Summary of the Thesis:
Regularity results for minimizers of $p(x)$-growth functionals

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Kunihiro USUBA
(Tokyo University of Science)

This thesis is concerned with regularity problems for solutions of variational problems, more precisely, we consider the regularity problems for a minimizer of a functional defined by

$$
\mathcal{F}(u) = \int_\Omega f(x, u(x), Du(x))dx,
$$

for $u : \Omega \subset \mathbb{R}^m \to \mathbb{R}^n$. Here $\Omega$ is a bounded domain and $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn} \to \mathbb{R}$ is a Carathéodory function satisfying the following so-called $(p,q)$-growth condition: there exist constants $\Lambda \geq \lambda > 0$ and $q \geq p \geq 1$ such that

$$
\lambda |\xi|^p \leq f(x, u, \xi) \leq \Lambda (1 + |\xi|^2)^{q/2}
$$

for all $(x, u, \xi) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}$, where $| \cdot |$ is the Euclidean norm. We call $\mathcal{F}$ a functional with standard growth or $p$-growth if $p = q$, and with non-standard growth if $q > p$. If the integrand $f = f_{p(x)}$ satisfies

$$
\lambda |\xi|^{p(x)} \leq f_{p(x)}(x, u, \xi) \leq \Lambda (1 + |\xi|^2)^{p(x)},
$$

for all $(x, u, \xi) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}$, then

$$
\mathcal{F}_{p(x)}(u) := \int_\Omega f_{p(x)}(x, u, Du)dx
$$

is called a functional with $p(x)$-growth.

In general, a minimizer for (1) is expected to satisfy the differential equation called Euler-Lagrange equation of (1). However, in general, even for standard
growth case, we can not expect to find such a minimizer or a solution of the Euler-Lagrange equation directly in the class of differentiable functions. These problems are often solved by the following two steps:

1. Find a minimizer of the functional or weak solution of the corresponding Euler-Lagrange equation in a suitable Sobolev space, a space consisting of weakly differentiable functions.
2. Show that the weak solution is sufficiently regular.

The problems dealing with the latter step are called “regularity problems”.

For the vector valued case \((n \geq 2)\), by counter examples due to De Giorgi [6], Giusti-Miranda [16], Nečas [22], Sòucek [30] etc., we know that the minimizers or weak solutions of nonlinear elliptic systems does not have everywhere regularity in general. So, it is worth studying “partial regularity”, i.e. regularity except (hopefully) sufficiently small.

We mainly consider the \(p(x)\)-energy type functional defined for maps \(u : \Omega \to \mathbb{R}^n\) as

\[
A(u) = \int_{\Omega} (A^{\alpha \beta}_{ij}(x, u) D_{\alpha u^i} D_{\beta u^j})^{\frac{p(x)}{2}} dx,
\]

where \(D_{\alpha} = ((D_{\alpha u^i})_{1 \leq i \leq m, 1 \leq \alpha \leq m})\) is the derivative of \(u\), and \(p(x)\) is a continuous function on \(\Omega\). We require that the coefficients \((A_{ij}^{\alpha \beta}(x, u))\) are defined and uniformly continuous, elliptic and bounded on \(\Omega \times \mathbb{R}^n\). Throughout this thesis, the Einstein summation convention is used: Greek indices \(\alpha, \beta, \ldots\) are to be summed from 1 to \(m\), and Latin indices \(i, j, \ldots\) from 1 to \(n\). The \(p(x)\)-energy type functional \(A\) is a \(p(x)\)-growth functional with a special structure.

The general \((p, q)\)-growth condition was introduced by Marcellini [17], and many important papers [18, 19, 20, 21] were published by him. About \((p, q)\)-growth problems, see also [2, 9, 10, 11, 27, 28, 29] and references therein. Especially, in 1995, Zhikov [35] studied Lavrentiev phenomenon for the functional

\[
D(u) := \int_{\Omega} |D_{\alpha} u^i|^{p(x)} dx.
\]

He also obtained higher integrability for the minimizers of \(D\) (see [36]).

In this thesis, we treat the regularity problem for the vector-valued case. For the scalar valued case \((n = 1)\), see, for example, Acerbi-Mingione [3], Eleuteri [8] and the references therein.

