DOCTORAL THESIS
DOCTOR OF SCIENCE

STUDIES OF TWO-SPECIES
CHEMOTAXIS-COMPETITION
SYSTEMS AND RELATED PROBLEMS
(2種走化性・競合系及び関連する問題の研究)

March, 2019

MASAAKI MIZUKAMI
(水上 雅昭)

DEPARTMENT OF MATHEMATICS
GRADUATE SCHOOL OF SCIENCE
TOKYO UNIVERSITY OF SCIENCE
Acknowledgments

First of all I would like to express my sincere and profound gratitude to my supervisor, Professor Tomomi Yokota. My recent five years are quite fruitful thanks to him. Five years ago, when I considered which laboratories I would be in, he told me that “I would do my best for you”. I believed this promise and decided to be in his group; now I am sure that his promise is exactly kept. This became one of turning points in my life; since I belonged to his group I have spent short but productive time with him. In particular, whenever he had time not only in weekdays but also in holidays, he taught me many things, not only about mathematical things but also about lives. Moreover, he gave me many chances for letting me grow up; he recommended and supported me to attend many not only domestic conferences but also international conferences; especially, thanks to his support, I could obtain “Honorable Mentions” of Student Paper Competition of 11th AIMS conference; he gave me chances to coordinate international conferences named “iWMAC” (three times). Furthermore, aided by his grant I could visit Paderborn (Germany) again and again; every time at Paderborn I could spend marvellous time and have much mathematical discussion; I guessed that these experiences enable me to be “strong”. Without him, I could not become what I am today. In future, I would like to do my best for students as he did.

I would like to thank Professor Michael Winkler for welcoming me to Paderborn. Until now I visited Paderborn for six times; the first time is January 31–February 20, 2016 (about 3 weeks); the second time is January 11–February 20, 2017 (about 1 month); the third time is June 6–17, 2017 (about 2 weeks); the fourth time is from November 4–25, 2017 (about 3 weeks); the fifth time is January 8–31, 2018 (about 3 weeks); the sixth time is May 21–August 19, 2018 (about 3 months). Even though he was busy, he made much time to discuss with me. In every discussions he taught me interesting mathematical topics and German things including potatoes and sausages. When I was in Paderborn in 2018, he kindly invited not only his students but also me to his house, and he really entertained with amazing environment. Actually, every time before I visited Paderborn, I felt nervous because I would be alone in a different environment; however, since I was strongly encouraged by his kindness and his marvellous support directly and indirectly, I could be positive and enjoy German lives. I would like to visit him again and again together with Japanese rice!

Also I would like to thank my German collaborators: Dr. Johannes Lankeit, Mr. Tobias Black for taking care of my German lives, and writing “BLM (Black–Lankeit–Mizukami)” papers with me. When I visited Germany, every time I really enjoyed discussion (not only mathematics, but also other things) with them. I thought that experiences with them
might extremely improve my abilities in the mathematical skill and the English skill.

Dr. Lankeit is a really clever man, and the boss of the “BLM” team (and moreover, I should note that he is also “Cooking Party Venue Provider” of Cooking Party with Japanese Food!); thus, for me, he is like my another supervisor. In particular, thanks to him, whenever I write a paper, I use many time to write “a good story” for readers in an introduction. He is also a humorous guy; I felt that, when I stayed Paderborn, I met his interesting jokes everyday. Thanks to him, I could meet “Leibniz butter keks” as a good souvenir for mathematicians.

Mr. Black is one of experts in the field of a chemotaxis-fluid system; thus thanks to him I learned new estimates for solutions of a problem. He is also a very kind and humorous man. I remember that, when he found something which has “black” in its name, e.g., “Black gate” at Ueno, “Cool black” which is one of mascots of Taipei 101, we took pictures of a combination of “Mr. Black” and them. He is also a big fan of the soccer; I still remember that, on a day after Germany lost the game with Korea in 2018 FIFA World Cup, he cried when he found “Korean pork” in a menu of “Mensa”.

I would be grateful to the other foreign collaborator: Dr. Xinru Cao for helping me by doing hard works in the hard schedules. Sometimes I could use an office with her; thus I enjoyed the comfortable German lives and “orange time” in that room. Thanks to her, I could learn how to use the maximal Sobolev regularity. I hope that my knowledge of the pen spinning lets her get techniques of it. I could easily recall good memories of the cooking party with her cooked curry rice (which is, of course, very delicious!).

I would express my gratitude to Mr. Marcel Freitag and Mr. Mario Fuest; I was very happy to spend many time with them in Paderborn and Tokyo; and I was very happy that they liked my cooked Japanese foods in Cooking Party with Japanese Food.

I would be deeply thankful to Professor Jose Ignacio Tello for welcoming me to Madrid. Even though our study does not become a fruit, but I enjoyed brilliant discussion with him. He not only is a professional of mathematics, but also seems to be a professional of histories; he told me precise information of places and statues when we walked together in Madrid. Aided by his help, I could spend really comfortable time in Madrid. I would like to continue to discuss our problem with him.

I am deeply thankful to all of the participants of iWMAC: Professors Tomasz Cieslak, Elio Espejo, Akio Ito, Yoshifumi Mimura, Etsushi Nakaguchi, Koichi Osaki, Takasi Senba, Christian Stinner, Yoshie Sugiyama, Youshau Tao, Yulan Wang, Tian Xiang. I really enjoyed talks about their new results, discussions with them, and excursions. Not only meeting with them but also the experience that I could obtain through these conferences are my treasure!

I would like to thank Professor Ryuichi Suzuki for teaching me many things (including about Paderborn, and about how to cook good curry!) at Paderborn in November, 2017 and in January, 2018. Frequently we went to cafe and “Real” (one of big supermarkets) together, and moreover, in every weekends we went to town and enjoy Paderborn and good European sweets. These were good healing for me. I now would like to go to some cafe in Japan together.
I would like to thank Professors Keiichi Kato, Naoyuki Koike, Naoko Kunugi, Shizuo Miyajima, Masahito Ohta, Noboru Okazawa, Katsunori Sanada and Mieko Tanaka for teaching me an interesting mathematics world through the bachelor course. When I entered the university, actually, I wanted to be a mathematics teacher. Nevertheless, thanks to their interesting classes, I thought I wanted to be mathematician and decided to enter the Ph.D. course. Moreover, in several seminars they gave me many kind comments; these are quite helpful for my study.

I would like to express my gratitude to Professors Kentarou Yoshii, Motohiro Sobajima, Toshiyuki Suzuki, Sachiko Ishida, Yutaka Tsuzuki, Kentarou Fujie. Thanks to their kind encouragement, I could do my best. Especially, as a pioneer of a chemotaxis system in our group, I would like to thank Professor Ishida. Professor Ishida was the first Ph.D. student in our group. When I became a graduate student, she had already played an active role in the study of a chemotaxis system; when I read her papers, I thought that she did important parts in the field of a chemotaxis system; when I worked as an assistant of her class and organization of iWMAC, I was deeply impressed by her communication skill and her hospitality. Therefore she is one of my role model! I also would like to thank Professor Fujie for imposing tough guidance on me. Professor Fujie is a strict but powerful and humorous man. He showed me behavior as a senior of the group. Things he left exceedingly helped me; thus I am no match for him. He did essentially creative works in the field of a chemotaxis system; moreover, students said that his class is easy to understand and interesting. In future I want to be a teacher like him.

I wish to express my thanks to Ms. Chihiro Nishiyama, Mr. Takuya Tomidokoro, Mr. Noriaki Yoshino, Ms. Ayaka Matsubara, Mr. Takahiro Miyashige, Mr. Takahiro Hashira, Ms. Misaki Hirata, Mr. Shunsuke Kurima, Mr. Hirohiko Otsuka Mr. Ryosuke Osawa, Ms. Yuka Chiyoda, Mr. Tatsuhiko Ono, Mr. Teruto Nishino, Mr. Tsukasa Ogawa, Ms. Natsumi Yoshimiya. Thanks to “existence” of them, I could really enjoy my life. Ms. Nishiyama is my master of writing on white boards, and also making beautiful documents. Thanks to her, I could obtain JSPS Research Fellowship for Young Scientists; thus she is one of my lifesaver. Mr. Tomidokoro is a honest and kind man. Mr. Yoshino is a humorous and clever man; his knowledge is like a library.

Ms. Matsubara seems to be very shy, but is very beautiful woman like “Ms. Keiko Kitagawa”. Meeting with her is really good experience in my life. Mr. Miyashige is very interesting and clever man. When once one of our group did/said stupid things, he soon said interesting responses. Moreover, he could think about other things in that situation; he frequently notice what I forgot/did not notice, and did appropriate behavior. Mr. Hashira is a “mood maker” and humorous man. After his graduating the university, we noticed that meetings without him became extremely silent. He finally attained (two!) good fruits after he tried again and again. I learned enthusiasm of study from him. Ms. Hirata is an urbane (not only in work time but also in night and morning) and faithful woman. Thanks to her high understanding, we could obtain two HKMY papers. Mr. Kurima is a honest man, and he can do his jobs really correctly. Even though sometimes he let my jobs increase, but he frequently he reduced my jobs. Mr. Otsuka has wide knowledges. Thus he might talk with other people for arbitrary topics. Moreover, he has high computing power. Thanks to him, we could attain MOY paper. Mr. Osawa is an
earnest man and has much concentration. Moreover, he is one of my best friend. When I was in Germany, we kept in touch via LINE. Thanks to him, I never forgot my Japanese language!

Ms. Chiyoda is one of my sweets friends. She sometimes said interesting words like “Mazi respect (very respect)”. And she is “Mizukami Kikitori Kentei (listening examination of Mizukami’s fast speaking Japanese)” grade pre-1; I think I will give her grade 1 of it soon. Mr. Ono is a honest man and has high power obtained from Kendo. He is a diet mate; thanks to him I could continue to do diet for a long period.

Mr. Nishino, Mr. Ogawa and Ms. Yoshimiya are called “Nishiteru”, “Juckie” and “Natchan”, respectively, and are new generations in the group; I would expect their success.

I would like to thank the dissertation committee, the chief of the committee: Professor Tomomi Yokota and the members of the committee: Professors Emiko Ishiwata, Keiichi Kato, Naoyuki Koike, Masahito Ohta for using their busy time for me and reviewing this thesis.

I also would like to appreciate my own body for doing hard works everyday. Without this I could not work at all!

This research is partially supported by JSPS Research Fellowships for Young Scientists (No. 17J00101).

I now would like to express my gratitude to my dear girl friend, Ms. Madoka Yamazaki. She supported my hard works in my doctoral course. Moreover, she taught me how to enjoy Tokyo Disney Land/Sea (including the cuteness of Stitch). She is one of my reason to do my best.

Finally I am deeply thankful to my dear family, and my dear friends (especially: to Mr. Genki Koda, Ms. Yuko Miyamoto, Ms. Natsuki Noya, Mr. Taiga Tamagawa, Ms. Miho Tatsuhara). I was really supported by them.

Masaaki MIZUKAMI
Tokyo, February 2019
Contents

1 Introduction .......................... 1
  1.1 Background .................................. 1
  1.2 Overview .................................. 3

2 Preliminaries ......................... 5
  2.1 Gagliardo–Nirenberg type inequalities .............. 5
  2.2 Standard estimates for the Neumann heat semigroup ............... 5

I Two-species chemotaxis systems ...... 7

3 Global existence and asymptotic stability of solutions to a two-species chemotaxis system with logistic term ........ 9
  3.1 Problem and results .................................. 9
  3.2 Local existence .................................. 12
  3.3 Global existence and boundedness ...................... 12
  3.4 Asymptotic behavior .................................. 18

4 Remarks on smallness of chemotactic effect for asymptotic stability in a two-species chemotaxis system with logistic term 25
  4.1 Motivation and the main result ...................... 25
  4.2 Proof of the main result ...................... 27

5 On the weakly competitive case in a two-species chemotaxis model 33
  5.1 Background and result .................................. 33
  5.2 Global existence .................................. 37
  5.3 Global asymptotic stability ...................... 38

6 Boundedness and stabilization in a two-species chemotaxis-competition system of parabolic-parabolic-elliptic type 45
  6.1 Motivation and results .................................. 45
  6.2 Global existence and boundedness ...................... 50
  6.3 Stabilization .................................. 55
    6.3.1 Convergence. Case 1: $a_1, a_2 \in (0, 1)$ ................. 58
    6.3.2 Convergence. Case 2: $a_1 \geq 1 > a_2$ .................. 61
7 Boundedness and stabilization in a two-species chemotaxis model with
signal-dependent sensitivity and competitive term 65
7.1 Problem and results .................................................. 65
7.2 Global existence and boundedness ................................. 70
7.3 Asymptotic behavior. Case 1: $a_1, a_2 \in (0, 1)$ .................. 76
7.4 Asymptotic behavior. Case 2: $a_1 \geq 1 > a_2 > 0$ .............. 81

8 Improvement of conditions for asymptotic stability in a two-species
chemotaxis-competition model with signal-dependent sensitivity 85
8.1 Problem and results .................................................. 85
8.2 Proof of Theorem 8.1.1 .............................................. 90
8.3 Discussions .......................................................... 93

II Two-species chemotaxis systems with fluid environment 95

9 Global existence and asymptotic behaviour in a two-dimensional two-
species chemotaxis-Navier–Stokes system with competitive kinetics 97
9.1 Background and results .............................................. 97
9.2 Local existence ...................................................... 100
9.3 Boundedness. Proof of Theorem 9.1.1 ...................... 101
9.4 Stabilization. Proof of Theorem 9.1.2 ...................... 108
  9.4.1 Key estimate. Case 1: $a_1, a_2 \in (0, 1)$ .................. 108
  9.4.2 Key estimate. Case 2: $a_1 \geq 1 > a_2$ .................. 109
  9.4.3 Proof of Theorem 9.1.2 .................................. 111

10 Global existence and asymptotic behaviour in a three-dimensional two-
species chemotaxis-Navier–Stokes system with competitive kinetics 113
10.1 Problem and results .............................................. 113
10.2 Proof of Theorem 10.1.1(Global existence) .................. 116
10.3 Proof of Theorem 10.1.2(Eventual smoothness and stabilization) ... 121

11 Global existence and asymptotic behavior of classical solutions for a 3D
two-species chemotaxis-Stokes system with competitive kinetics 125
11.1 Motivation and results ............................................ 125
11.2 Local existence and basic inequality .......................... 128
11.3 Boundedness. Proof of Theorem 11.1.1 .................... 130
11.4 Stabilization. Proof of Theorem 11.1.2 .................... 140

12 Global existence and asymptotic behavior of classical solutions for a 3D
two-species Keller–Segel-Stokes system with competitive kinetics 145
12.1 Background and results ............................................ 145
12.2 Local existence and basic inequality .......................... 149
12.3 Boundedness. Proof of Theorem 12.1.1 .................... 151
12.4 Asymptotic behavior. Proof of Theorem 12.1.2 .......... 157
  12.4.1 Case 1: $a_1, a_2 \in (0, 1)$ .............................. 157
12.4.2 Case 2: $a_1 \geq 1 > a_2$ ........................................... 158
12.4.3 Convergence for $u$ ............................................. 159
12.4.4 Proof of Theorem 12.1.2 ........................................ 160

III Related works: Single-species chemotaxis systems 161

13 A unified method for boundedness in Keller–Segel systems with signal-dependent sensitivity 163
13.1 Problem and results ............................................. 163
13.2 Local existence and basic inequalities ............................................. 166
13.3 A unified view point in energy estimates ............................................. 167
13.4 Global existence and boundedness ............................................. 170
  13.4.1 Energy estimate in the case $k > 1$ ............................................. 170
  13.4.2 Energy estimates in the case $k = 1$ ............................................. 173
  13.4.3 Proof of Theorem 13.1.1 ............................................. 176

14 Global existence and boundedness in a fully parabolic chemotaxis system with signal-dependent sensitivity and logistic term 177
14.1 Problem and results ............................................. 177
14.2 Proof of the main result ............................................. 179

15 The fast signal diffusion limit in a chemotaxis system with strong signal sensitivity 185
15.1 Motivation and results ............................................. 185
15.2 Local existence and basic estimates ............................................. 189
15.3 Global existence ............................................. 193
15.4 Convergence ............................................. 197

16 The fast signal diffusion limit in a Keller–Segel system 199
16.1 Motivation and results ............................................. 199
16.2 Local existence and basic property ............................................. 205
16.3 Uniform-in-$\lambda$ boundedness ............................................. 206
  16.3.1 The higher-dimensional setting ............................................. 206
  16.3.2 The 2-dimensional setting ............................................. 211
16.4 Convergence ............................................. 213

17 Convergence of solutions to chemotaxis system to solutions of Fisher–KPP equation 217
17.1 Motivation and results ............................................. 217
17.2 Global existence and uniform-in-$\varepsilon$ boundedness ............................................. 220
17.3 Local-in-time convergence to the Fisher–KPP equation ............................................. 222
17.4 Large time behaviour in both systems ............................................. 225
17.5 Global-in-time convergence: Proof of Theorem 17.1.1 ............................................. 229
18 Global existence and boundedness in a chemotaxis-haptotaxis system with signal-dependent sensitivity

18.1 Motivation and results ............................................. 231

18.2 Local existence and basic facts .................................. 234

18.3 Global existence and boundedness ............................... 236

19 Singular sensitivity in a Keller–Segel–fluid system

19.1 Introduction ............................................................. 247

19.1.1 Chemotaxis–fluid models ...................................... 247

19.1.2 Singular sensitivity .............................................. 248

19.1.3 The combination of fluid and singular sensitivity. Results of this work 248

19.1.4 Technical challenges and plan of this chapter ............... 250

19.2 Basic properties and estimates .................................... 250

19.3 Boundedness for $u$ .................................................. 255

19.3.1 The case $\kappa = 0$ .............................................. 255

19.3.2 The case $\kappa = 1$ and $N = 2$ ............................... 257

19.4 Boundedness for $n$ .................................................. 260

20 Existence of global weak solutions to a 3-dimensional degenerate and singular chemotaxis-Navier–Stokes system with logistic source

20.1 Background and results ............................................. 265

20.2 Global existence in an approximate problem .................... 270

20.3 Uniform-in-$\varepsilon$ estimates .................................... 274

20.4 Convergence: Proof of Theorem 20.1.1 .......................... 280

List of original papers .................................................... 298

Reference ................................................................. 263

List of original papers .................................................... 278
Chapter 1

Introduction

1.1. Background

This thesis is concerned with some systems of partial differential equations modeling chemotaxis. More precisely, we consider mathematical models which describe a situation where two different species, which compete each other for limited resources, react on and move towards some chemical substance. Here chemotaxis is the property such that species move towards higher concentration of a chemical substance when they plunge into hunger. One of examples of species which have the chemotaxis is *Dictyostelium discoideum*. In 1970 and 1971 Keller and Segel studied movement of *Dictyostelium discoideum* and proposed the following system of partial differential equations ([89, 90]):

\[
\begin{align*}
    u_t &= \Delta u - \nabla \cdot (u \chi(v) \nabla v), \\
    v_t &= \Delta v - v + u,
\end{align*}
\]

(1.1.1)

where \( \chi \) is some function which implies a situation such that the chemotactic interaction is determined by the chemical stimulus, \( \tau = 0 \) (a parabolic-elliptic system) or \( \tau > 0 \) (a parabolic-parabolic system), which is called a Keller–Segel system and chemotaxis system. This system was studied intensively during the past twenty years. In this system \( u \) denotes the density of the species and \( v \) shows the chemical concentration. Here \( \Delta u \) is called the diffusion term and describes the spreading of the species, and \( -\nabla \cdot (u \chi(v) \nabla v) \) is called a chemotaxis term and represents the movement of the species according to the chemotactic effect. One of main themes of the study of the chemotaxis system is concerned with the question:

*Does the chemotactic aggregation dominate the diffusion phenomena?*

About some answer to this question, it is known that there are many blow-up solutions of (1.1.1) under some conditions; we can find several results in surveys (Horstmann [74], Hillen–Painter [71], Bellomo et. al. [5, Section 3.2])

As to one of generalizations of the chemotaxis system (1.1.1), Mimura–Tsujikawa [125] proposed the following chemotaxis system with logistic term

\[
\begin{align*}
    u_t &= \Delta u - \nabla \cdot (u \chi(v) \nabla v) + ru - \mu u^2, \\
    v_t &= \Delta v - v + u,
\end{align*}
\]

(1.1.2)

where \( r, \mu > 0 \), which describes a combination of the chemotactic movement and the population growth. An important theme of the study of the chemotaxis system with logistic term is based on the question:
Does the logistic term suppress the blow-up phenomena?

Previous works about the chemotaxis system with logistic term can be found in, e.g., [5, Section 3.3]. Recently, the generalization of the chemotaxis system (1.1.2)

\[
\begin{align*}
    u_t &= \Delta u - \nabla \cdot (u \chi_1(w)\nabla w) + \mu_1 u(1 - u - a_1 v), \\
    v_t &= \Delta v - \nabla \cdot (v \chi_2(w)\nabla w) + \mu_2 v(1 - a_2 u - v), \\
    \tau w_t &= \Delta w + u + v - w,
\end{align*}
\]

where \(\chi_1, \chi_2\) are functions and \(\mu_1, \mu_2 > 0, a_1, a_2 > 0\), which is called a two-species chemotaxis system, was proposed by Tello–Winkler [178]. This system shows evolution of two competing species which react on a single chemoattractant: Thus the two-species chemotaxis system is a more realistic problem in some sense.

On the other hand, recently, the other generalization of the system (1.1.1)

\[
\begin{align*}
    n_t + u \cdot \nabla n &= \Delta n - \nabla \cdot (n \chi(c)\nabla c), \\
    c_t + u \cdot \nabla c &= \Delta c + g(n, c), \\
    u_t + \kappa(u \cdot \nabla)u &= \Delta u + \nabla P + n\nabla \Phi, \quad \nabla \cdot u = 0,
\end{align*}
\]

where

\[g(n, c) = -nc \quad \text{(chemotaxis-(Navier–)Stokes system)}\]

or

\[g(n, c) = -c + n \quad \text{(Keller–Segel-(Navier–)Stokes system)},\]

and \(\Phi\) is a given function and \(\kappa = 0, 1\), which is a combination of the chemotaxis system and the (Navier–)Stokes equation, was proposed by Tuval et. al. [179] and was studied intensively during the past ten years. Here, in this problem, \(n\) represents the population density of species, \(c\) shows the concentration of oxygen, \(u\) stands for the fluid velocity field and \(P\) denotes the pressure of the fluid. This system is called a chemotaxis-fluid system, and describes evolution of a species which move toward higher concentration of oxygen (e.g. Bacillus subtilis) in a drop of water. One of themes of the study of the chemotaxis-fluid system is come from the question:

**How does the fluid affect regularities of solutions to the chemotaxis system?**

Recent results about the chemotaxis-fluid system can be found in [5, Section 4]. Based on this chemotaxis-fluid system, in [72], we introduced the following two-species chemotaxis-fluid system

\[
\begin{align*}
    (n_1)_t + u \cdot \nabla n_1 &= \Delta n_1 - \nabla \cdot (n_1 \chi_1(c)\nabla c) + \mu_1 (1 - n_1 - a_1 n_2), \\
    (n_2)_t + u \cdot \nabla n_2 &= \Delta n_2 - \nabla \cdot (n_2 \chi_2(c)\nabla c) + \mu_2 (1 - a_2 n_2 - n_1), \\
    c_t + u \cdot \nabla c &= \Delta c + g(n_1, n_2, c), \\
    u_t + \kappa(u \cdot \nabla)u &= \Delta u + \nabla P + (n_1 + n_2)\nabla \Phi, \quad \nabla \cdot u = 0,
\end{align*}
\]

which is a mixed problem of the two-species chemotaxis system and the chemotaxis-fluid system.

In this thesis we show global existence and asymptotic behavior of solutions to the two-species chemotaxis systems with/without fluid environment.
1.2. Overview

This thesis is organized as follows. In the next chapter we collect basic inequalities which is often used in this thesis. The main part of this thesis consists of the following three parts.

Part I

In Part I we deal with the two-species chemotaxis system (1.1.3) in \( \Omega \times (0, \infty) \) under Neumann boundary conditions

\[
\nabla u \cdot \nu = \nabla v \cdot \nu = \nabla w \cdot \nu = 0 \quad \text{on } \partial \Omega \times (0, \infty)
\]

and initial conditions

\[
u(\cdot, 0) = u_0, \quad v(\cdot, 0) = v_0, \quad \tau w(\cdot, 0) = \tau w_0 \quad \text{in } \Omega,
\]

where \( \Omega \subset \mathbb{R}^n \) \((n \in \mathbb{N})\) is a bounded smooth domain and \( u_0, v_0, w_0 \) are smooth functions; we obtain global existence of classical solutions to the non-competitive two-species chemotaxis system \(((1.1.3)-(1.2.1)-(1.2.2))\) with \( a_1 = a_2 = 0 \) and \( \tau = 1 \) in Chapter 3 (which is based on Mizukami–Yokota [137]), and their stabilization in Chapters 3 and 4 (which is grounded on Mizukami [128]); in Chapters 5 and 6 (which are originated from Black–Lankeit–Mizukami [19] and Mizukami [131], respectively) we establish global existence and stabilization in the two-species chemotaxis system of parabolic–parabolic–elliptic type \(((1.1.3)-(1.2.1)-(1.2.2))\) with \( \tau = 0 \); Chapters 7 and 8 (which are originated from Mizukami [129] and [135], respectively) are devoted to showing global existence and stabilization in the fully parabolic two-species chemotaxis system \(((1.1.3)-(1.2.1)-(1.2.2))\) with \( \tau = 1 \).

Part II

In Part II we deal with the two-species chemotaxis-fluid system (1.1.5) in \( \Omega \times (0, \infty) \) under boundary conditions

\[
\nabla n_1 \cdot \nu = \nabla n_2 \cdot \nu = \nabla c \cdot \nu = 0, \quad u = 0 \quad \text{on } \partial \Omega \times (0, \infty)
\]

and initial conditions

\[
n(\cdot, 0) = n_0, \quad c(\cdot, 0) = c_0, \quad u(\cdot, 0) = u_0 \quad \text{in } \Omega,
\]

where \( \Omega \subset \mathbb{R}^N \) \((N = 2, 3)\) is a bounded smooth domain and \( n_0, c_0, u_0 \) are given smooth functions; we obtain global existence of classical solutions and their stabilization in the two-dimensional chemotaxis-Navier–Stokes system \(((1.1.5)-(1.2.3)-(1.2.4))\) with \( \kappa = 1 \) in Chapter 9 (which is grounded on Hirata–Kurima–Mizukami–Yokota [72]), and global existence of weak solutions and their large time behavior in the three-dimensional setting in Chapter 10 (which is grounded on Hirata–Kurima–Mizukami–Yokota [73]); Chapters 11 and 12 (which are based on Cao–Kurima–Mizukami [28] and [29], respectively) are devoted to establishing global existence of classical solutions to the three-dimensional two-species chemotaxis-fluid system \(((1.1.5)-(1.2.3)-(1.2.4))\) with \( \kappa = 0 \), and their asymptotic behavior.
Part III

In Part III we consider related problems; in Chapters 13 and 14 (which are originated from Mizukami–Yokota [138] and Mizukami [130], respectively) we obtain results on the chemotaxis system with signal-dependent sensitivity: (1.1.1) (or (1.1.2)) in $\Omega \times (0, \infty)$ under homogeneous Neumann boundary conditions and initial conditions; in Chapters 15 and 16 (which are grounded on Mizukami [132] and [134], respectively) we show convergences of solutions to a parabolic–parabolic chemotaxis system (1.1.1) in $\Omega \times (0, \infty)$ under homogeneous Neumann boundary conditions and initial conditions as $\tau \searrow 0$. Chapter 17 (which is based on Lankeit–Mizukami [103]) is concerned with a relation between classical solutions of the chemotaxis system and those of the Fisher–KPP equation: We study the chemotaxis system with logistic term

\begin{equation}
(1.2.5)
\begin{cases}
(u_{\varepsilon})_t = \Delta u_{\varepsilon} - \varepsilon \nabla \cdot (u_{\varepsilon} \nabla v_{\varepsilon}) + \mu u_{\varepsilon}(1 - u_{\varepsilon}), & x \in \Omega, \ t > 0, \\
v_{\varepsilon} = \Delta v_{\varepsilon} - v_{\varepsilon} + u_{\varepsilon}, & x \in \Omega, \ t > 0, \\
\nabla u_{\varepsilon} \cdot \nu = \nabla v_{\varepsilon} \cdot \nu = 0, & x \in \partial \Omega, \ t > 0, \\
u_{\varepsilon}(\cdot, 0) = u_{\text{init}}, \ v_{\varepsilon}(\cdot, 0) = v_{\text{init}}, & x \in \Omega,
\end{cases}
\end{equation}

where $\Omega \subset \mathbb{R}^n \ (n \in \mathbb{N})$ is a bounded convex smooth domain, and verify that solutions of (1.2.5) converge to those of Fisher–KPP equation (which is (1.2.5) with $\varepsilon = 0$) as $\varepsilon \searrow 0$; Chapter 18 (which is based on Mizukami–Otsuka–Yokota [136]) is devoted to deriving global existence of classical solutions to a chemotaxis-haptotaxis system:

\begin{equation}
(1.2.6)
\begin{cases}
u_t = \Delta \nu - \nu \chi(\nu) \nabla \nu - \xi \nabla \cdot (\nu \nabla w) + \mu \nu(1 - \nu - w), & x \in \Omega, \ t > 0, \\
v_t = \Delta \nu - \nu + u, \ & x \in \Omega, \ t > 0, \\
w_t = -\nu w, \ & x \in \Omega, \ t > 0, \\
\nabla u \cdot \nu - \nu \chi(\nu) \nabla \nu \cdot \nu - \xi \nabla w \cdot \nu = \nabla v \cdot \nu = 0, & x \in \partial \Omega, \ t > 0, \\
u(\cdot, 0) = u_0, \ v(\cdot, 0) = v_0, \ w(\cdot, 0) = w_0, & x \in \Omega,
\end{cases}
\end{equation}

where $\Omega \subset \mathbb{R}^n \ (n \in \mathbb{N})$ is a bounded smooth domain; in Chapters 19 and 20 (which are originated from Black–Lankeit–Mizukami [20] and Kurima–Mizukami [95], respectively) we investigate the generalization of the chemotaxis-fluid system

\begin{equation}
(1.2.7)
\begin{cases}
u_t = \Delta \nu - \nu \chi(c) \nabla c + r \nu - \mu \nu^2, & x \in \Omega, \ t > 0, \\
\nu_t = \Delta \nu + \nu \nabla P + \mu \nabla \Phi, \ \nabla \cdot \nu = 0, & x \in \Omega, \ t > 0, \\
\nabla c \cdot \nu = 0, \ u = 0, & x \in \partial \Omega, \ t > 0, \\
u(\cdot, 0) = n_0, \ c(\cdot, 0) = c_0, \ u(\cdot, 0) = u_0, & x \in \Omega,
\end{cases}
\end{equation}

where $\Omega \subset \mathbb{R}^N \ (N = 2, 3)$ and $m > 0$, and show existence of global solutions; in Chapter 19 we study (1.2.6) with $m = 1$, $\chi(c) = \frac{\chi_0}{c}$ with some $\chi_0 > 0$, $r = \mu = 0$ and $g(n, c) = -c + n$; in Chapter 20 we study (1.2.6) with $m > 0$, $\chi(c) = \chi_0$ with some $\chi_0 > 0$, $r, \mu > 0$ and $g(n, c) = -cn$.
Chapter 2

Preliminaries

In this chapter we state some lemmas which are often used in this thesis. We first recall Gagliardo–Nirenberg type inequalities, and then we collect standard estimates for the Neumann heat semigroup on bounded domains.

2.1. Gagliardo–Nirenberg type inequalities

The standard Gagliardo–Nirenberg inequality is stated in e.g., [48, Theorem 10.1] and [77, Section 2]. In this thesis the following less common version of the Gagliardo–Nirenberg inequality is frequently used. Here the precise proof of the following lemma can be found in the proof of [113, Lemma 2.3].

Lemma 2.1.1. Let \( \Omega \subset \mathbb{R}^d \) be a bounded domain with smooth boundary and let \( r \geq 1, \, 0 < q < p \leq \infty, \, s > 0 \) be such that \( \frac{1}{r} \leq \frac{1}{d} + \frac{1}{p} \). Then there exist \( C_1, C_2 > 0 \) such that

\[
\|w\|_{L^p(\Omega)} \leq C_1\|w\|_{W^{1,r}(\Omega)}\|w\|_{L^q(\Omega)}^{1-a}
\]

and

\[
\|w\|_{L^p(\Omega)} \leq C_2(\|\nabla w\|_{L^r(\Omega)}\|w\|_{L^q(\Omega)}^{1-a} + \|w\|_{L^r(\Omega)}^a)
\]

hold for all \( w \in W^{1,r}(\Omega) \cap L^q(\Omega) \), where \( a = \frac{\frac{1}{r} - \frac{1}{p}}{\frac{1}{r} + \frac{1}{s} - \frac{1}{p}} \).

2.2. Standard estimates for the Neumann heat semigroup

In this section we collect standard estimates for the heat semigroup under homogeneous Neumann boundary conditions. We first recall the following well-known facts concerning the Laplacian in \( \Omega \) supplemented with homogeneous Neumann boundary conditions (for details, see [68, 77]).

Lemma 2.2.1. Let \( \Omega \subset \mathbb{R}^d \) be a bounded domain with smooth boundary and suppose that \( k > 0 \). Let \( \Delta \) denote the realization of the Laplacian in \( L^s(\Omega) \) with domain \( \{z \in W^{2,s}(\Omega) \mid \nabla z \cdot \nu = 0 \text{ on } \partial \Omega\} \) for \( s \in (1, \infty) \). Then the operator \(-\Delta + k\) is sectorial and possesses closed fractional powers \((-\Delta + k)^\eta, \, \eta \in (0, 1)\), with dense domain \( D((-\Delta + k)^\eta) \). Moreover, the following holds.
(i) If \( m \in \{0, 1\} \), \( p \in [1, \infty] \) and \( q \in (1, \infty) \), then there exists a constant \( c_1 > 0 \) such that for all \( w \in \mathcal{D}((-\Delta + k)^n) \),

\[
\|w\|_{W^{m,p}(\Omega)} \leq c_1\|(-\Delta + k)^nw\|_{L^q(\Omega)},
\]

provided that \( m < 2q \) and \( m - d/p < 2q - d/q \).

(ii) Suppose that \( p \in [1, \infty) \). Then the associated heat semigroup \((e^{t\Delta})_{t \geq 0}\) maps \( L^p(\Omega) \) into \( \mathcal{D}((-\Delta + k)^n) \) in any of the space \( L^q(\Omega) \), \( q \geq p \), and there are \( c_2 > 0 \) and \( \lambda > 0 \) such that for all \( w \in L^p(\Omega) \),

\[
\|(-\Delta + k)^ne^{t\Delta-k}w\|_{L^q(\Omega)} \leq c_2e^{-\eta\frac{t}{2}}e^{-\lambda t}\|w\|_{L^p(\Omega)} \quad (t > 0).
\]

(iii) Let \( p \in (1, \infty) \). Then there exists \( \lambda > 0 \) such that for every \( \varepsilon > 0 \) there exists \( c_3 > 0 \) satisfying

\[
(2.2.1) \quad \|(-\Delta + k)^ne^{t\Delta-k}\nabla \cdot w\|_{L^p(\Omega)} \leq c_3e^{-\eta\varepsilon t}\|w\|_{L^p(\Omega)} \quad (t > 0)
\]

for all \( \mathbb{R}^d \)-valued \( w \in C^{0,\varepsilon}_0(\Omega) \). Accordingly, the operator \((-\Delta + k)^ne^{t\Delta-k}\nabla \cdot \) admits a unique extension to all of \( L^p(\Omega) \) which, again denoted by \((-\Delta + k)^ne^{t\Delta-k}\nabla \cdot \), satisfies (2.2.1) for all \( \mathbb{R}^d \)-valued \( w \in L^p(\Omega) \).

We next introduce \( L^p-L^q \) estimates for the heat semigroup under homogeneous Neumann boundary conditions (see [194, Lemma 1.3] and [25, Lemma 2.1]).

**Lemma 2.2.2.** Suppose that \( \Omega \subset \mathbb{R}^d \) \((d \in \mathbb{N})\) is a bounded domain with smooth boundary. Let \((e^{t\Delta})_{t \geq 0}\) be the Neumann heat semigroup in \( \Omega \), and let \( \lambda_1 > 0 \) denote the first nonzero eigenvalue of \(-\Delta \) in \( \Omega \) under Neumann boundary conditions. Then there exists \( k_1, \ldots, k_4 > 0 \) which only depend on \( \Omega \) and which have the following properties:

(i) If \( 1 \leq p \leq q \leq \infty \), then

\[
\|e^{t\Delta}w\|_{L^p(\Omega)} \leq k_1(1 + t^{-\frac{d}{2}}(\frac{1}{q} - \frac{1}{p})\lambda_1 t)e^{-\lambda_1 t}\|w\|_{L^q(\Omega)} \quad (t > 0)
\]

holds for all \( w \in L^q(\Omega) \) with \( \int_\Omega w = 0 \).

(ii) If \( 1 \leq q \leq p \leq \infty \), then

\[
\|\nabla e^{t\Delta}w\|_{L^p(\Omega)} \leq k_2(1 + t^{-\frac{d}{2}}(\frac{1}{q} - \frac{1}{p})\lambda_1 t)e^{-\lambda_1 t}\|w\|_{L^q(\Omega)} \quad (t > 0)
\]

holds for each \( w \in L^q(\Omega) \).

(iii) If \( 2 \leq q \leq p < \infty \), then

\[
\|\nabla e^{t\Delta}w\|_{L^p(\Omega)} \leq k_3(1 + t^{-\frac{d}{2}}(\frac{1}{q} - \frac{1}{p})\lambda_1 t)e^{-\lambda_1 t}\|\nabla w\|_{L^q(\Omega)} \quad (t > 0)
\]

is true for all \( w \in W^{1,q}(\Omega) \).

