - p_T(t, x) = 0 \subset \mathbb{C}^3,$$

and the covering morphism $f'$ is given by the restriction of the projection $(t, x, y) \mapsto (t, x)$. The covering transformation $\sigma_f'$ is given by $(t, x, y) \mapsto (t, x, -y)$. Thus we infer that the generic fiber of $\varphi : S \to \mathbb{P}^1$ is an elliptic curve $E$ over $\mathbb{C}(t)$ given by the above Weierstrass equation. Note that if $s \in \text{MW}(S)$ is integral, then the corresponding point $P_s \in E(\mathbb{C}(t))$ has polynomial coordinate components whose degrees are at most $d$ (resp. $3d/2$) for the $x$-coordinate (resp. the $y$-coordinate). In what follows, we say $P = (x(t), y(t))$ is integral if $x(t), y(t) \in \mathbb{C}[t], \deg x(t) \leq d, \deg y(t) \leq 3d/2,$.

Let $P_o = (x_o(t), y_o(t)) \in E(\mathbb{C}(t))$ be an integral point of the elliptic curve $E$ as in Introduction. Assume $y_o(t) \neq 0$ and let

$$y = l(t, x), \quad l(t, x) = m(t)(x - x_o(t)) + y_o(t)$$

be the tangent line at $P_o$ and put $[2]P_o = (x_1(t), y_1(t)).$

**Lemma 2.2.** If $[2]P_o$ is also an integral point, then $m(t) \in \mathbb{C}[t]$.

**Proof.** From the definition of addition, we have

$$p_T(t, x) - \{l(t, x)\}^2 = (x - x_o(t))^2(x - x_1(t)).$$

By comparing the coefficients of $x^2$ of the above equality, we have

$$a_1 - \{m(t)\}^2 = -2x_o(t) - x_1(t).$$

This implies $m(t) \in \mathbb{C}[t]$ \qed

**Corollary 2.1.** Under the assumption of Lemma 2.2, $p(t, x)$ has a decomposition

$$p_T(t, x) = (x - x_o(t))^2(x - x_1(t)) + \{l(t, x)\}^2.$$ 

Since any element of finite order in $E(\mathbb{C}(t))$ is always integral under our assumption, we have
Corollary 2.2. If \( P \) is an element of finite order in \( E(\mathbb{C}(t)) \), \( p(t, x) \) has a decomposition

\[
p_T(t, x) = (x - x_o(t))^2(x - x_1(t)) + \{l(t, x)\}^2.
\]

In particular, if \( P \) is an element of order three, as the \( x \)-coordinates of \([2]P\) and \(-P\) are the same, we have

\[
p_T(t, x) = (x - x_o(t))^3 + \{l(t, x)\}^2.
\]

Proof of Proposition 1. The half of Proposition 1 follows from Corollary 2.2, as the degree of \( l(t, x) \) with respect to \( x \) is equal to 1. Conversely, if \( p_T(t, x) \) has the decomposition described in Proposition 1, \((x_o(t), \pm(c_0(t)x_o(t) + c_1(t)))\) are 3-torsions of \( E(\mathbb{C}(t)) \). Thus we have Proposition 1. \( \square \)

§3. Rational elliptic surface \( S_{Q, z_o} \)

An elliptic surface is said to be rational if it is a rational surface. Any rational elliptic surface obtained as a double cover of \( \Sigma_2 \) described in §1. Let \( Q \) be a 3-cuspidal quartic as before and let \( z_o \) be a smooth point on \( Q \). Likewise in the second author’s article (e.g., [15, 1.3]), we associate a rational elliptic surface with \( Q \) and \( z_o \), which we denote by \( \varphi : S_{Q, z_o} \to \mathbb{P}^1 \). The tangent line \( l_{z_o} \) gives rise to a singular fiber of \( \varphi \) whose type is determined by how \( l_{z_o} \) intersects with \( Q \) as follows:

<table>
<thead>
<tr>
<th>( l_{z_o} ) and the corresponding singular fiber</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) ( l_{z_o} ) meets ( Q ) with two other distinct points.</td>
</tr>
<tr>
<td>(ii) ( l_{z_o} ) is a 3-fold tangent point.</td>
</tr>
<tr>
<td>(iii) ( l_{z_o} ) is a bitangent line.</td>
</tr>
<tr>
<td>(iv) ( l_{z_o} ) is a 4-fold tangent point.</td>
</tr>
<tr>
<td>(v) ( l_{z_o} ) passes through a cusp of ( Q )</td>
</tr>
</tbody>
</table>