For the case that \(A_{ij}^{\alpha \beta}(x, u)\) are given by

\[
A_{ij}^{\alpha \beta}(x, u) = g^{\alpha \beta}(x) G_{ij}(u),
\]

namely the functional is given by

\[
E_1(u) = \int_{\Omega} (g^{\alpha \beta}(x) G_{ij}(u) D_{\alpha u^i} D_{\beta u^j})^{\frac{p(x)}{2}} dx,
\]

the functional \(E_1\) is called \(p(x)\)-energy and its critical points are called \(p(x)\)-harmonic maps.

For the vector-valued case \((n \geq 2)\), Coscia-Mingione [5] showed that a minimizer \(u\) of the functional \(D\), is of class \(C^{1, \alpha}\) for some \(\alpha \in (0, 1)\), assuming that \(p(x)\) is Hölder continuous. Comparing with the standard growth case (the case when \(p(x) \equiv p\)), we can say that this regularity result corresponds to those by Uhlenbeck.
results. For the variable exponent $p$ of the exponent $p$, consider the following conditions:

$$p > 1.$$

On the other hand, for the general form of $p(x)$-growth functionals $F_p(x)$, Acerbi-Mingione [4] proved almost everywhere regularity. Namely, they showed that a minimizer $u$ of $F_p(x)$ is in the class $C^{1, \alpha}(\Omega_0)$ for an open set $\Omega_0 \subset \Omega$ with $|\Omega - \Omega_0| = 0$. Here, and in what follows, for a measurable set $D \subset \mathbb{R}^m$, $|D|$ stands for the the Lebesgue measure of $D$. For the standard $p$-growth case, this type of partial regularity results have been proved by Giaquinta-Giusti [13] for $p \geq 2$, and by Acerbi-Fusco [1] for $1 < p < 2$.

For the standard $p$-energy functionals, by the result given by Giaquinta-Modica [15, Theorem 4.2] (see also [12]) for functionals with a special structure which include $p$-energy, we see that a $p$-energy minimizing map $u$ is in the class $C^{1, \alpha}(\Omega_0)$ for some open set $\Omega_0 \subset \Omega$ with $\mathcal{H}^{m-p}(\Omega - \Omega_0) = 0$. Here and in what follows, $\mathcal{H}^s$ denotes $s$-dimensional Hausdorff measure. If $u$ is also assumed to be bounded, the estimate for the singular set can be improved. More precisely, it was shown that $\mathcal{H}^{m-p}(\Omega - \Omega_0) = 0$ by Giaquinta-Giusti [14] for $p = 2$, and by Fusco-Hutchinson [12, Theorem 8.1] for general $p \geq 2$. Also, Duzzar-Grotowski-Kronz [7] considered the partial and full boundary regularity for minimizers of

$$\int_{\Omega} [G^{\alpha \beta}(x)g_{ij}(x, u)D_{\alpha}u^iD_{\beta}u^j]^{p/2}dx.$$

For the functional $\mathcal{E}_1$ with $p(x) \geq 2$, Ragusa-Tachikawa-Takabayashi [26] showed partial regularity of local minimizers. The estimate for the Hausdorff dimension of the singular set is improved in Tachikawa [31], and boundary regularity is shown in Ragusa-Tachikawa [24]. Besides these partial regularity results, everywhere regularity is obtained in Ragusa-Tachikawa [25] under the so-called one-sided condition.

By some slight modifications, we can see easily that the results of [26] hold for more general functionals of having from of the $A$, and the results of [31, 24] for the case when $G_{ij}$ depend also on $x$.

The partial regularity results by Ragusa-Tachikawa-Takabayashi [26] tell us that a minimizer $u$ of the functional $A$ is of class $C^{1, \alpha}(\Omega_0)$ for some $\alpha \in (0, 1)$ and an open set $\Omega_0 \subset \Omega$ with $\mathcal{H}^{m-\inf p(x)}(\Omega - \Omega_0) = 0$. These have been the preliminary results. In this thesis we mainly deal with the unsolved problem of the regularity of a map that minimizes the $p(x)$-energy type functional for $1 < p(x) < 2$, and also the more general condition of $p(x) > 1$.