(iv) If \( 1 < q \leq p < \infty \), then

\[
(2.2.2) \quad \|e^{t\Delta}\nabla \cdot w\|_{L^p(\Omega)} \leq k_4(1 + t^{-\frac{d}{2}}(\frac{1}{q} - \frac{1}{p})\lambda_1 t)e^{-\lambda_1 t}\|w\|_{L^q(\Omega)} \quad (t > 0)
\]

is valid for all \( w \in (C^0_0(\Omega))^d \). Consequently, for all \( t > 0 \) the operator \( e^{t\Delta}\nabla \cdot \) possesses a uniquely determined extension to an operator from \( L^q(\Omega) \) into \( L^p(\Omega) \), which norm controlled according to (2.2.2).
Part I
Two-species chemotaxis systems
Chapter 3

Global existence and asymptotic stability of solutions to a two-species chemotaxis system with logistic term

3.1. Problem and results

In this chapter we consider the two-species chemotaxis system

\[
\begin{cases}
    u_t = \Delta u - \nabla \cdot (u \chi_1(w) \nabla w) + \mu_1 u(1 - u), & x \in \Omega, \ t > 0, \\
    v_t = \Delta v - \nabla \cdot (v \chi_2(w) \nabla w) + \mu_2 v(1 - v), & x \in \Omega, \ t > 0, \\
    w_t = d \Delta w + h(u, v, w), & x \in \Omega, \ t > 0, \\
    \nabla u \cdot \nu = \nabla v \cdot \nu = \nabla w \cdot \nu = 0, & x \in \partial \Omega, \ t > 0, \\
    u(x, 0) = u_0(x), v(x, 0) = v_0(x), w(x, 0) = w_0(x), & x \in \Omega,
\end{cases}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) (\( n \in \mathbb{N} \)) with smooth boundary \( \partial \Omega \) and \( \nu \) is the outward normal vector to \( \partial \Omega \). The initial data \( u_0, v_0 \) and \( w_0 \) are assumed to be nonnegative functions. The unknown functions \( u(x, t) \) and \( v(x, t) \) represent the population densities of two species and \( w(x, t) \) shows the concentration of the substance at place \( x \) and time \( t \).

In a mathematical view, global existence and behavior of solutions are fundamental theme. However, the problem (3.1.1) has some difficult points caused by the logistic term and by generalization of \( \chi_1 \) and \( h \). For example, we cannot use the Lyapunov function. To overcome the difficulty, Negreanu–Tello [150, 151] built a technical way to prove global existence and asymptotic behavior of solutions to (3.1.1). In [151] they dealt with (3.1.1) when \( d = 0, \mu_i > 0 \) under the condition

\[
\exists \overline{w} \geq w_0; \ h(\overline{u}, \overline{v}, \overline{w}) \leq 0,
\]

where \( \overline{u}, \overline{v} \) satisfy some representations determined by \( \overline{w} \). In [150] they studied (3.1.1) when \( 0 < d < 1, \mu_i = 0 \) under similar conditions as in [151] and

\[
\chi_i' + \frac{1}{1-d} \chi_i^2 \leq 0 \quad (i = 1, 2).
\]
They supposed in [150, 151] that the functions $h, \chi_i$ for $i = 1, 2$ generalize of the prototypical case $\chi_i(w) = \frac{x_{0,i}}{(1+w)^{\sigma_i}}$ ($\chi_{0,i} > 0, \sigma_i > 1$), $h(u, v, w) = u + v - w$. As to the special case that $d = 1$ and $h(u, v, w) = u + v - w$, Zhang–Li [213] proved global existence of solutions to (3.1.1) under the assumption that $\mu_i$ is small and $\chi_i(w) \leq \frac{x_{0,i}}{(1+w)^{\sigma_i}}$ for $\sigma_i > 1, x_{0,i} > 0$ being small enough.

The purpose of the present chapter is to obtain global existence and asymptotic stability of solutions to (3.1.1) without the restriction of $0 \leq d < 1$. We shall suppose throughout this chapter that $h, \chi_i (i = 1, 2)$ satisfy the following conditions:

$$\begin{align*}
(3.1.3) & \quad \chi_i \in C^{1+\theta}([0, \infty)) \cap L^1([0, \infty)) (0 < \exists \theta < 1), \quad \chi_i > 0 \quad (i = 1, 2), \\
(3.1.4) & \quad h \in C^1([0, \infty) \times [0, \infty) \times [0, \infty)), \quad h(0, 0, 0) \geq 0, \\
(3.1.5) & \quad \exists \gamma > 0; \quad \frac{\partial h}{\partial u}(u, v, w) \geq 0, \quad \frac{\partial h}{\partial v}(u, v, w) \geq 0, \quad \frac{\partial h}{\partial w}(u, v, w) \leq -\gamma, \\
(3.1.6) & \quad \exists \delta > 0, \exists M > 0; \quad |h(u, v, w) + \delta w| \leq M(u + v + 1), \\
(3.1.7) & \quad \exists k_i > 0; \quad -\chi_i(w)h(0, 0, w) \leq k_i \quad (i = 1, 2).
\end{align*}$$

We also assume that

$$\begin{align*}
(3.1.8) & \quad \exists p > n; \quad 2d\chi'_i(w) + \left( (d-1)p + \sqrt{(d-1)^2p^2 + 4dp} \right) [\chi_i(w)]^2 \leq 0 \quad (i = 1, 2).
\end{align*}$$

The above conditions cover the prototypical example $\chi_i(w) = \frac{x_{0,i}}{(1+w)^{\sigma_i}}$ ($\chi_{0,i} > 0, \sigma_i > 1$), $h(u, v, w) = u + v - w$. Asymptotic behavior of solutions to (3.1.1) will be discussed under the following additional condition: There exists $\tau > 0$ such that

$$\begin{align*}
(3.1.9) & \quad \max\{C_u(\tau)\chi_1(0)\|h_u\|_{L^\infty(\Omega)}, C_v(\tau)\chi_2(0)\|h_v\|_{L^\infty(\Omega)}\} < \frac{1}{2}\gamma, \\
(3.1.10) & \quad C_w(\tau)^2\chi_i(0)^2 < 4\mu_i \quad (i = 1, 2),
\end{align*}$$

where $h_u := \partial h/\partial u$, $h_v := \partial h/\partial v$, $I_\tau := (0, C_u(\tau)) \times (0, C_v(\tau)) \times (0, C_w(\tau))$ and $C_u(\tau)$, $C_v(\tau)$, $C_w(\tau)$ will be defined in the proof of Theorem 3.1.1. We assume that the initial data $u_0, v_0, w_0$ satisfy

$$\begin{align*}
(3.1.11) & \quad 0 \leq u_0 \in C(\overline{\Omega}) \setminus \{0\}, \quad 0 \leq v_0 \in C(\overline{\Omega}) \setminus \{0\}, \quad 0 \leq w_0 \in W^{1,q}(\Omega) \quad (\exists q > n).
\end{align*}$$

Now the main results read as follows. The first theorem is concerned with global existence and boundedness in (3.1.1).

**Theorem 3.1.1.** Let $d \geq 0$, $\mu_i > 0$ ($i = 1, 2$). Assume that $h, \chi_i$ satisfy (3.1.3)–(3.1.8). Then for any $u_0, v_0, w_0$ satisfying (3.1.11) for some $q > n$, there exists an exactly one pair $(u, v, w)$ of nonnegative functions

$$\begin{align*}
&u, \quad v, \quad w \in C(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)) \quad \text{when} \ d > 0, \\
&u, \quad v, \quad w \in C([0, \infty); W^{1,q}(\Omega)) \cap C^1((0, \infty); W^{1,q}(\Omega)) \quad \text{when} \ d = 0,
\end{align*}$$

which satisfy (3.1.1). Moreover, the solution $(u, v, w)$ is uniformly bounded, i.e., there exists a constant $C > 0$ such that

$$\begin{align*}
\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all} \ t \geq 0.
\end{align*}$$
Remark 3.1.1. When $0 < d < 1$, we note that the condition (3.1.8) in Theorem 3.1.1 relaxes (3.1.2) assumed in [150], because the following relation holds:

$$
\frac{(d-1)p + \sqrt{(d-1)^2p^2 + 4dp}}{2d} < \frac{1}{1 - d}.
$$

Remark 3.1.2. We excluded the case $\mu_i = 0$ $(i = 1, 2)$ in Theorem 3.1.1. It seems to be indefinite whether our proof of boundedness is applicable to such case.

The second one gives asymptotic behavior of solutions to (3.1.1).

Theorem 3.1.2. Let $d > 0$, $\mu_i > 0$ $(i = 1, 2)$. Under the conditions (3.1.3)–(3.1.8) and (3.1.9), (3.1.10), the unique global solution $(u, v, w)$ of (3.1.1) has the following asymptotic behavior:

$$
\|u(\cdot, t) - 1\|_{L^\infty(\Omega)} \to 0, \quad \|v(\cdot, t) - 1\|_{L^\infty(\Omega)} \to 0, \quad \|w(\cdot, t) - \tilde{w}\|_{L^\infty(\Omega)} \to 0 \quad (t \to \infty),
$$

where $\tilde{w} \geq 0$ is such that $h(1, 1, \tilde{w}) = 0$.

Remark 3.1.3. From (3.1.4)–(3.1.6) there exists $\tilde{w}$ such that $h(1, 1, \tilde{w}) = 0$. Indeed, if we choose $\tilde{w} \geq 3M/\delta$, then (3.1.6) yields that $h(1, 1, \tilde{w}) \leq 3M - \delta \tilde{w} \leq 0$. On the other hand, (3.1.4) and (3.1.5) imply that $h(1, 1, 0) \geq h(0, 0, 0) \geq 0$. Hence, by the intermediate value theorem there exists $\tilde{w} \geq 0$ such that $h(1, 1, \tilde{w}) = 0$.

Remark 3.1.4. Theorem 3.1.2 shows that the solutions $u$, $v$ have the same behavior. This theorem is a new result not only in two-species chemotaxis system but also in chemotaxis-growth system (single-species case); note that there are not any result about asymptotic behavior of solutions to the fully parabolic system. The difference between one-species and two-species appears only in dealing with $w$ because there are not competitive terms. However the difference is not so big because we can make an estimate for $h(u, v, w)$ by using some conditions. By using a different method we will study the case that the two-species compete with each other, e.g., $\mu_i u(1 - a_1 u - a_2 v)$ in Chapter 7 later.

The strategy for the proof of Theorem 3.1.1 is to construct estimates for $\int_\Omega u^p$ and $\int_\Omega v^p$. One of the keys for this strategy is to derive inequality

$$(3.1.12) \quad \frac{d}{dt} \int_\Omega u^p[f_1(w)]^{-r} \leq a \int_\Omega u^p[f_1(w)]^{-r} - b \left( \int_\Omega u^p[f_1(w)]^{-r} \right)^{\frac{r+1}{r}}$$

for some positive constants $a, b$, where $f_1(w) := \exp \left\{ \int_0^w \chi_1(s) \, ds \right\}$. Negreanu–Tello [150, 151] proved a similar differential inequality for “all” $p \geq 1$ and $r := \frac{(p-1)p}{p-d(p-1)}$. In this work we derive (3.1.12) for “some” $p > n$ and some $r = r(d, p) > 0$ by modifying the proof in [150, 151]. This enables us to improve the previous work and to remove the restriction of $0 \leq d < 1$. On the other hand, the strategy for the proof of Theorem 3.1.2 is to modify an argument in [151]. The key for this strategy is to obtain positive lower bounds for $\int_\Omega u$ and $\int_\Omega v$ by modifying an argument in [56].

This chapter is organized as follows. In Section 3.2 we collect basic facts which will be used later. In Section 3.3 we prove global existence and boundedness (Theorem 3.1.1). Section 3.4 is devoted to the proof of asymptotic stability (Theorem 3.1.2).
3.2. Local existence

In this section we recall the following result concerned with local existence of classical solutions to (3.1.1).

**Lemma 3.2.1.** Let \( d \geq 0, \mu_i \geq 0 \) \((i = 1, 2)\). Assume that \( h, \chi_i \) satisfy (3.1.3), (3.1.4), (3.1.6). Then for any \( u_0, v_0, w_0 \) satisfying (3.1.11) for some \( q > n \), there exist \( T_{\text{max}} \in (0, \infty) \) and an exactly one pair \((u, v, w)\) of nonnegative functions

\[
\begin{align*}
&u, v, w \in C(\Omega \times [0, T_{\text{max}})) \cap C^2(\Omega \times (0, T_{\text{max}})) \quad \text{when} \ d > 0, \\
&u, v, w \in C([0, T_{\text{max}}); W^{1,q}(\Omega)) \cap C^1((0, T_{\text{max}}); W^{1,q}(\Omega)) \quad \text{when} \ d = 0,
\end{align*}
\]

which satisfy (3.1.1). Moreover,

either \( T_{\text{max}} = \infty \) or \( \lim_{t \to T_{\text{max}}} (\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)}) = \infty. \)

**Proof.** We first consider the case \( d > 0 \). The proof of local existence of classical solutions to (3.1.1) is based on a standard contraction mapping argument, which can be found in [193, 195]. The case \( d = 0 \) is show in [151]. Finally the maximum principle is applied to yield \( u > 0, v > 0, w \geq 0 \) in \( \Omega \times (0, T_{\text{max}}). \)

3.3. Global existence and boundedness

Let \((u, v, w)\) be the solution to (3.1.1) on \([0, T_{\text{max}}]\) as in Lemma 3.2.1. We introduce the functions \( f_1 = f_1(w) \) and \( f_2 = f_2(w) \) by

\[
(3.3.1) \quad f_i(w) := \exp \left\{ \int_0^w \chi_i(s) \, ds \right\} \quad \text{for} \ i = 1, 2
\]

to prove the following lemma.

**Lemma 3.3.1.** Let \( d \geq 0, \mu_i \geq 0 \) \((i = 1, 2)\). Assume that \( \chi_i \) satisfy (3.1.3) and (3.1.8) with some \( p > n \). Then there exists \( r = r(d, p) > 0 \) such that

\[
\begin{align*}
(3.3.2) \quad &\frac{d}{dt} \int_\Omega u^p f_1^{-r} \leq p \mu_1 \int_\Omega u^p f_1^{-r}(1 - u) - r \int_\Omega u^p f_1^{-r} \chi_1(w) h(u, v, w), \\
(3.3.3) \quad &\frac{d}{dt} \int_\Omega v^p f_2^{-r} \leq p \mu_2 \int_\Omega v^p f_2^{-r}(1 - v) - r \int_\Omega v^p f_2^{-r} \chi_2(w) h(u, v, w).
\end{align*}
\]

**Proof.** We let \( p \geq 1 \) be fixed later. From the first and third equations in (3.1.1) we have

\[
\frac{d}{dt} \int_\Omega u^p f_1^{-r} = p \int_\Omega u^{p-1} f_1^{-r} \nabla \cdot (\nabla u - u \chi_1(w) \nabla w) + \mu_1 \int_\Omega u^p f_1^{-r}(1 - u) \\
- rd \int_\Omega u^p f_1^{-r} \chi_1(w) \Delta w - r \int_\Omega u^p f_1^{-r} \chi_1(w) h(u, v, w).
\]

Denoting by \( I_1 \) and \( I_2 \) the first and third terms on the right-hand side as

\[
\begin{align*}
I_1 &:= p \int_\Omega u^{p-1} f_1^{-r} \nabla \cdot (\nabla u - u \chi_1(w) \nabla w), \\
I_2 &:= -rd \int_\Omega u^p f_1^{-r} \chi_1(w) \Delta w,
\end{align*}
\]

12
we can write as

\begin{equation}
\frac{d}{dt} \int_{\Omega} u^p f_1^{-r} = I_1 + I_2 + p\mu_1 \int_{\Omega} u^p f_1^{-r} (1 - u) - r \int_{\Omega} u^p f_1^{-r} \chi_1(w) h(u,v,w). \tag{3.3.4}
\end{equation}

We shall show that the following inequality:

\[ \exists p > n, \exists r > 0; I_1 + I_2 \leq 0. \]

Noting that

\[ f_1 \nabla \left( \frac{u}{f_1} \right) = \nabla u - u \chi_1(w) \nabla w, \]

we obtain

\[
I_1 = p \int_{\Omega} u^{p-1} f_1^{-r} \nabla \cdot \left(f_1 \nabla \left( \frac{u}{f_1} \right) \right)
= p \int_{\Omega} \left( \frac{u}{f_1} \right)^{p-1} f_1^{-r+p-1} \nabla \cdot \left(f_1 \nabla \left( \frac{u}{f_1} \right) \right)
= -p(p - 1) \int_{\Omega} \left( \frac{u}{f_1} \right)^{p-2} f_1^{-r+p} \left| \nabla \left( \frac{u}{f_1} \right) \right|^2
- p(-r + p - 1) \int_{\Omega} \left( \frac{u}{f_1} \right)^{p-1} f_1^{-r+p} \chi_1(w) \nabla \left( \frac{u}{f_1} \right) \cdot \nabla w.
\]

Similarly, we see that

\[
I_2 = -rd \int_{\Omega} \left( \frac{u}{f_1} \right)^p f_1^{-r+p} \chi_1(w) \Delta w
= rdp \int_{\Omega} \left( \frac{u}{f_1} \right)^{p-1} f_1^{-r+p} \chi_1(w) \nabla \left( \frac{u}{f_1} \right) \cdot \nabla w
+ rd \int_{\Omega} \left( \frac{u}{f_1} \right)^p f_1^{-r+p} \left((-r + p)[\chi_1(w)]^2 + \chi_1(w) \right) |\nabla w|^2.
\]

Therefore it follows that

\[
I_1 + I_2
= -p(p - 1) \int_{\Omega} \left( \frac{u}{f_1} \right)^{p-2} f_1^{-r+p} \left| \nabla \left( \frac{u}{f_1} \right) \right|^2
- (p(p - 1) - (1 + d)p) \int_{\Omega} \left( \frac{u}{f_1} \right)^{p-1} f_1^{-r+p} \chi_1(w) \nabla \left( \frac{u}{f_1} \right) \cdot \nabla w
+ \int_{\Omega} \left( \frac{u}{f_1} \right)^p f_1^{-r+p} \left( dr(-r + p)[\chi_1(w)]^2 + dr\chi_1(w) \right) |\nabla w|^2
= -p(p - 1) \int_{\Omega} \left( \frac{u}{f_1} \right)^{p-2} f_1^{-r+p} \left| \nabla \left( \frac{u}{f_1} \right) \right|^2
+ \frac{p(p - 1) - (1 + d)p}{2p(p - 1)} \chi_1(w) \frac{u}{f_1} \nabla \left( \frac{u}{f_1} \right) \cdot \nabla w
+ \int_{\Omega} \left( \frac{u}{f_1} \right)^p f_1^{-r+p} \left( \frac{dr(-r + p)}{4p(p - 1)} \chi_1(w) \right)^2 + dr\chi_1(w) \right) |\nabla w|^2.
\]

13
Here we write as
\[
\left( \frac{(p(p-1)-(1+d)p\gamma)^2}{4p(p-1)} + dr(-r+p) \right) [\chi_1(w)]^2 + d\gamma'(w)
= \frac{1}{4p(p-1)}(a_1r^2 + 2a_2r + a_3),
\]
where \(a_1, a_2, a_3\) are given by
\[
a_1 := ((d-1)^2p + 4d)[\chi_1(w)]^2,
a_2 := (p-1)(p(d-1)[\chi_1(w)]^2 + 2d\gamma'(w)),
a_3 := p(p-1)^2[\chi_1(w)]^2.
\]
Then there exists \(p > n\) such that the discriminant
\[
D_r = 4(p-1)^2[(px_1^2(d-1) + 2d\gamma'(w))^2 - p\gamma_1^4(p(d-1)^2 + 4d)]
\]
is nonnegative in view of (3.1.8). Therefore we have that there exists \(r > 0\) such that
\[
I_1 + I_2 \leq 0.
\]
Hence (3.3.4) implies
\[
\frac{d}{dt} \int_{\Omega} u^p f_1^{-r} \leq p\mu_1 \int_{\Omega} u^p f_1^{-r}(1-u) - r \int_{\Omega} u^p f_1^{-r}\chi_1 h(u,v,w).
\]
This means that (3.3.2) holds. In the same way, we obtain (3.3.3). \(\square\)

**Lemma 3.3.2.** Let \(d \geq 0, \mu_i > 0\) \((i = 1, 2)\). Assume that \(h, \chi_i\) satisfy (3.1.3)–(3.1.5), (3.1.7), and (3.1.8) with some positive constants \(k_i\) \((i = 1, 2)\) and \(p > n\), then
\[
(3.3.5) \quad \|u(\cdot, t)\|_{L^p(\Omega)} \leq \left( e^{\|\chi_1\|_{L^1(0,\infty)}} \right)^{r/p} \max \left\{ \|u_0\|_{L^p(\Omega)}, \frac{p\mu_1 + r\gamma_1}{p\mu_1} |\Omega|^{1/p} \right\},
\]
\[
(3.3.6) \quad \|v(\cdot, t)\|_{L^p(\Omega)} \leq \left( e^{\|\chi_2\|_{L^1(0,\infty)}} \right)^{r/p} \max \left\{ \|v_0\|_{L^p(\Omega)}, \frac{p\mu_2 + r\gamma_2}{p\mu_2} |\Omega|^{1/p} \right\}.
\]

**Proof.** From the mean value theorem, the condition (3.1.5) and the fact that \(u, v > 0\), it follows that for some \(\xi_1, \xi_2\) satisfying \(0 \leq \xi_1 \leq u\) and \(0 \leq \xi_2 \leq v\),
\[
h(u,v,w) = \frac{\partial h}{\partial u}(\xi_1, v, w)u + \frac{\partial h}{\partial v}(0, \xi_2, w)v + h(0,0,w)
\geq h(0,0,w).
\]
This together with the condition (3.1.7) leads to
\[
(3.3.7) \quad -r \int_{\Omega} u^p f_1^{-r}\chi_1(w)h(u,v,w) \leq -r \int_{\Omega} u^p f_1^{-r}\chi_1(w)h(0,0,w)
\leq k_1r \int_{\Omega} u^p f_1^{-r}.
\]

14
Combining (3.3.2) with (3.3.7), we obtain
\[ \frac{d}{dt} \int_{\Omega} u^p f_1^{-r} \leq (\mu_1 p + k_1 r) \int_{\Omega} u^p f_1^{-r} - \mu_1 p \int_{\Omega} u^{p+1} f_1^{-r}. \]
Hence the Hölder inequality gives
\[ \frac{d}{dt} \int_{\Omega} u^p f_1^{-r} \leq (\mu_1 p + k_1 r) \int_{\Omega} u^p f_1^{-r} - \mu_1 p |\Omega|^{-1/p} \left( \int_{\Omega} u^{p+1} f_1^{-r} \right)^{(p+1)/p}. \]
Solving this differential inequality, we infer
\[ \left( \int_{\Omega} u^p f_1^{-r} \right)^{1/p} \leq \max \left\{ \left( \int_{\Omega} u^p_0 f_1^{-r} \right)^{1/p}, \frac{p\mu_1 + r k_1}{p\mu_1} |\Omega|^{1/p} \right\}. \]
Recalling the definition (3.3.1), we notice the relation
\[ 1 \leq f_1(w) \leq e^{\|u\|_{L^1(\Omega)}}, \]
which yields (3.3.5). In the same way, we obtain (3.3.6).

**Remark 3.3.1.** When \( d = 0 \), (3.3.2), (3.3.3), (3.3.5) and (3.3.6) still hold for all \( p \geq 1 \). Indeed, we have only to choose \( r = p - 1 \) in the above proof.

**Proof of Theorem 3.1.1.** First consider the case \( d > 0 \). We let \( \tau \in (0, T_{\max}) \). In view of Lemma 3.2.1 it is sufficient to make sure that
\[ \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C_u(\tau), \quad \|v(\cdot, t)\|_{L^\infty(\Omega)} \leq C_v(\tau), \quad \|w(\cdot, t)\|_{L^\infty(\Omega)} \leq C_w(\tau), \quad t \in (\tau, T_{\max}) \]
holds with some \( C_u(\tau), C_v(\tau), C_w(\tau) > 0 \). We let \( \rho \in \left( \frac{p+n}{2p}, 1 \right) \). This means \( 1 < 2\rho - \frac{n}{p} \).
Writing as
\[ w_t = d(\Delta - \delta/d)w + h(u, v, w) + \delta w, \]
and applying the variation of constants formula for \( w \), we have
\[ w(\cdot, t) = e^{d(\Delta - \delta/d)} w_0 + \int_0^t e^{d(t-s)(\Delta - \delta/d)} (h(u(\cdot, s), v(\cdot, s), w(\cdot, s)) + \delta w(\cdot, s)) \, ds. \]
From Lemma 2.2.1 we obtain that for all \( t \in (\tau, T_{\max}) \),
\[ \|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq c_1 \|(-\Delta + \delta/d)^\rho w(\cdot, t)\|_{L^p(\Omega)} \]
\[ \leq c_1 c_2 t^{-\rho} e^{-\lambda t} \|w_0\|_{L^p(\Omega)} + c_1 c_2 \int_0^t (t-s)^{-\rho} e^{-\lambda(t-s)} \|h(u(\cdot, s), v(\cdot, s), w(\cdot, s)) + \delta w(\cdot, s)\|_{L^p(\Omega)} \, ds \]
Thus a combination of (3.1.6) and Lemma 3.3.2 entails that
\[ \|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq c_1 c_2 t^{-\rho} e^{-\lambda t} \|w_0\|_{L^p(\Omega)} + c_1 c_2 c_4 \int_0^t (t-s)^{-\rho} e^{-\lambda(t-s)} \, ds, \]
where
\[ c_4 := \sup_{0 \leq s < T_{\max}} \{ M(\|u(\cdot, s)\|_{L^p(\Omega)} + \|v(\cdot, s)\|_{L^p(\Omega)} + 1) \} \]
\( (c_4 < \infty \) by Lemma 3.3.2). Noting that
\[ \int_0^t (t - s)^{-\rho}e^{-\lambda(t-s)} ds \leq \int_0^\infty r^{-\rho}e^{-\lambda r} dr < \infty, \]
we deduce that
\[ (3.3.8) \quad \|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq c_1 c_2 \left( \tau^{-\rho}e^{-\lambda \tau} + c_4 \int_0^\infty r^{-\rho}e^{-\lambda r} dr \right) =: C_w(\tau). \]
Since (3.1.8) implies \( \chi'_1 < 0 \), it follows from (3.3.5) and (3.3.8) that for all \( t \in (\tau/2, T_{\max}) \),
\[ \|u(\cdot, t)\chi_1(w(\cdot, t))\nabla w(\cdot, t)\|_{L^p(\Omega)} \leq \chi_1(0)\|u(\cdot, t)\|_{L^p(\Omega)}\|\nabla w(\cdot, t)\|_{L^\infty(\Omega)} \leq \chi_1(0) \sup_{0 \leq t < T_{\max}} \|u(\cdot, t)\|_{L^p(\Omega)} C_w(\tau/2) =: c_5. \]
Employing the variation of constants formula for \( u \) yields
\[ u(\cdot, t) = e^{(t-\tau/2)(\Delta - 1)} u \left( \cdot, \frac{T}{2} \right) - \int_{\tau/2}^t e^{(t-s)(\Delta - 1)} \nabla \cdot (u(\cdot, s) \chi_1(w(\cdot, s)) \nabla w(\cdot, s)) ds \]
\[ + \int_{\tau/2}^t e^{(t-s)(\Delta - 1)}[(\mu_1 + 1)u(\cdot, s) - \mu_1 u(\cdot, s)^2] ds \]
\[ =: J_1 + J_2 + J_3, \quad t \in (\tau, T_{\max}). \]
Let \( \eta \in \left( \frac{\nu}{2p}, \frac{1}{2} \right) \) and \( \varepsilon \in (0, \frac{1}{2} - \eta) \). Then we observe that \( 0 < 2\eta - \frac{\nu}{p} \) and \( \eta + \varepsilon + \frac{1}{2} < 1 \). By Lemmas 2.2.1 and 3.3.2 we see that for all \( t \in (\tau, T_{\max}) \),
\[ \|J_1\|_{L^{\infty}(\Omega)} = \left\| e^{(t-\tau/2)(\Delta - 1)} u \left( \cdot, \frac{T}{2} \right) \right\|_{L^{\infty}(\Omega)} \]
\[ \leq c_1 \left\| (-\Delta + 1)^{\eta} e^{(t-\tau/2)(\Delta - 1)} u \left( \cdot, \frac{T}{2} \right) \right\|_{L^p(\Omega)} \]
\[ \leq c_1 c_2 \left( t - \frac{T}{2} \right)^{-\eta} e^{-\lambda t} \left\| u \left( \cdot, \frac{T}{2} \right) \right\|_{L^p(\Omega)} \]
\[ \leq 2^{\eta} c_1 c_2 \tau^{-\eta} e^{-\eta \tau} \sup_{0 \leq t < T_{\max}} \|u(\cdot, t)\|_{L^p(\Omega)}. \]
Using Lemma 2.2.1, we obtain
\[ \|J_2\|_{L^{\infty}(\Omega)} \leq \int_{\tau/2}^t \|e^{(t-s)(\Delta - 1)} \nabla \cdot (u(\cdot, s) \chi_1(w(\cdot, s)) \nabla w(\cdot, s))\|_{L^{\infty}(\Omega)} ds \]
\[ \leq c_1 \int_{\tau/2}^t \|(-\Delta + 1)^{\eta} e^{(t-s)(\Delta - 1)} \nabla \cdot (u(\cdot, s) \chi_1(w(\cdot, s)) \nabla w(\cdot, s))\|_{L^p(\Omega)} ds \]
\[ \leq c_1 c_3 \int_{\tau/2}^t (t - s)^{-\eta-\varepsilon-1/2} e^{-(\nu+1)(t-s)} \|u(\cdot, s) \chi_1(w(\cdot, s)) \nabla w(\cdot, s)\|_{L^p(\Omega)} ds, \]
which with (3.3.9) yields that
\[ \|J_2\|_{L^\infty(\Omega)} \leq c_1c_3c_5 \int_0^\infty r^{-(\eta+\varepsilon+1/2)} e^{-(\nu+1)r} \, dr. \]

Since the Neumann heat semigroup \((e^{t\Delta})_{t \geq 0}\) has the order preserving property, we infer from the inequality
\[ -\mu_1 \left( u(\cdot, s) - \frac{\mu_1 + 1}{2\mu_1} \right)^2 + \frac{(\mu_1 + 1)^2}{4\mu_1} \leq \frac{(\mu_1 + 1)^2}{4\mu_1} \]
that
\[ J_3 = \int_{\tau/2}^t e^{(t-s)(\Delta-1)} \left[ -\mu_1 \left( u(\cdot, s) - \frac{\mu_1 + 1}{2\mu_1} \right)^2 + \frac{(\mu_1 + 1)^2}{4\mu_1} \right] \, ds \]
\[ \leq \frac{(\mu_1 + 1)^2}{4\mu_1} \int_{\tau/2}^t e^{(t-s)\Delta} e^{-(t-s)} \, ds, \]
and moreover, by the maximum principle we have
\[ J_3 \leq \frac{(\mu_1 + 1)^2}{4\mu_1} \int_{\tau/2}^t e^{-(t-s)} \, ds \]
\[ \leq \frac{(\mu_1 + 1)^2}{4\mu_1} (1 - e^{-\tau/2}). \]

Therefore we obtain that there exists \(C_u(\tau) > 0\) such that
\[ u(\cdot, t) \leq \|J_1\|_{L^\infty(\Omega)} + \|J_2\|_{L^\infty(\Omega)} + J_3 \]
\[ \leq C_u(\tau), \quad t \in (\tau, T_{\text{max}}). \]

The positivity of \(u\) yields that
\[ \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C_u(\tau), \quad t \in (\tau, T_{\text{max}}). \]

The same argument as for \(u\) gives the \(L^\infty(\Omega)\) bound for \(v\). This completes the proof in the case \(d > 0\). Next consider the case \(d = 0\). From Remark 3.3.1 we have
\[ \|u(\cdot, t)\|_{L^p(\Omega)} \leq \exp\{\|\chi_1\|_{L^1(0,\infty)}^{(p-1)/p} \max \left\{ \|u_0\|_{L^p(\Omega)}, \frac{p\mu_1 + (p - 1)k_1}{pp_1} |\Omega|^{1/p} \right\} \}
\]
for all \(p \geq 1\). Taking the limits as \(p \to \infty\), we obtain the \(L^\infty(\Omega)\) bound for \(u\), and similarly for \(v\). The \(L^\infty\) bound for \(w\) follows from
\[ w(\cdot, t) = e^{-\delta t}w_0 + \int_0^t e^{-\delta(t-s)} \left( h(u(\cdot, s), v(\cdot, s), w(\cdot, s)) + \delta w(\cdot, s) \right) \, ds. \]
This completes the proof when \(d = 0\). \(\square\)
3.4. Asymptotic behavior

In this section we prove the asymptotic stability of solutions to (3.1.1). We denote by \( u^*, v^* \) and \( w^* \) the functions defined as

\[
  u^*(t) := \frac{1}{|\Omega|} \int_{\Omega} u(x, t) \, dx, \quad v^*(t) := \frac{1}{|\Omega|} \int_{\Omega} v(x, t) \, dx
\]

and

\[
  w^*(t) := \frac{1}{|\Omega|} \int_{\Omega} w(x, t) \, dx.
\]

We also put \( \Omega_{\infty} := \Omega \times (0, \infty), \Omega_T := \Omega \times (\tau, T) \) for \( \tau \in [0, \infty), \, T \in (\tau, \infty] \).

**Lemma 3.4.1.** Let \( (u, v, w) \) be a solution to (3.1.1). Under the condition (3.1.9), the following inequality holds:

\[
\int_{\Omega_{\infty}} \left| \nabla u \right|^2 + \int_{\Omega_{\infty}} \left| \nabla v \right|^2 + \int_{\Omega_{\infty}} \left| \nabla w \right|^2 + \int_{\Omega_{\infty}} [u(u - 1)^2 + v(v - 1)^2] \leq C
\]

with some constant \( C > 0 \).

**Proof.** Let \( \tau > 0 \) be a constant fixed later. We multiply the first equation in (3.1.1) by \( u - 1 \) and the second equation in (3.1.1) by \( v - 1 \), and integrate them over \( \Omega_T \) to obtain

\[
\frac{1}{2} \int_{\Omega} [u(\cdot, t) - 1]^2 \, d\tau = - \int_{\Omega_T} \left| \nabla u \right|^2 + \int_{\Omega_T} u \chi_w \nabla u \cdot \nabla w - \mu_1 \int_{\Omega_T} u(u - 1)^2,
\]

\[
\frac{1}{2} \int_{\Omega} [v(\cdot, t) - 1]^2 \, d\tau = - \int_{\Omega_T} \left| \nabla v \right|^2 + \int_{\Omega_T} v \chi_w \nabla v \cdot \nabla w - \mu_2 \int_{\Omega_T} v(v - 1)^2.
\]

Multiplying the third equation in (3.1.1) by \(-\Delta w\) and integrating it over \( \Omega_T \), we have

\[
\frac{1}{2} \int_{\Omega} \left| \nabla w(\cdot, t) \right|^2 \, d\tau = -d \int_{\Omega_T} \left| \Delta w \right|^2 - \int_{\Omega_T} h(u, v, w) \Delta w
\]

\[
= -d \int_{\Omega_T} \left| \Delta w \right|^2 - \int_{\Omega_T} h_u \nabla u \cdot \nabla w + \int_{\Omega_T} h_v \nabla v \cdot \nabla w + \int_{\Omega_T} h_w |\nabla w|^2
\]

\[
\leq \int_{\Omega_T} h_u \nabla u \cdot \nabla w + \int_{\Omega_T} h_v \nabla v \cdot \nabla w + \int_{\Omega_T} (-\gamma) |\nabla w|^2.
\]

Adding this to the above identities multiplied by positive constants \( \lambda_1 \) and \( \lambda_2 \) (fixed later), we see that

\[
\lambda_1 \left( \int_{\Omega_T} \left| \nabla u \right|^2 + \mu_1 \int_{\Omega_T} u(u - 1)^2 \right)
\]

\[
+ \lambda_2 \left( \int_{\Omega_T} \left| \nabla v \right|^2 + \mu_2 \int_{\Omega_T} v(v - 1)^2 \right) + \gamma \int_{\Omega_T} |\nabla w|^2
\]

\[
\leq \int_{\Omega_T} \left( \lambda_1 u \chi_w + h_u \right) \nabla u \cdot \nabla w + \int_{\Omega_T} \left( \lambda_2 v \chi_2 + h_v \right) \nabla v \cdot \nabla w + O(1)
\]

18
as $T \to \infty$. Since (3.1.8) implies $\chi_1' < 0$, it follows that
\[
\int_\Omega (\lambda_1 u \chi_1(w) + h_u) \nabla u \cdot \nabla w \leq \left( \lambda_1 C_u(\tau) \chi_1(0) + \|h_u\|_{L^\infty(I_r)} \right) \int_\Omega \nabla u \cdot \nabla w
\]
\[
\leq \left( \lambda_1 C_u(\tau) \chi_1(0) + \|h_u\|_{L^\infty(I_r)} \right)^2 \int_\Omega |\nabla u|^2 + \frac{\gamma}{2} \int_\Omega |\nabla w|^2,
\]
where $I_r := (0, C_u(\tau)) \times (0, C_v(\tau)) \times (0, C_w(\tau))$. In the same way as above, we obtain
\[
\int_\Omega (\lambda_2 u \chi_2(w) + h_v) \nabla v \cdot \nabla w \leq \left( \lambda_2 C_v(\tau) \chi_2(0) + \|h_v\|_{L^\infty(I_r)} \right)^2 \int_\Omega |\nabla v|^2 + \frac{\gamma}{2} \int_\Omega |\nabla w|^2.
\]
Therefore we have
\[
(3.4.3) \quad \left( \lambda_1 - \frac{(\lambda_1 C_u(\tau) \chi_1(0) + \|h_u\|_{L^\infty(I_r)})^2}{2\gamma} \right) \int_\Omega |\nabla u|^2
\]
\[
+ \left( \lambda_2 - \frac{(\lambda_2 C_v(\tau) \chi_2(0) + \|h_v\|_{L^\infty(I_r)})^2}{2\gamma} \right) \int_\Omega |\nabla v|^2 \leq O(1)
\]
as $T \to \infty$. Here we note that
\[
\lambda_1 - \frac{(\lambda_1 C_u(\tau) \chi_1(0) + \|h_u\|_{L^\infty(I_r)})^2}{2\gamma} > 0
\]
holds if
\[
(C_u(\tau) \chi_1(0))^2 \lambda_1^2 + 2 (C_u(\tau) \chi_1(0) \|h_u\|_{L^\infty(I_r)} - \gamma) \lambda_1 + \|h_u\|_{L^\infty(I_r)}^2 < 0.
\]
Since the discriminant
\[
D_{\lambda_1} = 4\gamma (\gamma - 2C_u(\tau) \chi_1(0) \|h_u\|_{L^\infty(I_r)}^2)
\]
is positive from the condition (3.1.9), we can find $\lambda_1 > 0$ such that
\[
\lambda_1 - \frac{(\lambda_1 C_u(\tau) \chi_1(0) + \|h_u\|_{L^\infty(I_r)})^2}{2\gamma} > 0.
\]
The same argument yields that there exists $\lambda_2 > 0$ such that
\[
\lambda_2 - \frac{(\lambda_2 C_v(\tau) \chi_2(0) + \|h_v\|_{L^\infty(I_r)})^2}{2\gamma} > 0.
\]
Therefore (3.4.3) implies that
\[
\int_\Omega|\nabla u|^2 + \int_\Omega|\nabla v|^2 \leq C
\]
for some $C > 0$. Thanks to this estimate, we obtain from (3.4.2) that
\[
\int_\Omega|u(u - 1)|^2 + \int_\Omega|v(v - 1)|^2 + \int_\Omega|\nabla w|^2 \leq C
\]
for some $C > 0$. This completes the proof. \qed
Lemma 3.4.2. Let \((u, v, w)\) be a solution to (3.1.1). Under the condition (3.1.10), the following inequalities hold:

\[ \int_{\Omega} u(\cdot, t) \geq C, \quad \int_{\Omega} v(\cdot, t) \geq C \]

for all \(t > 0\) with some constant \(C > 0\).