By [8, Table 6.2] and Table 1 as above, possible configurations of singular fibers of \( S_{Q, z_o} \) are as follows:

<table>
<thead>
<tr>
<th>Case</th>
<th>Singular fibers</th>
<th>the position of ( l_{z_o} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 3I_3, I_2, I_1 )</td>
<td>(i)</td>
</tr>
<tr>
<td>2</td>
<td>( IV, 2I_3, I_2 )</td>
<td>(ii)</td>
</tr>
<tr>
<td>3</td>
<td>( 3I_3, III )</td>
<td>(ii)</td>
</tr>
<tr>
<td>4</td>
<td>( 4I_3 )</td>
<td>(iii)</td>
</tr>
<tr>
<td>5</td>
<td>( 3I_3, IV )</td>
<td>(iv)</td>
</tr>
<tr>
<td>6</td>
<td>( 5, 2I_3, I_1 )</td>
<td>(v)</td>
</tr>
</tbody>
</table>
The Table 2 give us possible cases, but by [11], the Cases 3, 5 and 6 in Table 2 do not occur. Let \( \mathcal{C} \) be the conic described in Theorem 1. Note that \( \mathcal{C} \) exists by Lemma 2.1. Then by our construction of \( S_{Q,z_0} \), \( \mathcal{C} \) gives rise to two sections, \( s^\pm_\mathcal{C} \), which meets singular fibers as in the following figures if we label irreducible components of singular fibers suitably. Let \( P_{\mathcal{C}+} \) and \( P_{\mathcal{C}-} \) be the corresponding rational points to \( s_{\mathcal{C}+} \) and \( s_{\mathcal{C}-} \), respectively. Then we have \( \langle P_{\mathcal{C}+}, P_{\mathcal{C}-} \rangle = 0 \) and \( P_{\mathcal{C}+} \) are torsions and their orders are 3 by [11] or [9].

![Figure 1: Case 1](image1)

![Figure 2: Case 2](image2)
§4. Proof of Theorem 1

Choose homogeneous coordinates $[T, X, Z]$ of $\mathbb{P}^2$ such that $l_{z_0} : Z = 0$ and $z_0 = [0, 1, 0]$. Then $F_Q$ and $F_C$ are of the form

\[ F_Q(T, X, Z) = X^3Z + b_2(T, Z)X^2 + b_3(T, Z)X + b_4(T, Z), \]
\[ F_C(T, X, Z) = XZ - c_0t^2 - c_1TZ - c_2Z^2, \quad c_i \in \mathbb{C}(i = 0, 1, 2), c_0 \neq 0 \]

where $b_i (i = 2, 3, 4)$ are homogeneous polynomial of degree $\leq i$. Put $p_Q(t, x) = F_Q(t, x, 1)$ and $x_o(t) = c_0t^2 + c_1t + c_2$. Then the elliptic curve $E_Q$ given by $y^2 = p_Q(t, x)$ has a 3 torsion point $P_C^+ \in E_Q(\mathbb{C}(t))$ and $x_o(t)$ is its $x$-coordinate. Hence by Proposition 1, we have

\[ F_Q(t, x, 1) = (x - c_0t^2 - c_1t - c_2)^3 + \left\{ m(t)\{x - c_0t^2 - c_1t - c_2\} + y_o(t)\right\}^2, \]

where $y_o(t)$ is the $y$-coordinate of $P_C^+$ and $y = m(t)(x - c_0t^2 - c_1t - c_2) + y_o(t)$ is the tangent line at $P_C^+$. By comparing the coefficients of both hand side with respect to $x$, we have

\[ b_2(t, 1) = \{m(t)\}^2 - 3(c_0t^2 + c_1t + c_2), \]
\[ b_4(t, 1) = \{-m(t)(c_0t^2 + c_1t + c_2) + y_o(t)\}^2 - (c_0t^2 + c_1t + c_2)^3. \]