Now, let us introduce some conditions and definitions in order to state the main results. For the variable exponent $p(x)$ and coefficients $A(x, u) = (A^{\alpha \beta}_{ij}(x, u))$ we consider the following conditions:

1. The exponent $p(x)$ is Hölder continuous; namely there exist positive constants $L_p$ and $\sigma \in (0, 1)$ such that

$$|p(x) - p(y)| \leq L_p|x - y|^{\sigma} =: \omega_p(|x - y|/2),$$

for any $x, y \in \Omega$. 

[33] for $p \geq 2$ and by Acerbi-Fusco [1] for $1 < p < 2$. They showed $C^{1, \alpha}$-regularity of minimizers for the functionals of the form

$$\int_{\Omega} h(|Du|)dx,$$

where $h(t)$ is a smooth function satisfying

$$\lambda t^p \leq h(t) \leq \Lambda(1 + t^p) \quad (t \geq 0)$$

for some constants $\Lambda \geq \lambda > 0$ and $p > 1$. 
(p2) $p(x)$ satisfies

$$1 < \gamma_1 := \inf_{\Omega} \frac{p(x)}{\Omega} \leq \sup_{\Omega} p(x) =: \gamma_2. \quad (8)$$

(A1) $A$ satisfies the following symmetry.

$$A_{ij}^{\alpha\beta}(x, u) = A_{ji}^{\beta\alpha}(x, u).$$

(A2) There exist positive constants $\lambda_\alpha < \Lambda_\alpha$ such that

$$\sup_{x, u} |A_{ij}^{\alpha\beta}(x, u)| \leq \Lambda_\alpha, \quad A_{ij}^{\alpha\beta}(x, u)\xi_\alpha^i \xi_\beta^j \geq \lambda_\alpha |\xi|^2, \quad (9)$$

for all $\xi \in \mathbb{R}^m$, $x \in \Omega$ and $u \in \mathbb{R}^n$.

(A3) There exists a concave nondecreasing function $\omega_A : [0, \infty) \to [0, \infty)$ with $\lim_{t \to 0^+} \omega_A(t) = 0$ for which $A(x, u) = (A_{ij}^{\alpha\beta}(x, u))$ satisfy

$$|A(x, u) - A(y, v)| \leq \omega_A(|x - y|^{\gamma_1} + |u - v|^{\gamma_1}) \quad (10)$$

for any $(x, u), (y, v) \in \Omega \times \mathbb{R}^n$. As we can see easily, if $A(x, u)$ is bounded and uniformly continuous on $\Omega \times \mathbb{R}^n$, then we can find $\omega_A$ so that (A3) is fulfilled.

In Chapter 2, we prove partial regularity of minimizers $u$ for $p(x)$-energy type functional $A$ assuming that $A_{ij}^{\alpha\beta}$ and $p(x)$ satisfy conditions (A1), (A2), (A3), (p1), (p2) and $\gamma_2 < 2$.

In the proofs for the case of $p(x) \geq 2$, it is important to estimate the quantity $\int_{B(z_1, R)} |Du - Dv|^2(2R) dx$ with minimizer $v$ of $A_0$ by considering a frozen functional $A_0$ which is sufficiently close to the functional $A$ and of which the regularity of minimizer is known. However, in the case of $1 < p(x) < 2$, a problem in estimating this quantity was that there exists negative exponent terms which may diverge in the process of calculation. Here, for $p(x) = p > 1$ ($p : \text{constant}$), referring to the method of Duzaar-Grotowski-Kronz [7], we define $V_\mu(z) := (\mu^2 + |z|^2)^{(p/2(2R) - 2)/4} z$ for any $\mu \geq 0$ and estimate $\int_{B(z_1, R)} |V_\mu(Du) - V_\mu(Dv)|^2 dx$ instead of $\int_{B(z_1, R)} |Du - Dv|^2(2R) dx$, and by $\mu \to 0$, it became possible to prevent divergence and prove the following theorem.