Proof. We shall prove that there exist \(0 < \beta < 1\), \(C > 0\) and \(\tau > 0\) such that

(3.4.4) \[ \int_{\Omega} u^{-\beta}(\cdot, t) \leq C \quad (t > \tau). \]

Let \(0 < \beta < 1\) and \(\tau > 0\) be fixed later. In view of (3.3.8) we have that for all \(t > \tau\),

(3.4.5) \[ \frac{d}{dt} \int_{\Omega} u^{-\beta} = -\beta(\beta + 1) \int_{\Omega} u^{-\beta-2}|\nabla u|^2 + \beta(\beta + 1) \int_{\Omega} u^{-\beta-1} \chi_1(w) \nabla u \cdot \nabla w \\
- \mu_1 \beta \int_{\Omega} u^{-\beta}(1 - u) \\
\leq \beta(\beta + 1) \int_{\Omega} u^{-\beta} \chi_1(w)^2 |\nabla w|^2 - \mu_1 \beta \int_{\Omega} u^{-\beta}(1 - u) \\
\leq \beta \left( \frac{\beta + 1}{4} C_w(\tau)^2 \chi_1(0)^2 - \mu_1 \right) \int_{\Omega} u^{-\beta} + \mu_1 \beta \int_{\Omega} u^{1-\beta}. \]

By the Hölder inequality we have

(3.4.6) \[ \int_{\Omega} u^{1-\beta}(\cdot, t) \leq |\Omega|^{1/\beta} \left( \int_{\Omega} u(\cdot, t) \right)^{1-\beta}, \]

where the right-hand side is uniformly bounded from the estimate for \(u\) in \(L^\infty\) norm (see Theorem 3.1.1) or in \(L^1\) norm (see [151, Lemma 2.4]). From (3.1.10) we can choose \(\tau > 0\) and \(0 < \beta < 1\) such that

\[ \frac{\beta + 1}{4} C_w(\tau)^2 \chi_1(0)^2 - \mu_1 < 0. \]

Hence applying (3.4.6) to (3.4.5) gives

\[ \frac{d}{dt} \int_{\Omega} u^{-\beta} \leq -b_1 \int_{\Omega} u^{-\beta} + b_2 \]

with some constants \(b_1, b_2 > 0\). Solving this differential inequality, we can obtain (3.4.4). Finally, noting by the Hölder inequality that

\[ |\Omega|^{(\beta+1)/\beta} \left( \int_{\Omega} u^{-\beta}(\cdot, t) \right)^{-1/\beta} \leq \int_{\Omega} u(\cdot, t), \]

we have a positive lower bound for \(\int_{\Omega} u\) from (3.4.4). In the same way, we arrive at a lower bound for \(\int_{\Omega} v\). \qed
Lemma 3.4.3. Let \((u,v,w)\) be a solution to (3.1.1). Under the conditions (3.1.9) and (3.1.10), it holds that

\[ u^*(t) \to 1 \]

and

\[ v^*(t) \to 1 \quad \text{as} \quad t \to \infty. \]

Proof. Using the same argument as in [151, Lemma 3.3], we see that

\[ u^*(t) \to 0 \quad \text{or} \quad u^*(t) \to 1 \quad \text{as} \quad t \to \infty. \]

In light of Lemma 3.4.2 we conclude that \( u^*(t) \to 1 \) as \( t \to \infty \). Similarly, we have

\[ v^*(t) \to 1 \]

as \( t \to \infty. \)

Lemma 3.4.4. Let \((u,v,w)\) be a solution to (3.1.1). Under the conditions (3.1.9) and (3.1.10), it holds that

\[ \|u(\cdot,t) - u^*(t)\|_{L^2(\Omega)} \to 0 \]

and

\[ \|v(\cdot,t) - v^*(t)\|_{L^2(\Omega)} \to 0 \quad \text{as} \quad t \to 0. \]

Proof. The proof is the same as in [151, Lemma 3.4].

Proof of Theorem 3.1.2. Firstly boundedness of \(u, v, \nabla w\) and standard parabolic regularity theory ([98]) yield that there exist \(\alpha \in (0,1)\) and \(C > 0\) such that

\[ \|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\Omega \times [1,t])} + \|v\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\Omega \times [1,t])} + \|w\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\Omega \times [1,t])} \leq C \quad \text{for all} \quad t \geq 1. \]

Therefore in view of the Arzelà–Ascoli theorem or the Gagliardo–Nirenberg inequality

\[ \|\varphi\|_{L^\infty(\Omega)} \leq c\|\varphi\|_{W^{1,\infty}(\Omega)}^{\frac{2}{n}}\|\varphi\|_{L^2(\Omega)}^{\frac{n}{2}} \quad \text{for all} \quad \varphi \in W^{1,\infty}(\Omega) \]

(see Lemma 2.1.1), it is sufficient to show that

\[ \|u(\cdot,t) - 1\|_{L^2(\Omega)} \to 0 \]

and

\[ \|v(\cdot,t) - 1\|_{L^2(\Omega)} \to 0 \]

as well as

\[ \|w(\cdot,t) - \bar{w}\|_{L^2(\Omega)} \to 0 \]
as $t \to \infty$. Here the convergences of $u$ and $v$ come from Lemmas 3.4.3 and 3.4.4. As to $w$, we shall show that

\begin{equation}
\| w(\cdot, t) - w^*(t) \|_{L^2(\Omega)} \to 0 \quad \text{as } t \to \infty
\end{equation}

and

\begin{equation}
\quad w^*(t) \to \tilde{w} \quad \text{as } t \to \infty,
\end{equation}

where $\tilde{w} \geq 0$ such that $h(1, 1, \tilde{w}) = 0$. We denote by $q$ the function defined as

\[ q(t) := \int_{\Omega} |w(\cdot, t) - w^*(t)|^2. \]

Then the Poincaré–Wirtinger inequality gives

\[ \int_0^\infty q(t) \, dt \leq c \iint_{\Omega_{\infty}} |\nabla w|^2 < \infty. \]

To use [49, Lemma 5.1] we prove that there exists $C > 0$ such that

\[ |q'(t)| \leq C. \]

Indeed, using the third equation in (3.1.1), we have

\[ |q'(t)| = 2 \left| \int_{\Omega} (w - w^*) \left( d\Delta w + h(u, v, w) - w^*_t \right) \right| \]

\[ \leq 2d \int_{\Omega} |\nabla w|^2 + \int_{\Omega} |w - w^*| \, |h(u, v, w) - w^*_t| \]

\[ \leq C \]

for some $C > 0$. From [49, Lemma 5.1] it follows that

\[ q(t) \to 0 \quad \text{as } t \to \infty. \]

This means (3.4.7). In the rest of the proof, we shall show (3.4.8). We denote by $\varepsilon$ the function defined as

\[ \varepsilon(t) := \frac{1}{|\Omega|} \int_{\Omega} h(u, v, w) - h(u^*, v^*, w^*). \]

By the mean value theorem we have

\[ |\varepsilon(t)| \leq \frac{1}{|\Omega|} \int_{\Omega} |h(u, v, w) - h(u^*, v^*, w^*)| \]

\[ \leq \frac{C}{|\Omega|} \int_{\Omega} (|u - u^*| + |v - v^*| + |w - w^*|) \]

22
for some $C > 0$. Thus we obtain from the Poincaré–Wirtinger inequality and Lemma 3.4.1 that
\begin{equation}
\int_0^\infty |\varepsilon(t)|^2 dt < \infty.
\end{equation}

Since $h(1, 1, \tilde{w}) = 0$, it follows from the mean value theorem that
\[
\frac{d}{dt} w^* = \frac{1}{|\Omega|} \int_\Omega h(u, v, w) \\
= h(u^*, v^*, w^*) - h(1, 1, \tilde{w}) + \varepsilon \\
= h_u(u^* - 1) + h_v(v^* - 1) + h_w(w^* - \tilde{w}) + \varepsilon
\]
for some derivatives $h_u, h_v, h_w$. Multiplying this equation by $w^* - \tilde{w}$, we have
\[
\frac{1}{2} \frac{d}{dt} |w^* - \tilde{w}|^2 = h_u|w^* - \tilde{w}|^2 + (h_u(u^* - 1) + h_v(v^* - 1) + \varepsilon)(w^* - \tilde{w}) \\
\leq -\gamma|w^* - \tilde{w}|^2 + C(|u^* - 1| + |v^* - 1| + |\varepsilon|)|w^* - \tilde{w}|
\]
for some $C > 0$. Putting
\[
\tilde{\varepsilon}(t) := |u^*(t) - 1| + |v^*(t) - 1| + |\varepsilon(t)|
\]
and solving the above differential inequality, we see that
\begin{equation}
|w^*(t) - \tilde{w}| \leq e^{-\gamma t} \left| \frac{1}{|\Omega|} \int_\Omega w_0 - \tilde{w} \right| + C \int_0^t e^{-\gamma(t-s)}|\tilde{\varepsilon}(s)| ds.
\end{equation}

It remains to show that
\[
\int_0^t e^{-\gamma(t-s)}|\tilde{\varepsilon}(s)| ds \to 0 \quad \text{as } t \to \infty.
\]

From [151, Lemma 3.3] we know that
\[
\int_0^\infty u^*(t)(u^*(t) - 1)^2 dt < \infty
\]
and
\[
\int_0^\infty v^*(t)(v^*(t) - 1)^2 dt < \infty,
\]
which together with Lemma 3.4.2 implies
\begin{equation}
\int_0^\infty (u^*(t) - 1)^2 dt < \infty
\end{equation}
and
\begin{equation}
\int_0^\infty (v^*(t) - 1)^2 dt < \infty.
\end{equation}
Collecting (3.4.9) and (3.4.11)–(3.4.12), we have

\[ \int_0^\infty |\tilde{\varepsilon}(s)|^2 ds < \infty. \]

Therefore it follows that

\[
\int_0^t e^{-\gamma(t-s)}|\tilde{\varepsilon}(s)| ds = \int_0^{t/2} e^{-\gamma(t-s)}|\tilde{\varepsilon}(s)| ds + \int_{t/2}^t e^{-\gamma(t-s)}|\tilde{\varepsilon}(s)| ds \\
\leq e^{-\frac{1}{2}\gamma t} \int_0^{t/2} |\tilde{\varepsilon}(s)| ds + e^{-\gamma t} \left( \int_{t/2}^t e^{2\gamma s} ds \right)^{1/2} \left( \int_{t/2}^t |\tilde{\varepsilon}(s)|^2 ds \right)^{1/2} \\
\leq \sqrt{\frac{t}{2}} e^{-\frac{1}{2}\gamma t} \left( \int_0^{\infty} |\tilde{\varepsilon}(s)|^2 ds \right)^{1/2} + \sqrt{\frac{1}{2\gamma}} \left( \int_{t/2}^t |\tilde{\varepsilon}(s)|^2 ds \right)^{1/2},
\]

which means that

\[ \int_0^t e^{-\gamma(t-s)}|\tilde{\varepsilon}(s)| ds \to 0 \]

as \( t \to \infty \). In view of (3.4.10) this means \( w^*(t) \to \tilde{w} \) as \( t \to \infty \), the end of the proof. \( \square \)
Chapter 4

Remarks on smallness of chemotactic effect for asymptotic stability in a two-species chemotaxis system with logistic term

4.1. Motivation and the main result

In this chapter we consider the two-species chemotaxis system

\[
\begin{aligned}
  u_t &= \Delta u - \nabla \cdot (u \chi_1(w) \nabla w) + \mu_1 u(1 - u), & x \in \Omega, & t > 0, \\
  v_t &= \Delta v - \nabla \cdot (v \chi_2(w) \nabla w) + \mu_2 v(1 - v), & x \in \Omega, & t > 0, \\
  w_t &= d \Delta w + h(u, v, w), & x \in \Omega, & t > 0, \\
  \nabla u \cdot \nu &= \nabla v \cdot \nu = \nabla w \cdot \nu = 0, & x \in \partial \Omega, & t > 0, \\
  u(x, 0) &= u_0(x), & v(x, 0) &= v_0(x), & w(x, 0) &= w_0(x), & x \in \Omega,
\end{aligned}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n (n \in \mathbb{N}) \) with smooth boundary \( \partial \Omega \) and \( \nu \) denotes the outward normal vector of \( \partial \Omega \). The initial data \( u_0, v_0 \) and \( w_0 \) are assumed to be nonnegative functions. The unknown functions \( u(x, t) \) and \( v(x, t) \) represent the population densities of two species and \( w(x, t) \) shows the concentration of the substance at place \( x \) and time \( t \).

In mathematical view, global existence and behavior of solutions are fundamental themes. Recently, Negreanu–Tello [150, 151] built a technical way to prove global existence and asymptotic behavior of solutions to (4.1.1). In [151] they dealt with (4.1.1) when \( d = 0, \mu_i > 0 \) under the condition

\[ \exists \bar{w} \geq w_0; \quad h(\bar{u}, \bar{v}, \bar{w}) \leq 0, \]

where \( \bar{u}, \bar{v} \) satisfy some representations determined by \( \bar{w} \). In [150] they studied (4.1.1) when \( 0 < d < 1, \mu_i = 0 \) under similar conditions as in [151] and

\[
\chi_i' + \frac{1}{1-d} \chi_i^2 \leq 0 \quad (i = 1, 2).
\]
They supposed in [150, 151] that the functions \( h, \chi_i \) for \( i = 1, 2 \) generalize of the prototypical case \( \chi_i(w) = \frac{K_i}{(1+w)^{\sigma_i}} \) \( (K_i > 0, \sigma_i \geq 1) \), \( h(u, v, w) = u + v - w \). In Chapter 3 the restriction of \( 0 \leq d < 1 \) for global existence is completely removed and asymptotic stability of solutions to (4.1.1) is established for the first time under a smallness condition for the function \( \chi_i \) generalizing of \( \chi_i(w) = \frac{K_i}{(1+w)^{\sigma_i}} \) \( (K_i > 0, \sigma_i > 1) \).

The purpose of this chapter is to improve a way in Chapter 3 for obtaining asymptotic stability of solutions to (4.1.1) under a more general and sharp smallness condition for the sensitivity function \( \chi_i(w) \). We shall suppose throughout this chapter that \( h, \chi_i \) \( (i = 1, 2) \) satisfy the following conditions:

\[
\begin{align*}
(4.1.3) & \quad \chi_i \in C^{1+\omega}([0, \infty)) \cap L^1(0, \infty) \quad (0 < \exists \omega < 1), \quad \chi_i > 0 \quad (i = 1, 2), \\
(4.1.4) & \quad h \in C^1([0, \infty) \times [0, \infty), \quad h(0, 0, 0) \geq 0, \\
(4.1.5) & \quad \exists \gamma > 0; \quad \frac{\partial h}{\partial u}(u, v, w) \geq 0, \quad \frac{\partial h}{\partial v}(u, v, w) \geq 0, \quad \frac{\partial h}{\partial w}(u, v, w) \leq -\gamma, \\
(4.1.6) & \quad \exists \delta > 0, \quad \exists M > 0; \quad |h(u, v, w) + \delta w| \leq M(u + v + 1), \\
(4.1.7) & \quad \exists k_i > 0; \quad -\chi_i(w)h(0, 0, w) \leq k_i \quad (i = 1, 2).
\end{align*}
\]

We also assume that

\[
(4.1.8) \quad \exists p > N; \quad 2d\chi_i'(w) + \left( (d-1)p + \sqrt{(d-1)^2p^2 + 4dp} \right) [\chi_i(w)]^2 \leq 0 \quad (i = 1, 2).
\]

The above conditions cover the prototypical example \( \chi_i(w) = \frac{K_i}{(1+w)^{\sigma_i}} \) \( (K_i > 0, \sigma_i > 1) \), \( h(u, v, w) = u + v - w \). We assume that the initial data \( u_0, v_0, w_0 \) satisfy

\[
(4.1.9) \quad 0 \leq u_0 \in C(\bar{\Omega}) \setminus \{0\}, \quad 0 \leq v_0 \in C(\bar{\Omega}) \setminus \{0\}, \quad 0 \leq w_0 \in W^{1,q}(\Omega) \quad (\exists q > N).
\]

Since Theorem 3.1.1 guarantees that \( u, v \) and \( w \) exist globally and are bounded and nonnegative, it is possible to define nonnegative numbers \( \alpha, \beta \) by

\[
(4.1.10) \quad \alpha := \max_{(u,v,w) \in I} h_u(u, v, w), \quad \beta := \max_{(u,v,w) \in I} h_v(u, v, w),
\]

where \( I = (0, C_1)^3 \) and \( C_1 \) is defined in Theorem 3.1.1.

Now the main result reads as follows. The main theorem is concerned with asymptotic stability in (4.1.1).

**Theorem 4.1.1.** Let \( d > 0 \), \( \mu_i > 0 \) \( (i = 1, 2) \). Under the conditions (4.1.3)–(4.1.9) and

\[
(4.1.11) \quad \alpha > 0, \quad \beta > 0, \quad \chi_1(0)^2 < \frac{16 \mu_1 d^\gamma}{\alpha^2 + \beta^2 + 2\alpha \beta}, \quad \chi_2(0)^2 < \frac{16 \mu_2 d^\gamma}{\alpha^2 + \beta^2 + 2\alpha \beta},
\]

the unique global solution \( (u, v, w) \) of (4.1.1) satisfies that there exist \( C > 0 \) and \( \lambda > 0 \) such that

\[
\|u(\cdot, t) - 1\|_{L^\infty(\Omega)} + \|v(\cdot, t) - 1\|_{L^\infty(\Omega)} + \|w(\cdot, t) - \tilde{w}\|_{L^\infty(\Omega)} \leq Ce^{-\lambda t} \quad (t > 0),
\]

where \( \tilde{w} \geq 0 \) such that \( h(1, 1, \tilde{w}) = 0 \).
Remark 4.1.1. This result improves the previous result (Theorem 3.1.2). Indeed, the condition (4.1.11) is sharper than “\( \chi_i(0) \) are suitably small” assumed in Theorem 3.1.2. Moreover, this result attains to show the convergence rate which cannot be given in Theorem 3.1.2.

Remark 4.1.2. From (4.1.4)–(4.1.6) there exists \( \bar{w} \geq 0 \) such that \( h(1, 1, \bar{w}) = 0 \). Indeed, if we choose \( \bar{w} \geq 3M/\delta \), then (4.1.6) yields that \( h(1, 1, \bar{w}) \leq 3M - \delta \bar{w} \leq 0 \). On the other hand, (4.1.4) and (4.1.5) imply that \( h(1, 1, 0) \geq h(0, 0, 0) \geq 0 \). Hence, by the intermediate value theorem there exists \( \bar{w} \geq 0 \) such that \( h(1, 1, \bar{w}) = 0 \).

The strategy for the proof of Theorem 4.1.1 is to construct the following energy estimate which was not given in Chapter 3:

\[
\frac{d}{dt} E(t) \leq -\varepsilon \left( \int_{\Omega} (u-1)^2 + \int_{\Omega} (v-1)^2 + \int_{\Omega} (w-\bar{w})^2 \right)
\]

with some function \( E(t) \geq 0 \) and some \( \varepsilon > 0 \). This strategy enables us to improve the conditions assumed in Theorem 3.1.2.

4.2. Proof of the main result

In this section we will establish asymptotic stability of solutions to (4.1.1). For the proof of Theorem 4.1.1, we shall prepare some elementary results.

Lemma 4.2.1 ([4, Lemma 3.1]). Suppose that \( f : (1, \infty) \to \mathbb{R} \) is a uniformly continuous nonnegative function satisfying \( \int_1^\infty f(t) \, dt < \infty \). Then \( f(t) \to 0 \) as \( t \to \infty \).

Lemma 4.2.2. Let \( a_1, a_2, a_3, a_4, a_5 \in \mathbb{R} \). Suppose that

(4.2.1) \quad a_1 > 0, \quad a_3 > 0, \quad a_5 - \frac{a_2^2}{4a_1} - \frac{a_4^2}{4a_3} > 0.

Then

(4.2.2) \quad a_1 x^2 + a_2 x z + a_3 y^2 + a_4 y z + a_5 z^2 \geq 0

holds for all \( x, y, z \in \mathbb{R} \).

Proof. From straightforward calculations we obtain

\[
a_1 x^2 + a_2 x z + a_3 y^2 + a_4 y z + a_5 z^2 = a_1 \left( x + \frac{a_2 z}{2a_1} \right)^2 + a_3 \left( y + \frac{a_4 z}{2a_3} \right)^2 + \left( a_5 - \frac{a_2^2}{4a_1} - \frac{a_4^2}{4a_3} \right) z^2.
\]

In view of the above equation, (4.2.1) leads to (4.2.2).

Now we will prove the key estimate for the proof of Theorem 4.1.1.
Lemma 4.2.3. Let \((u, v, w)\) be a solution to (4.1.1). Under the conditions (4.1.3)–(4.1.9) and (4.1.11), there exist \(\delta_1, \delta_2 > 0\) and \(\varepsilon > 0\) such that the nonnegative functions \(E_1\) and \(F_1\) defined by

\[
E_1(t) := \int_{\Omega} (u - 1 - \log u) + \frac{\delta_1}{\mu_2} \int_{\Omega} (v - 1 - \log v) + \frac{\delta_2}{2} \int_{\Omega} (w - \tilde{w})^2
\]

and

\[
F_1(t) := \int_{\Omega} (u - 1)^2 + \int_{\Omega} (v - 1)^2 + \int_{\Omega} (w - \tilde{w})^2
\]

satisfy

\[
\frac{d}{dt} E_1(t) \leq -\varepsilon F_1(t) \quad (t > 0).
\]

Proof. Thanks to (4.1.11), we can choose \(\delta_1 = \frac{\beta}{\alpha} > 0\) and \(\delta_2 > 0\) satisfying

\[
\max\left\{ \frac{\chi_1(0)^2(1 + \delta_1)}{4\alpha d}, \frac{\mu_1 \chi_2(0)^2(1 + \delta_1)}{4\mu_2 d} \right\} < \delta_2 < \frac{4\mu_1 \gamma \delta_1}{\alpha^2 \delta_1 + \beta^2}.
\]

We denote by \(A_1(t)\), \(B_1(t)\), \(C_1(t)\) the functions defined as

\[
A_1(t) := \int_{\Omega} (u - 1 - \log u), \quad B_1(t) := \int_{\Omega} (v - 1 - \log v), \quad C_1(t) := \frac{1}{2} \int_{\Omega} (w - \tilde{w})^2,
\]

and we write as

\[
E_1(t) = A_1(t) + \frac{\delta_1}{\mu_2} B_1(t) + \delta_2 C_1(t).
\]

The Taylor formula applied to \(H(s) = s - \log s (s \geq 0)\) yields \(A_1(t) = \int_{\Omega} (H(u) - H(1))\) is a nonnegative function for \(t > 0\) (more detail, see [4, Lemma 3.2]). Similarly, we have that \(B_1(t)\) is a positive function. By straightforward calculations we infer

\[
\frac{d}{dt} A_1(t) = -\mu_1 \int_{\Omega} (u - 1)^2 - \int_{\Omega} \frac{\nabla u}{u} \nabla u - \frac{\chi_1(w)}{u} \nabla u \cdot \nabla w
\]

and

\[
\frac{d}{dt} B_1(t) = -\mu_2 \int_{\Omega} (v - 1)^2 - \int_{\Omega} \frac{\nabla v}{v} \nabla v - \frac{\chi_2(w)}{v} \nabla v \cdot \nabla w
\]

as well as

\[
\frac{d}{dt} C_1(t) = \int_{\Omega} h_u (u - 1)(w - \tilde{w}) + \int_{\Omega} h_v (v - 1)(w - \tilde{w}) + \int_{\Omega} h_w (w - \tilde{w})^2 - d \int_{\Omega} |\nabla w|^2
\]

with some derivatives \(h_u\), \(h_v\) and \(h_w\). Hence we have

\[
\frac{d}{dt} E_1(t) = I_1(t) + I_2(t), \tag{4.2.5}
\]

28
where
\[
I_1(t) := -\mu_1 \int_{\Omega} (u - 1)^2 - \delta_1 \mu_1 \int_{\Omega} (v - 1)^2 + \delta_2 \int_{\Omega} h_u (u - 1) (w - \bar{w}) \\
+ \delta_2 \int_{\Omega} h_v (v - 1) (w - \bar{w}) + \delta_2 \int_{\Omega} h_w (w - \bar{w})^2
\]
and
\[
(4.2.6) \quad I_2(t) := -\int_{\Omega} \frac{|\nabla u|^2}{u^2} + \int_{\Omega} \frac{\chi_1(w)}{u} \nabla u \cdot \nabla w - \delta_1 \frac{\mu_1}{\mu_2} \int_{\Omega} \frac{|\nabla v|^2}{v^2} \\
+ \delta_1 \frac{\mu_1}{\mu_2} \int_{\Omega} \frac{\chi_1(w)}{v} \nabla v \cdot \nabla w - d\delta_2 \int_{\Omega} |\nabla w|^2.
\]
At first, we shall show from Lemma 4.2.2 that there exists \( \varepsilon_1 > 0 \) such that
\[
(4.2.7) \quad I_1(t) \leq -\varepsilon_1 \left( \int_{\Omega} (u - 1)^2 + \int_{\Omega} (v - 1)^2 + \int_{\Omega} (w - \bar{w})^2 \right).
\]
To see this, we put
\[
g_1(\varepsilon) := \mu_1 - \varepsilon, \\
g_2(\varepsilon) := \delta_1 \mu_1 - \varepsilon
\]
and
\[
g_3(\varepsilon) := (-\delta_2 h_w - \varepsilon) - \frac{h_u^2}{4(\mu_1 - \varepsilon)} \delta_2^2 - \frac{h_v^2}{4(\delta_1 \mu_1 - \varepsilon)} \delta_2^2.
\]
Since \( \mu_1 > 0 \) and \( \delta_1 = \frac{\beta}{\alpha} > 0 \), we have
\[
g_1(0) = \mu_1 > 0
\]
and
\[
g_2(0) = \delta_1 \mu_1 > 0.
\]
In light of (4.1.5) and the definitions of \( \delta_2, \alpha, \beta > 0 \) (see (4.1.10) and (4.2.4)) we obtain
\[
g_3(0) = \delta_2 \left( -h_w - \left( \frac{h_u^2}{4\mu_1} + \frac{h_v^2}{4\delta_1 \mu_1} \right) \delta_2 \right) \\
\geq \delta_2 \left( \gamma - \left( \frac{\alpha^2}{4\mu_1} + \frac{\beta^2}{4\delta_1 \mu_1} \right) \delta_2 \right) \\
\geq \delta_2 \left( \gamma - \left( \frac{\alpha^2 \delta_1 + \beta}{4\delta_1 \mu_1} \right) \delta_2 \right) > 0.
\]
Combination of the above inequalities and the continuity of \( g_i \) for \( i = 1, 2, 3 \) yields that there exists \( \varepsilon_1 > 0 \) such that
\[
g_i(\varepsilon_1) > 0
\]
hold for $i = 1, 2, 3$. Thanks to Lemma 4.2.2 with
\begin{align*}
a_1 &= \mu_1 - \varepsilon_1, \quad a_2 = -\delta_2 h_u, \quad a_3 = \delta_1 \mu_1 - \varepsilon_1, \\
a_4 &= -\delta_2 h_v, \quad a_5 = -\delta_2 h_w - \varepsilon_1
\end{align*}
and
\begin{align*}x &= u(\cdot, t) - 1, \quad y = v(\cdot, t) - 1, \quad z = w(\cdot, t) - \tilde{w},
\end{align*}
we obtain (4.2.7) with $\varepsilon_1 > 0$. Lastly we will prove
\begin{equation}
I_2(t) \leq 0.
\end{equation}
Noting that $\chi_i' < 0$ (from (4.1.8)) and then using the Young inequality, we have
\begin{align*}
\int_{\Omega} \frac{\chi_1(w)}{u} \nabla u \cdot \nabla w &\leq \chi_1(0) \int_{\Omega} \frac{\nabla u \cdot \nabla w}{u} \\
&\leq \frac{\chi_1(0)^2(1 + \delta_1)}{4d\delta_2} \int_{\Omega} \frac{\nabla u^2}{u^2} + \frac{d\delta_2}{1 + \delta_1} \int_{\Omega} |\nabla w|^2
\end{align*}
and
\begin{align*}
\delta_1 \frac{\mu_1}{\mu_2} \int_{\Omega} \frac{\chi_2(w)}{v} \nabla v \cdot \nabla w &\leq \chi_2(0) \delta_1 \frac{\mu_1}{\mu_2} \int_{\Omega} \frac{\nabla v \cdot \nabla w}{v} \\
&\leq \frac{\chi_2(0)^2 \delta_1 (1 + \delta_1)}{4d\delta_2} \left( \frac{\mu_1}{\mu_2} \right)^2 \int_{\Omega} \frac{\nabla v^2}{v^2} + \frac{d\delta_1 \delta_2}{1 + \delta_1} \int_{\Omega} |\nabla w|^2.
\end{align*}
Plugging these into (4.2.6), we infer
\begin{align*}
I_2(t) &\leq - \left( 1 - \frac{\chi_1(0)^2(1 + \delta_1)}{4d\delta_2} \right) \int_{\Omega} \frac{\nabla u^2}{u^2} - \delta_1 \frac{\mu_1}{\mu_2} \left( 1 - \frac{\mu_1 \chi_2(0)^2(1 + \delta_1)}{4d\mu_2 \delta_2} \right) \int_{\Omega} \frac{\nabla v^2}{v^2}.
\end{align*}
We note from the definition of $\delta_2 > 0$ that
\begin{align*}
1 - \frac{\chi_1(0)^2(1 + \delta_1)}{4d\delta_2} > 0
\end{align*}
and
\begin{align*}
1 - \frac{\mu_1 \chi_2(0)^2(1 + \delta_1)}{4d\mu_2 \delta_2} > 0.
\end{align*}
Thus we have (4.2.8). Combination of (4.2.5), (4.2.7) and (4.2.8) implies the end of the proof.

**Lemma 4.2.4.** Let $(u, v, w)$ be a solution to (4.1.1). Under the conditions (4.1.3)–(4.1.9) and (4.1.11), $(u, v, w)$ has the following asymptotic behavior:
\begin{align*}
\|u(\cdot, t) - 1\|_{L^\infty(\Omega)} &\to 0, \quad \|v(\cdot, t) - 1\|_{L^\infty(\Omega)} \to 0, \quad \|w(\cdot, t) - \tilde{w}\|_{L^\infty(\Omega)} \to 0 \quad (t \to \infty).
\end{align*}
Proof. Firstly the boundedness of $u$, $v$, $\nabla w$ and a standard parabolic regularity theory ([98]) yield that there exist $\theta \in (0,1)$ and $C > 0$ such that

$$
\|u\|_{C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega} \times [1,t])} + \|v\|_{C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega} \times [1,t])} + \|w\|_{C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega} \times [1,t])} \leq C \quad \text{for all } t \geq 1.
$$

Therefore in view of the Gagliardo–Nirenberg inequality

$$(4.2.9) \quad \|\varphi\|_{L^\infty(\Omega)} \leq C \|\varphi\|_{W^{1,\infty}(\Omega)} \|\varphi\|_{L^2(\Omega)} \quad (\varphi \in W^{1,\infty}(\Omega))$$

(see Lemma 2.1.1), it is sufficient to show that

$$
\|u(\cdot,t) - 1\|_{L^2(\Omega)} \to 0, \quad \|v(\cdot,t) - 1\|_{L^2(\Omega)} \to 0, \quad \|w(\cdot,t) - \tilde{w}\|_{L^2(\Omega)} \to 0 \quad (t \to \infty).
$$

We let

$$
f_1(t) := \int_{\Omega} (u - 1)^2 + \int_{\Omega} (v - 1)^2 + \int_{\Omega} (w - \tilde{w})^2.
$$

We have that $f_1$ is a nonnegative function, and thanks to the regularity of $u,v,w$ we can see that $f_1$ is uniformly continuous. Moreover, integrating (4.2.3) over $(1,\infty)$, we infer from the positivity of $E_1(t)$ that

$$
\int_{1}^{\infty} f_1(t) \, dt \leq \frac{1}{\varepsilon} E_1(1) < \infty.
$$

Therefore we conclude from Lemma 4.2.1 that $f_1(t) \to 0 \ (t \to \infty)$, which means

$$
\int_{\Omega} (u - 1)^2 + \int_{\Omega} (v - 1)^2 + \int_{\Omega} (w - \tilde{w})^2 \to 0 \quad (t \to \infty).
$$

This implies the end of the proof. \(\square\)

Lemma 4.2.5. Let $(u,v,w)$ be a solution to (4.1.1). Under the conditions (4.1.3)–(4.1.9) and (4.1.11), there exist $C > 0$ and $\lambda > 0$ such that

$$
\|u(\cdot,t) - 1\|_{L^\infty(\Omega)} + \|v(\cdot,t) - 1\|_{L^\infty(\Omega)} + \|w(\cdot,t) - \tilde{w}\|_{L^\infty(\Omega)} \leq C e^{-\lambda t} \quad (t > 0).
$$

Proof. From the L'Hôpital theorem applied to $H(s) := s - \log s$ we can see

$$(4.2.10) \quad \lim_{s \to 1} \frac{H(s) - H(1)}{(s - 1)^2} = \lim_{s \to 1} \frac{H''(s)}{2} = \frac{1}{2}.
$$

In view of a combination of (4.2.10) and $\|u - 1\|_{L^\infty(\Omega)} \to 0$ from Lemma 4.2.4 we obtain that there exists $t_0 > 0$ such that

$$(4.2.11) \quad \frac{1}{4} \int_{\Omega} (u - 1)^2 \leq A_1(t) = \int_{\Omega} (H(u) - H(1)) \leq \int_{\Omega} (u - 1)^2 \quad (t > t_0).$$
A similar argument, for the function $v$, yields that there exists $t_1 > t_0$ such that

\begin{equation}
\frac{1}{4} \int_{\Omega} (v - 1)^2 \leq B_1(t) \leq \int_{\Omega} (v - 1)^2 \quad (t > t_1).
\end{equation}

We infer from (4.2.11) and the definitions of $E_1(t)$, $F_1(t)$ that

$$E_1(t) \leq c_6 F_1(t)$$

for all $t > t_1$ with some $c_6 > 0$. Plugging this into (4.2.3), we have

$$\frac{d}{dt} E_1(t) \leq -\varepsilon F_1(t) \leq -\frac{\varepsilon}{c_6} E_1(t) \quad (t > t_1),$$

which implies that there exist $c_7 > 0$ and $\ell > 0$ such that

$$E_1(t) \leq c_7 e^{-\ell t} \quad (t > t_1).$$

Thus we obtain from (4.2.11) and (4.2.12) that

$$\int_{\Omega} (u - 1)^2 + \int_{\Omega} (v - 1)^2 + \int_{\Omega} (w - \tilde{w})^2 \leq c_8 E_1(t) \leq c_7 c_8 e^{-\ell t}$$

for all $t > t_1$ with some $c_8 > 0$. From the Gagliardo–Nirenberg inequality (4.2.9) with the regularity of $u, v, w$, we achieve that there exist $C > 0$ and $\lambda > 0$ such that

$$\|u(\cdot, t) - 1\|_{L^\infty(\Omega)} + \|v(\cdot, t) - 1\|_{L^\infty(\Omega)} + \|w(\cdot, t) - \tilde{w}\|_{L^\infty(\Omega)} \leq C e^{-\lambda t} \quad (t > 0).$$

This completes the proof of Lemma 4.2.5.

**Proof of Theorem 4.1.1.** Theorem 4.1.1 follows directly from Lemma 4.2.5.
Chapter 5

On the weakly competitive case in a two-species chemotaxis model

5.1. Background and result

The questions how different populations living in the same habitat interact with each other and with their surroundings are central to mathematical biology.

The competition for resources by two species (in contrast to, e.g., one being prey to the other) is often modeled by means of Lotka–Volterra type competition terms, i.e. with coupling coefficients $a_1 > 0$, $a_2 > 0$ in

$$u_t = u(1 - u - a_1 v), \quad v_t = v(1 - v - a_2 u).$$

We refer to [64] for conditions for global stability of fixed points in such systems: If $a_1 > 0$ and $a_2 > 0$, for positive equilibria to be globally stable it is sufficient that they be locally stable.

Of course, spatial homogeneity is a highly idealized situation, and dependence of the population size on a spatial variable together with the effects of random motion of individuals has been incorporated into the model (see e.g. [139, Chapter 1.2] and references therein).

One way of even simplest lifeforms to react to their environment is chemotaxis, that is the tendency to move in the direction of higher concentrations of a signal substance. Inter alia, effects of chemotaxis on possible population size have been considered in [107].