Hence, we infer that $\deg m(t) \leq 1, \deg y_o(t) \leq 3$, and we have

\[ Z^2F_Q(T, X, Z) = F_C(T, X, Z)^3 + \{Zm(T/Z)F_C(T, X, Z) + Z^3y_o(T/Z)^2. \]

This implies Theorem 1. \hfill $\square$

**Remark 4.1.**  (i) Note that we also obtain a rational elliptic surface $S_{Q_1, z_0}$ from a reduced quartic $Q_1$, which is not concurrent 4 lines, and a distinguished smooth point. A 3-cuspidal quartic and a quartic consisting
of a cuspidal cubic and its unique inflectional tangent line are the only ones so that MW(S_{Q_1,z_0}) has a 3-torsion point for a general z_0. This explains why a 3-cuspidal quartic is so special and we have Theorem 1. We hope this point of view is new.

(ii) As for the case of a cuspidal cubic and its unique inflectional tangent line, the configurations of singular fibers of S_{Q_1,z_0} is either I_6, I_3, I_2, I_1, IV^*, I_3, I_1, or IV^*, IV.

§5. Example

Now let us consider an explicit example. Let C : T^2 - XZ = 0 and Q is the standard quadratic transformation with respect to \{-2T + X + Z = 0\}, \{2T + X + Z = 0\} and \{Z = 0\}.

If P = [a, a^2, 1], a \in \mathbb{C}, a \neq \pm 1, then tangent line at P is \(-2aT + x + a^2Z = 0\). Hence Q(C), Q(L) and Q(P) are given as follows:

\[
F_{Q(C)} = 16T^2X^2 - 8T^2XZ + T^2Z^2 - 8TX^2Z - 2TXZ^2 + X^2Z^2,
\]
\[
F_{Q(L)} = 2a^2TX + (1 + a)XZ + (1 - a)ZT - 2TX,
\]
\[
Q(P) = [(a + 1)^2, (a - 1)^2, (a + 1)^2(a - 1)^2].
\]

The tangent line, L_{Q(P)}, to Q(C) at Q(P) has the following equation:

\[
(a - 1)^3T - (a + 1)^3X + 2Z = 0.
\]

Let \Phi be a coordinate change such that L_{Q(P)} is transformed into the line Z = 0 and Q(P) is mapped to [0, 1, 0]. Then \Phi(Q(C)) and \Phi(Q(L)) are given as follows in the affine equations:

\[
F_{\Phi(Q(C))} = x^3 + \left(\frac{3(a + 1)}{2(a - 1)} t^2 + \frac{3}{2} t - \frac{(a + 3)^2}{8(a^2 - 1)}\right) x^2 + \left(\frac{2a(a + 1)}{(a - 1)^2} t^3 - \frac{3(a + 1)}{(a - 1)^2} t^2 + \frac{a + 3}{(a - 1)^2(a + 1)} t\right) x - \frac{2(a + 1)}{(a - 1)^3} t^4 + \frac{4}{(a - 1)^3} t^3 - \frac{2}{(a - 1)^3(a + 1)} t^2 = 0,
\]
\[
F_{\Phi(Q(L))} = x + \frac{2(a + 1)}{a - 1} t^2 - \frac{2}{a - 1} t = 0,
\]

where t = T/Z and x = X/Z.
Then we have

\[
F_{\Phi(Q(C))} = F_{\Phi(Q(L))}^3 + l_a(t, x)^2,
\]

\[
l_a(t, x) = \frac{6(a + 1)t - (a + 3)x}{\sqrt{-8(a - 1)(a + 1)}} + \frac{4(a + 1)^2t^3 - 6(a + 1)t^2 + 2t}{\sqrt{-2(a - 1)^3(a + 1)}}.
\]

If we first homogenize these equations, then apply \(\Phi^{-1}\), we have the following degenerated (2, 3) torus decomposition:

\[
L_a^2F_{Q(C)} = -8F_{Q(L)}^3 + G^2,
\]

\[
L_a = -(a - 1)^3T + (a + 1)^3X - 2Z,
\]

\[
G = \frac{4(a - 1)^3T^2X - (a - 1)^3T^2Z + 4(a + 1)^3TX^2 - (a + 1)^3X^2Z + 2a(a^2 - 9)TXZ + 2TZ^2 - 2XZ^2}{2(a - 1)(a + 1)}.
\]

References


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