**Theorem 2.2.** Let $\Omega \subset \mathbb{R}^m$ be a bounded open set with sufficiently smooth boundary $\partial\Omega$. Assume that $A_{ij}^{\alpha\beta}$ satisfies the conditions (A1), (A2) and (A3). Let $u \in W^{1,p(x)}(\Omega)$ be a local minimizer of the functional

$$A(u) = \int_{\Omega} (A_{ij}^{\alpha\beta}(x, u)D_x u^i D_x u^j)^{p(x)/2} dx,$$

where $p : \Omega \to (1, 2)$ satisfies (p1) and

$$1 < \gamma_1 := \inf_{\Omega} \frac{p(x)}{\Omega} \leq \sup_{\Omega} p(x) =: \gamma_2 < 2 \quad \text{for all } x \in \Omega.$$

Then $u \in C^{0, \alpha}(\Omega_0)$ for some $0 < \alpha < 1$, where $\Omega_0$ is an open set of $\Omega$ with $\mathcal{H}^{m-\gamma_1}(\Omega - \Omega_0) = 0$.

In Chapter 3, we show partial regularity up to the boundary $\partial\Omega$ of a bounded open set $\Omega \subset \mathbb{R}^m$ for minimizers $u$ for $p(x)$-energy type functional $A$ assuming that $A_{ij}^{\alpha\beta}$ and $p(x)$ satisfy the conditions (A1), (A2), (A3), (p1) and (p2). Moreover,
when \( A_{ij}^\alpha(x, u) \) are given as \( A_{ij}^\alpha(x, u) = g^{\alpha \beta}(x) G_{ij}(x, u) \), we can also prove that minimizers have no singular points on the boundary.

Instead of (A2), we consider the following condition on \((g^{\alpha \beta}(x))\) and \((G_{ij}(x, u))\).

(G1) There exists positive constant \( \lambda_g, \Lambda_g, \lambda_G, \Lambda_G \) such that

\[
\lambda_g |\xi|^2 \leq g^{\alpha \beta}(x) \zeta_\alpha \zeta_\beta \leq \Lambda_g |\xi|^2, \quad \lambda_G |\eta|^2 \leq G_{ij}(u) \eta^i \eta^j \leq \Lambda_G |\eta|^2
\]

for any \((x, \zeta, u, \eta) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n \).

Comparing the proof of the partial regularity result of [26] for \( p(x) \geq 2 \) with the result of Theorem 2.2 in Chapter 2 for \( 1 < p(x) < 2 \), (see also [34]), we have similar main estimates for both cases. More precisely, for \( p > 1 \) and \( \xi, \eta \in \mathbb{R}^k \) \((k \geq 1)\), we define a new function

\[
W(p; \xi, \eta) := \begin{cases} 
|\xi - \eta|^p, & (p \geq 2), \\
|V(\xi) - V(\eta)|^2, & (1 < p < 2),
\end{cases}
\]

where \( V \) is defined by

\[
V(\xi) := \begin{cases} 
|\xi|^{(p-2)/2}, & (\xi \neq 0) \\
0, & (\xi = 0),
\end{cases}
\]

and estimate \( \int_{B^+(2R/10)} W(p; Dv, Dw) dx \), for the case of both \( p \geq 2 \) and \( 1 < p < 2 \), we can obtain a decay estimate for \( \int_{B^+} |Du|^2(v^r) dx \), and thus get the following result.

**Theorem 3.1.** Let \( \Omega \subset \mathbb{R}^m \) be a bounded domain with \( C^1 \)-boundary \( \partial \Omega \). Assume that \( p(x) \) and \( A \) satisfy (p1), (p2), (A1), (A2), and (A3). Let \( u \in W^{1,p(x)}(\Omega) \) be a minimizer of the functional \( A \) defined by

\[
A(u) = \int_\Omega (A_{ij}^\alpha(x, u) D_\alpha u^i D_\beta u^j)^{(p(x)/2)} dx,
\]

in the class

\( h + W^{1,p(x)}_0(\Omega) := \{ v \in W^{1,p(x)}(\Omega) ; v - h \in W^{1,p(x)}_0(\Omega) \} \),

for a given boundary data \( h \in W^{1,s}(\Omega) \) for some \( s > m \). Then \( u \in C^{0,\gamma}(\Omega_0) \) for a relatively open subset \( \Omega_0 \subset \Omega \). Moreover,

\[
dim^H(\Omega \setminus \Omega_0) < m - \gamma_1.
\]

Moreover, if \( A^\alpha_{ij}(x, u) \) are given as \( A^\alpha_{ij}(x, u) = g^{\alpha \beta}(x) G_{ij}(x, u) \), we obtain a convergence lemma for \( p(x) > 1 \), and, by virtue of this lemma and the result of Duzaar-Grotowski-Kruzik [7, Theorem 5.4], we obtain that minimizers have no singular points on the boundary.