Exploitation of chemotaxis for biotechnological purposes in mind and envisioning applications in e.g. agriculture (like nitrogen fixation or denitrification), or in mammalian intestinal microbial ecology, in [108] the authors compare species that undergo growth with different rates and diffusive versus diffusive plus chemotactic motion. They conclude: “Thus, chemotaxis can, when the response is sufficiently strong, overcome both disadvantages of inferior growth kinetics and random motility. [...] At any rate, these results suggest that chemotactic responses might provide a useful means for controlling population dynamics in nonmixed systems. In particular, they provide a way to permit slowly growing populations to coexist with or outcompete faster growing species. Such a situation might be highly desirable in many environmental or biotechnological applications.” For the effectiveness of nutrient-taxis as advantageous dispersal strategy for populations in heterogeneous environments see e.g. [121, 23].
Chemotaxis terms in combination with several populations also appear in the context of the host-parasite interaction modelled and analyzed in [155] and [166], the food-chain model of [117], or the system in [8] that deals with the spread of an epidemic disease.

For one single species, chemotaxis is described by the celebrated model by Keller and Segel

\[ u_t = \Delta u - \nabla \cdot (u \nabla v), \quad \tau v_t = \Delta v - v + u, \]  

(5.1.1)

which has been treated intensively over the last decades. We refer to the surveys [71, 74, 5].

Incorporating growth terms into this model, that is, adding \( u^2 \) to the first equation in (5.1.1) (studied in [177, 192, 195, 100]), gives rise to colorful dynamics as witnessed in e.g. [154], emphasized by attractor results in [97, 145, 2, 147] or illustrated by recent results on transient growth phenomena in [200], [99].

One of the most straightforward generalizations of this model to the situation of several species is to consider two species (or two subpopulations of one species) that react to the same signalling substance they both produce, as occurring in the differentiation of cell-types during slime mold formation in populations of Dictyostelium discoideum, cf. e.g. [123, 180].

The pure two-species chemotaxis model without growth or competition effects has been introduced in [208] and in particular blow-up of solutions in finite time, known to occur in the single-species situation, has been investigated also for the several species model (see e.g. [75]), both as to the question of occurrence of blow-up versus global existence (see [13, 14, 43, 96, 37, 114]) and of qualitative features of the former, for example whether it occurs simultaneously ([46, 44]) in both species, or nonsimultaneously ([45]); for numerical observations pertaining to these models also confer [94]. Furthermore, the long term behavior of globally existent solutions emanating from small initial data ([211, 112]) or for chemosensitivity functions capturing saturation effects ([150]) has been investigated.

In this chapter we consider the two-species chemotaxis-competition model

\[
\begin{align*}
    u_t &= d_1 \Delta u - \chi_1 \nabla \cdot (u \nabla w) + \mu_1 u(1 - u - a_1 v), \quad x \in \Omega, \; t > 0, \\
    v_t &= d_2 \Delta v - \chi_2 \nabla \cdot (v \nabla w) + \mu_2 v(1 - a_2 u - v), \quad x \in \Omega, \; t > 0, \\
    \tau w_t &= d_3 \Delta w + \alpha u + \beta v - \gamma w, \quad x \in \Omega, \; t > 0, \\
    \nabla u \cdot \nu = \nabla v \cdot \nu = \nabla w \cdot \nu &= 0, \quad x \in \partial \Omega, \; t > 0, \\
    u(\cdot,0) &= u_0, \quad v(\cdot,0) = v_0, \quad \tau w(\cdot,0) = \tau w_0, \quad x \in \Omega,
\end{align*}
\]

(5.1.2)

where \( u \) and \( v \) denote the population densities of two species undergoing chemotaxis in reaction to the signal having concentration \( w \), posed in a bounded smooth domain \( \Omega \subset \mathbb{R}^n \). Herein, the diffusion rates of species and signal are given by \( d_1, d_2, d_3 > 0 \), whereas \( \mu_i > 0, \chi_i \geq 0, a_i \geq 0 \; (i \in \{1, 2\}) \) are used to denote the strengths of chemotaxis, growth kinetics and competition for each species, whereas the size of \( \alpha > 0, \beta > 0 \) and \( \gamma > 0 \) regulates the production of the signal by the first and second species and its decay, respectively. This model has first been considered in [178] for \( \tau = 0 \), where it was shown...
for the weakly competitive case, i.e.

\[ a_1, a_2 \in [0, 1), \]

that solutions to (5.1.2) exist and converge to the coexistence steady state

\[ (u_*, v_*, w_*) = \left( \frac{1 - a_1}{1 - a_1 a_2}, \frac{1 - a_2}{1 - a_1 a_2}, \frac{2 - a_1 - a_2}{1 - a_1 a_2} \right), \]

provided that

\[ 2(\chi_1 + \chi_2) + a_1 \mu_2 < \mu_1 \quad \text{and} \quad 2(\chi_1 + \chi_2) + a_2 \mu_1 < \mu_2. \]

Result and proof of [178] have successfully been extended to a setting of even more species in [184], the condition being analogous to (5.1.4).

This condition seems quite unnatural, because it is not automatically satisfied in the absence of chemotaxis \((\chi_1 = \chi_2 = 0)\) and it is the goal of the present chapter to replace this condition by a smallness condition on \(q_i = \frac{\chi_i}{\mu_i}\), \(i \in \{1, 2\}\) alone and to thus remove the additional condition on \(a_1, a_2, \mu_1, \mu_2\) implicitly posed by (5.1.4):

**Theorem 5.1.1.** Let \(n \geq 1\) and \(\Omega \subset \mathbb{R}^n\) be a bounded domain with smooth boundary. Let \(\tau = 0\) and \(d_1, d_2, d_3, \alpha, \beta, \gamma \in (0, \infty)\). Let \(a_1, a_2\) fulfil (5.1.3) and let \(\chi_1 \geq 0, \chi_2 \geq 0\) and \(\mu_1, \mu_2 > 0\) be such that \(q_1 := \frac{\chi_1}{\mu_1}, q_2 := \frac{\chi_2}{\mu_2}\) satisfy the conditions:

\[ q_1 \in \left[0, \frac{d_1}{2\alpha}\right] \cap \left[0, \frac{a_1 d_1}{\beta}\right], \quad q_2 \in \left[0, \frac{d_2}{2\beta}\right] \cap \left[0, \frac{a_2 d_1}{\alpha}\right], \]

\[ a_1 a_2 d_2^3 < (d_3 - 2\alpha q_1)(d_3 - 2\beta q_2). \]

Then the following holds:

\(\text{(i)}\) For all nonnegative functions \(u_0, v_0 \in C(\overline{\Omega})\) satisfying \(u_0 \neq 0\), \(v_0 \neq 0\), there exists a unique global-in-time classical solution

\[(u, v, w) \in \left(C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty))\right)^3\]

of (5.1.2) such that \(u > 0, v > 0\) and \(w > 0\) in \(\overline{\Omega} \times (0, \infty)\).

\(\text{(ii)}\) The unique global solution \((u, v, w)\) of (5.1.2) has the following asymptotic behavior:

\[ u(\cdot, t) \to u^*, \quad v(\cdot, t) \to v^*, \quad w(\cdot, t) \to \frac{\alpha u^* + \beta v^*}{\gamma} \quad \text{as} \ t \to \infty, \]

uniformly in \(\Omega\), where

\[ u^* = \frac{1 - a_1}{1 - a_1 a_2}, \quad \text{and} \quad v^* = \frac{1 - a_2}{1 - a_1 a_2}. \]

We want to emphasize that (5.1.5) can indeed be viewed as a smallness condition on the relative chemotactic strength only and, in particular, satisfied in the chemotaxis-free
case ($\chi_1 = \chi_2 = 0$).

The case of partially strong competition ($a_1 > 1 > a_2 > 0$) was considered in [164]. It was shown that solutions exist globally and satisfy $(u(\cdot, t), v(\cdot, t)) \to (0, 1)$ if $q_1 \leq a_1$, $q_2 < \frac{1}{2}$, $\alpha q_1 + \max\{q_2, \frac{a_2-q_2}{1-q_2}, \frac{a_2-q_2}{1-q_2}\} < 1$.

In the fully parabolic system ((5.1.2) with $\tau = 1$), it was proven in [4] that for sufficiently large values of $\mu_1$, $\mu_2$ global classical solutions converge to the unique positive homogeneous equilibrium exponentially for $a_1 > 1 > a_2 > 0$ (and with an algebraic rate if $a_1 = 1$ and $\mu_2$ is large), and moreover that there are global bounded classical solutions for $n \leq 2$ even if the parameters of the system are merely positive.

Furthermore, in spatially one-dimensional domains more insight into qualitative behavior of the system has been obtained in [182] and [183], where global existence of solutions was shown as well as existence of nonconstant steady states by bifurcation analysis. The stability of the bifurcating solutions is investigated there, too, and a time-periodic solution has been found. Additionally, the findings have also been illustrated by numerical experiments ([183, Section 4]).

In [213], for different sensitivity functions satisfying $\chi_i(w) \leq \frac{K_i}{(1+a_i w)^{k_i}}$ for some $k_i > 1$, $i \in \{1, 2\}$, global bounded classical solutions were proven to exist under the condition of $\chi, \mu, K$ being sufficiently small (where the meaning of 'sufficiently small' depends on the initial mass).

In the competition-free case ($a_1 = a_2 = 0$) for sensitivity functions generalizing $\chi_i(w) \leq \frac{K_i}{(1+w)^{k_i}}$ ($k > 1$), global existence and boundedness of solutions were obtained, together with a result on asymptotic stability of steady states (Chapter 3).

A system where the chemoattractant is not a signal substance produced by the population itself but a nutrient which is consumed, (but which is otherwise similar), has been treated in [215, 185].

Let us finally mention that also (parabolic-elliptic) Keller–Segel type systems of two species and two chemicals have been studied where the signal for each species is produced by the other ([175]).

A parabolic-parabolic-ODE model with connections to chemotaxis-haptotaxis models (see e.g. [31]) has been treated in [151].

Cross-diffusive effects like those added to the present system by means of the chemotaxis term pose a serious threat to any monotonicity properties one would like to employ and usually should render comparison arguments useless. Nevertheless, in some situations closely related, comparison methods were employable, for example the proof of the result in [178] relies on comparison with solutions to a system of four coupled ODEs. (In fact, a system closely related to (5.1.2) was considered as example in [149], where comparison with solutions of ODE systems has been developed more systematically. However, there the essential coupling arising from the appearance of $u$ and $v$ as source terms for the third equation was not included.)

Most successfully, comparison arguments were utilized in deriving the global asymptotic stability of in (5.1.2) for the case of strong competition in [164].

The situation considered there is, in a certain sense, easier than the present case, since the limit of one component being 0 makes lower estimates for this component unnecessary.
and simplifies the system of inequalities that has to be dealt with during the proof.

Nevertheless this method of ‘eventual comparison’, as we would like to call it, turns out to be a powerful tool also in the present context and we will use it to derive the desired result.

More precisely, we shall proceed as follows: At first we will establish global existence and boundedness of the solutions to (5.1.2). Then, knowing that limes inferior and limes superior of each component exist and are not infinite, we find differential inequalities that are eventually (that is, on time intervals of the form \((T, \infty)\) for some \(T > 0\) ) satisfied and that are accessible to comparison arguments, the corresponding ODE solution converging (almost) regardless of its initial value. This will give us enough information to deduce the precise value of limits superior and inferior and to prove convergence of the solution to the coexistence steady state.

5.2. Global existence

**Lemma 5.2.1.** Let \(n \geq 1\) and \(\Omega \subset \mathbb{R}^n\) be a bounded domain with smooth boundary, let \(\chi_1, \chi_2, a_1, a_2 \geq 0\), \(d_1, d_2, d_3, \alpha, \beta, \gamma, \mu_1, \mu_2 \in (0, \infty)\). Suppose that \(u_0, v_0 \in C(\overline{\Omega})\) are nonnegative such that \(u_0 \neq 0\), \(v_0 \neq 0\). Then there exist \(T_{\text{max}} \in (0, \infty)\) and a unique classical solution \((u, v, w) \in (C(\overline{\Omega} \times [0, T_{\text{max}}])) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\text{max}})))^3\) of (5.1.2) on \(\Omega \times [0, T_{\text{max}}]\), such that moreover the following extensibility criterion holds:

Either \(T_{\text{max}} = \infty\) or \(\limsup_{t \uparrow T_{\text{max}}} (\|u(\cdot, t)\|_{L^{\infty}(\Omega)} + \|v(\cdot, t)\|_{L^{\infty}(\Omega)}) = \infty\).

Furthermore, \(u, v\) and \(w\) are positive in \(\overline{\Omega} \times (0, T_{\text{max}})\).

**Proof.** The proof is the same as \([164, \text{Lemma 2.1}]\). \(\square\)

In the following lemma we infer boundedness of the solutions by a comparison argument and hence, in accordance with the above extensibility criterion, global existence. In its proof and also later on, given \(d_1 > 0\), \(d_2 > 0\) and \(w\), we denote by \(\mathcal{L}_1\) and \(\mathcal{L}_2\) the operators defined by

\[
\mathcal{L}_1 \tilde{u} := d_1 \Delta \tilde{u} - \chi_1 \nabla \tilde{u} \cdot \nabla w, \quad \mathcal{L}_2 \tilde{v} := d_2 \Delta \tilde{v} - \chi_2 \nabla \tilde{v} \cdot \nabla w.
\]

**Lemma 5.2.2.** Suppose that the assumptions of Theorem 5.1.1 are satisfied and that \(T_{\text{max}}\), \(u, v, w\) are as given by Lemma 5.2.1. Then \(T_{\text{max}} = \infty\) and both \(u\) and \(v\) are bounded in \(\Omega \times (0, \infty)\).

**Proof.** Making use of (5.2.1), from the first and third equation of (5.1.2) we obtain

\[
u_t = \mathcal{L}_1 u - \chi_1 u \frac{\gamma w - \alpha u - \beta v}{d_3} + \mu_1 u(1 - u - a_1 v)
= \mathcal{L}_1 u + \mu_1 u \left(1 - \left(1 - \frac{\alpha q_1}{d_3}\right)u - \left(a_1 - \frac{\beta q_1}{d_3}\right)v - \frac{\gamma q_1}{d_3} w\right),
\]

37
wherein the choice of \( q_1 \) then implies

\[
u_t \leq \mathcal{L}_1 u + \mu_1 u \left( 1 - \left(1 - \frac{\alpha q_1}{d_3}\right) u \right).
\]

We choose \( \overline{u} \in (0, \infty) \) such that \( \|u_0\|_{L^\infty(\Omega)} \leq \overline{u} \), and denote by \( y_1 : [0, \infty) \to \mathbb{R} \) the function solving

\[
\begin{cases}
y_1' = \mu_1 y_1 \left( 1 - \left(1 - \frac{\alpha q_1}{d_3}\right) y_1 \right), \\
y_1(0) = \overline{u}
\end{cases}
\]

which satisfies

\[
y_1(t) \to \frac{d_3}{d_3 - \alpha q_1} \quad \text{as } t \to \infty.
\]

By a comparison theorem we obtain

\[
\limsup_{t \to T_{\text{max}}} u(\cdot, t) \leq \limsup_{t \to T_{\text{max}}} y_1(t) = \frac{d_3}{d_3 - \alpha q_1}.
\]

Treating the second equation of (5.1.2) in a similar fashion we get

\[
v_t \leq \mathcal{L}_2 v + \mu_2 v \left( 1 - \left(1 - \frac{\beta q_2}{d_3}\right) v \right),
\]

and analogously conclude

\[
\limsup_{t \to T_{\text{max}}} v(\cdot, t) \leq \frac{d_3}{d_3 - \beta q_2}.
\]

By the extensibility criterion we obtain \( T_{\text{max}} = \infty \).

**5.3. Global asymptotic stability**

Since Lemma 5.2.2 guarantees that \( u \) and \( v \) exist globally and are bounded and nonnegative, it is possible to define nonnegative finite real numbers \( L_1, l_1, L_2, l_2 \) by

\[
L_1 := \limsup_{t \to \infty} \left( \max_{x \in \Omega} u(x, t) \right),
\]

\[
l_1 := \liminf_{t \to \infty} \left( \min_{x \in \Omega} u(x, t) \right),
\]

\[
L_2 := \limsup_{t \to \infty} \left( \max_{x \in \Omega} v(x, t) \right),
\]

\[
l_2 := \liminf_{t \to \infty} \left( \min_{x \in \Omega} v(x, t) \right).
\]

From the definition we have that for all \( \varepsilon > 0 \) there exists \( T_\varepsilon > 0 \) such that

\[
l_1 - \varepsilon < u(x, t) < L_1 + \varepsilon, \quad l_2 - \varepsilon < v(x, t) < L_2 + \varepsilon
\]
hold for all $t > T$ and all $x \in \Omega$. By the maximum principle applied to
\[
\begin{cases} 
-d_3 \Delta w + \gamma w = \alpha u + \beta v & \text{in } \Omega, \\
\nabla w \cdot \nu = 0 & \text{on } \partial \Omega,
\end{cases}
\]
we have
\[
\min_{\xi \in \Omega}(\alpha u(\xi, t) + \beta v(\xi, t)) \leq \gamma w(x, t) \leq \max_{\xi \in \Omega}(\alpha u(\xi, t) + \beta v(\xi, t))
\]
for all $t > 0$ and all $x \in \Omega$. Consequently, we obtain from (5.3.2) that for all $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that
\[
\alpha l_1 + \beta l_2 - 2\varepsilon < \gamma w(x, t) < \alpha L_1 + \beta L_2 + 2\varepsilon
\]
for all $t > T_\varepsilon$ and all $x \in \Omega$. Employing comparison arguments on ultimate time intervals, we derive first estimates for the quantities defined in (5.3.1).

**Lemma 5.3.1.** Under the assumptions of Theorem 5.1.1 and with (5.3.1), the following inequalities hold:
\[
L_1 \leq \frac{(d_3 - \alpha q_1 l_1 - a_1 d_3 l_2)_+}{d_3 - \alpha q_1} \quad \text{and} \quad l_1 \geq \frac{d_3 - \alpha q_1 L_1 - a_1 d_3 L_2}{d_3 - \alpha q_1}.
\]

**Proof.** Recalling that from the first and third equation of (5.1.2) and using the same notation as in (5.2.1) we have
\[
u \frac{\partial}{\partial t} u + \alpha q_1 l_1 + a_1 d_3 l_2 - \frac{\partial}{\partial t} v - \gamma q_1 d_3 w = 0
\]
we let $\varepsilon > 0$ and make use of (5.3.2) and (5.3.3) to find $T_\varepsilon > 0$ such that
\[
u \frac{\partial}{\partial t} u + \alpha q_1 l_1 + a_1 d_3 l_2 - \frac{\partial}{\partial t} v - \gamma q_1 d_3 w = 0
\]
on $(T_\varepsilon, \infty)$, where for the estimates we relied on nonnegativity of $a_1 - \frac{\beta q_1}{d_3}$ as guaranteed by (5.1.5).

We choose $\bar{\omega}_\varepsilon \in (0, \infty)$ such that $u(\cdot, T_\varepsilon) \leq \bar{\omega}_\varepsilon$ in $\Omega$ and we denote by $\bar{\omega} : [T_\varepsilon, \infty) \to \mathbb{R}$ the function solving
\[
\begin{cases} 
\bar{\omega}' = \mu_1 \bar{\omega} \left(1 - \left(1 - \frac{\alpha q_1}{d_3}\right) \bar{\omega} - \frac{\alpha q_1}{d_3} l_1 - a_1 l_2 + \left(a_1 + (2 - \beta) \frac{q_1}{d_3}\right) \varepsilon\right) & \text{in } (T_\varepsilon, \infty), \\
\bar{\omega}(T_\varepsilon) = \bar{\omega}_\varepsilon,
\end{cases}
\]
which satisfies
\[
\bar{\omega}(t) \to \frac{(d_3 - \alpha q_1 l_1 - a_1 d_3 l_2 + \left(a_1 d_3 + (2 - \beta) q_1\right) \varepsilon)_+}{d_3 - \alpha q_1} \quad \text{as } t \to \infty.
\]
By comparison we obtain

\[
L_1 = \limsup_{t \to \infty} \left( \max_{x \in \Omega} u(x, t) \right) \\
\leq \limsup_{t \to \infty} \bar{z}(t) \\
= \left( \frac{d_3 - \alpha q_1 l_1 - a_4 d_3 l_2 + \left( a_4 d_3 + (2 - \beta) q_1 \right) \varepsilon}{d_3 - \alpha q_1} \right) +.
\]

On the other hand, making use of the other estimates in (5.3.2) and (5.3.3) and again of (5.1.5), we have

\[
u_t - L_1 u = \mu_1 u \left( 1 - \left( 1 - \frac{\alpha q_1}{d_3} \right) u - \left( a_1 - \frac{\beta q_1}{d_3} \right) v - \frac{\gamma q_1}{d_3} w \right) \\
\geq \mu_1 u \left( 1 - \left( 1 - \frac{\alpha q_1}{d_3} \right) u - \left( a_1 - \frac{\beta q_1}{d_3} \right) (L_2 + \varepsilon) - \frac{q_1}{d_3} (\alpha L_1 + \beta L_2 + 2\varepsilon) \right) \\
= \mu_1 u \left( 1 - \left( 1 - \frac{\alpha q_1}{d_3} \right) u - \frac{\alpha q_1}{d_3} L_1 - a_1 L_2 - \left( a_1 - (2 - \beta) \frac{q_1}{d_3} \right) \varepsilon \right) \text{ on } (T_\varepsilon, \infty).
\]

Choosing \( u_\varepsilon > 0 \) such that \( u(\cdot, T_\varepsilon) \geq u_\varepsilon \) in \( \Omega \) and denoting by \( \tilde{z} : [T_\varepsilon, \infty) \to \mathbb{R} \) the function solving

\[
\begin{cases}
\tilde{z}' = \mu_1 \tilde{z} \left( 1 - \left( 1 - \frac{\alpha q_1}{d_3} \right) \tilde{z} - \frac{\alpha q_1}{d_3} L_1 - a_1 L_2 - \left( a_1 - (2 - \beta) \frac{q_1}{d_3} \right) \varepsilon \right) \text{ in } (T_\varepsilon, \infty), \\
\tilde{z}(T_\varepsilon) = u_\varepsilon,
\end{cases}
\]

which satisfies

\[
\tilde{z}(t) \to \left( \frac{d_3 - \alpha q_1 L_1 - a_4 d_3 L_2 - \left( a_4 d_3 + (2 - \beta) q_1 \right) \varepsilon}{d_3 - \alpha q_1} \right) + \text{ as } t \to \infty.
\]

Thus we obtain from the comparison theorem that

\[
l_1 = \liminf_{t \to \infty} \left( \min_{x \in \Omega} u(x, t) \right) \\
\geq \liminf_{t \to \infty} \tilde{z}(t) \\
\geq \frac{d_3 - \alpha q_1 L_1 - a_4 d_3 L_2 - \left( a_4 d_3 + (2 - \beta) q_1 \right) \varepsilon}{d_3 - \alpha q_1}
\]

holds. Because \( \varepsilon > 0 \) was arbitrary, (5.3.6) and (5.3.7) entail (5.3.4).

\[
\square
\]

**Lemma 5.3.2.** Under the assumptions of Theorem 5.1.1 and with notation as in (5.3.1), the following inequalities hold:

\[
L_2 \leq \left( \frac{d_3 - \alpha q_1 L_1 - \beta q_2 L_2}{d_3 - \beta q_2} \right) + \text{ and } l_2 \geq \frac{d_3 - a_2 d_3 L_1 - \beta q_2 L_2}{d_3 - \beta q_2}.
\]

40
Proof. Repeating the arguments from the proof of Lemma 5.3.1, this time with
\[ v_t - L_v = \mu v \left( 1 - \left( 1 - \frac{\beta q_2}{d_3} \right) v - \left( a_2 - \frac{\alpha q_2}{d_3} \right) u - \frac{\gamma q_2}{d_3} w \right), \]
instead of (5.3.5), leads to Lemma 5.3.2.

Before we continue, let us briefly verify that the differences appearing in the numerators of the upper bounds for \( L_1 \) and \( L_2 \) are already nonnegative, so that we can neglect the positive part operator \((\cdot)_+\). Later on, this will allow us to conclude convergence to the non-zero equilibrium state.

Lemma 5.3.3. Under the assumptions of Theorem 5.1.1 and with notation as in (5.3.1),
\[ d_3 - \alpha q_1 l_1 - a_1 d_3 l_2 \geq 0 \]
and
\[ d_3 - a_2 d_3 l_1 - \beta q_2 l_2 \geq 0. \]

Proof. We work along the lines of a contradiction argument to show that the undesired cases can in fact not appear. If
\[ (5.3.8) \quad d_3 - \alpha q_1 l_1 - a_1 d_3 l_2 < 0 \quad \text{and} \quad d_3 - a_2 d_3 l_1 - \beta q_2 l_2 < 0, \]
we obtain from Lemmas 5.3.1 and 5.3.2, and the nonnegativity of \( u \) and \( v \) that
\[ 0 \leq l_1 \leq L_1 \leq 0, \]
and
\[ 0 \leq l_2 \leq L_2 \leq 0. \]

Plugging this back into (5.3.8) yields the contradiction \( d_3 < 0 \). In the case
\[ d_3 - \alpha q_1 l_1 - a_1 d_3 l_2 \geq 0 \quad \text{and} \quad d_3 - a_2 d_3 l_1 - \beta q_2 l_2 < 0, \]
Lemmas 5.3.1 and 5.3.2, and the nonnegativity of \( v \) show that
\[ (5.3.9) \quad L_2 = l_2 = 0 \quad \text{and} \quad L_1 \leq \frac{d_3 - \alpha q_1 l_1}{d_3 - \alpha q_1}, \quad l_1 \geq \frac{d_3 - \alpha q_1 L_1}{d_3 - \alpha q_1}. \]

Herein, the first two inequalities lead to
\[ (d_3 - \alpha q_1) L_1 \leq d_3 - \alpha q_1 l_1, \]
\[ (d_3 - \alpha q_1) l_1 \geq d_3 - \alpha q_1 L_1, \]
which implies
\[ (d_3 - 2\alpha q_1)(L_1 - l_1) \leq 0. \]
Due to $q_1 < \frac{d_3}{2a}$ (by (5.1.5)) this yields $L_1 = l_1$. Together with (5.3.9) this equality shows

$$L_1 = \frac{d_3 - \alpha q_1 L_1}{d_3 - \alpha q_1},$$

which leads to $l_1 = L_1 = 1$ because of $d_3 > 0$. Making use of this and (5.3.9), we conclude that the second inequality from Lemma 5.3.2 implies the contradiction

$$0 = l_2 \geq \frac{d_3 - \alpha_2 d_3 l_1 - \beta q_2 L_2}{d_3 - \beta q_2} = \frac{(1 - \alpha_2)d_3}{d_3 - \beta q_2} > 0,$$

since $d_3 > 0$, $d_3 > \beta q_2$ due to (5.1.5) and, by (5.1.3), $1 > \alpha_2$. The case

$$d_3 - \alpha q_1 l_1 - a_1 d_3 l_2 < 0 \quad \text{and} \quad d_3 - a_2 d_3 l_1 - \beta q_2 l_2 \geq 0$$

can be treated in a similar fashion relying on the facts that $q_2 < \frac{d_3}{2a}$ by (5.1.5) and $a_1 < 1$ by (5.1.3) to obtain the contradiction $0 = l_1 > 0$. \hfill $\square$

Having explicit bounds for $L_1$ and $L_2$ at hand, we can now calculate the exact values of $L_1$ and $L_2$.

**Lemma 5.3.4.** Let the assumptions of Theorem 5.1.1 be satisfied. Then

$$L_1 = l_1 = u^*$$

and

$$L_2 = l_2 = v^*,$$

where

$$u^* := \frac{1 - a_1}{1 - a_1 a_2}, \quad v^* := \frac{1 - a_2}{1 - a_1 a_2}.$$

Moreover the solution of (5.1.2) converges to nontrivial steady states, i.e.,

$$u(\cdot, t) \to u^*, \quad v(\cdot, t) \to v^*, \quad w(\cdot, t) \to \frac{\alpha u^* + \beta v^*}{\gamma}$$

as $t \to \infty$, uniformly in $\Omega$.

**Proof.** At first, we shall prove that in fact the solutions converge, namely that

$$L_1 = l_1, \quad L_2 = l_2$$

hold. Thanks to Lemmas 5.3.1, 5.3.2 and 5.3.3 we know that the inequalities

\begin{align*}
(5.3.10) \quad & (d_3 - \alpha q_1) L_1 \leq d_3 - \alpha q_1 l_1 - a_1 d_3 l_2, \\
(5.3.11) \quad & (d_3 - \beta q_2) L_2 \leq d_3 - a_2 d_3 l_1 - \beta q_2 l_2,
\end{align*}
and

\[(d_3 - \alpha q_1) l_1 \geq d_3 - \alpha q_1 L_1 - a_1 d_3 L_2,\]
\[(d_3 - \beta q_2) l_2 \geq d_3 - a_2 d_3 L_1 - \beta q_2 L_2\]

hold. From (5.3.10) and (5.3.12) we extract

\[(d_3 - \alpha q_1) (L_1 - l_1) \leq \alpha q_1 (L_1 - l_1) + a_1 d_3 (L_2 - l_2).\]

Re-ordering this inequality while paying attention to (5.1.5), we see that

\[(5.3.14)\]
\[L_1 - l_1 \leq \frac{a_1 d_3}{d_3 - 2\alpha q_1} (L_2 - l_2).\]

Taking into account (5.3.11) and (5.3.13), from a similar argument we obtain

\[(5.3.15)\]
\[L_2 - l_2 \leq \frac{a_2 d_3}{d_3 - 2\beta q_2} (L_1 - l_1).\]

Combination of (5.3.14) and (5.3.15) shows

\[L_1 - l_1 \leq \frac{a_1 d_3}{d_3 - 2\alpha q_1} \cdot \frac{a_2 d_3}{d_3 - 2\beta q_2} (L_1 - l_1).\]

Noting from the smallness condition on \(q_1, q_2\) in (5.1.6) that

\[\frac{a_1 d_3}{d_3 - 2\alpha q_1} \cdot \frac{a_2 d_3}{d_3 - 2\beta q_2} < 1,\]

we obtain

\[L_1 = l_1.\]

Plugging this result into (5.3.15) also yields

\[L_2 = l_2.\]

Lastly, we shall prove

\[l_1 = u^* \quad \text{and} \quad l_2 = v^*.\]

From (5.3.10), (5.3.12), \(L_1 = l_1\) and \(L_2 = l_2\) we have

\[(d_3 - \alpha q_1) l_1 = d_3 - \alpha q_1 l_1 - a_1 d_3 l_2,\]

which, after re-ordering, leads to

\[(5.3.16)\]
\[l_1 = 1 - a_1 l_2.\]

Similarly we obtain from (5.3.11), (5.3.13), \(L_1 = l_1\) and \(L_2 = l_2\) that

\[(5.3.17)\]
\[l_2 = 1 - a_2 l_1.\]
Combination of (5.3.16) and (5.3.17) therefore leads to

\[ l_1 = \frac{1 - a_1}{1 - a_1 a_2} = u^* \]

and

\[ l_2 = \frac{1 - a_2}{1 - a_1 a_2} = v^*. \]

Let \( l_1 = u^* \) and \( l_2 = v^* \) imply that

\[ u(\cdot, t) \rightarrow u^* \quad \text{and} \quad v(\cdot, t) \rightarrow v^* \]

as \( t \rightarrow \infty \), uniformly in \( \Omega \). Finally, accordance with (5.3.3), for any \( \varepsilon > 0 \) there is \( T_\varepsilon > 0 \) such that

\[ \alpha u^* + \beta v^* - 2\varepsilon < \gamma w(x, t) < \alpha u^* + \beta v^* + 2\varepsilon \quad \text{for all} \quad (x, t) \in \Omega \times (T_\varepsilon, \infty) \]

and hence

\[ w(\cdot, t) \rightarrow \frac{\alpha u^* + \beta v^*}{\gamma} \]

as \( t \rightarrow \infty \), uniformly in \( \Omega \).

With this lemma, we actually have completed the proof of Theorem 5.1.1.

**Proof of Theorem 5.1.1.** Part (i) follows from Lemma 5.2.2, while (ii) is contained in Lemma 5.3.4.
Chapter 6

Boundedness and stabilization in a two-species chemotaxis-competition system of parabolic-parabolic-elliptic type

6.1. Motivation and results

A mathematical problem which describes a part of the life cycle of cellular slime molds with chemotaxis is called the Keller–Segel system:

\[
\begin{align*}
  u_t &= \Delta u - \chi \nabla \cdot (u \nabla v), \\
  \tau v_t &= \Delta v + u - v,
\end{align*}
\]

where \( \chi > 0 \) and \( \tau \in \{0, 1\} \). Moreover, the chemotaxis system with growth terms

\[
\begin{align*}
  u_t &= \Delta u - \chi \nabla \cdot (u \nabla v) + \kappa u - \mu u^2, \\
  \tau v_t &= \Delta v + u - v
\end{align*}
\]

was proposed by [125, 152], where \( \chi, \kappa, \mu > 0 \) and \( \tau \in \{0, 1\} \). After the pioneering work of Keller–Segel [89], the Keller–Segel system and the chemotaxis system are intensively studied (see e.g., [5, 71, 74]). A generalized problem of Keller–Segel systems, which means a two-species chemotaxis system, was proposed in [208] and was studied (see e.g., [13, 14, 35, 36, 46, 114, 211]; global existence was proved in [35, 36, 211]; and their asymptotic stability was shown in [211]; related works which deal with blow-up of solutions can be seen in [13, 14, 35, 36, 46, 114]). Recently, a two-species chemotaxis system with competitive kinetics

\[
\begin{align*}
  u_t &= \Delta u - \chi_1 \nabla \cdot (u \nabla w) + \mu_1 u(1 - u - a_1 v), \\
  v_t &= \Delta v - \chi_2 \nabla \cdot (v \nabla w) + \mu_2 v(1 - a_2 u - v), \\
  \tau w_t &= \Delta w + \alpha u + \beta v - \gamma w
\end{align*}
\]

with some \( \chi_1, \chi_2, \mu_1, \mu_2, a_1, a_2 > 0 \) and \( \tau \in \{0, 1\} \), which describes evolution of two competing species which react on a single chemoattractant, was proposed by Tello–Winkler [178] and was studied (see [4, 115, 150, 151, 213] and Chapters 3, 4). About this problem with \( \tau = 1 \), global existence and boundedness was obtained in the 2-dimensional case.
([4]) and the n-dimensional setting ([115]); moreover, asymptotic behavior of solutions was established in [4]. Related works which deal with global existence and boundedness in this two-species problem with sensitivity functions can be found in [213]; and related works which treat the non-competition case are in [150, 151] and Chapters 3, 4. These results in the case \( \tau = 1 \) are motivated by the results ([164, 178] and Chapter 5) in the case \( \tau = 0 \). Therefore it seems to be meaningful to study the parabolic-parabolic-elliptic problem reduced by letting \( \tau = 0 \).

In this chapter we consider the two-species chemotaxis system with competitive kinetics of parabolic-parabolic-elliptic type

\[
\begin{align*}
\dot{u} &= d_1 \Delta u - \chi_1 \nabla \cdot (u \nabla w) + \mu_1 u (1 - u - a_1 v), \quad x \in \Omega, \ t > 0, \\
\dot{v} &= d_2 \Delta v - \chi_2 \nabla \cdot (v \nabla w) + \mu_2 v (1 - a_2 u - v), \quad x \in \Omega, \ t > 0, \\
0 &= d_3 \Delta w + \alpha u + \beta v - \gamma w, \quad x \in \Omega, \ t > 0, \\
\nabla u \cdot \nu &= \nabla v \cdot \nu = \nabla w \cdot \nu = 0, \quad x \in \partial \Omega, \ t > 0, \\
u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \Omega,
\end{align*}
\]

(6.1.1)

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) \((n \geq 2)\) with smooth boundary \( \partial \Omega \) and \( \nu \) is the outward normal vector to \( \partial \Omega \). The constants \( d_1, d_2, d_3, \chi_1, \chi_2, \mu_1, \mu_2, a_1, a_2 \) and \( \alpha, \beta, \gamma \) are positive. The initial data \( u_0, v_0 \) are assumed to be nonnegative functions. The unknown functions \( u(x, t) \) and \( v(x, t) \) represent the population densities of two species and \( w(x, t) \) shows the concentration of the chemical substance at place \( x \) and time \( t \).

The problem (6.1.1) is a problem on account of the influence of chemotaxis, diffusion, and the Lotka–Volterra competitive kinetics, i.e., with coupling coefficients \( a_1, a_2 > 0 \) in

\[
\begin{align*}
\dot{u} &= u(1 - u - a_1 v), \quad v_t = v(1 - a_2 u - v).
\end{align*}
\]

(6.1.2)

The mathematical difficulties of the problem (6.1.1) are to deal with the chemotaxis term \( \nabla \cdot (u \nabla w) \) and the competition term \( u(1 - u - a_1 v) \). To overcome these difficulties, in the case that \( a_1, a_2 \in (0, 1) \) and \( d_3 = \alpha = \beta = 1 \) in (6.1.1), Tello–Winkler [178] applied comparison methods to this problem and obtained global existence of classical bounded solutions and their asymptotic behavior under the conditions that

\[
\begin{align*}
2(\chi_1 + \chi_2) + a_2 \mu_1 &< \mu_2 \quad \text{and} \quad 2(\chi_1 + \chi_2) + a_1 \mu_2 < \mu_1.
\end{align*}
\]

(6.1.3)

However, if \( \mu_1 \to \infty \) or \( \mu_2 \to \infty \), then these conditions break down; indeed, these conditions do not hold when \( \mu_1 \) or \( \mu_2 \) is extremely large. In Chapter 5, it was shown that the conditions

\[
\begin{align*}
\chi_1 \mu_1 &< \min \left\{ \frac{d_3}{2 \alpha}, \frac{a_1 d_3}{\beta} \right\}, \quad \chi_2 \mu_2 < \min \left\{ \frac{d_3}{2 \beta}, \frac{a_2 d_3}{\alpha} \right\}, \\
\end{align*}
\]

(6.1.4)

\[
\begin{align*}
a_1 a_2 d_3^2 &< \left( d_3 - \frac{2 \alpha \chi_1}{\mu_1} \right) \left( d_3 - \frac{2 \beta \chi_2}{\mu_2} \right)
\end{align*}
\]

(6.1.5)

lead to global existence and asymptotic stability in (6.1.1) in the case that \( a_1, a_2 \in (0, 1) \). The conditions (6.1.4)–(6.1.5) partially relax (6.1.3) in view of the point mentioned above;
indeed, (6.1.4)–(6.1.5) are satisfied even though \( \mu_1 \) or \( \mu_2 \) is extremely large. On the other hand, in the case that \( a_1 > 1 > a_2 \) and \( d_3 = \beta = 1 \) in (6.1.1) Stinner–Tello–Winkler [164] established global existence and stabilization of global classical solutions when

\[
\frac{\chi_1}{\mu_1} \leq a_1, \quad \frac{\chi_2}{\mu_2} < \frac{1}{2} \quad \text{and} \quad \frac{\alpha \chi_1}{\mu_1} + \max \left\{ \frac{\chi_2}{\mu_2} \frac{a_2 (\mu_2 - \chi_2)}{\mu_2 - 2 \chi_2}, \frac{(\alpha - a_2) \chi_2}{\mu_2 - 2 \chi_2} \right\} < 1
\]

are satisfied. In summary the two-species chemotaxis-competition model (6.1.1) were studied in the cases that \( a_1, a_2 \in (0, 1) \) and \( a_1 > 1 > a_2 \), and it was proved that global existence and same asymptotic behavior as solutions to the Lotka–Volterra competition model (6.1.2) hold when \( \frac{\chi_2}{\mu_2} \) are small. However, the conditions in the above two cases strongly depend on \( a_1, a_2 \), and have not been obtained in the case that \( a_1, a_2 \geq 1 \). Moreover, convergence rates in the cases that \( a_1, a_2 \in (0, 1) \) and \( a_1 > 1 > a_2 \) have not been studied.

The purpose of this work is to construct conditions which derive global existence of classical bounded solutions for all \( a_1, a_2 > 0 \) which covers the case that \( a_1, a_2 \geq 1 \), and lead to convergence rates for solutions of (6.1.1) in the cases that \( a_1, a_2 \in (0, 1) \) and \( a_1 > 1 > a_2 \).

For establishing global existence and boundedness we shall suppose that \( \chi_1, \chi_2 \) and \( \mu_1, \mu_2 \) satisfy the following conditions:

\[
(6.1.6) \quad n = 2, \quad \text{or} \quad n \geq 3, \quad \frac{\chi_1}{\mu_1} < \frac{nd_3}{n - 2} \min \left\{ \frac{1}{\alpha}, \frac{\alpha_1}{\beta} \right\} \quad \text{and} \quad \frac{\chi_2}{\mu_2} < \frac{nd_3}{n - 2} \min \left\{ \frac{1}{\beta}, \frac{\alpha_2}{\alpha} \right\}.
\]

We assume that the initial data \( u_0, v_0 \) satisfy

\[
(6.1.7) \quad 0 \leq u_0 \in C(\Omega) \setminus \{0\}, \quad 0 \leq v_0 \in C(\Omega) \setminus \{0\}.
\]

Now the main results read as follows. The first one is concerned with global existence and boundedness in (6.1.1).

**Theorem 6.1.1.** Let \( d_1, d_2, d_3 > 0, \mu_1, \mu_2 > 0, a_1, a_2 > 0, \chi_1, \chi_2 > 0, \alpha, \beta, \gamma > 0 \) and let \( \Omega \subset \mathbb{R}^n \) \((n \geq 2)\) be a bounded domain with smooth boundary. Assume that (6.1.6) are satisfied. Then for any \( u_0, v_0 \) satisfying (6.1.7), there exists an exactly one triplet \((u, v, w)\) of nonnegative functions

\[
u, v, w \in C(\Omega \times [0, \infty)) \cap C^{2,1}(\Omega \times (0, \infty)),
\]

which satisfy (6.1.1). Moreover, the solutions \( u, v, w \) are uniformly bounded, i.e., for all \( q \in [1, \infty) \) there exists a constant \( C > 0 \) such that

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{W^{1,q}(\Omega)} \leq C \quad \text{for all} \ t \geq 0,
\]

and the solutions \( u, v, w \) are the Hölder continuous functions, i.e., there exist \( \theta \in (0, 1) \) and \( M > 0 \) such that

\[
\|u\|_{C^{\theta, \frac{q}{2}}(\Omega \times [t, t+1])} + \|v\|_{C^{\theta, \frac{q}{2}}(\Omega \times [t, t+1])} + \|w\|_{C^{\theta, \frac{q}{2}}(\Omega \times [t, t+1])} \leq M \quad \text{for all} \ t \geq 1.
\]
Remark 6.1.1. This result gives the existence of global classical bounded solutions in the case that $a_1, a_2 \geq 1$. Moreover, the condition (6.1.6) relaxes (6.1.4) which is assumed for global existence of solutions in Theorem 5.1.1. Indeed, if $\chi_1, \chi_2$ and $\mu_1, \mu_2$ satisfy the condition (6.1.4), then $\chi_1, \chi_2$ and $\mu_1, \mu_2$ satisfy the condition (6.1.6). However, the condition (6.1.6) does not always relax those assumed in [164] and [178]; in the case that $a_1, a_2 \in (0, 1)$ the condition (6.1.6) relaxes (6.1.3) under the condition

$$n = 2, \quad \text{or} \quad n \geq 3, \quad \chi_1 < \frac{2n a_1 (\chi_1 + \chi_2) (1 + a_1)}{(n - 2)(1 - a_1 a_2)} \quad \text{and} \quad \chi_2 < \frac{2n a_2 (\chi_1 + \chi_2) (1 + a_2)}{(n - 2)(1 - a_1 a_2)},$$

and in the case that $a_1 > 1 > a_2$ the condition (6.1.6) relaxes the condition

$$\frac{\alpha \chi_1}{\mu_1} + \frac{\chi_2}{\mu_2} < 1,$$

which was used to obtain global existence in [164], when

$$\frac{\alpha (n - 2)}{n} < \min\{1, a_1 \alpha, a_2\}$$

hold.

The main theorem tells us the following result in the 2-dimensional case.

**Corollary 6.1.2.** Let $d_1, d_2, d_3 > 0$, $\mu_1, \mu_2 > 0$, $a_1, a_2 > 0$, $\chi_1, \chi_2 > 0$, $\alpha, \beta, \gamma > 0$ and let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Then for any $u_0, v_0$ satisfying (6.1.7), (6.1.1) possesses a unique global bounded classical solution. In the case that $a_1, a_2 \in (0, 1)$ asymptotic behavior of solutions to (6.1.1) will be discussed under the following additional conditions: there exists $\delta_1 > 0$ such that

$$4 \delta_1 - a_1 a_2 (1 + \delta_1)^2 > 0 \quad (6.1.8)$$

and

$$\mu_1 > \frac{\chi_1^2 (1 + \delta_1) (1 - a_1) (\alpha^2 a_1 \delta_1 + \beta^2 a_2 - \alpha \beta a_1 a_2 (1 + \delta_1))}{4d_1 d_1 d_3 (1 - a_1 a_2) (4 \delta_1 - a_1 a_2 (1 + \delta_1)^2)}, \quad (6.1.9)$$

$$\mu_2 > \frac{\chi_2^2 (1 + \delta_1) (1 - a_2) (\alpha^2 a_1 \delta_1 + \beta^2 a_2 - \alpha \beta a_1 a_2 (1 + \delta_1))}{4a_2 d_2 d_3 (1 - a_1 a_2) (4 \delta_1 - a_1 a_2 (1 + \delta_1)^2)}. \quad (6.1.10)$$

The second theorem gives asymptotic behavior in (6.1.1) in the case that $a_1, a_2 \in (0, 1)$.

**Theorem 6.1.3.** Let $d_1, d_2, d_3 > 0$, $\mu_1, \mu_2 > 0$, $a_1, a_2 \in (0, 1)$, $\chi_1, \chi_2 > 0$, $\alpha, \beta, \gamma > 0$ and let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded domain with smooth boundary. Assume that there exists a unique global classical solution $(u, v, w)$ of (6.1.1) satisfying

$$\|u\|_{C^{\alpha}((\Omega \times [t, t + 1]))} + \|v\|_{C^{\alpha}((\Omega \times [t, t + 1]))} + \|w\|_{C^{\alpha}((\Omega \times [t, t + 1]))} \leq M \quad \text{for all } t > 0 \quad (\text{with some } M > 0)$$

with $M > 0$. Then under the conditions (6.1.8)–(6.1.10), $(u, v, w)$ satisfies that there exist $C > 0$ and $\ell > 0$ such that

$$\|u(\cdot, t) - u^*\|_{L^\infty(\Omega)} + \|v(\cdot, t) - v^*\|_{L^\infty(\Omega)} + \|w(\cdot, t) - w^*\|_{L^\infty(\Omega)} \leq Ce^{-\alpha t} \quad \text{for all } t > 0,$$

where

$$u^* := \frac{1 - a_1}{1 - a_1 a_2}, \quad v^* := \frac{1 - a_2}{1 - a_1 a_2}, \quad w^* := \frac{\alpha u^* + \beta v^*}{\gamma}.$$
Remark 6.1.2. If the assumption of Theorem 6.1.1 and (6.1.8)–(6.1.10) are satisfied, then Theorem 6.1.3 gives the convergence rates for solutions of (6.1.1) in the case that \( a_1, a_2 \in (0, 1) \). Here we note that the conditions (6.1.8)–(6.1.10) with \( \delta_1 = 1 \) are similar to those assumed in [4]. However, \( \delta_1 = 1 \) is not the best choice; indeed, \( \delta_1 = 1 \) is not a minimizer of the right-hand sides of (6.1.9)–(6.1.10) except the case that \( \beta^2 a_2 = \alpha^2 a_1 \).

In the case that \( a_1 \geq 1 > a_2 \) asymptotic behavior of solutions to (6.1.1) will be discussed under the following additional conditions: there exist \( \delta_1 > 0 \) and \( a_1' \in [1, a_1] \) such that

\[
(6.1.11) \quad 4\delta_1 - a_1' a_2 (1 + \delta_1)^2 > 0,
\]

\[
(6.1.12) \quad \mu_2 > \frac{\chi_2 \delta_1 (\alpha^2 a_1' \delta_1 + \beta^2 a_2 - \alpha \beta a_1' a_2 (1 + \delta_1))}{4a_2 d_2 d_3 (4\delta_1 - a_1' a_2 (1 + \delta_1)^2)}.
\]

The third one gives asymptotic behavior in (6.1.1) in the case that \( a_1 \geq 1 > a_2 \).

Theorem 6.1.4. Let \( d_1, d_2, d_3 > 0, \mu_1, \mu_2 > 0, a_1 \geq 1, a_2 \in (0, 1), \chi_1, \chi_2 > 0, \alpha, \beta, \gamma > 0 \) and let \( \Omega \subset \mathbb{R}^n \) \((n \geq 2)\) be a bounded domain with smooth boundary. Assume that there exists a unique global classical solution \((u, v, w)\) of (6.1.1) such that

\[
\|u\|_{C^0 \left( \overline{\Omega} \times [0, t+1] \right)} + \|v\|_{C^0 \left( \overline{\Omega} \times [0, t+1] \right)} + \|w\|_{C^0 \left( \overline{\Omega} \times [0, t+1] \right)} \leq M \quad \text{for all } t > 0
\]

with some \( M > 0 \). Then under the conditions (6.1.11)–(6.1.12), \((u, v, w)\) has the following properties:

(i) If \( a_1 > 1 \) and \( a_1' \in (1, a_1) \) in (6.1.11)–(6.1.12), then there exist \( C > 0 \) and \( \ell > 0 \) satisfying

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t) - 1\|_{L^\infty(\Omega)} + \|w(\cdot, t) - \frac{\beta}{\gamma}\|_{L^\infty(\Omega)} \leq C e^{-\ell t} \quad \text{for all } t > 0.
\]

(ii) If \( a_1 = 1 \), then there exist \( C > 0 \) and \( \ell > 0 \) satisfying

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t) - 1\|_{L^\infty(\Omega)} + \|w(\cdot, t) - \frac{\beta}{\gamma}\|_{L^\infty(\Omega)} \leq C(t + 1)^{-\ell} \quad \text{for all } t > 0.
\]

Remark 6.1.3. If the assumption of Theorem 6.1.1 and (6.1.11)–(6.1.12) are satisfied, then Theorem 6.1.4 gives the convergence rates for solutions in the cases that \( a_1 > 1 > a_2 \) and \( a_1 = 1 > a_2 \).

Remark 6.1.4. Stabilization in the case that \( a_1, a_2 \geq 1 \) is a still open question. In the case that \( a_1, a_2 > 1 \) a Lotka–Volterra competition model with diffusion term was studied; however, its analysis is difficult and it is known that solutions have complicated structures: spatial homogeneity of stable equilibria ([91]); studies of a travelling wave ([87, 88, 122, 124]); dynamics of interfaces ([42]); diffusion-induced extinction ([79]).
The strategy of the proof of Theorem 6.1.1 is to extend a method in [177] to a two-species case. We first aim to establish the $L^p$-estimate for $u$ with some $p > \frac{n}{2}$ from the following derivative of $\int_{\Omega} u^p$:

\begin{equation}
\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p \leq (p-1)\chi_1 \int_{\Omega} u^{p-1} \nabla u \cdot \nabla w + \mu_1 \int_{\Omega} u^p (1 - u - a_1 v).
\end{equation}

(6.1.13)

Since the third equation in (6.1.1) derives that

\begin{equation}
(p-1)\chi_1 \int_{\Omega} u^{p-1} \nabla u \cdot \nabla w = \frac{(p-1)\chi_1}{d_3^p} \int_{\Omega} u^p (\alpha u + \beta v - \gamma w),
\end{equation}

(6.1.14)

we shall show that a combination of (6.1.13) and (6.1.14), along with the condition (6.1.6) implies that there are $c_1, c_2 > 0$ such that

\begin{equation}
\frac{d}{dt} \int_{\Omega} u^p \leq -c_1 \left(\int_{\Omega} u^p\right)^{\frac{p+1}{p}} + c_2 \int_{\Omega} u^p,
\end{equation}

which leads to $L^p$-estimate for $u$. Then aided by standard semigroup estimates, we can obtain the $L^\infty$-estimate for $u$. On the other hand, one of the keys for the proof of Theorems 6.1.3 and 6.1.4 is to derive the following energy estimate:

\begin{equation}
\frac{d}{dt} E(t) \leq -\varepsilon \int_{\Omega} [(u(\cdot, t) - \overline{u})^2 + (v(\cdot, t) - \overline{v})^2 + (w(\cdot, t) - \overline{w})^2]
\end{equation}

(6.1.15)

for all $t > 0$ with some positive function $E$ (see (6.3.7) and (6.3.14)) and some constant $\varepsilon > 0$, where $(\overline{u}, \overline{v}, \overline{w}) \in \mathbb{R}^3$ is a solution of (6.1.1). Thanks to (6.1.15), we can obtain that there exists $C > 0$ such that

\begin{align*}
\int_0^\infty \int_{\Omega} (u - \overline{u})^2 + \int_0^\infty \int_{\Omega} (v - \overline{v})^2 + \int_0^\infty \int_{\Omega} (w - \overline{w})^2 \leq C,
\end{align*}

which together with the regularity of the solution leads to Theorems 6.1.3 and 6.1.4. This strategy comes from a method for the fully parabolic system established in [4], which means that the fully parabolic system is also helpful to analyse the parabolic-parabolic-elliptic system.

This chapter is organized as follows. In Section 6.2 we prove global existence and boundedness (Theorem 6.1.1) through a series of lemmas. Section 6.3 is devoted to the proof of asymptotic stability (Theorems 6.1.3 and 6.1.4); we first provide some lemmas which will be used later, and we next divide the section into Sections 6.3.1 and 6.3.2 according to the proof of Theorem 6.1.3 and that of Theorem 6.1.4, respectively.

6.2. Global existence and boundedness

In this section we shall show global existence and boundedness in (6.1.1). First we will recall the known result about local existence of solutions to (6.1.1) ([164, Lemma 2.1]).
Lemma 6.2.1. Let \(d_1, d_2, d_3 > 0\), \(\mu_1, \mu_2 > 0\), \(a_1, a_2 > 0\), \(\chi_1, \chi_2 \in \mathbb{R}\), \(\alpha, \beta, \gamma > 0\) and let \(\Omega \subset \mathbb{R}^n\) (\(n \in \mathbb{N}\)) be a bounded domain with smooth boundary. Then for any \(u_0, v_0\) satisfying (6.1.7), there exist \(T_{\text{max}} \in (0, \infty)\) and an exactly one triplet \((u, v, w)\) of nonnegative functions

\[u, \ v, \ w \in C(\overline{\Omega} \times [0, T_{\text{max}})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\text{max}}))\]

which satisfy (6.1.1). Moreover,

either \(T_{\text{max}} = \infty\) or \(\lim_{t \to T_{\text{max}}} (\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)}) = \infty\).

We next give the \(L^p\)-estimate for \(u\) with some \(p > \frac{n}{2}\) which plays an important role in deriving \(L^\infty\)-estimate for \(u\).

Lemma 6.2.2. Assume that (6.1.6)–(6.1.7) are satisfied. Then for all \(p \in I_1\), there exists \(C(p) > 0\) such that

\[\|u(\cdot, t)\|_{L^p(\Omega)} \leq C(p)\]

for all \(t \in (0, T_{\text{max}})\), where

\[I_1 := \left(\frac{n}{2}, \min \left\{ \frac{\alpha \chi_1}{(\alpha \chi_1 - d_3 \mu_1)_+}, \frac{\beta \chi_1}{(\beta \chi_1 - a_1 d_3 \mu_1)_+} \right\} \right)\].

Proof. We fix \(p \in I_1\). Here we note from the condition (6.1.6) that \(I_1 \neq \emptyset\). Multiplying the first equation in (6.1.1) by \(u^{p-1}\) and integrating it over \(\Omega\), we obtain that

\[\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p + d_1 (p - 1) \int_{\Omega} u^{p-2} |\nabla u|^2 = (p - 1) \chi_1 \int_{\Omega} u^{p-1} \nabla u \cdot \nabla w + \mu_1 \int_{\Omega} u^p (1 - u - a_1 v).\]

Then integration by parts and the third equation in (6.1.1) imply that

\[(p - 1) \chi_1 \int_{\Omega} u^{p-1} \nabla u \cdot \nabla w = - \frac{(p - 1) \chi_1}{p} \int_{\Omega} u^p \Delta w = \frac{(p - 1) \chi_1}{d_3 p} \int_{\Omega} u^p (\alpha u + \beta v - \gamma w).\]

Therefore a combination of (6.2.1) with (6.2.2) yields that

\[\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p \leq \mu_1 \int_{\Omega} u^p - \left(\mu_1 - \frac{(p - 1) \chi_1}{d_3 p}\right) \int_{\Omega} u^{p+1} - \left(a_1 \mu_1 - \frac{\beta p - 1 \chi_1}{d_3 p}\right) \int_{\Omega} u^p v.\]

Recalling from \(p \in I_1 = \left(\frac{n}{2}, \min \left\{ \frac{\alpha \chi_1}{(\alpha \chi_1 - d_3 \mu_1)_+}, \frac{\beta \chi_1}{(\beta \chi_1 - a_1 d_3 \mu_1)_+} \right\} \right)\) that

\[\mu_1 - \frac{(p - 1) \chi_1}{d_3 p} > 0\] and \(a_1 \mu_1 - \frac{\beta p - 1 \chi_1}{d_3 p} > 0\),
we establish that
\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p \leq \mu_1 \int_{\Omega} u^p - \left( \mu_1 - \frac{\alpha(p-1)\chi_1}{d_3 p} \right) \int_{\Omega} u^{p+1},
\]
and then the Hölder inequality
\[
\int_{\Omega} u^p \leq |\Omega|^{\frac{p}{p+1}} \left( \int_{\Omega} u^{p+1} \right)^{\frac{p}{p+1}}
\]
leads to
\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p \leq \mu_1 \int_{\Omega} u^p - \varepsilon \left( \int_{\Omega} u^p \right)^{\frac{p+1}{p}}
\]
with \( \varepsilon := |\Omega|^{-1} \left( \mu_1 - \frac{\alpha(p-1)\chi_1}{d_3 p} \right) \), which implies that
\[
\|u(\cdot, t)\|_{L^p(\Omega)} \leq \max \left\{ \|u_0\|_{L^p(\Omega)}, \frac{\mu_1}{\varepsilon} \right\} \quad \text{for all } t \in (0, T_{\text{max}}).
\]
Thus we can attain the conclusion of this lemma.

Similarly, we can confirm the following \( L^p \)-estimate for \( v \) with some \( p > \frac{n}{2} \).

**Lemma 6.2.3.** Assume that (6.1.6)–(6.1.7) are satisfied. Then for all \( p \in I_2 \), there exists \( C(p) > 0 \) such that
\[
\|v(\cdot, t)\|_{L^p(\Omega)} \leq C(p) \quad \text{for all } t > 0,
\]
where \( I_2 := \left( \frac{n}{2}, \min \left\{ \frac{\beta \chi_2}{(\beta \chi_2 - d_3 \mu_2^+)^+}, \frac{\alpha \chi_2}{(\alpha \chi_2 - d_3 \mu_2^+)^+} \right\} \right) \).

**Proof.** A similar argument as in the proof of Lemma 6.2.2 derives this lemma.

Now we could construct all estimates which will enable us to obtain the estimate for the solution; Lemmas 6.2.2 and 6.2.3 lead to the following lemma. The proof is based on a known argument involving semigroup estimates which derive the \( L^\infty \)-estimate for \( u \) from \( L^p \)-estimate with \( p > \frac{n}{2} \) (see e.g., [5, Lemma 3.2]).

**Lemma 6.2.4.** Assume that (6.1.6)–(6.1.7) are satisfied. Then for all \( q > n \) there exists \( C > 0 \) such that
\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{W^{1,q}(\Omega)} \leq C \quad \text{for all } t > 0.
\]
Moreover, there exist \( M > 0 \) and \( \theta \in (0, 1) \) such that
\[
\|u\|_{C^{\theta, \frac{q}{2}}(\overline{\Omega} \times [t, t+1])} + \|v\|_{C^{\theta, \frac{q}{2}}(\overline{\Omega} \times [t, t+1])} + \|w\|_{C^{\theta, \frac{q}{2}}(\overline{\Omega} \times [t, t+1])} \leq M \quad \text{for all } t \geq 1.
\]
Proof. We fix \( p \in I_1 \cap I_2 \cap (0,n) \), where \( I_1 \) and \( I_2 \) are the intervals defined in Lemmas 6.2.2 and 6.2.3. Then thanks to Lemmas 6.2.2 and 6.2.3, we can find \( C_1 > 0 \) such that

\[
\|u(\cdot, t)\|_{L^p(\Omega)} + \|v(\cdot, t)\|_{L^p(\Omega)} \leq C_1 \quad \text{for all } t \in (0,T_{\text{max}}).
\]

(6.2.4)

We first verify the \( L^{\frac{np}{n-r}} \)-estimate for \( \nabla w \). Here for all \( q \in (1, \infty) \), the standard elliptic regularity argument (see e.g., [48, Theorem 19.1]) leads to the existence of a constant \( C_E(q) > 0 \) satisfying

\[
\|w(\cdot, t)\|_{W^{2,q}(\Omega)} \leq C_E(q)(\|u(\cdot, t)\|_{L^q(\Omega)} + \|v(\cdot, t)\|_{L^q(\Omega)}) \quad \text{for all } t \in (0,T_{\text{max}}).
\]

(6.2.5)

Therefore a combination of (6.2.5) with (6.2.4) yields from the Sobolev embedding theorem that there exists \( C_2 > 0 \) such that

\[
\|\nabla w(\cdot, t)\|_{L^{\frac{np}{n-r}}(\Omega)} \leq C_2 \quad \text{for all } t \in (0,T_{\text{max}})
\]

since \( p < n \). We next establish the \( L^1 \)-estimate for \( u \). Since \( p > \frac{n}{2} \), we can take \( r \in (n,q) \) such that

\[
p > \frac{nr}{n+r}.
\]

We take \( \vartheta > 1 \) satisfying

\[
\frac{1}{\vartheta} < \min \left\{ 1 - \frac{r(n-p)}{np}, \frac{q-r}{q} \right\}.
\]

Then \( \vartheta' := \frac{\vartheta}{\vartheta-1} \) satisfies

\[
r\vartheta' < \frac{np}{n-p}.
\]

Now for all \( T \in (0,T_{\text{max}}) \) we note that

\[
A(T) := \sup_{t \in (0,T)} \|u(\cdot, t)\|_{L^\infty(\Omega)}
\]

is finite. To obtain the estimate for \( A(T) \) we put \( t_0 := (t-1)_+ \) and represent \( u \) according to

\[
u(\cdot, t) = e^{(t-t_0)d_1 \Delta}u(t_0) - \chi_1 \int_{t_0}^t e^{(t-s)d_1 \Delta} \cdot (u(\cdot, s)\nabla w(\cdot, s)) \, ds
\]

\[+ \mu_1 \int_{t_0}^t e^{(t-s)d_1 \Delta}u(\cdot, s)(1-u(\cdot, s) - a_1 v(\cdot, s)) \, ds
\]

\[=: u_1(\cdot, t) + u_2(\cdot, t) + u_3(\cdot, t)
\]

for \( t \in (0,T_{\text{max}}) \). In the case that \( t \leq 1 \), i.e., \( t_0 = 0 \), from the order preserving property of the Neumann heat semigroup we see that

\[
\|u_1(\cdot, t)\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)} \quad \text{for all } t \in (0,1] \cap (0,T).
\]

(6.2.7)
In the case that \( t > 1 \) using the \( L^p-L^q \) estimate for \( (e^r\Delta)^{-\frac{p}{2}} \) (see Lemma 2.2.2 (i)) yields that there is \( C_3 > 0 \) such that
\[
\|u_1(\cdot,t)\|_{L^\infty(\Omega)} \leq C_3 \|u(\cdot,t_0)\|_{L^p(\Omega)} \leq C_1 C_3 \quad \text{for all } t \in (1, T).
\]

(6.2.8) \( \|u_1(\cdot,t)\|_{L^\infty(\Omega)} \leq C_3 \|u(\cdot,t_0)\|_{L^p(\Omega)} \leq C_1 C_3 \quad \text{for all } t \in (1, T). \)

Next due to a known smoothing property of \( (e^r\Delta)^{-\frac{p}{2}} \) (see Lemmas [52, Lemma 3.3]), we can find \( C_4 > 0 \) such that
\[
\|u_2(\cdot,t)\|_{L^\infty(\Omega)} \leq C_4 \sup_{t \in (0,T)} \|u(\cdot,t)\|_{L^p(\Omega)} \int_0^1 \sigma^{-\frac{1}{2} - \frac{p}{n}} \, d\sigma.
\]

Noting from \( r \vartheta' < \frac{np}{n-p} \) and (6.2.4) that
\[
\|u(\cdot,t)\|_{L^r(\Omega)} \leq \|u(\cdot,t)\|_{L^p(\Omega)} \|
abla w(\cdot,t)\|_{L^p(\Omega)}
\leq C_5 \|u(\cdot,t)\|_{L^p(\Omega)} \|
abla w(\cdot,t)\|_{L^p(\Omega)}
\leq C_5 C_2 C_3 A(T)^{1 - \frac{p}{n}} \quad \text{for all } t \in (0, T)
\]
with some \( C_5 > 0 \), we establish that there exists \( C_6 > 0 \) such that

(6.2.9) \( \|u_2(\cdot,t)\|_{L^\infty(\Omega)} \leq C_6 \quad \text{for all } t \in (0, T). \)

Finally, the maximum principle together with the elementary inequality
\[
\mu_1 u(1 - u - a_1 v) \leq -\mu_1 \left( u - \frac{1 + \mu_1}{2\mu_1} \right)^2 + \frac{(1 + \mu_1)^2}{4\mu_1} \leq \frac{(1 + \mu_1)^2}{4\mu_1}
\]
implies that there exists \( C_7 > 0 \) such that

(6.2.10) \( u_3(\cdot,t) \leq C_7 \quad \text{for all } t \in (0, T). \)

Therefore a combination of (6.2.6), the nonnegativity of \( u \) with (6.2.7), (6.2.8), (6.2.9), (6.2.10) tells us that there exist \( C_8, C_9 > 0 \) such that
\[
A(T) \leq C_8 + C_9 A(T)^{1 - \frac{p}{n}},
\]
which implies from \( p < r \vartheta \) that
\[
A(T) \leq C_10 \quad \text{for all } T \in (0, T_{\max})
\]
with some \( C_{10} > 0 \). Thus we obtain the \( L^\infty \)-estimate for \( u \). Similarly, we can verify the \( L^\infty \)-estimate for \( v \). Then invoking (6.2.5), we see that there exists \( C_{11} > 0 \) such that
\[
\|w(\cdot,t)\|_{W^{1,q}(\Omega)} \leq C_{11} \quad \text{for all } t \in (0, T_{\max}),
\]
which implies (6.2.3). Moreover, known regularity arguments (see arguments and references in the proof of [24, Proposition 2.3]) enable us to find \( C_{12} > 0 \) and \( \theta \in (0, 1) \) satisfying
\[
\|u\|_{C^{\frac{\theta}{n} \frac{n}{2}(\overline{\Omega} \times [t,t+1])}} + \|v\|_{C^{\frac{\theta}{n} \frac{n}{2}(\overline{\Omega} \times [t,t+1])}} + \|w\|_{C^{\frac{\theta}{n} \frac{n}{2}(\overline{\Omega} \times [t,t+1])}} \leq C_{12} \quad \text{for all } t \geq 1,
\]
which implies the end of the proof.

\( \square \)

**Proof of Theorem 6.1.1.** Lemma 6.2.4 directly shows Theorem 6.1.1.

\( \square \)
6.3. Stabilization

In this section we will establish stabilization of solutions to (6.1.1). Here we assume that there exists a unique global classical solution \((u, v, w)\) of (6.1.1) satisfying

\[
\|u\|_{C^0, \frac{1}{2}}(\Omega \times [t, t+1]) + \|v\|_{C^0, \frac{1}{2}}(\Omega \times [t, t+1]) + \|w\|_{C^0, \frac{1}{2}}(\Omega \times [t, t+1]) \leq M \quad \text{for all } t \geq 1
\]

with some \(M > 0\). For the proof of Theorems 6.1.3 and 6.1.4 we first show an important lemma which summarizes a common argument in some cases.

**Lemma 6.3.1.** Let \(n \in C^0(\Omega \times [0, \infty))\) satisfy that there exist constants \(C^* > 0\) and \(\theta^* > 0\) such that

\[
(6.3.1) \quad \|n\|_{C^0, \frac{\theta^*}{\theta^* - 1}}(\Omega \times [t, t+1]) \leq C^* \quad \text{for all } t \geq 1.
\]

Assume that

\[
(6.3.2) \quad \int_0^\infty \int_\Omega (n(x, t) - N^*)^2 \, dx \, dt < \infty
\]

with some constant \(N^* > 0\). Then

\[
n(\cdot, t) \to N^* \quad \text{in } C^0(\Omega) \quad \text{as } t \to \infty.
\]

**Proof.** Putting

\[
n[j](x, s) := n(x, j + s) \quad \text{for } x \in \Omega, \quad s \in [0, 1],
\]

we will first show that \((n[j])_{j \in \mathbb{N}}\) is relatively compact in \(C^0(\Omega \times [0, 1])\). From (6.3.1) the inequality

\[
|n[j](x_1, t_1) - n[j](x_2, t_2)| \leq C^* \left( |x_1 - x_2|^{\frac{1}{\theta^*}} + |t_1 - t_2|^{\frac{1}{\theta^*}} \right)
\]

holds for all \(x_1, x_2 \in \Omega, \ t_1, t_2 \in [0, 1]\) and \(j \in \mathbb{N}\). Thanks to this inequality, we can confirm that for all \(\varepsilon > 0\) there exists \(\delta > 0\) such that if \((|x_1 - x_2|^2 + |t_1 - t_2|^2)^{\frac{1}{2}} < \delta\), then

\[
|n[j](x_1, t_1) - n[j](x_2, t_2)| < \varepsilon
\]

holds for all \(x_1, x_2 \in \Omega, \ t_1, t_2 \in [0, 1]\) and \(j \in \mathbb{N}\), which means from the Arzelà-Ascoli theorem that \((n[j])_{j \in \mathbb{N}}\) is relatively compact in \(C^0(\Omega \times [0, 1])\). We shall next claim that \(n[j] \to N^* \) in \(C^0(\Omega \times [0, 1])\) as \(j \to \infty\). If this does not hold, then we can find \(\varepsilon_0 > 0\) and a sequence \((j_k)_{k \in \mathbb{N}} \subset \mathbb{N}, \ j_k \to \infty\) as \(k \to \infty\), such that

\[
(6.3.3) \quad \|n[j_k] - N^*\|_{C^0(\Omega \times [0, 1])} > \varepsilon_0 \quad \text{for all } k \in \mathbb{N}.
\]

Since \((n[j])_{j \in \mathbb{N}}\) is relatively compact in \(C^0(\Omega \times [0, 1])\), we can further select a subsequence \(n[j_k]\) and a function \(n_\infty \in C^0(\Omega \times [0, 1])\) such that

\[
(6.3.4) \quad \|n[j_k] - n_\infty\|_{C^0(\Omega \times [0, 1])} \to 0 \quad \text{as } \ell \to \infty.
\]

55
It follows from (6.3.2) that
\[
\int_0^1 \int_\Omega |n[j](x,s) - N^*|^2 \, dx \, ds = \int_j^{j+1} \int_\Omega |n(x,t) - N^*|^2 \, dx \, dt \to 0 \quad \text{as } j \to \infty.
\]
On the other hand, (6.3.4) yields
\[
\int_0^1 \int_\Omega |n[j_k](x,s) - n_\infty(x,s)|^2 \, dx \, ds \leq |\Omega||n[j_k] - n_\infty|^2_{C^0(\overline{\Omega} \times [0,1])} \to 0 \quad \text{as } \ell \to \infty,
\]
which implies \(n_\infty = N^*\). Therefore (6.3.4) together with \(n_\infty = N^*\) contradicts (6.3.3). Thus
\[
\sup_{s \in [0,1]} n[j](\cdot, s) - N^* \to 0 \quad \text{as } j \to \infty,
\]
which derives that
\[
n(\cdot, t) \to N^* \quad \text{in } C^0(\overline{\Omega})
\]
as \(t \to \infty\).

We next provide the following lemma which will be used to confirm that the assumption of Lemma 6.3.1 is satisfied.

**Lemma 6.3.2.** Let \(a, b, c, d, e, f \in \mathbb{R}\). Suppose that
\[
a > 0, \quad d - \frac{b^2}{4a} > 0, \quad f - \frac{c^2}{4a} - \frac{(2ae - bc)^2}{4a(4ad - b^2)} > 0.
\]
Then
\[
a x^2 + bxy + cxz + dy^2 + eyz + fz^2 \geq 0
\]
holds for all \(x, y, z \in \mathbb{R}\).

**Proof.** From straightforward calculations we obtain
\[
a x^2 + bxy + cxz + dy^2 + eyz + fz^2
\]
\[
= a \left( x + \frac{by + cz}{2a} \right)^2 + \left( d - \frac{b^2}{4a} \right) \left( y + \frac{2ae - bc}{4ad - b^2} \right)^2 + \left( f - \frac{c^2}{4a} - \frac{(2ae - bc)^2}{4a(4ad - b^2)} \right) z^2.
\]
In view of the above equation, (6.3.5) leads to (6.3.6). \(\square\)

Finally, we give the following lemma which enables us to upgrade the \(L^2\)-convergence rate to \(L^\infty\)-convergence rate.

**Lemma 6.3.3.** Let \((\overline{u}, \overline{v}, \overline{w}) \in \mathbb{R}^3\) be a solution to (6.1.1). Assume that there exists a decreasing function \(h : [0, \infty) \to \mathbb{R}\) satisfying
\[
\|u(\cdot, t) - \overline{u}\|_{L^2(\Omega)} + \|v(\cdot, t) - \overline{v}\|_{L^2(\Omega)} \leq h(t) \quad \text{for all } t > 0.
\]
Then there exists \(C > 0\) such that
\[
\|u(\cdot, t) - \overline{u}\|_{L^\infty(\Omega)} + \|v(\cdot, t) - \overline{v}\|_{L^\infty(\Omega)} + \|w(\cdot, t) - \overline{w}\|_{L^\infty(\Omega)} \leq C(h(t - 1))^{\frac{1}{n+1}}
\]
for all \(t > 1\).

56
Proof. For all $p > 2$ we first obtain from the H"older inequality that
\[
\|f\|_{L^p(\Omega)} \leq \|f\|_{L^\infty(\Omega)}^{1 - \frac{2}{p}} \|f\|_{L^2(\Omega)}^{\frac{2}{p}}
\]
holds for all $f \in L^\infty(\Omega)$, which means from the boundedness of $u, v$ that
\[
\|u(\cdot, t) - \overline{u}\|_{L^p(\Omega)} + \|v(\cdot, t) - \overline{v}\|_{L^p(\Omega)} \leq C_1(p)(h(t))^{\frac{2}{p}}
\]
for all $t > 0$ with some $C_1(p) > 0$. Here (6.2.5) enables us to see that
\[
\|w(\cdot, t) - \overline{w}\|_{W^{2,2n+2}(\Omega)} \leq C_E(2n + 2)C_1(2n + 2)(h(t))^{\frac{1}{n+1}}
\]
for all $t > 0$.

Thus we have that there is $C_2 > 0$ such that
\[
\|\nabla w(\cdot, t)\|_{L^{2n+2}(\Omega)} \leq C_2(h(t))^{\frac{1}{n+1}}
\]
for all $t > 0$.

Then noting from a variation-of-constants representation that
\[
u(\cdot, t) - \overline{v} = e^{d_1 \Delta}(v(\cdot, t - 1) - \overline{v}) - \chi_1 \int_{t-1}^t e^{d_1(t-s) \Delta} \nabla \cdot (v(\cdot, s) \nabla w(\cdot, s)) \, ds
\]
\[+ \mu_1 \int_{t-1}^t e^{d_1(t-s) \Delta} (1 - u(\cdot, s) - a_1 v(\cdot, s)) \, ds
\]
and
\[
v(\cdot, t) - \overline{v} = e^{d_2 \Delta}(v(\cdot, t - 1) - \overline{v}) - \chi_1 \int_{t-1}^t e^{d_2(t-s) \Delta} \nabla \cdot (v(\cdot, s) \nabla w(\cdot, s)) \, ds
\]
\[+ \mu_2 \int_{t-1}^t e^{d_2(t-s) \Delta} (1 - a_2 u(\cdot, s) - v(\cdot, s)) \, ds,
\]
by using a similar argument as in the proof of [4, Lemma 3.6] we infer that there exists $C_3 > 0$ such that
\[
\|u(\cdot, t) - \overline{u}\|_{L^\infty(\Omega)} + \|v(\cdot, t) - \overline{v}\|_{L^\infty(\Omega)} \leq C_3(h(t-1))^{\frac{1}{n+1}}
\]
for all $t > 1$.