**Theorem 3.2.** Let \( \Omega \subset \mathbb{R}^m \) be a bounded domain with \( C^1 \)-boundary \( \partial \Omega \). Assume that \( p(x) \), \( g^{\alpha \beta}(x) \), \( G_{ij}(x, u) \) satisfy the conditions (p1), (p2), (G1) and \( A^\alpha_{ij}(x, u) := g^{\alpha \beta}(x) G_{ij}(x, u) \) satisfy (A3). Let \( u \in W^{1,p(x)}(\Omega) \) be a bounded minimizer of the functional \( \mathcal{E}(v; \Omega) \) defined by

\[
\mathcal{E}(v; \Omega) := \int_\Omega (g^{\alpha \beta}(x) G_{ij}(x, v) D_\alpha v^i (x) D_\beta v^j (x))^{p(x)/2} dx,
\]

in the class \( h + W^{1,p(x)}_0(\Omega) \) for a given boundary data \( h \in W^{1,s}(\Omega) \) for some \( s > m \). Then \( u \) is Hölder continuous near the boundary \( \partial \Omega \).
The contents of Chapter 3 is a joint work with Professor Atsushi Tachikawa.

In Chapter 4, we show partial regularity of minimizers $u$ for functionals of the following type

$$A_g(u) = \int_{\Omega} \left[ (A^{\alpha\beta}_{ij}(x,u)D_{\alpha}u^iD_{\beta}u^j)^{p(x)/2} + g(x,u, Du) \right] dx,$$

assuming that $A^{\alpha\beta}_{ij}(x,u), p(x)$ and $g$ satisfy the conditions (A1), (A2), (A3), (p1), (p2) and (g1) $g$ satisfies the following growth condition.

$$|g(x,u,\xi)| \leq A \left( 1 + |\xi|^2 \right)^{q(x)/2},$$

for some constant $A > 0$, where the function $q : \Omega \to (0, +\infty)$ satisfies $p(x) - q(x) > 0,$

for all $x \in \Omega$.

For the standard growth case ($p(x) \equiv 2$), Giaquinta-Giusti [13] proved partial regularity of minimizers of the functional defined by

$$\int_{\Omega} A^{\alpha\beta}_{ij}(x)D_{\alpha}u^iD_{\beta}u^j dx + \int_{\Omega} g(x,u, Du) dx.$$

The result of this Chapter is an extension of the above result by Giaquinta-Giusti.

For the method of the proof of the following main result in this Chapter (see also [23]), we used the method of Giaquinta-Giusti [13], Ragusa-Tachikawa-Takabayashi [26] and the one shown in the result of Chapter 3 (see also [32]).

**Theorem 4.1.** Let $\Omega \subset \mathbb{R}^m$ be a bounded open set with sufficiently smooth boundary $\partial \Omega$. Assume that $A^{\alpha\beta}_{ij}$ satisfies the conditions (A1), (A2) and (A3). Let $u \in W^{1,p(x)}(\Omega)$ be a local minimizer of the functional

$$A_g(u) = \int_{\Omega} \left[ (A^{\alpha\beta}_{ij}(x,u)D_{\alpha}u^iD_{\beta}u^j)^{p(x)/2} + g(x,u, Du) \right] dx,$$

where $p : \Omega \to (1, 2)$ and $g$ satisfies (p1), (p2) and (g1). Then $u \in C^{0,\alpha}(\Omega_0)$ for some $0 < \alpha < 1$, where $\Omega_0$ is an open set of $\Omega$ with $H^{m-1}(\Omega - \Omega_0) = 0$.

The contents of Chapter 4 is a joint work with Ms. Erika Nio.

**References**


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