Finally, since $(\overline{u}, \overline{v}, \overline{w})$ satisfies
\[
\alpha \overline{u} + \beta \overline{v} - \gamma \overline{w} = 0,
\]
we can apply the maximum principle to
\[-\Delta (w - \overline{w}) + \gamma (w - \overline{w}) = \alpha (u - \overline{u}) + \beta (v - \overline{v}),
\]
and hence obtain the existence of a constant $C_4 > 0$ such that
\[
\|w(\cdot, t) - \overline{w}\|_{L^\infty(\Omega)} \leq C_4(\|u(\cdot, t) - \overline{u}\|_{L^\infty(\Omega)} + \|v(\cdot, t) - \overline{v}\|_{L^\infty(\Omega)})
\]
\[\leq C_3C_4(h(t-1))^{\frac{1}{n+1}}
\]
for all $t > 1$,

which concludes the proof of this lemma. \qed
6.3.1. Convergence. Case 1: \( a_1, a_2 \in (0, 1) \)

In this subsection we establish stabilization in the case that \( a_1, a_2 \in (0, 1) \). We first confirm that the assumption of Lemma 6.3.1 are satisfied.

Lemma 6.3.4. Assume that (6.1.8)–(6.1.10) are satisfied and let \( \delta_1 \) be a constant defined in (6.1.8)–(6.1.10). Then the function \( E_1 : (0, \infty) \to \mathbb{R} \) defined as

\[
E_1 := \int_{\Omega} \left( u - u^* - u^* \log \frac{u}{u^*} \right) + \frac{a_1 \mu_1 \delta_1}{a_2 \mu_2} \int_{\Omega} \left( v - v^* - v^* \log \frac{v}{v^*} \right)
\]

satisfies that

\[
\frac{d}{dt} E_1(t) \leq -\varepsilon \int_{\Omega} [ (u(\cdot, t) - u^*)^2 + (v(\cdot, t) - v^*)^2 + (w(\cdot, t) - w^*)^2 ]
\]

for all \( t > 0 \) with some \( \varepsilon > 0 \). Moreover, there exists \( C > 0 \) satisfying

\[
\int_0^\infty \int_{\Omega} (u - u^*)^2 + \int_0^\infty \int_{\Omega} (v - v^*)^2 + \int_0^\infty \int_{\Omega} (w - w^*)^2 \leq C.
\]

Proof. Let \( \delta_1 > 0 \) be a constant defined in (6.1.8)–(6.1.10). First we shall show that the function \( E_1 \) satisfies that (6.3.8) holds for all \( t > 0 \) with some \( \varepsilon > 0 \). From straightforward calculations we infer that

\[
\frac{d}{dt} E_1(t) = -\mu_1 \int_{\Omega} (u(\cdot, t) - u^*)^2 - (1 + \delta_1) a_1 \mu_1 \int_{\Omega} (u(\cdot, t) - u^*)(v(\cdot, t) - v^*) - \frac{a_1 \mu_1 \delta_1}{\mu_2} \int_{\Omega} (v(\cdot, t) - v^*)^2 - d_1 u^* \int_{\Omega} \frac{\nabla u(\cdot, t)}{u^2} + u^* \chi_1 \int_{\Omega} \frac{\nabla u(\cdot, t) \cdot \nabla w(\cdot, t)}{u} - \frac{2 a_1 \mu_1 \delta_1}{a_2 \mu_2} \int_{\Omega} \frac{\nabla v(\cdot, t)}{v^2} + \frac{a_1 \mu_1 v^*_2 \chi_2 \delta_1}{a_2 \mu_2} \int_{\Omega} \frac{\nabla v(\cdot, t) \cdot \nabla w(\cdot, t)}{v} \quad \text{for all } t > 0.
\]

Here in light of (6.1.8)–(6.1.10) we can take \( \delta_2 > 0 \) satisfying

\[
\frac{u^* \chi_1}{4 \delta_1} < \delta_2 < \frac{d_2 a_1 \mu_1 \gamma (4 \delta_1 - (1 + \delta_1)^2 a_1 a_2)}{(1 + \delta_1)(a_1 \alpha^2 \delta_1 + a_2 \beta^2 - (1 + \delta_1)a_1 a_2 \alpha \beta)},
\]

\[
\frac{a_1 \mu_1 v^*_2 \chi_2 \delta_1}{4 \delta_2} < \delta_2 < \frac{d_2 a_1 \mu_1 \gamma (4 \delta_1 - (1 + \delta_1)^2 a_1 a_2)}{(1 + \delta_1)(a_1 \alpha^2 \delta_1 + a_2 \beta^2 - (1 + \delta_1)a_1 a_2 \alpha \beta)}.
\]

Invoking the Young inequality, we obtain that

\[
\frac{u^* \chi_1}{4 \delta_1} \int_{\Omega} \frac{\nabla u \cdot \nabla w}{u} \leq \frac{u^* \chi_1}{4 \delta_2} \int_{\Omega} \frac{\nabla u}{u^2} + \delta_2 \int_{\Omega} |\nabla w|^2
\]

and

\[
\frac{a_1 \mu_1 v^*_2 \chi_2 \delta_1}{4 \delta_2} \int_{\Omega} \frac{\nabla v \cdot \nabla w}{v} \leq \frac{a_1 \mu_1 v^*_2 \chi^2 \delta_1}{4 \delta_2} \int_{\Omega} \frac{\nabla v}{v^2} + \delta_1 \delta_2 \int_{\Omega} |\nabla w|^2.
\]
Therefore since the definition of $\delta_2$ yields
\[
d_1 - \frac{u^*\lambda_1^2}{4\delta_2} > 0 \quad \text{and} \quad d_2 - \frac{a_1\mu_1v^*\lambda_2^2}{4a_2\mu_2\delta_2} > 0,
\]
a combination of (6.3.9) with (6.3.10) and (6.3.11) implies
\[
\frac{d}{dt}E_1(t) \leq -\mu_1 \int_\Omega (u(\cdot, t) - u^*)^2 - (1 + \delta_1)a_1\mu_1 \int_\Omega (u(\cdot, t) - u^*)(v(\cdot, t) - v^*)
\]
\[
\quad - \frac{a_1\mu_1\delta_1}{\mu_2} \int_\Omega (v(\cdot, t) - v^*)^2 + \frac{\alpha(1 + \delta_1)\delta_2}{d_3} \int_\Omega (u(\cdot, t) - u^*)(w(\cdot, t) - w^*)
\]
\[
\quad + \frac{\beta(1 + \delta_1)\delta_2}{d_3} \int_\Omega (v(\cdot, t) - v^*)(w(\cdot, t) - w^*) - \frac{\gamma(1 + \delta_1)\delta_2}{d_3} \int_\Omega (w(\cdot, t) - w^*)^2
\]
for all $t > 0$.

Noting from the third equation in (6.1.1) that
\[
\int_\Omega |\nabla w|^2 = \frac{\alpha}{d_3} \int_\Omega (u - u^*)(w - w^*) + \frac{\beta}{d_3} \int_\Omega (v - v^*)(w - w^*) - \frac{\gamma}{d_3} \int_\Omega (w - w^*)^2,
\]
we establish that
\[
\frac{d}{dt}E_1(t) \leq F_1(t) \quad \text{for all } t > 0,
\]
where
\[
F_1(t) := -\mu_1 \int_\Omega (u(\cdot, t) - u^*)^2 - (1 + \delta_1)a_1\mu_1 \int_\Omega (u(\cdot, t) - u^*)(v(\cdot, t) - v^*)
\]
\[
\quad - \frac{a_1\mu_1\delta_1}{\mu_2} \int_\Omega (v(\cdot, t) - v^*)^2 + \frac{\alpha(1 + \delta_1)\delta_2}{d_3} \int_\Omega (u(\cdot, t) - u^*)(w(\cdot, t) - w^*)
\]
\[
\quad + \frac{\beta(1 + \delta_1)\delta_2}{d_3} \int_\Omega (v(\cdot, t) - v^*)(w(\cdot, t) - w^*) - \frac{\gamma(1 + \delta_1)\delta_2}{d_3} \int_\Omega (w(\cdot, t) - w^*)^2
\]
for all $t > 0$. In order to see (6.3.8) we will show that
\[
F_1(t) \leq -\varepsilon \int_\Omega [(u(\cdot, t) - u^*)^2 + (v(\cdot, t) - v^*)^2 + (w(\cdot, t) - w^*)^2]
\]
with some $\varepsilon > 0$ by using Lemma 6.3.2. To confirm that the assumption of Lemma 6.3.2 is satisfied we put
\[
g_1(\varepsilon) := \mu_1 - \varepsilon, \quad g_2(\varepsilon) := \frac{a_1\mu_1\delta_1}{a_2} - \varepsilon - \frac{(1 + \delta_1)^2a_1^2\mu_1^2}{4(\mu_1 - \varepsilon)},
\]
\[
g_3(\varepsilon) := \frac{\gamma(1 + \delta_1)\delta_2}{d_3} - \varepsilon - \frac{\alpha^2(1 + \delta_1)^2\delta_2^2}{4d_3^2(\mu_1 - \varepsilon)}
\]
\[
\quad - \frac{(2(\mu_1 - \varepsilon)\beta(1 + \delta_1)\delta_2 - (1 + \delta_1)^2a_1\mu_1\alpha\delta_1)^2}{4d_3^2(\mu_1 - \varepsilon)(4(\mu_1 - \varepsilon)(\frac{a_1\mu_1\delta_1}{a_2} - \varepsilon) - (1 + \delta_1)^2a_1\mu_1)}
\]
for $\varepsilon > 0$, and shall see that there exists $\varepsilon_1 > 0$ such that $g_i(\varepsilon_1) > 0$ for $i = 1, 2, 3$. Here $g_i(0) = \mu_1 > 0$ obviously holds, and the condition (6.1.8) implies that
\[
g_2(0) = \frac{a_1\mu_1(4\delta_1 - (1 + \delta_1)^2a_1a_2)}{4a_2} > 0.
\]
Moreover, aided by the definition of $\delta_2$, we can obtain that
\[
g_3(0) = \frac{\gamma(1 + \delta_1)\delta_2}{d_3} - \frac{\alpha^2(1 + \delta_1)^2\delta_2^2}{4d_3^2\mu_1} - \frac{(2\mu_1\beta(1 + \delta_1)\delta_2 - (1 + \delta_1)^2a_1\mu_1\alpha\delta_1)^2}{4d_3^2\mu_1 (4\mu_1\alpha\delta_1 - (1 + \delta_1)^2a_1\mu_1)}
= (1 + \delta_1)\delta_2 \left( \frac{\gamma}{d_3} - \frac{(1 + \delta_1)(\alpha^2a_1\delta_1 + a_2\beta^2 - (1 + \delta_1)a_1a_2\alpha\beta)}{d_3^2a_1\mu_1(4\delta_1 - (1 + \delta_1)^2a_1a_2)} \right) > 0.
\]
Therefore a combination of the above inequalities and the continuity argument implies that there exists $\varepsilon_1 > 0$ such that $g_i(\varepsilon_1) > 0$ hold for $i = 1, 2, 3$. Thus Lemma 6.3.2 derives that
\[
F_1(t) \leq -\varepsilon_1 \int_{\Omega} [(u(\cdot,t) - u^*)^2 + (v(\cdot,t) - v^*)^2 + (w(\cdot,t) - w^*)^2] \quad \text{for all } t > 0,
\]
which yields that (6.3.8) holds for all $t > 0$. Then since from the Taylor formula $E_1$ is a nonnegative function for $t > 0$ (more details, see [4, Lemma 3.2]) and a continuous function for $t \geq 0$ (from the regularity of the solution), integrating (6.3.8) over $[0, \infty)$ concludes the proof of this lemma. \qed

Lemma 6.3.5. Assume that (6.1.8)--(6.1.10) are satisfied. Then
\[
\|u(\cdot,t) - u^*\|_{L^2(\Omega)} + \|v(\cdot,t) - v^*\|_{L^2(\Omega)} + \|w(\cdot,t) - w^*\|_{L^2(\Omega)} \to 0 \quad \text{as } t \to \infty.
\]
Proof. A combination of Lemmas 6.3.1 and 6.3.4 implies this lemma. \qed

Next we desire to establish convergence rates for the solution of (6.1.1). We note that in view of Lemma 6.3.3 it is sufficient to confirm the $L^2$-convergence rates for the solution.

Lemma 6.3.6. Assume that (6.1.8)--(6.1.10) are satisfied. Then there exist $C > 0$ and $\ell > 0$ such that
\[
\|u(\cdot,t) - u^*\|_{L^2(\Omega)} + \|v(\cdot,t) - v^*\|_{L^2(\Omega)} \leq Ce^{-\ell t} \quad \text{for all } t > 0.
\]
Proof. Aided by Lemma 6.3.5 and the L'Hôpital theorem, a similar argument as in the proof of [4, Lemma 3.7] derives that there exist $C_1, C_2 > 0$ and $t_0 > 0$ such that for all $t > t_0$,
\[
(6.3.12) \quad C_1 \left( \int_{\Omega} (u - u^*)^2 + \int_{\Omega} (v - v^*)^2 \right) \leq E_1 \leq C_2 \left( \int_{\Omega} (u - u^*)^2 + \int_{\Omega} (v - v^*)^2 \right),
\]
where $E_1$ is the function defined as (6.3.7). Therefore we obtain from (6.3.8) that
\[
\frac{d}{dt} E_1(t) \leq -C_3 E_1(t) \quad \text{for all } t > t_0
\]
with some $C_3 > 0$, which implies that there exists $C_4 > 0$ such that
\[
(6.3.13) \quad E_1(t) \leq C_4 e^{-C_3 t} \quad \text{for all } t > 0.
\]
Thus a combination of (6.3.12) and (6.3.13) yields that
\[
\int_{\Omega} (u - u^*)^2 + \int_{\Omega} (v - v^*)^2 \leq \frac{C_4}{C_1} e^{-C_3 t},
\]
which concludes the proof of this lemma. \qed
6.3.2. Convergence. Case 2: $a_1 \geq 1 > a_2$

In this subsection we will obtain stabilization in the case that $a_1 \geq 1 > a_2$. In this case we also have to confirm that the assumption of Lemma 6.3.1 is satisfied.

**Lemma 6.3.7.** Assume that (6.1.11)–(6.1.12) are satisfied and let $\delta_1 > 0$ and $a'_1 \in [1, a_1]$ be constants defined in (6.1.11)–(6.1.12). Then the function $E_2 : (0, \infty) \to \mathbb{R}$ defined as

$$E_2 := \int_\Omega u + \frac{a'_1 \mu_1 \delta_1}{a_2 \mu_2} \int_\Omega (v - 1 - \log v)$$

satisfies that

$$\frac{d}{dt} E_2(t) \leq -\varepsilon_1 \int_\Omega u(\cdot, t) - \varepsilon_2 \int_\Omega \left[ u(\cdot, t)^2 + (v(\cdot, t) - 1)^2 + \left( w(\cdot, t) - \frac{\beta}{\gamma} \right)^2 \right]$$

for all $t > 0$ with some $\varepsilon_1 \geq 0$ and $\varepsilon_2 > 0$. Moreover, there exists $C > 0$ satisfying

$$\int_0^\infty \int_\Omega u^2 + \int_0^\infty \int_\Omega (v - 1)^2 + \int_0^\infty \int_\Omega \left( w - \frac{\beta}{\gamma} \right)^2 \leq C.$$

**Proof.** Let $\delta_1 > 0$ and $a'_1 \in [1, a_1]$ be constants defined in (6.1.11)–(6.1.12). We first show that the function $E_2$ fulfills (6.3.15) for all $t > 0$ with $\varepsilon_1 := a'_1 - 1$ and some $\varepsilon_2 > 0$. Noting from the relation $a'_1 \leq a_1$ that

$$u(1 - u - a_1 v) \leq u(1 - u - a'_1 v)$$

$$= -\mu_1 u^2 - a'_1 \mu_1 u(v - 1) - (a'_1 - 1) \mu_1 u,$$

from straightforward calculations we derive that

$$\frac{d}{dt} E_2(t) = -\mu_1 \int_\Omega u(\cdot, t)^2 - (1 + \delta_1) a'_1 \mu_1 \int_\Omega u(\cdot, t)(v(\cdot, t) - 1)$$

$$- \frac{a'_1 \mu_1 \delta_1}{\mu_2} \int_\Omega (v(\cdot, t) - 1)^2 - (a'_1 - 1) \int_\Omega u(\cdot, t)$$

$$- \frac{d_2 a'_1 \mu_1 \delta_1}{a_2 \mu_2} \int_\Omega \frac{|\nabla v(\cdot, t)|^2}{v^2} + \frac{a'_1 \mu_1 \chi_2 \delta_1}{a_2 \mu_2} \int_\Omega \frac{|\nabla v(\cdot, t)|}{v} \cdot \nabla w(\cdot, t)$$

for all $t > 0$. Here thanks to (6.1.11)–(6.1.12), we can take $\delta_2 > 0$ such that

$$\frac{a'_1 \mu_1 \chi_2 \delta_1}{4 d_2 a_2 \mu_2} < \delta_2 < \frac{d_3 a'_1 \mu_1 \gamma (4 \delta_2 - (1 + \delta_1) a'_1 a_2)}{a'_1 \alpha^2 \delta_1 + a_2 \beta^2 - (1 + \delta_1) a'_1 a_2 \alpha \beta}.$$

Invoking the Young inequality, we obtain that

$$\frac{a'_1 \mu_1 \chi_2 \delta_1}{a_2 \mu_2} \int_\Omega \frac{|\nabla v|}{v} \cdot \nabla w \leq \frac{a'_1 \mu_1 \chi_2 \delta_1}{4 \delta_2} \int_\Omega \frac{|\nabla v|^2}{v^2} + \delta_2 \int_\Omega |\nabla w|^2.$$
Therefore since the definition of $\delta_2$ yields $d_2 - \frac{a_1' \mu_1 x^2 \delta_1}{4a_2 \mu_2 d_2} > 0$, the equation (6.3.17) implies that
\[
\frac{d}{dt} E_2(t) \leq -\varepsilon_1 \int_{\Omega} u - \mu_1 \int_{\Omega} u(\cdot,t)^2 - (1 + \delta_1)a_1' \mu_1 \int_{\Omega} u(\cdot,t)(v(\cdot,t) - 1)
- \frac{a_1' \mu_1 \delta_1}{\mu_2} \int_{\Omega} (v(\cdot,t) - 1)^2 + \delta_2 \int_{\Omega} |\nabla w(\cdot,t)|^2
\]
for all $t > 0$, where $\varepsilon_1 = a'_1 - 1$. Noting from the third equation in (6.1.1) that
\[
\int_{\Omega} |\nabla w|^2 = \frac{\alpha}{d_3} \int_{\Omega} u \left( w - \frac{\beta}{\gamma} \right) + \frac{\beta}{d_3} \int_{\Omega} (v - 1) \left( w - \frac{\beta}{\gamma} \right) - \frac{\gamma}{d_3} \int_{\Omega} \left( w - \frac{\beta}{\gamma} \right)^2,
\]
we establish that for all $t > 0$,
\[
\frac{d}{dt} E_2(t) \leq -\varepsilon_1 \int_{\Omega} u + F_2(t) \quad \text{for all } t > 0,
\]
where
\[
F_2(t) := -\mu_1 \int_{\Omega} u(\cdot,t)^2 - (1 + \delta_1)a_1' \mu_1 \int_{\Omega} u(\cdot,t)(v(\cdot,t) - 1)
- \frac{a_1' \mu_1 \delta_1}{\mu_2} \int_{\Omega} (v(\cdot,t) - 1)^2 + \frac{\alpha(1 + \delta_1)\delta_2}{d_3} \int_{\Omega} u(\cdot,t) \left( w(\cdot,t) - \frac{\beta}{\gamma} \right)
+ \frac{\beta(1 + \delta_1)\delta_2}{d_3} \int_{\Omega} (v(\cdot,t) - 1) \left( w(\cdot,t) - \frac{\beta}{\gamma} \right) - \frac{\gamma(1 + \delta_1)\delta_2}{d_3} \int_{\Omega} \left( w(\cdot,t) - \frac{\beta}{\gamma} \right)^2.
\]
Then by using the same argument as in the proof of Lemma 6.3.4 we can see that
\[
F_2(t) \leq -\varepsilon_2 \int_{\Omega} \left[ u(\cdot,t)^2 + (v(\cdot,t) - 1)^2 + \left( w(\cdot,t) - \frac{\beta}{\gamma} \right)^2 \right] \quad \text{for all } t > 0
\]
with some $\varepsilon_2 > 0$, which means that (6.3.15) with $\varepsilon_1 = a'_1 - 1 \geq 0$ and $\varepsilon_2 > 0$ and (6.3.16) hold.

Then we will establish the convergence result for the solution to (6.1.1) in the case that $a_1 \geq 1 > a_2$.

**Lemma 6.3.8.** Assume that (6.1.11)–(6.1.12) are satisfied. Then we have
\[
\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t) - 1\|_{L^\infty(\Omega)} + \left\| w(\cdot,t) - \frac{\beta}{\gamma} \right\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty.
\]

**Proof.** A combination of Lemmas 6.3.1 and 6.3.7 implies this lemma. \qed

Finally, we shall show two lemmas which give asymptotic behavior in the case that $a_1 > 1 > a_2$. 

62
Lemma 6.3.9. Let \( a_1 > 1 \) and \( a_2 \in (0, 1) \). Assume that (6.1.11)–(6.1.12) are satisfied with \( \delta_1 > 0 \) and \( a'_1 > 0 \). Then there exist \( C > 0 \) and \( \ell > 0 \) satisfying
\[
\|u(\cdot, t)\|_{L^2(\Omega)} + \|v(\cdot, t) - 1\|_{L^2(\Omega)} \leq Ce^{-\ell t} \quad \text{for all } t > 0.
\]

Proof. In the case that \( a_1 > 1 \) and \( a_2 \in (0, 1) \) a combination of the inequalities
\[
\lim_{s \to 0} \frac{s}{s^2 + (a'_1 - 1)s} = \frac{1}{a'_1 - 1}
\]
and
\[
\lim_{s \to 1} \frac{H_1(s) - H_1(1)}{(s - 1)^2} = \lim_{s \to 1} \frac{H''_1(s)}{2} = \frac{1}{2}
\]
with \( H_1(s) := s - \log s \), and the fact \( u \to 0, \ v \to 1 \) in \( L^\infty(\Omega) \) as \( t \to \infty \) (from Lemma 6.3.8) enables us to see that there exist \( C_1, C_2 > 0 \) and \( t_0 > 0 \) such that
\[
C_1 h_1(t) \leq E_2(t) \leq C_2 h_1(t) \quad \text{for all } t > t_0,
\]
where \( E_2 \) is the function defined as (6.3.14) and
\[
h_1(t) := \int_\Omega u(\cdot, t)^2 + \int_\Omega (v(\cdot, t) - 1)^2 + (a'_1 - 1) \int_\Omega u(\cdot, t).
\]
Thus an argument similar to that in the proof of Lemma 6.3.6 leads to the conclusion of this lemma in the case that \( a_1 > 1 \) and \( a_2 \in (0, 1) \).

Lemma 6.3.10. Let \( a_1 = 1 \) and \( a_2 \in (0, 1) \). Assume that (6.1.11)–(6.1.12) are satisfied. Then there exist \( C > 0 \) and \( \ell > 0 \) satisfying
\[
\|u(\cdot, t)\|_{L^2(\Omega)} + \|v(\cdot, t) - 1\|_{L^2(\Omega)} \leq \frac{C}{\sqrt{t} + 2} \quad \text{for all } t > 0.
\]

Proof. First we can verify from an argument similar to that in the proof of Lemma 6.3.9 that there exist \( C_4, C_5 > 0 \) and \( t_1 > 0 \) such that
\[
(6.3.18) \quad C_4 \int_\Omega (v(\cdot, t) - 1)^2 \leq \int_\Omega (v(\cdot, t) - 1 - \log v(\cdot, t)) \leq C_5 \int_\Omega (v(\cdot, t) - 1)^2
\]
for all \( t > t_1 \). Hence it follows from the Cauchy–Schwarz inequality and the boundedness of \( v \) that
\[
E_2(t) \leq \int_\Omega u(\cdot, t) + \frac{a'_1 \mu_1 \delta_1}{a_2 \mu_2} \int_\Omega (v(\cdot, t) - 1)^2
\]
\[
\leq C_6 \left( \int_\Omega u(\cdot, t)^2 \right)^{\frac{1}{2}} + C_6 \left( \int_\Omega (v(\cdot, t) - 1)^2 \right)^{\frac{1}{2}}
\]
\[
\leq \sqrt{2} C_6 \left( \int_\Omega u(\cdot, t)^2 + \int_\Omega (v(\cdot, t) - 1)^2 \right)^{\frac{1}{2}} \quad \text{for all } t > t_1,
\]

63
which implies from (6.3.15) that
\[
\frac{d}{dt} E_2(t) \leq -C_7 E_2(t)^2
\]
for all \( t > t_1 \).

Therefore we can find \( C_8 > 0 \) such that \( E_2(t) \leq \frac{C_8}{t+2} \) for all \( t > t_1 \). Therefore thanks to the boundedness of \( u \) and (6.3.18), we obtain that
\[
\int_{\Omega} u(\cdot, t)^2 + \int_{\Omega} (v(\cdot, t) - 1)^2 \leq C_9 E_2(t) \leq \frac{C_8 C_9}{t + 2} \quad \text{for all } t > t_1,
\]
which proves this lemma.

**Proof of Theorems 6.1.3 and 6.1.4.** A combination of Lemmas 6.3.6, 6.3.9, 6.3.10 and 6.3.3 immediately leads to the conclusions of these theorems.

\[ \square \]
Chapter 7

Boundedness and stabilization in a two-species chemotaxis model with signal-dependent sensitivity and competitive term

7.1. Problem and results

In this chapter we consider the two-species chemotaxis system

\[
\begin{align*}
    u_t &= d_1 \Delta u - \nabla \cdot (u\chi_1(w)\nabla w) + \mu_1 u(1 - u - a_1 v), & x \in \Omega, & t > 0, \\
    v_t &= d_2 \Delta v - \nabla \cdot (v\chi_2(w)\nabla w) + \mu_2 v(1 - a_2 u - v), & x \in \Omega, & t > 0, \\
    w_t &= d_3 \Delta w + h(u, v, w), & x \in \Omega, & t > 0, \\
    \nabla u \cdot \nu = \nabla v \cdot \nu = \nabla w \cdot \nu = 0, & x \in \partial \Omega, & t > 0, \\
    u(x, 0) &= u_0(x), & v(x, 0) &= v_0(x), & w(x, 0) &= w_0(x), & x \in \Omega,
\end{align*}
\]

(7.1.1)

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) (\( n \in \mathbb{N} \)) with smooth boundary \( \partial \Omega \) and \( \nu \) is the outward normal vector to \( \partial \Omega \). The initial data \( u_0, v_0 \) and \( w_0 \) are assumed to be nonnegative functions. The unknown functions \( u(x, t) \) and \( v(x, t) \) represent the population densities of two species and \( w(x, t) \) shows the concentration of the substance at place \( x \) and time \( t \).

The problem (7.1.1) is an interesting problem on account of the influence of chemotaxis, diffusion, and the Lotka–Volterra kinetics. In mathematical view, global existence and behavior of solutions are fundamental theme. In the case \( \chi_i(w) = \chi_i \) and \( h(u, v, w) = \alpha u + \beta v - \gamma w \), Bai–Winkler [4] showed global existence of solutions to (7.1.1) when \( n = 2 \). Moreover, they considered asymptotic behavior of solutions to (7.1.1). When \( a_1, a_2 \in (0, 1) \), they proved that the solution \( (u, v, w) \) satisfies \( u(\cdot, t) \to u^* \), \( v(\cdot, t) \to v^* \), \( w(\cdot, t) \to \frac{\alpha u^* + \beta v^*}{\gamma} \) in \( L^\infty(\Omega) \) as \( t \to \infty \), where \( u^* = \frac{1-a_1}{1-a_1a_2} \), \( v^* = \frac{1-a_2}{1-a_1a_2} \), under the conditions

\[
\begin{align*}
    \mu_1 > \frac{d_2 \chi_1^2 u^*}{4\alpha_1\chi_1^2(1-a_1a_2)d_1d_2d_3 - d_1\chi_1^2 v^*}, \\
    \mu_2 > \frac{\chi_2^2 v^*(a_1\alpha^2 + a_2\beta^2 - 2a_1a_2\alpha\beta)}{16d_2d_3\gamma(1-a_1a_2)}.
\end{align*}
\]

(7.1.2)
These conditions are not natural because they are not symmetric. When \(a_1 \geq 1 \geq a_2 > 0\), they obtained that \(u(\cdot, t) \to 0\), \(v(\cdot, t) \to 1\), \(w(\cdot, t) \to \frac{1}{q}\) in \(L^\infty(\Omega)\) as \(t \to \infty\) under the condition that there exists \(a_1' \in [1, a_1]\) such that \(a_1'a_2 < 1\) and

\[
\mu_2 > \frac{\chi_2^2(a_1'\alpha^2 + a_2\beta^2 - 2a_1'a_2\alpha\beta)}{16d_2d_3a_2\gamma(1 - a_1'a_2)}.
\]

In the non-competitive case \((a_1 = a_2 = 0)\), global existence and asymptotic stability were established under some conditions for \(\chi_i(w)\) (Chapter 3). In the case that \(d_1 = d_2 = d_3 = 1\) and \(h(u, v, w) = u + v - w\), Zhang–Li [213] proved global existence of solutions to (7.1.1) under the assumption that \(\mu_i > 0\) is small and \(\chi_i(w) \leq \frac{K_i}{(1 + a_i(w))^{\gamma_i}}\) for \(\sigma_i > 1\) and \(K_i > 0\) being small enough.

The purpose of the present chapter is to improve the methods in [4] and Chapter 3 for obtaining global existence and asymptotic stability of solutions to (7.1.1) under a more general and sharp condition for the sensitivity function \(\chi_i(w)\). We shall suppose throughout this chapter that \(h, \chi_i (i = 1, 2)\) satisfy the following conditions:

\[
\begin{align*}
(7.1.4) & \quad \chi_i \in C^{1+\theta}([0, \infty)) \cap L^1(0, \infty) \ (0 < \exists \theta < 1), \quad \chi_i > 0 \quad (i = 1, 2), \\
(7.1.5) & \quad h \in C^1([0, \infty) \times [0, \infty) \times [0, \infty)), \quad h(0, 0, 0) \geq 0, \\
(7.1.6) & \quad \exists \gamma > 0; \quad \frac{\partial h}{\partial u}(u, v, w) \geq 0, \quad \frac{\partial h}{\partial v}(u, v, w) \geq 0, \quad \frac{\partial h}{\partial w}(u, v, w) \leq -\gamma, \\
(7.1.7) & \quad \exists \delta > 0, \quad \exists M > 0; \quad |h(u, v, w) + \delta w| \leq M(u + v + 1), \\
(7.1.8) & \quad \exists k_i > 0; \quad -\chi_i(w)h(0, 0, 0) \leq k_i \quad (i = 1, 2).
\end{align*}
\]

We also assume that there exists \(p > n\) such that

\[
2d_i d_3 \chi_i'(w) + \left((d_3 - d_i)p + \sqrt{(d_3 - d_i)^2p^2 + 4d_3 d i p}\right)[\chi_i(w)]^2 \leq 0 \quad (i = 1, 2).
\]

The above conditions cover the prototypical example \(\chi_i(w) = \frac{K_i}{(1 + a_i(w))^{\gamma_i}}\ (K_i > 0, \sigma_i > 1), h(u, v, w) = u + v - w\). We assume that the initial data \(u_0, v_0, w_0\) satisfy

\[
(7.1.10) \quad 0 \leq u_0 \in C(\overline{\Omega}) \setminus \{0\}, \quad 0 \leq v_0 \in C(\overline{\Omega}) \setminus \{0\}, \quad 0 \leq w_0 \in W^{1, q}(\Omega) \quad (\exists q > n).
\]

Now the main results read as follows. The first one is concerned with global existence and boundedness in (7.1.1).

**Theorem 7.1.1.** Let \(d_1, d_2, d_3 > 0, \mu_1, \mu_2 > 0, a_1, a_2 \geq 0\). Assume that \(h, \chi_1, \chi_2\) satisfy (7.1.4)–(7.1.9). Then for any \(u_0, v_0, w_0\) satisfying (7.1.10) for some \(q > n\), there exists an exactly one pair \((u, v, w)\) of nonnegative functions

\[
u, \quad w \in C(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)),
\]

which satisfy (7.1.1). Moreover, the solutions \(u, v, w\) are uniformly bounded, i.e., there exists a constant \(C_1 > 0\) such that

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{W^{1, \infty}(\Omega)} \leq C_1
\]
for all \( t \geq 0 \), and the solutions \( u, v, w \) are the Hölder continuous functions, i.e., there exist \( \alpha \in (0, 1) \) and \( C_2 > 0 \) such that

\[
\|u\|_{C^{2+\alpha,1+\frac{1}{2}}(\Omega \times [1,t])} + \|v\|_{C^{2+\alpha,1+\frac{1}{2}}(\Omega \times [1,t])} + \|w\|_{C^{2+\alpha,1+\frac{1}{2}}(\Omega \times [1,t])} \leq C_2
\]

for all \( t \geq 1 \).

**Remark 7.1.1.** Theorem 7.1.1 improves the result in [213] when \( \chi_i(w) = \frac{K_i}{(1+w)^p} \). Indeed, we do not need any condition for \( \mu_i \). Moreover, the condition in [213] is \( K_i \leq \frac{r}{r+1} \frac{a}{\sqrt{2p(1-w)^p}} \) with some \( r > 0 \), while the condition (7.1.9) becomes \( K_i \leq \frac{a}{\sqrt{B}} \) when \( \chi_i(w) = \frac{K_i}{(1+w)^p} \).

Since Theorem 7.1.1 guarantees that \( u, v \) and \( w \) exist globally and are bounded and nonnegative, it is possible to define nonnegative numbers \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) by

\[
\begin{align*}
\alpha_1 &:= \min_{(u,v,w) \in I} h_u(u,v,w), & \alpha_2 &:= \max_{(u,v,w) \in I} h_u(u,v,w), \\
\beta_1 &:= \min_{(u,v,w) \in I} h_v(u,v,w), & \beta_2 &:= \max_{(u,v,w) \in I} h_v(u,v,w),
\end{align*}
\]

where \( I = (0, C_1)^3 \) and \( C_1 \) is defined in Theorem 7.1.1.

In the case \( a_1, a_2 \in (0,1) \) asymptotic behavior of solutions to (7.1.1) will be discussed under the following additional conditions: there exists \( \delta_1 > 0 \) such that

\[
4\delta_1 - a_1 a_2 (1 + \delta_1)^2 > 0
\]

and

\[
\begin{align*}
\mu_1 &> \frac{\chi_1(0)^2 u^*(1 + \delta_1)(\alpha_2^2 a_1 \delta_1 + \beta_2^2 a_2 - \alpha_1 \beta_1 a_1 a_2 (1 + \delta_1))}{4a_1 d_1 d_3 \gamma (4\delta_1 - a_1 a_2 (1 + \delta_1)^2)}, \\
\mu_2 &> \frac{\chi_2(0)^2 v^*(1 + \delta_1)(\alpha_2^2 a_1 \delta_1 + \beta_2^2 a_2 - \alpha_1 \beta_1 a_1 a_2 (1 + \delta_1))}{4a_2 d_2 d_3 \gamma (4\delta_1 - a_1 a_2 (1 + \delta_1)^2)}.
\end{align*}
\]

The second theorem gives asymptotic behavior in (7.1.1) in the case \( a_1, a_2 \in (0,1) \).

**Theorem 7.1.2.** Let \( d_1, d_2, d_3 > 0, \mu_1, \mu_2 > 0 \) and \( a_1, a_2 \in (0,1) \). Under the conditions (7.1.4)–(7.1.10) and (7.1.12)–(7.1.14), the unique global solution \((u, v, w)\) of (7.1.1) satisfies that there exist \( C > 0 \) and \( \lambda > 0 \) such that

\[
\|u(\cdot, t) - u^*\|_{L^\infty(\Omega)} + \|v(\cdot, t) - v^*\|_{L^\infty(\Omega)} + \|w(\cdot, t) - w^*\|_{L^\infty(\Omega)} \leq C e^{-\lambda t} \quad (t > 0),
\]

where

\[
\begin{align*}
u^* &:= \frac{1 - a_1}{1 - a_1 a_2}, & v^* &:= \frac{1 - a_2}{1 - a_1 a_2}
\end{align*}
\]

and \( w^* \geq 0 \) such that

\[
h(u^*, v^*, w^*) = 0.
\]

67
Remark 7.1.2. The methods in the proof of Theorem 7.1.2 (and Theorem 7.1.4, see below) can be applied to the case \( \chi_i(w) = \chi_i \) and \( h(u, v, w) = \alpha u + \beta v - \gamma w \). Then the conditions (7.1.12)–(7.1.14) have symmetry and relax the condition (7.1.2) assumed in [4]. Indeed, the conditions (7.1.2) are stronger than (7.1.12)–(7.1.14) when \( \delta_1 = 1 \). Moreover, in view of considering the function

\[
f(x) = \frac{a_1(\alpha^2 - \alpha \beta a_2)x^2 + (\beta^2 a_2 - \alpha^2 a_1)x}{-a_1 a_2 x^2 + 4x - 4}
\]

(we put \( x = 1 + \delta_1 \), \( x = 2 (\delta_1 = 1) \) is not a minimizer of the right-hand sides of (7.1.13) and (7.1.14) except the case \( \beta_2 a_2 = \alpha^2 a_1 \). Thus the conditions (7.1.12)–(7.1.14) relax (7.1.2).

This remark implies the following result which improves the previous work in [4].

Theorem 7.1.3. Let \( d_1, d_2, d_3 > 0 \), \( \mu_1, \mu_2 > 0 \) and \( a_1, a_2 \in (0, 1) \). Assume that there exists a unique global solution \((u, v, w)\) of (7.1.1) satisfying that there exists \( C > 0 \) such that

\[
\|u\|_{C^{2+\alpha,1+\frac{\gamma}{2}}(\Omega \times [1,t])} + \|v\|_{C^{2+\alpha,1+\frac{\gamma}{2}}(\Omega \times [1,t])} + \|w\|_{C^{2+\alpha,1+\frac{\gamma}{2}}(\Omega \times [1,t])} \leq C
\]

for all \( t \geq 1 \) and \( \chi_1, \chi_2 \) satisfies that there exist the positive constants \( M_1, M_2 > 0 \) and \( \delta_1 > 0 \) such that

\[
\chi_i(w) \leq M_i \quad \text{for all } w \geq 0 \quad (i = 1, 2),
\]

\[
4\delta_1 - a_1 a_2 (1 + \delta_1)^2 > 0,
\]

and

\[
\mu_i > \frac{M_i^2(1 + \delta_1)(1 - a_i)(\alpha_2^2 a_1 \delta_1 + \beta_2^2 a_2 - \alpha_1 \beta_1 a_1 a_2 (1 + \delta_1))}{4a_1 d_1 d_2 d_3 \gamma(1 - a_1 a_2)(4 \delta_1 - a_1 a_2 (1 + \delta_1)^2)} \quad (i = 1, 2).
\]

Then the same conclusion as in Theorem 7.1.2 holds.

In the case \( a_1 \geq 1 > a_2 > 0 \) asymptotic behavior of solutions to (7.1.1) will be discussed under the following additional conditions: there exist \( \delta_1 > 0 \) and \( a'_1 \in [1, a_1] \) such that

\[
(7.1.15) \quad 4\delta_1 - a'_1 a_2 (1 + \delta_1)^2 > 0,
\]

\[
(7.1.16) \quad \mu_2 > \frac{\chi_2(0)^2 \delta_1 (\alpha_2^2 a_1 \delta_1 + \beta_2^2 a_2 - \alpha_1 \beta_1 a'_1 a_2 (1 + \delta_1))}{4a_2 d_2 d_3 \gamma(4 \delta_1 - a'_1 a_2 (1 + \delta_1)^2)}.
\]

The third one is concerned with asymptotic behavior of solutions to (7.1.1) in the case that \( a_1 \geq 1 > a_2 > 0 \).

Theorem 7.1.4. Let \( d_1, d_2, d_3 > 0 \), \( \mu_1, \mu_2 > 0 \) and \( a_1 \geq 1, a_2 \in (0, 1) \). Under the conditions (7.1.4)–(7.1.9) and (7.1.15)–(7.1.16), the unique global solution \((u, v, w)\) of (7.1.1) has the following properties:
(i) Let \( a_1 > 1 \) and take \( a'_1 \in (1, a_1] \) in (7.1.15)–(7.1.16). Then there exist \( C > 0 \) and \( \lambda > 0 \) satisfying
\[
\|u(\cdot, t)\|_{L_\infty(\Omega)} + \|v(\cdot, t) - 1\|_{L_\infty(\Omega)} + \|w(\cdot, t) - \bar{w}\|_{L_\infty(\Omega)} \leq Ce^{-\lambda t}
\]
for all \( t > 0 \), where \( \bar{w} \geq 0 \) is such that \( h(0, 1, \bar{w}) = 0 \).

(ii) Let \( a_1 = 1 \). Then there exist \( C > 0 \) and \( \lambda > 0 \) satisfying
\[
\|u(\cdot, t)\|_{L_\infty(\Omega)} + \|v(\cdot, t) - 1\|_{L_\infty(\Omega)} + \|w(\cdot, t) - \bar{w}\|_{L_\infty(\Omega)} \leq C(t + 1)^{-\lambda}
\]
for all \( t > 0 \), where \( \bar{w} \geq 0 \) is such that \( h(0, 1, \bar{w}) = 0 \).

**Remark 7.1.3.** \( \delta_1 = 1 \) and \( a'_1 = \frac{1 + \min(a_1, a_2^{-1})}{2} \) satisfy (7.1.15). Thus there exist some \( \delta_1, a'_1, \mu_2 \) satisfying (7.1.15)–(7.1.16). Moreover, from the same argument as in Remark 7.1.2 the conditions (7.1.15)–(7.1.16) relax (7.1.3).

From this remark we can also established the following result which improves the previous work in [4].

**Theorem 7.1.5.** Let \( d_1, d_2, d_3 > 0 \), \( \mu_1, \mu_2 > 0 \) and \( a_1 \geq 1, a_2 \in (0, 1) \). Assume that there exists a unique global solution \((u, v, w)\) of (7.1.1) satisfying that there exists \( C > 0 \) such that
\[
\|u\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{\Omega} \times [1, t])} + \|v\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{\Omega} \times [1, t])} + \|w\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{\Omega} \times [1, t])} \leq C
\]
for all \( t \geq 1 \) and \( \chi_1, \chi_2 \) satisfies that there exist the positive constants \( M_1, M_2 > 0 \), \( a'_1 \in [1, a_1] \) and \( \delta_1 > 0 \) such that
\[
\chi_i(w) \leq M_i \quad \text{for all } w \geq 0 \quad (i = 1, 2),
\]
\[
4\delta_1 - a'_1 a_2 (1 + \delta_1)^2 > 0,
\]
and
\[
\mu_2 > \frac{M_2^2 \delta_1 (a_2^2 a'_1 \delta_1 + \beta_2 a_2 - \alpha_1 a'_1 a_2 (1 + \delta_1))}{4a_2 d_2 d_3 \gamma (4\delta_1 - a'_1 a_2 (1 + \delta_1)^2)}.
\]
Then the same conclusion as in Theorem 7.1.4 holds.

**Remark 7.1.4.** In Theorems 7.1.2 and 7.1.4 we can find \( w^* \geq 0 \) satisfying \( h(u^*, v^*, w^*) = 0 \) and \( \bar{w} \geq 0 \) satisfying \( h(0, 1, \bar{w}) = 0 \). Indeed, from (7.1.5)–(7.1.7) for every \( a, b \geq 0 \) there exists \( \bar{w} \) such that \( h(a, b, \bar{w}) = 0 \). Indeed, if we choose \( w_1 \geq \frac{M(a + b + 1)}{\delta} \), then (7.1.7) yields \( h(a, b, w_1) \leq M(a + b + 1) - \delta w_1 \leq 0 \). On the other hand, (7.1.5) and (7.1.6) imply that \( h(a, b, 0) \geq h(0, 0, 0) \geq 0 \). Hence, by the intermediate value theorem there exists \( \bar{w} \geq 0 \) such that \( h(a, b, \bar{w}) = 0 \).

The strategy for the proof of Theorem 7.1.1 is to construct estimates for \( \int_\Omega u^p \) and \( \int_\Omega v^p \) by modifying a method in Chapter 3. One of the keys for this strategy is to derive inequality
\[
\frac{d}{dt} \int_\Omega u^p[f_1(w)]^{-r} \leq a \int_\Omega u^p[f_1(w)]^{-r} - b \left( \int_\Omega u^p[f_1(w)]^{-r} \right)^{\frac{p+1}{p}}.
\]

69
for some positive constants $a, b$, where $f_1(w) := \exp\left\{ \frac{1}{d_1} \int_0^w \chi_1(s) \, ds \right\}$. Applying the new method in Chapter 3 for obtaining the above inequality, we can improve the result in [213]. On the other hand, the strategy for the proof of Theorems 7.1.2 and 7.1.4 is to modify an argument in [4]. The key for this strategy is to construct the following energy estimate:

$$\frac{d}{dt} E(t) \leq -\varepsilon \left( \int_{\Omega} (u - \overline{u})^2 + \int_{\Omega} (v - \overline{v})^2 + \int_{\Omega} (w - \overline{w})^2 + \int_{\Omega} |\nabla w|^2 \right)$$

with some function $E(t) \geq 0$ and some $\varepsilon > 0$, where $(\overline{u}, \overline{v}, \overline{w}) \in \mathbb{R}^3$ is a solution of (7.1.1). For finding the above inequality we apply more “suitable” estimates for

$$\int_{\Omega} \frac{\chi_1(w)}{u} \nabla u \cdot \nabla w \quad \text{and} \quad \int_{\Omega} \frac{\chi_2(w)}{v} \nabla v \cdot \nabla w.$$

These enable us to improve the conditions (7.1.2) and (7.1.3).

This chapter is organized as follows. In Section 7.2 we prove global existence and boundedness (Theorem 7.1.1). Sections 7.3 and 7.4 are devoted to the proof of asymptotic stability (Theorems 7.1.2, 7.1.4).

**7.2. Global existence and boundedness**

In this section we shall show global existence and boundedness in (7.1.1). Firstly we will recall the well-known result about local existence of solutions to (7.1.1).

**Lemma 7.2.1.** Let $d_1, d_2, d_3 > 0$, $\mu_1, \mu_2 \geq 0$ and $a_1, a_2 \geq 0$. Assume that $h, \chi_1, \chi_2$ satisfy (7.1.4), (7.1.5), (7.1.7). Then for any $u_0, v_0, w_0$ satisfying (7.1.10) for some $q > n$, there exist $T_{\text{max}} \in (0, \infty]$ and an exactly one pair $(u, v, w)$ of nonnegative functions

$$u, v, w \in C(\overline{\Omega} \times [0, T_{\text{max}}]) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\text{max}}))$$

which satisfy (7.1.1). Moreover,

either $T_{\text{max}} = \infty$ or $\lim_{t \to T_{\text{max}}} (\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)}) = \infty$.

**Proof.** The proof of local existence of classical solutions to (7.1.1) is based on a standard contraction mapping argument, which can be found in [193, 195]. Finally the maximum principle is applied to yield $u > 0, v > 0, w \geq 0$ in $\Omega \times (0, T_{\text{max}})$.

Let $(u, v, w)$ be the solution of (7.1.1) on $[0, T_{\text{max}})$ as in Lemma 7.2.1. We introduce the functions $f_1 = f_1(w)$ and $f_2 = f_2(w)$ by

$$f_i(w) := \exp \left\{ \frac{1}{d_i} \int_0^w \chi_i(s) \, ds \right\} \quad \text{for } i = 1, 2$$

to prove the following lemma.
Lemma 7.2.2. Let \( d_1, d_2, d_3 > 0, \mu_1, \mu_2 > 0 \) and \( a_1, a_2 > 0 \). Assume that \( \chi_1, \chi_2 \) satisfy (7.1.4) and (7.1.9) with some \( p > n \). Then there exist positive constants \( r_1 = r_1(d_1, d_3, p) \) and \( r_2 = r_2(d_2, d_3, p) \) such that

\[
\frac{d}{dt} \int_{\Omega} u^p f_1^{-r_1} \leq p\mu_1 \int_{\Omega} u^p f_1^{-r_1}(1 - u) - \frac{r_1}{d_1} \int_{\Omega} u^p f_1^{-r_1} \chi_1(w) h(u, v, w)
\]

and

\[
\frac{d}{dt} \int_{\Omega} v^p f_2^{-r_2} \leq p\mu_2 \int_{\Omega} v^p f_2^{-r_2}(1 - v) - \frac{r_2}{d_2} \int_{\Omega} v^p f_2^{-r_2} \chi_2(w) h(u, v, w).
\]

Proof. We let \( p \geq 1 \) and \( r > 0 \) be fixed later. From the first and third equations in (7.1.1) we have

\[
\frac{d}{dt} \int_{\Omega} u^p f_1^{-r} = p \int_{\Omega} u^{p-1} f_1^{-r} \nabla \cdot (d_1 \nabla u - u \chi_1(w) \nabla w) + p\mu_1 \int_{\Omega} u^p f_1^{-r}(1 - u - a_1 v)
\]

\[
- \frac{d_3 r}{d_1} \int_{\Omega} u^p f_1^{-r} \chi_1(w) \Delta w - \frac{r}{d_1} \int_{\Omega} u^p f_1^{-r} \chi_1(w) h(u, v, w).
\]

Denoting by \( I_1 \) and \( I_2 \) the first and third terms on the right-hand side as

\[
I_1 := p \int_{\Omega} u^{p-1} f_1^{-r} \nabla \cdot (d_1 \nabla u - u \chi_1(w) \nabla w),
\]

\[
I_2 := - \frac{d_3 r}{d_1} \int_{\Omega} u^p f_1^{-r} \chi_1(w) \Delta w,
\]

we can write as

\[
\frac{d}{dt} \int_{\Omega} u^p f_1^{-r} = I_1 + I_2 + p\mu_1 \int_{\Omega} u^p f_1^{-r}(1 - u - a_1 v)
\]

\[
- \frac{r}{d_1} \int_{\Omega} u^p f_1^{-r} \chi_1(w) h(u, v, w).
\]

We shall show the following inequality:

\[
\exists p > n, \exists r > 0; I_1 + I_2 \leq 0.
\]

Noting that

\[
d_1 f_1 \nabla \left( \frac{u}{f_1} \right) = d_1 \nabla u - u \chi_1(w) \nabla w,
\]

we obtain

\[
I_1 = d_1 p \int_{\Omega} u^{p-1} f_1^{-r} \nabla \cdot \left( f_1 \nabla \left( \frac{u}{f_1} \right) \right)
\]

\[
= d_1 p \int_{\Omega} \left( \frac{u}{f_1} \right)^{p-1} f_1^{-r+p-1} \nabla \cdot \left( f_1 \nabla \left( \frac{u}{f_1} \right) \right)
\]

\[
= -d_1 p(p - 1) \int_{\Omega} \left( \frac{u}{f_1} \right)^{p-2} f_1^{-r+p} \left| \nabla \left( \frac{u}{f_1} \right) \right|^2
\]

\[
- p(-r + p - 1) \int_{\Omega} \left( \frac{u}{f_1} \right)^{p-1} f_1^{-r+p} \chi_1(w) \nabla \left( \frac{u}{f_1} \right) \cdot \nabla w.
\]
Similarly, we see that
\[ I_2 = -\frac{d_3 r}{d_1} \int_{\Omega} \left( \frac{u}{f_1} \right)^p f_1^{-r+p} \chi_1(w) \Delta w \]
\[ = \frac{d_3 r}{d_1} \int_{\Omega} \left( \frac{u}{f_1} \right)^{p-1} f_1^{-r+p} \chi_1(w) \nabla \left( \frac{u}{f_1} \right) \cdot \nabla w \]
\[ + \frac{d_3 r}{d_1} \int_{\Omega} \left( \frac{u}{f_1} \right)^p f_1^{-r+p} \left( \frac{p - r}{d_1} \chi_1(w)^2 + \chi_1'(w) \right) |\nabla w|^2. \]

Therefore it follows that
\[ I_1 + I_2 \]
\[ = -d_1 p(p - 1) \int_{\Omega} \frac{\left( \frac{u}{f_1} \right)^p f_1^{-r+p} }{f_1} \left. \nabla \left( \frac{u}{f_1} \right) \right|^2 \]
\[ - \left( \frac{p(p - 1) - (1 + \frac{d_3}{d_1}) p r}{f_1} \right) \int_{\Omega} \frac{\left( \frac{u}{f_1} \right)^{p-1} f_1^{-r+p} \chi_1(w) }{f_1} \nabla \left( \frac{u}{f_1} \right) \cdot \nabla w \]
\[ + \int_{\Omega} \frac{\left( \frac{u}{f_1} \right)^p f_1^{-r+p} }{f_1} \left( \frac{d_3 r(-r + p)}{d_1^2} \chi_1(w)^2 + \frac{d_3 r}{d_1} \chi_1'(w) \right) |\nabla w|^2 \]
\[ = -d_1 p(p - 1) \int_{\Omega} \frac{\left( \frac{u}{f_1} \right)^p f_1^{-r+p} }{f_1} \left. \nabla \left( \frac{u}{f_1} \right) + \frac{p(p - 1) - (1 + \frac{d_3}{d_1}) p r}{2d_1 p(p - 1) f_1} \chi_1(w) \frac{u}{f_1} \nabla w \right|^2 \]
\[ + \int_{\Omega} \frac{\left( \frac{u}{f_1} \right)^p f_1^{-r+p} }{f_1} \left[ \frac{a_1 r^2 + 2a_2 r + a_3}{4d_1(p - 1)} \right] |\nabla w|^2, \]

where \( a_1, a_2, a_3 \) are given by
\[ a_1 := \left( \frac{d_3}{d_1} - 1 \right)^2 + \frac{4d_3}{d_1} |\chi_1(w)|^2, \]
\[ a_2 := (p - 1) \left( \frac{d_3}{d_1} - 1 \right) |\chi_1(w)|^2 + 2d_3 \chi_1'(w), \]
\[ a_3 := p(p - 1)^2 |\chi_1(w)|^2. \]

Then there exists \( p > n \) such that the discriminant of \( a_1 r^2 + 2a_2 r + a_3 \)
\[ D := 4(p - 1)^2 \left[ p \chi_1^2 \left( \frac{d_3}{d_1} - 1 \right) + 2d_3 \chi_1' \right]^2 - p \chi_1^4 \left( p \left( \frac{d_3}{d_1} - 1 \right)^2 + \frac{4d_3}{d_1} \right) \]
is nonnegative in view of (7.1.9). Therefore we have that there exists \( r > 0 \) such that \( a_1 r^2 + 2a_2 r + a_3 \leq 0 \) and hence
\[ I_1 + I_2 \leq 0. \]

On the other hand, we can see from the positivity of \( u \) and \( v \) that
\[ \mu_1 u(1 - u - a_1 v) \leq \mu_1 u(1 - u). \]
Hence (7.2.3) implies
\[
\frac{d}{dt} \int_{\Omega} u^p f_1^{-r} \leq p \mu_1 \int_{\Omega} u^p f_1^{-r}(1 - u) - \frac{r}{d_1} \int_{\Omega} u^p f_1^{-r} \chi_i h(u, v, w).
\]
This means that (7.2.1) holds. In the same way, we obtain (7.2.2).

**Lemma 7.2.3.** Let \( d_1, d_2, d_3 > 0, \mu_1, \mu_2 > 0 \) and \( a_1, a_2 > 0 \). Assume that \( h, \chi_i \) satisfy (7.1.4)–(7.1.6), (7.1.8), and (7.1.9) with some positive constants \( k_i \) \((i = 1, 2)\) and \( p > n \), then

\[
\|u(\cdot, t)\|_{L^p(\Omega)} \leq \left( e^{\|x_1\|_{L^1(0, \infty)}} \right)^{\frac{\eta}{2p}} \max \left\{ \|u_0\|_{L^p(\Omega)}, \frac{d_1 \mu_1 + r_1 k_1}{d_1 \mu_1} |\Omega|^{1/p} \right\},
\]

\[
\|v(\cdot, t)\|_{L^p(\Omega)} \leq \left( e^{\|x_2\|_{L^1(0, \infty)}} \right)^{\frac{\eta}{2p}} \max \left\{ \|v_0\|_{L^p(\Omega)}, \frac{d_2 \mu_2 + r_2 k_2}{d_2 \mu_2} |\Omega|^{1/p} \right\}.
\]

**Proof.** The proof is same as that of Lemma 3.3.2.

For the proof of Theorem 7.1.1, we put \( k > 0 \) and let \( \Delta \) denote the realization of the Laplacian in \( L^s(\Omega) \) with domain \( \{z \in W^{2,s}(\Omega) \mid \nabla z \cdot \nu = 0 \text{ on } \partial \Omega\} \) for \( s \in (1, \infty) \). Then it is known that the operator \( -\Delta + k \) is sectorial and possesses closed fractional powers \((-\Delta + k)^\eta, \eta \in (0, 1)\), with dense domain \( D((-\Delta + k)^\eta) \). Moreover, there exist \( c_1, c_2 > 0 \) such that if \( m \in \{0, 1\}, p \in [1, \infty], q \in (1, \infty) \) satisfy \( m < 2\eta \) and \( m - n/p < 2\eta - n/q \), then

\[
\|z\|_{W^{m,p}(\Omega)} \leq c_1 \|(-\Delta + k)^\eta z\|_{L^n(\Omega)}
\]
for every \( z \in D((-\Delta + k)^\eta) \), and if \( p \in [1, \infty), q \geq p \), then there exists \( \lambda > 0 \) such that

\[
\|(-\Delta + k)^\eta e^{\lambda \Delta - k} z\|_{L^q(\Omega)} \leq c_2 t^{-\eta - \frac{\eta}{p}(\frac{1}{p} - \frac{1}{q})} e^{-\lambda t} \|z\|_{L^p(\Omega)} \quad (t > 0)
\]
for all \( z \in L^p(\Omega) \) (see Lemma 2.2.1).

**Proof of Theorem 7.1.1.** We let \( \tau \in (0, T_{\max}) \). In view of Lemma 7.2.1 it is sufficient to make sure that

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)} \leq C(\tau), \quad t \in (\tau, T_{\max})
\]
holds with some \( C(\tau) > 0 \). We let \( \rho \in \left( \frac{p + n}{2p}, 1 \right) \). This means \( 1 < 2\rho - \frac{n}{p} \). Writing as

\[
w_t = d_3(\Delta - \delta/d_3)w + h(u, v, w) + \delta w,
\]
and applying the variation of constants formula for \( w \), we have

\[
w(\cdot, t) = e^{d_3(t-\delta/d_3)}w_0 + \int_0^t e^{d_3(t-s)\Delta - \delta/d_3}(h(u(\cdot, s), v(\cdot, s), w(\cdot, s)) + \delta w(\cdot, s)) ds.
\]
From (7.2.5), (7.2.6) and (7.1.7) we obtain that for all \( t \in (\tau, T_{\max}) \),

\[
\|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq c_1 \|(-\Delta + \delta/d_3)^{\rho} w(\cdot, t)\|_{L^p(\Omega)}
\leq c_1 c_2 t^{-\rho} e^{-\lambda t} \|w_0\|_{L^p(\Omega)}
+ c_1 c_2 \int_0^t (t - s)^{-\rho} e^{-\lambda(t-s)} \|h(u(\cdot, s), v(\cdot, s), w(\cdot, s)) + \delta w(\cdot, s)\|_{L^p(\Omega)} \, ds
\leq c_1 c_2 \tau^{-\rho} e^{-\lambda \tau} \|w_0\|_{L^p(\Omega)}
+ c_1 c_2 c_3 \int_0^\tau (t - s)^{-\rho} e^{-\lambda(t-s)} \, ds,
\]

where \( c_3 := \sup_{0 \leq s < T_{\max}} \{ M(\|u(\cdot, s)\|_{L^p(\Omega)} + \|v(\cdot, s)\|_{L^p(\Omega)} + 1) \} \) (< \infty by Lemma 7.2.3). Noting that

\[
\int_0^t (t - s)^{-\rho} e^{-\lambda(t-s)} \, ds \leq \int_0^\infty r^{-\rho} e^{-\lambda r} \, dr < \infty,
\]

we deduce that

(7.2.7) \[ \|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq c_1 c_2 \left( \tau^{-\rho} e^{-\lambda \tau} + c_3 \int_0^\infty r^{-\rho} e^{-\lambda r} \, dr \right) =: C_w(\tau). \]

Since (7.1.9) implies \( \chi'_1 < 0 \), it follows from (7.2.4) and (7.2.7) that for all \( s \in (\tau/2, T_{\max}) \),

(7.2.8) \[ \|u(\cdot, s)\chi_1(w(\cdot, s)) \nabla w(\cdot, s)\|_{L^p(\Omega)} \leq \chi_1(0) \|u(\cdot, s)\|_{L^p(\Omega)} \|\nabla w(\cdot, s)\|_{L^\infty(\Omega)}
\leq \chi_1(0) \sup_{0 \leq t < T_{\max}} \|u(\cdot, t)\|_{L^p(\Omega)} C_w(\tau/2) =: c_4. \]

Employing the variation of constants formula for \( u \) yields

\[
u(\cdot, t) = e^{d_1(t-\tau/2)(\Delta-1)u(\cdot, \frac{\tau}{2})} - \int_{\tau/2}^t e^{d_1(t-s)(\Delta-1)\nabla w(\cdot, s)} \nabla \cdot (u(\cdot, s)\chi_1(w(\cdot, s)) \nabla w(\cdot, s)) \, ds
+ \int_{\tau/2}^t e^{d_1(t-s)(\Delta-1)} [(\mu_1 + d_1)u(\cdot, s) - \mu_1 u(\cdot, s)^2 - \mu_1 u(\cdot, s)v(\cdot, s)] \, ds
=: J_1 + J_2 + J_3, \quad t \in (\tau, T_{\max}). \]

Let \( \eta \in \left( \frac{n}{2p}, \frac{1}{2} \right) \) and \( \varepsilon \in (0, \frac{1}{2} - \eta) \). Then we see that \( 0 < 2\eta - \frac{n}{p} \) and \( \eta + \varepsilon + \frac{1}{2} < 1 \). By (7.2.5), (7.2.6) and Lemma 7.2.3 we observe that for all \( t \in (\tau, T_{\max}) \),

\[
\|J_1\|_{L^\infty(\Omega)} = \left\| e^{d_1(t-\tau/2)(\Delta-1)u(\cdot, \frac{\tau}{2})} \right\|_{L^\infty(\Omega)}
\leq c_1 \left\| (-\Delta + 1)^{\eta} e^{d_1(t-\tau/2)(\Delta-1)u(\cdot, \frac{\tau}{2})} \right\|_{L^p(\Omega)}
\leq c_1 c_2 \left( t - \frac{\tau}{2} \right)^{-\eta} e^{-\lambda t} \left\| u(\cdot, \frac{\tau}{2}) \right\|_{L^p(\Omega)}
\leq 2^n c_1 c_2 \tau^{-\eta} e^{-\eta \tau} \sup_{0 \leq t < T_{\max}} \|u(\cdot, t)\|_{L^p(\Omega)}. \]

74
Now, in light of Lemma 2.2.1, for all \( p \in (1, \infty) \) we can find \( \lambda > 0 \) such that for every \( \varepsilon > 0 \) we can find \( c_5 > 0 \) satisfying
\[
(7.2.9) \quad \|(-\Delta + k)\eta e^{\varepsilon \Delta} \nabla \cdot \omega\|_{L^p(\Omega)} \leq c_5 t^{-\eta - \varepsilon - \frac{1}{2}} e^{-\lambda t} \|\omega\|_{L^p(\Omega)} \quad (t > 0)
\]
for all \( \mathbb{R}^n \)-valued \( \omega \in L^p(\Omega) \). Using (7.2.5), (7.2.8) and (7.2.9), we obtain
\[
\|J_2\|_{L^\infty(\Omega)} \leq \int_0^t \|e^{d_1(t-s)\Delta-1} \nabla \cdot (u(\cdot, s) \chi_1(w(\cdot, s)) \nabla w(\cdot, s))\|_{L^\infty(\Omega)} \, ds
\]
\[
\leq c_1 \int_0^t \|(-\Delta + 1)\eta e^{d_1(t-s)\Delta-1} \nabla \cdot (u(\cdot, s) \chi_1(w(\cdot, s)) \nabla w(\cdot, s))\|_{L^p(\Omega)} \, ds
\]
\[
\leq c_1 c_5 \int_0^t (t-s)^{-\eta - \varepsilon - 1/2} e^{-(\nu + d_1)(t-s)} \|u(\cdot, s) \chi_1(w(\cdot, s)) \nabla w(\cdot, s)\|_{L^p(\Omega)} \, ds
\]
\[
\leq c_1 c_4 c_5 \int_0^t r^{-(\eta + \varepsilon + 1/2)} e^{-(\nu + d_1)r} \, dr.
\]
Since the Neumann heat semigroup \((e^{t\Delta})_{t \geq 0}\) has the order preserving property, we infer
\[
J_3 \leq \int_0^t e^{d_1(t-s)\Delta-1} \left[ -\mu_1 \left( u(\cdot, s) - \frac{\mu_1 + d_1}{2\mu_1} \right)^2 + \frac{(\mu_1 + d_1)^2}{4\mu_1} \right] \, ds
\]
\[
\leq \frac{(\mu_1 + d_1)^2}{4\mu_1} \int_0^t e^{d_1(t-s)\Delta} e^{-d_1(t-s)} \, ds,
\]
and moreover, by the maximum principle we have
\[
J_3 \leq \frac{(\mu_1 + d_1)^2}{4\mu_1} \int_0^t e^{-d_1(t-s)} \, ds
\]
\[
\leq \frac{(\mu_1 + d_1)^2}{4d_1\mu_1} (1 - e^{-d_1T}).
\]
Therefore we obtain that there exists \( C_u(\tau) > 0 \) such that
\[
u(\cdot, t) \leq \|J_1\|_{L^\infty(\Omega)} + \|J_2\|_{L^\infty(\Omega)} + J_3
\]
\[
\leq C_u(\tau), \quad t \in (\tau, T_{\max}).
\]
The positivity of \( u \) yields that
\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C_u(\tau) \quad \text{for all } t \in (\tau, T_{\max}).
\]
The same argument as for \( u \) gives the \( L^\infty(\Omega) \) bound for \( v \). Finally, the Hölder continuity of the solution \((u, v, w)\) comes from standard parabolic regularity theory ([98]). This completes the proof of Theorem 7.1.1. 
\[\square\]
7.3. Asymptotic behavior. Case 1: \( a_1, a_2 \in (0, 1) \)

In this section we will establish asymptotic stability of solutions to (7.1.1) in the case \( a_1, a_2 \in (0, 1) \). For the proof of Theorem 7.1.2, we shall recall some elementary results.

**Lemma 7.3.1** (see [4, Lemma 3.1]). Suppose that \( f : (1, \infty) \to \mathbb{R} \) is a uniformly continuous nonnegative function satisfying \( \int_1^{\infty} f(t) \, dt < \infty \). Then \( f(t) \to 0 \) as \( t \to \infty \).

**Lemma 7.3.2** (see Lemma 6.3.2). Let \( a, b, c, d, e, f \in \mathbb{R} \). Suppose that

\[
\begin{align*}
a &> 0, \quad d - \frac{b^2}{4a} > 0, \quad f - \frac{c^2}{4a} - \frac{(2ae - bc)^2}{4a(4ad - b^2)} > 0.
\end{align*}
\]

Then

\[
a x^2 + bxy + cxz + dy^2 + eyz + fz^2 \geq 0
\]

holds for all \( x, y, z \in \mathbb{R} \).

Now we will prove the key estimate for the proof of Theorem 7.1.2.

**Lemma 7.3.3.** Let \( a_1, a_2 \in (0, 1) \) and \( (u, v, w) \) a solution to (7.1.1). Under the conditions (7.1.4)–(7.1.10) and (7.1.12)–(7.1.14), there exist \( \delta_1, \delta_2 > 0 \) and \( \varepsilon > 0 \) such that the nonnegative functions \( E_1 \) and \( F_1 \) defined by

\[
E_1(t) := \int_{\Omega} \left( u - u^* - u^* \log \frac{u}{u^*} \right) + \delta_1 \frac{a_1 \mu_1}{a_2 \mu_2} \int_{\Omega} \left( v - v^* - v^* \log \frac{v}{v^*} \right) + \frac{\delta_2}{2} \int_{\Omega} (w - w^*)^2
\]

and

\[
F_1(t) := \int_{\Omega} (u - u^*)^2 + \int_{\Omega} (v - v^*)^2 + \int_{\Omega} (w - w^*)^2 + \int_{\Omega} |\nabla w|^2
\]

satisfy

\[
\frac{d}{dt} E_1(t) \leq -\varepsilon F_1(t) \quad (t > 0).
\]

**Proof.** Thanks to (7.1.12)–(7.1.14) we can choose \( \delta_1 > 0 \) defined in (7.1.12)–(7.1.14) and \( \delta_2 > 0 \) satisfying

\[
\frac{\chi_1(0)^2 u^*(1 + \delta_1)}{4d_1 d_3} < \delta_1 < \frac{a_1 \mu_1 \gamma (4d_1 - a_1 a_2 (1 + \delta_1)^2)}{\alpha_2^2 a_1 \beta_1 + \beta_2^2 a_2 - \alpha_1 \beta_1 a_1 a_2 (1 + \delta_1)}
\]

and

\[
\frac{a_1 \mu_1 \chi_2(0)^2 v^*(1 + \delta_1)}{4a_2 \mu_2 d_2 d_3} < \delta_2 < \frac{a_1 \mu_1 \gamma (4d_1 - a_1 a_2 (1 + \delta_1)^2)}{\alpha_2^2 a_1 \beta_1 + \beta_2^2 a_2 - \alpha_1 \beta_1 a_1 a_2 (1 + \delta_1)}
\]

76
We denote by \( A_1(t), B_1(t), C_1(t) \) the functions defined as

\[
A_1(t) := \int_\Omega (u - u^* - u^* \log \frac{u}{u^*}) , \quad B_1(t) = \int_\Omega (v - v^* - v^* \log \frac{v}{v^*}) , \\
C_1(t) := \frac{1}{2} \int_\Omega (w - w^*)^2 ,
\]

and we write as

\[
E_1(t) = A_1(t) + \delta_1 \frac{\alpha_1 \mu_1}{a_2 \mu_2} B_1(t) + \delta_2 C_1(t).
\]

The Taylor formula applied to \( H(s) = s - u^* \log s \) \((s \geq 0)\) yields \( A_1 = \int_\Omega (H(u) - H(u^*)) \) is a nonnegative function for \( t > 0 \) (more detail, see [4, Lemma 3.2]). Similarly, we have that \( B_1 \) is a positive function. By the straightforward calculations we infer

\[
\frac{d}{dt} A_1(t) = -\mu_1 \int_\Omega (u - u^*)^2 - a_1 \mu_1 \int_\Omega (u - u^*)(v - v^*) - d_1 u^* \int_\Omega \frac{|\nabla u|^2}{u^2} \\
\quad + u^* \int_\Omega \frac{\chi_1(w)}{u} \nabla u \cdot \nabla w ,
\]

\[
\frac{d}{dt} B_1(t) = -\mu_2 \int_\Omega (v - v^*)^2 - a_2 \mu_2 \int_\Omega (u - u^*)(v - v^*) - d_2 v^* \int_\Omega \frac{|\nabla v|^2}{v^2} \\
\quad + v^* \int_\Omega \frac{\chi_2(w)}{v} \nabla v \cdot \nabla w ,
\]

\[
\frac{d}{dt} C_1(t) = \int_\Omega h_u(u - u^*)(w - w^*) + \int_\Omega h_v(v - v^*)(w - w^*) + \int_\Omega h_w(w - w^*)^2 \\
\quad - d_3 \int_\Omega |\nabla w|^2
\]

with some derivatives \( h_u, h_v, \) and \( h_w \). Hence we have

\[
(7.3.4) \quad \frac{d}{dt} E_1(t) = I_3(t) + I_4(t) ,
\]

where

\[
I_3(t) := -\mu_1 \int_\Omega (u - u^*)^2 - a_1 \mu_1 \int_\Omega (u - u^*)(v - v^*) - \delta_1 \frac{\alpha_1 \mu_1}{a_2 \mu_2} \int_\Omega (v - v^*)^2 \\
\quad + \delta_2 \int_\Omega h_u(u - u^*)(w - w^*) + \delta_2 \int_\Omega h_v(v - v^*)(w - w^*) \\
\quad + \delta_2 \int_\Omega h_w(w - w^*)^2
\]

and

\[
(7.3.5) \quad I_4(t) := -d_1 u^* \int_\Omega \frac{|\nabla u|^2}{u^2} + u^* \int_\Omega \frac{\chi_1(w)}{u} \nabla u \cdot \nabla w - d_2 v^* \delta_1 \frac{a_1 \mu_1}{a_2 \mu_2} \int_\Omega \frac{|\nabla v|^2}{v^2} \\
\quad + v^* \delta_1 \frac{a_1 \mu_1}{a_2 \mu_2} \int_\Omega \frac{\chi_2(w)}{v} \nabla v \cdot \nabla w - d_3 \delta_2 \int_\Omega |\nabla w|^2.
\]

77
At first, we shall show from Lemma 7.3.2 that there exists $\varepsilon_1 > 0$ such that

$$ I_3(t) \leq -\varepsilon_1 \left( \int_{\Omega} (u - u^*)^2 + \int_{\Omega} (v - v^*)^2 + \int_{\Omega} (w - w^*)^2 \right). $$

(7.3.6)

To see this, we put

$$ g_1(\varepsilon) := \mu_1 - \varepsilon, \quad g_2(\varepsilon) := \left( \frac{a_1}{a_2} \mu_1 \delta_1 - \varepsilon \right) - \frac{a_1^2 \mu_1^2 (1 + \delta_1)^2}{4(\mu_1 - \varepsilon)}, $$

$$ g_3(\varepsilon) := (-\delta_2 h_w - \varepsilon) - \frac{h_w^2}{4(\mu_1 - \varepsilon)} \delta_2^2 $$

$$ - \frac{(2h_w(\mu_1 - \varepsilon) - h_w a_1 \mu_1 (1 + \delta))^2}{4(\mu_1 - \varepsilon)(4a_1 \mu_1 \delta_1 (\mu_1 - \varepsilon) - a_2^2 \mu_1^2 (1 + \delta_1)^2)} \delta_2^2. $$

Since $\mu_1 > 0$, we have $g_1(0) = \mu_1 > 0$. Due to (7.1.12), we infer

$$ g_2(0) = \frac{a_1 \mu_1}{4a_2} (4\delta_1 - a_1 a_2 (1 + \delta_1)^2) > 0. $$

In light of (7.1.6) and the definitions of $\delta_2 > 0$, $\alpha_i, \beta_i \geq 0$ (defined in (7.1.11)) we obtain

$$ g_3(0) = \delta_2 \left( -h_w - \left( \frac{h_w^2}{4\mu_1} + \frac{a_2 (2h_w - h_w a_1 (1 + \delta_1)^2)}{4a_1 \mu_1 (4\delta_1 - a_1 a_2 (1 + \delta_1)^2)} \right) \delta_2 \right) $$

$$ \geq \delta_2 \left( \gamma - \left( \frac{\alpha_2^2 a_1 \delta_1 + \beta_2^2 a_2 - \alpha_1 \beta_1 a_1 a_2 (1 + \delta_1)}{a_1 \mu_1 (4\delta_1 - a_1 a_2 (1 + \delta_1)^2)} \right) \delta_2 \right) > 0. $$

Combination of the above inequalities and the continuity argument yields that there exists $\varepsilon_1 > 0$ such that $g_i(\varepsilon_1) > 0$ hold for $i = 1, 2, 3$. Thanks to Lemma 7.3.2 with

$$ a = \mu_1 - \varepsilon_1, \quad b = a_1 \mu_1 (1 + \delta_1), \quad c = -\delta_2 h_w, $$

$$ d = \delta_1 \frac{a_1 \mu_1}{a_2} - \varepsilon_1, \quad e = -\delta_2 h_w, \quad f = -\delta_2 h_w - \varepsilon_1, $$

$$ x = u(\cdot, t) - u^*, \quad y = v(\cdot, t) - v^*, \quad z = w(\cdot, t) - w^*, $$

we obtain (7.3.6) with $\varepsilon_1 > 0$. Lastly we will find $\varepsilon_2 > 0$ satisfying

$$ I_4(t) \leq -\varepsilon_2 \int_{\Omega} |\nabla w|^2. $$

(7.3.7)

By virtue of the definition of $\delta_2 > 0$, we can find $\delta_3 \in \left( \frac{\chi_i(0)^2 a^*(1 + \delta_1)}{4d_3 \delta_2}, 1 \right)$. Noting that $\chi'_i < 0$ (from (7.1.9)) and then using the Young inequality, we have

$$ u^* \int_{\Omega} \frac{\chi_1(w u)}{u} \nabla u \cdot \nabla w \leq \frac{\chi_1(0)^2 a^2 (1 + \delta_1)}{4d_3 \delta_2 \delta_3} \int_{\Omega} |\nabla u|^2 + \frac{d_3 \delta_2 \delta_3}{1 + \delta_1} \int_{\Omega} |\nabla w|^2; $$

$$ v^* \delta_1 \frac{a_1 \mu_1}{a_2 \mu_2} \int_{\Omega} \frac{\chi_2(w v)}{v} \nabla v \cdot \nabla w \leq \frac{\chi_2(0)^2 a^2 (1 + \delta_1)}{4d_3 \delta_2 \delta_3} \left( \frac{a_1 \mu_1}{a_2 \mu_2} \right)^2 \int_{\Omega} |\nabla v|^2 + \frac{d_3 \delta_1 \delta_2}{1 + \delta_1} \int_{\Omega} |\nabla w|^2. $$
Then there exist classical solution to (7.1.1) from the positivity of $E(0)$. Lemma 7.3.5. In order to complete the proof of Theorem 7.1.2, we will prepare the following lemma.

Proof. We let $u; v; w$ be any solution of (7.1.4)–(7.1.10) and (7.3.5) we infer $u; v; w$ to be nonnegative function, and thanks to the regularity of $f$, $v$, and $w$ we let $u; v; w$ is uniformly continuous. Moreover, integrating (7.3.3) over (1, $\infty$), we infer from the positivity of $E_1$ that

$$\int_1^\infty f_1(t) \, dt \leq \frac{1}{c} E_1(t) < \infty.$$ 

Therefore we obtain from Lemma 7.3.1 that $f_1(t) \to 0$. Lemma 7.3.4. Let $a_1, a_2 \in (0, 1)$ and let $(u, v, w)$ be a solution to (7.1.1). Under the conditions (7.1.4)–(7.1.10) and (7.1.12)–(7.1.14), $(u, v, w)$ has the following asymptotic behavior:

$$\|u(\cdot, t) - u^*\|_{L^\infty(\Omega)} \to 0, \quad \|v(\cdot, t) - v^*\|_{L^\infty(\Omega)} \to 0, \quad \|w(\cdot, t) - w^*\|_{L^\infty(\Omega)} \to 0 \quad (t \to \infty).$$

Proof. We let $f_1(t) := \int_\Omega (u - u^*)^2 + \int_\Omega (v - v^*)^2 + \int_\Omega (w - w^*)^2 \geq 0$. We have $f_1$ is a nonnegative function, and thanks to the regularity of $u, v, w$ (see Theorem 7.1.1) we can see that $f_1$ is uniformly continuous. Moreover, integrating (7.3.3) over (1, $\infty$), we infer from the positivity of $E_1$ that

$$\int_1^\infty f_1(t) \, dt \leq \frac{1}{c} E_1(t) < \infty.$$ 

Therefore we obtain from Lemma 7.3.1 that $f_1(t) \to 0$. In order to complete the proof of Theorem 7.1.2, we will prepare the following lemma.

Lemma 7.3.5. Let $(\pi, \bar{v}, \bar{w}) \in \mathbb{R}^3$ be any solution of (7.1.1) and $(u, v, w)$ a global bounded classical solution to (7.1.1). Suppose that there exist two decreasing functions $h_1, h_2$ on (0, $\infty$) and $t_0 > 0$ such that

$$\int_\Omega (u - \bar{u})^2 + \int_\Omega (v - \bar{v})^2 + \int_\Omega (w - \bar{w})^2 \leq h_1(t),$$

$$\left(\int_{t_0}^t \int_\Omega |\nabla w|^2 \right)^{1/2} \leq h_2(t) \quad \text{for all } t > t_0.$$ 

Then there exist $C > 0$ and $t_1 > 0$ such that

$$\|u(\cdot, t) - \bar{u}\|_{L^\infty(\Omega)} + \|v(\cdot, t) - \bar{v}\|_{L^\infty(\Omega)} + \|w(\cdot, t) - \bar{w}\|_{L^\infty(\Omega)} \leq C([h_1(t) - 1]^{1/2} + h_2(t))$$ 

for all $t > t_1$. 

79
Proof. The arguments in [4, Lemma 3.6] and Theorem 7.1.1 lead to the proof of this lemma.

**Proof of Theorem 7.1.2.** From the L'Hôpital theorem applied to $H_1(s) := s - u^* \log s$ we can see

$$(7.3.8) \quad \lim_{s \to u^*} \frac{H_1(s) - H_1(u^*)}{(s - u^*)^2} = \lim_{s \to u^*} \frac{H_1''(s)}{2} = \frac{1}{2u^*}.$$ 

In view of a combination of (7.3.8) and $\|u - u^*\|_{L^\infty(\Omega)} \to 0$ from Lemma 7.3.4 we obtain that there exists $t_0 > 0$ such that

$$(7.3.9) \quad \frac{1}{4u^*} \int_{\Omega} (u - u^*)^2 \leq A_1(t) = \int_{\Omega} (H(u) - H(u^*)) \leq \frac{1}{u^*} \int_{\Omega} (u - u^*)^2 \quad (t > t_0).$$

A similar argument yields that there exists $t_1 > t_0$ such that

$$(7.3.10) \quad \frac{1}{4v^*} \int_{\Omega} (v - v^*)^2 \leq B_1(t) \leq \frac{1}{v^*} \int_{\Omega} (v - v^*)^2 \quad (t > t_1).$$

We infer from (7.3.9) and the definitions of $E_1, F_1$ that $E_1(t) \leq c_6 F_1(t)$ for all $t > t_1$ with some $c_6 > 0$. Plugging this into (7.3.3), we have

$$\frac{d}{dt} E_1(t) \leq -\varepsilon F_1(t) \leq -\frac{\varepsilon}{c_6} E_1(t) \quad (t > t_1),$$

which implies that there exist $c_7 > 0$ and $\ell > 0$ such that

$$E_1(t) \leq c_7 e^{-\alpha t} \quad (t > t_1).$$

Thus we obtain from (7.3.9) and (7.3.10) that

$$\int_{\Omega} (u - u^*)^2 + \int_{\Omega} (v - v^*)^2 + \int_{\Omega} (w - w^*)^2 \leq c_8 E_1(t) \leq c_7 c_8 e^{-\alpha t}$$

for all $t > t_1$ with some $c_8 > 0$. Moreover, there exists $c_9 > 0$ such that

$$\int_{t_{l-1}}^{t} \int_{\Omega} |\nabla w|^2 \leq \int_{t_{l-1}}^{t} F(t) \leq -\frac{1}{\varepsilon} \int_{t_{l-1}}^{t} \frac{d}{dt} E_1(t) \leq -\frac{1}{\varepsilon} E(t - 1) \leq c_9 e^{-\alpha t}.$$

Thanks to Lemma 7.3.5, we achieve that there exist $C > 0$ and $\lambda > 0$ such that

$$\|u(\cdot, t) - u^*\|_{L^\infty(\Omega)} + \|v(\cdot, t) - v^*\|_{L^\infty(\Omega)} + \|w(\cdot, t) - w^*\|_{L^\infty(\Omega)} \leq C e^{-\lambda t} \quad (t > 0).$$

This completes the proof of Theorem 7.1.2. \qed
7.4. Asymptotic behavior. Case 2: \( a_1 \geq 1 > a_2 > 0 \)

In this section we will prove asymptotic stability in the case that \( a_1 \geq 1 > a_2 > 0 \).

**Lemma 7.4.1.** Let \( a_1 \geq 1 > a_2 > 0 \) and let \((u, v, w)\) be a solution to (7.1.1). Under the conditions (7.1.4)–(7.1.10) and (7.1.15)–(7.1.16), there exist \( \delta_1, \delta_2 > 0, \varepsilon > 0 \) and \( \alpha_1' \geq 1 \) such that the nonnegative functions \( E_2 \) and \( F_2 \) defined by

\[
E_2(t) := \int_{\Omega} u + \delta_1 \frac{\alpha_1' \mu_1}{a_2 \mu_2} \int_{\Omega} (v - 1 - \log v) + \frac{\delta_2}{2} \int_{\Omega} (w - \tilde{w})^2
\]

and

\[
F_2(t) := \int_{\Omega} u^2 + \int_{\Omega} (v - 1)^2 + \int_{\Omega} (w - \tilde{w})^2 + \int_{\Omega} |\nabla w|^2
\]

satisfy

(7.4.1) \[
\frac{d}{dt} E_2(t) \leq -\varepsilon F_2(t) - \mu_1 (\alpha_1' - 1) \int_{\Omega} u \quad (t > 0).
\]

**Proof.** Thanks to (7.1.15)–(7.1.16) we can take \( \delta_1 > 0, \alpha_1' \geq 1 \) defined in (7.1.15)–(7.1.16) and choose \( \delta_2 > 0 \) such that

\[
\frac{\alpha_1' \mu_1 \chi_2(0) \delta_1}{4a_2 \mu_2 d_2 \delta_3} < \delta_2 < \frac{\alpha_1' \mu_1 \gamma (4 \delta_1 - \alpha_1' a_2 (1 + \delta_1)^2)}{\alpha_1' a_1^2 \delta_1 + \beta_2^2 a_2 - \alpha_1 \beta_1 a_1' a_2 (1 + \delta_1)}.
\]

We denote by \( A_2(t), B_2(t), C_2(t) \) the nonnegative functions defined as

\[
A_2(t) := \int_{\Omega} u, \quad B_2(t) = \int_{\Omega} (v - 1 - \log v),
\]

\[
C_2(t) := \frac{1}{2} \int_{\Omega} (w - \tilde{w})^2
\]

and we write as

\[
E_2(t) = A_2(t) + \delta_1 \frac{\alpha_1' \mu_1}{a_2 \mu_2} B_2(t) + \delta_2 C_2(t).
\]

Then by the straightforward calculations we infer

\[
\frac{d}{dt} A_2(t) \leq -\mu_1 \int_{\Omega} u^2 - \alpha_1' \mu_1 \int_{\Omega} u(v - 1) - \mu_1 (\alpha_1' - 1) \int_{\Omega} u,
\]

\[
\frac{d}{dt} B_2(t) = -\mu_2 \int_{\Omega} (v - 1)^2 - a_2 \mu_2 \int_{\Omega} u(v - 1) - d_2 \int_{\Omega} |\nabla v|^2
\]

\[
+ \int_{\Omega} \frac{\chi_2(w)}{v} \nabla v \cdot \nabla w,
\]

\[
\frac{d}{dt} C_2(t) = \int_{\Omega} h_u u (w - \tilde{w}) + \int_{\Omega} h_v (v - 1) (w - \tilde{w}) + \int_{\Omega} h_w (w - \tilde{w})^2
\]

\[
- d_3 \int_{\Omega} |\nabla w|^2
\]
with some derivatives $h_u, h_v$ and $h_w$. Hence we have

$$\frac{d}{dt} E_2(t) \leq I_5(t) + I_6(t) - \mu_1(a'_1 - 1) \int u,$$

where

$$I_5(t) := -\mu_1 \int u^2 - a'_1 \mu_1 (1 + \delta_1) \int u(v - 1) - \delta_1 \frac{a'_1 \mu_1}{a_2} \int (v - 1)^2$$
$$+ \delta_2 \int h_u u(w - \bar{w}) + \delta_2 \int h_v (v - 1)(w - \bar{w}) + \delta_2 \int h_w (w - \bar{w})^2$$

and

$$I_6(t) := -d_3 \delta_1 \frac{a'_1 \mu_1}{a_2 \mu_2} \int \frac{|v|^2}{v} + \delta_1 \frac{a'_1 \mu_1}{a_2 \mu_2} \int \frac{\chi_2(w)}{v} \nabla v \cdot \nabla w - d_3 \delta_2 \int |\nabla w|^2.$$

From the same argument as in the proof of Lemma 7.3.3 we obtain that there exists $\varepsilon_1 > 0$ such that

$$I_5(t) \leq -\varepsilon_1 \left( \int u^2 + \int (v - 1)^2 + \int (w - \bar{w})^2 \right).$$

On the other hand, thanks to $\chi'_2 < 0$ and the Young inequality, we infer that

$$\delta_1 \frac{a'_1 \mu_1}{a_2 \mu_2} \int \frac{\chi_2(w)}{v} \nabla v \cdot \nabla w \leq \delta_1 \chi_2(0) \frac{a'_1 \mu_1}{a_2 \mu_2} \int \frac{|\nabla v \cdot \nabla w|}{v}$$
$$\leq d_3 \delta_1 \frac{a'_1 \mu_1}{a_2 \mu_2} \int \frac{|v|^2}{v} + \frac{a'_1 \mu_1 \delta_1 \chi_2(0)^2}{4 a_2 \mu_2 \delta_2} \int |\nabla w|^2.$$

Plugging this into (7.4.3), we have

$$I_6(t) \leq - \left( d_3 \delta_2 - \frac{a'_1 \mu_1 \chi_2(0)^2 \delta_1}{4 a_2 \mu_2 \delta_2} \right) \int |\nabla w|^2.$$

Noting by the definition of $\delta_2 > 0$ that

$$d_3 \delta_2 - \frac{a'_1 \mu_1 \chi_2(0)^2 \delta_1}{4 a_2 \mu_2 \delta_2} > 0,$$

we obtain that there exists $\varepsilon_2 > 0$ such that

$$I_6(t) \leq -\varepsilon_2 \int |\nabla w|^2.$$

Combination of (7.4.2), (7.4.4) and (7.4.5) implies the end of the proof.

**Lemma 7.4.2.** Let $a_1 \geq 1 > a_2 > 0$ and let $(u, v, w)$ be a solution to (7.1.1). Under the conditions (7.1.4)–(7.1.10) and (7.1.15)–(7.1.16), $(u, v, w)$ has the following asymptotic behavior:

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \to 0, \|v(\cdot, t) - 1\|_{L^\infty(\Omega)} \to 0, \|w(\cdot, t) - \bar{w}\|_{L^\infty(\Omega)} \to 0 \quad (t \to \infty).$$
Proof. From the same argument as in Lemma 7.3.4 we can obtain (7.4.6).

Lemma 7.4.3. Let $a_1 > 1$, $a_2 \in (0,1)$ and let $(u,v,w)$ be a solution to (7.1.1). Under the conditions (7.1.4)–(7.1.10) and (7.1.15)–(7.1.16), there exist $C > 0$ and $\lambda > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t) - 1\|_{L^\infty(\Omega)} + \|w(\cdot,t) - \tilde{w}\|_{L^\infty(\Omega)} \leq Ce^{-\lambda t} \quad (t > 0).$$

Proof. Combination of $u \to 0$ in $L^\infty(\Omega)$ as $t \to \infty$ (from Lemma 7.3.4) and

$$\lim_{s \to 0} \frac{s}{s^2 + (a'_1 - 1)s} = \frac{1}{a'_1 - 1}$$

implies that there exists $t_0 > 0$ such that

$$\frac{1}{2(a'_1 - 1)} \int_\Omega u^2 + \frac{1}{2} \int_\Omega u \leq A_2(t) \leq \frac{2}{a'_1 - 1} \int_\Omega u^2 + 2 \int_\Omega u \quad (t > t_0). \quad (7.4.7)$$

In light of a similar argument to seeing (7.3.9) we obtain that there exists $t_1 > t_0$ such that

$$\frac{1}{4} \int_\Omega (v - 1)^2 \leq B_2(t) \leq \int_\Omega (v - 1)^2 \quad (t > t_1). \quad (7.4.8)$$

The definitions of $E_2$, $F_2$ and (7.4.7), (7.4.8) yield that

$$E_2(t) \leq c_{10} \left( F_2(t) + (a'_1 - 1) \int_\Omega u \right) \quad (t > t_1)$$

with some $c_{10} > 0$. Plugging this into (7.4.1), we have

$$\frac{d}{dt} E_2(t) \leq -\varepsilon F_2(t) - \mu_1 (a'_1 - 1) \int_\Omega u \leq -\varepsilon \frac{c_{10}}{c_{11}} E_2(t) \quad (t > t_1),$$

which implies that there exist $c_{11}$ and $\ell > 0$ such that

$$E_2(t) \leq c_{11} e^{-\ell t} \quad (t > t_1).$$

Therefore from (7.4.7) and (7.4.8) we can find $c_{12} > 0$ satisfying

$$\int_\Omega u^2 + \int_\Omega (v - 1)^2 + \int_\Omega (w - \tilde{w})^2 \leq c_{12} E_2(t) \leq c_{11} c_{12} e^{-\ell t} \quad (t > t_1).$$

Moreover, there exists $c_{13} > 0$ such that

$$\int_{t-1}^t \int_\Omega |\nabla w|^2 \leq \int_{t-1}^t F(t) \leq -\frac{1}{\varepsilon} \int_{t-1}^t \frac{d}{dt} E_1(t) \leq \frac{1}{\varepsilon} E(t - 1) \leq c_{13} e^{-\ell t}.$$

Thanks to Lemma 7.3.5, we achieve that there exist $C > 0$ and $\lambda > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t) - 1\|_{L^\infty(\Omega)} + \|w(\cdot,t) - \tilde{w}\|_{L^\infty(\Omega)} \leq Ce^{-\lambda t} \quad (t > 0),$$

which implies the end of the proof. 

83
Lemma 7.4.4. Let $a_1 = 1$, $a_2 \in (0, 1)$ and let $(u, v, w)$ be a solution to (7.1.1). Under the conditions (7.1.4)–(7.1.10) and (7.1.15)–(7.1.16), there exist $C > 0$ and $\lambda > 0$ such that

$$
\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t) - 1\|_{L^\infty(\Omega)} + \|w(\cdot, t) - \bar{w}\|_{L^\infty(\Omega)} \leq C(t + 1)^{-\lambda} \quad (t > 0).
$$

Proof. We have already known that there exists $t_0 > 0$ such that (7.4.8) holds for all $t > t_0$. Hence the Cauchy–Schwarz inequality and the boundedness of $(u, v, w)$ imply that there exists $c_{14} > 0$ satisfying

$$
E_2(t) \leq \int_\Omega u + \int_\Omega (v - 1)^2 + \int_\Omega (w - \bar{w})^2
$$

$$
\leq c_{14} \left( \int_\Omega u^2 \right)^{\frac{1}{2}} + c_{14} \left( \int_\Omega (v - 1)^2 \right)^{\frac{1}{2}} + c_{14} \left( \int_\Omega (w - \bar{w})^2 \right)^{\frac{1}{2}}
$$

$$
\leq \sqrt{3} c_{14} \left( \int_\Omega u^2 + \int_\Omega (v - 1)^2 + \int_\Omega (w - \bar{w})^2 \right)^{\frac{1}{2}} = c_{14} \sqrt{3} F_2(t)
$$

for all $t > t_0$. Thus from (7.4.1) we can find $c_{15} > 0$ such that

$$
\frac{d}{dt} E_2(t) \leq -c_{15} E_2(t)^2 \quad (t > t_0),
$$

which implies that there exists $c_{16} > 0$ satisfying

$$
E_2(t) \leq \frac{c_{16}}{t + 1} \quad (t > t_0).
$$

Therefore we have

$$
\int_\Omega u^2 + \int_\Omega (v - 1)^2 + \int_\Omega (w - \bar{w})^2 \leq c_{16} E_2(t) \leq \frac{c_{16} c_{17}}{t + 1} \quad (t > t_0)
$$

with some $c_{17} > 0$. Moreover, (7.4.1) yields

$$
\int_{t-1}^t \int_\Omega |\nabla w|^2 \leq \int_{t-1}^t F_2(t) \leq -\frac{1}{\varepsilon} \int_{t-1}^t \frac{d}{dt} E_2(t) \leq \frac{1}{\varepsilon} E_2(t - 1) \leq \frac{c_{16}}{\varepsilon(t + 1)} \quad (t > t_0).
$$

From Lemma 7.3.5 we obtain that there exist $C > 0$ and $\lambda > 0$ such that

$$
\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t) - 1\|_{L^\infty(\Omega)} + \|w(\cdot, t) - \bar{w}\|_{L^\infty(\Omega)} \leq C(t + 1)^{-\lambda} \quad (t > 0),
$$

which means the end of the proof.

Proof of Theorem 7.1.4. Part (i) follows from Lemma 7.4.3, while (ii) is contained in Lemma 7.4.4.
Chapter 8

Improvement of conditions for asymptotic stability in a two-species chemotaxis-competition model with signal-dependent sensitivity

8.1. Problem and results

This chapter presents improvement of results in [4] and in Chapter 7. In this chapter we consider the two-species chemotaxis system with competitive kinetics

\[
\begin{align*}
    u_t &= d_1 \Delta u - \nabla \cdot (u \chi_1(w) \nabla w) + \mu_1 u (1 - u - a_1 v), \quad x \in \Omega, \; t > 0, \\
    v_t &= d_2 \Delta v - \nabla \cdot (v \chi_2(w) \nabla w) + \mu_2 v (1 - a_2 u - v), \quad x \in \Omega, \; t > 0, \\
    w_t &= d_3 \Delta w + \alpha u + \beta v - \gamma w, \quad x \in \Omega, \; t > 0, \\
    \nabla u \cdot \nu &= \nabla v \cdot \nu = \nabla w \cdot \nu = 0, \quad x \in \partial \Omega, \; t > 0, \\
    u(x,0) &= u_0(x), \; v(x,0) = v_0(x), \; w(x,0) = w_0(x) \quad x \in \Omega,
\end{align*}
\]

(8.1.1)

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) (\( n \geq 2 \)) with smooth boundary \( \partial \Omega \) and \( \nu \) is the outward normal vector to \( \partial \Omega \); \( d_1, d_2, d_3, \mu_1, \mu_2, a_1, a_2 \) and \( \alpha, \beta, \gamma \) are positive constants; \( \chi_1, \chi_2, u_0, v_0, w_0 \) are assumed to be nonnegative functions. The unknown functions \( u(x,t) \) and \( v(x,t) \) represent the population densities of two species and \( w(x,t) \) shows the concentration of the chemical substance at place \( x \) and time \( t \).

The problem (8.1.1), which is proposed by Tello–Winkler [178], is a problem on account of the influence of chemotaxis, diffusion, and the Lotka–Volterra competitive kinetics, i.e., with coupling coefficients \( a_1, a_2 > 0 \) in

\[
\begin{align*}
    u_t &= u (1 - u - a_1 v), \quad v_t = v (1 - a_2 u - v).
\end{align*}
\]

(8.1.2)

The mathematical difficulties of the problem (8.1.1) are to deal with the chemotaxis term \( \nabla \cdot (u \nabla w) \) and the competition term \( u (1 - u - a_1 v) \). To overcome these difficulties, firstly, the parabolic-parabolic-elliptic problem (i.e., \( w_t \) is replaced with 0 in (8.1.1)) was studied and some conditions for global existence and stabilization in (8.1.1) were established.
([164, 178] and Chapter 5). In the parabolic-parabolic-elliptic case global existence of classical solutions to (8.1.1) and their asymptotic behavior were obtained in the case that \(a_1, a_2 \in (0, 1)\) ([178] and Chapter 5) and the case that \(a_1 > 1 > a_2\) ([164]). These results which give global existence and stabilization in (8.1.1) were improved in some cases (Chapter 6).

On the other hand, in general, the fully parabolic problem (8.1.1) is a more difficult problem than the parabolic-parabolic-elliptic case; because we cannot use the relation

\[
\Delta w = \alpha u + \beta v - \gamma w.
\]

About this problem, global existence and boundedness were shown in the 2-dimensional case ([4]) and the \(n\)-dimensional case ([115]). Moreover, in the case that \(a_1, a_2 \in (0, 1)\), Bai–Winkler [4] obtained asymptotic stability in (8.1.1) under the conditions

\[
\begin{align*}
\mu_1 &> \frac{d_2 \chi_2^2 u^*}{4 a_1 (1-a_1 a_2) d_3 d_2 + (a_1^2 + a_2^2 - 2a_1a_2a_3)}, \\
\mu_2 &> \frac{\chi_2^2 v^* (a_1 \alpha_2^2 + a_2 \beta_2^2 - 2a_1a_2a_3)}{16 d_2 d_3 a_2 (1 - a_1 a_2)}.
\end{align*}
\]

In Chapter 7 the conditions (8.1.3) were improved; asymptotic behavior of solutions holds when there exists \(\delta_1 > 0\) satisfying \(4 \delta_1 - a_1 a_2(1+\delta_1)^2 > 0\),

\[
\begin{align*}
\mu_1 &> \frac{M_1^2 u^* (1 + \delta_1) (a_1^2 a_1 \delta_1 + \beta_2^2 a_2 - \alpha_1 \beta_1 a_1 a_2 (1 + \delta_1))}{4 a_1 d_1 d_3 \gamma (4 \delta_1 - a_1 a_2 (1 + \delta_1)^2)}, \\
\mu_2 &> \frac{M_2^2 v^* (1 + \delta_1) (a_1^2 a_1 \delta_1 + \beta_2 a_2 - \alpha_1 \beta_1 a_1 a_2 (1 + \delta_1))}{4 a_2 d_2 d_3 \gamma (4 \delta_1 - a_1 a_2 (1 + \delta_1)^2)},
\end{align*}
\]

where \(M_1, M_2 > 0\) are some constants satisfying \(\chi_1(s) \leq M_1, \chi_2(s) \leq M_2\) for all \(s \geq 0\). Here we note that the conditions (8.1.3) and (8.1.4)–(8.1.5) can be rewritten as

\[
\begin{align*}
\left( \frac{u^* \chi_1^2}{4 a_1 d_3 a_2 \mu_1}, \frac{v^* \chi_2^2}{4 a_2 d_3 a_2 \mu_2} \right) &\in \left\{ (s, t) \in \mathbb{R}^2 \mid s, t \geq 0, s + t < f(1) \right\},
\end{align*}
\]

and

\[
\begin{align*}
\left( \frac{u^* M_1^2}{4 a_1 d_3 a_2 \mu_1}, \frac{v^* M_2^2}{4 a_2 d_3 a_2 \mu_2} \right) &\in \bigcup_{q \in \mathbb{R}} \left\{ (s, t) \in \mathbb{R}^2 \mid 0 \leq s < \frac{f(q)}{1+q}, 0 \leq t < \frac{f(q)}{1+q} \right\} \\
&= \left\{ (s, t) \in \mathbb{R}^2 \mid 0 \leq s < \frac{f(q_0)}{1+q_0}, 0 \leq t < \frac{f(q_0)}{1+q_0} \right\},
\end{align*}
\]

respectively, where

\[
\begin{align*}
I &:= \{ q > 0 \mid 4 q - (1 + q)^2 a_1 a_2 > 0 \}, \\
f(q) &:= \frac{\gamma (4 q - (1 + q)^2 a_1 a_2)}{a_1 \alpha_2^2 q + a_2 \beta_2^2 - a_1 a_2 \alpha_3 (1 + q)}
\end{align*}
\]

and \(q_0 \in I\) is a maximizer of \(\frac{f(q)}{1+q}\). The regions derived from (8.1.6) and (8.1.7) are described in Figure 1.

These results in [4] and in Chapter 7 were also concerned with asymptotic stability in (8.1.1) in the case that \(a_1 \geq 1 > a_2\). More related works can be found in [150, 151, 213].
and Chapters 3, 4, 7; global existence and boundedness in (8.1.1) with general sensitivity functions can be found in [213] and Chapter 7; related works which treated the non-competition case are in [150, 151] and Chapters 3, 4.

In summary the conditions for asymptotic stability in (8.1.1) are known; however, there is a gap between the conditions (8.1.3) and (8.1.4)–(8.1.5) (for more details, see Figure 1). The purpose of this work is to improve the conditions assumed in [4] and in Theorem 7.1.2 for asymptotic behavior in the case that $a_1, a_2 \in (0, 1)$. In order to attain this purpose we shall assume throughout this chapter that there exists $M_1, M_2 > 0$ satisfying

\[(8.1.9) \quad \chi_1(s) \leq M_1, \quad \chi_2(s) \leq M_2 \text{ for all } s > 0,\]

\[(8.1.10) \quad \left(\frac{u^* M_1^2}{4d_1d_3a_1\mu_1}, \frac{v^* M_2^2}{4d_2d_3a_2\mu_2}\right) \in \bigcup_{q \in I} \{(s, t) \in \mathbb{R}^2 | s, t \geq 0, \quad s + qt < f(q)\},\]

where the interval $I$ and the function $f$ are defined as (8.1.8). The region derived from the condition (8.1.10) is described in Figure 2, and include the regions derived from (8.1.6) and (8.1.7).
Now the main results read as follows. We suppose that the initial data \( u_0, v_0, w_0 \) satisfy
\[
(8.1.11) \quad 0 \leq u_0 \in C(\overline{\Omega}) \setminus \{0\}, \quad 0 \leq v_0 \in C(\overline{\Omega}) \setminus \{0\}, \quad 0 \leq w_0 \in W^{1,q}(\Omega) \ (\exists q > n).
\]
The first theorem is concerned with asymptotic behavior in (8.1.1) in the case that \( a_1, a_2 \in (0, 1) \).

**Theorem 8.1.1.** Let \( d_1, d_2, d_3, \mu_1, \mu_2, \alpha, \beta, \gamma > 0, \ a_1, a_2 \in (0, 1) \) be constants, let \( \chi_1, \chi_2 \) be nonnegative functions and let \( \Omega \subset \mathbb{R}^n \ (n \geq 2) \) be a bounded domain with smooth boundary. Assume that there exists a unique global classical solution \((u, v, w)\) of (8.1.1) satisfying
\[
\|u\|_{C^{2+\theta,1+\frac{\theta}{2}}(\Omega \times [t, t+1])} + \|v\|_{C^{2+\theta,1+\frac{\theta}{2}}(\Omega \times [t, t+1])} + \|w\|_{C^{2+\theta,1+\frac{\theta}{2}}(\Omega \times [t, t+1])} \leq M \quad \text{for all } t \geq 1
\]
with some \( \theta \in (0, 1) \) and \( M > 0 \). Then under the conditions \((8.1.9) - (8.1.11)\), the solution \((u, v, w)\) satisfies that there exist \( C > 0 \) and \( \ell > 0 \) such that
\[
\|u(\cdot, t) - u^*\|_{L^\infty(\Omega)} + \|v(\cdot, t) - v^*\|_{L^\infty(\Omega)} + \|w(\cdot, t) - w^*\|_{L^\infty(\Omega)} \leq Ce^{-\ell t} \quad \text{for all } t > 0,
\]
where
\[
w^* := \frac{1 - a_1}{1 - a_1a_2}, \quad v^* := \frac{1 - a_2}{1 - a_1a_2}, \quad w^* := \frac{\alpha u^* + \beta v^*}{\gamma}.
\]

**Remark 8.1.1.** The condition \((8.1.10)\) improves the conditions assumed in [4] and Chapter 7 (for more details, see Section 8.3). Moreover, from the careful calculations we have that
\[
\bigcup_{q \in I} \{ (s, t) \in \mathbb{R}^2 \mid s, t \geq 0, \ s + qt < f(q) \}
\]

\[
eq \left\{ (s, t) \in \mathbb{R}^2 \mid s \geq 0, \ 0 \leq t < \frac{a_2 \gamma}{\alpha(a_2 \beta - \alpha)} , \ h_1(s, t) > 0, \ h_2^+(s, t) > 0, \ h_2^-(s, t) < 0 \right\},
\]

where
\[
h_1(s, t) := a_1^2 \alpha^2 (\alpha - a_2 \beta)^2 s^2 + a_2^2 \beta^2 (\beta - a_1 \alpha)^2 t^2 - 2a_1a_2 \alpha \beta (\alpha - a_2 \beta)(\beta - a_1 \alpha)st
\]
\[
- 4\gamma(2a_1 \alpha^2 - 2a_1a_2 \alpha \beta + a_1a_2 \beta^2 - a_1^2a_2^2 \alpha^2)s
\]
\[
- 4\gamma(2a_2 \beta^2 \gamma - 2a_1a_2 \alpha \beta + a_1^2a_2^2 \alpha^2 - a_1^2a_2^2 \beta^2)t + 16\gamma^2(1 - a_1a_2)
\]

and
\[
h_2^\pm(s, t) := a_1 \alpha (\alpha - a_2 \beta)s + \left( \frac{\alpha(4 - 2a_1a_2 \pm 4\sqrt{1 - a_1a_2})(\alpha - a_2 \beta)}{a_2} + a_2 \beta (\beta - a_1 \alpha) \right)t
\]
\[
\pm 4\gamma\sqrt{1 - a_1a_2}.
\]

**Remark 8.1.2.** We note from \( q \in I \) that
\[
a_1 \alpha^2 q + a_2 \beta^2 - a_1a_2 \alpha \beta (1 + q) > 0
\]
holds. Indeed, the discriminant of \( a_1 q \alpha^2 - a_1a_2 \alpha \beta (1 + q) \alpha + a_2 \beta^2 \) is negative:
\[
D_{\alpha} := -a_1a_2 \beta^2 (4q - (1 + q)^2 a_1a_2) < 0.
\]
Then a combination of results concerned with global existence and boundedness in (8.1.1) ([4, 115] and Chapter 7) and Theorem 8.1.1 implies the following results.

**Theorem 8.1.2.** Let $d_1, d_2, d_3 > 0$, $\mu_1, \mu_2 > 0$, $a_1, a_2 \in (0, 1)$, $\alpha, \beta, \gamma > 0$ and let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded domain with smooth boundary. Assume that $\chi_1, \chi_2 > 0$ are constants and one of the following two properties is satisfied:

(i) $n = 2,$

(ii) $\Omega$ is a convex domain and $\mu_1 > \frac{nx_1}{4}$, $\mu_2 > \frac{nx_2}{4}$, $\mu_1 + \frac{a_1\mu_1}{2} + \frac{a_2\mu_2\chi_1}{2\chi_2} > \frac{nx_1}{2}$ and $\mu_2 + \frac{a_2\mu_2}{2} + \frac{a_2\mu_2\chi_2}{2\chi_2} > \frac{nx_2}{2}$ hold.

Then under the conditions (8.1.9)–(8.1.11), the same conclusion as in Theorem 8.1.1 holds.

**Theorem 8.1.3.** Let $d_1, d_2, d_3 > 0$, $\mu_1, \mu_2 > 0$, $a_1, a_2 \in (0, 1)$, $\alpha, \beta, \gamma > 0$ and let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded domain with smooth boundary. Assume that the functions $\chi_1, \chi_2$ satisfy the following conditions:

\[
\chi_i \in C^{1+\eta}([0, \infty)) \cap L^1(0, \infty),
\]

\[
2d_1d_3\chi_i'(w) + \left((d_3 - d_i)p + \sqrt{(d_3 - d_i)^2p^2 + 4d_1d_3p}\right)[\chi_i(w)]^2 \leq 0,
\]

\[
w\chi_i(w) \leq C_\chi \quad \text{for all } w \geq 0 \text{ and } i = 1, 2, 3
\]

with some $\eta \in (0, 1)$, $p > n$ and $C_\chi > 0$. Then under the conditions (8.1.9)–(8.1.11), the same conclusion as in Theorem 8.1.1 holds.

The strategy of the proof of Theorem 8.1.1 is to modify the methods in [4] and in Chapter 7. One of the keys for the proof of Theorem 8.1.1 is to derive the following energy estimate:

\[
\frac{d}{dt}E(t) \leq -\varepsilon \int_\Omega \left[(u(\cdot, t) - u^*)^2 + (v(\cdot, t) - v^*)^2 + (w(\cdot, t) - w^*)^2\right]
\]

for all $t > 0$ with some positive function $E$ and some constant $\varepsilon > 0$. Thanks to (8.1.12), we can obtain Theorem 8.1.1. The key for the improvement is to provide the best estimate for the terms

\[
\int_\Omega \frac{\chi_1(w)}{u} \nabla u \cdot \nabla w \quad \text{and} \quad \int_\Omega \frac{\chi_2(w)}{v} \nabla v \cdot \nabla w.
\]

This chapter is organized as follows. In Section 8.2 we prove asymptotic stability in the case that $a_1, a_2 \in (0, 1)$ (Theorems 8.1.1, 8.1.2 and 8.1.3). Section 8.3 is devoted to discussions; we confirm that the condition (8.1.10) improves the conditions assumed in [4] and Theorem 7.1.2.
8.2. Proof of Theorem 8.1.1

In this section we prove stabilization in (8.1.1) in the case that \( a_1, a_2 \in (0, 1) \). Here we assume that there exists a unique global classical solution \((u, v, w)\) of (8.1.1) satisfying

\[
\|u\|_{C^{2+\theta,1}((\Omega \times [t, t+1]))} + \|v\|_{C^{2+\theta,1}((\Omega \times [t, t+1]))} + \|w\|_{C^{2+\theta,1}((\Omega \times [t, t+1]))} \leq M
\]

for all \( t \geq 1 \) with some \( \theta \in (0, 1) \) and \( M > 0 \). We first provide the following lemma which will be used later.

Lemma 8.2.1. Let \( a, b, c, d, e, f \in \mathbb{R} \). Suppose that

\[
a > 0, \quad ad - \frac{b^2}{4} > 0, \quad adf + \frac{bce}{4} - \frac{cd^2}{4} - \frac{b^2f}{4} - \frac{ae^2}{4} > 0.
\]

Then there exists \( \varepsilon > 0 \) such that

\[
y_1^2 + by_1y_2 + cy_1y_3 + dy_2^2 + ey_2y_3 + fy_3^2 \geq \varepsilon (y_1^2 + y_2^2 + y_3^2)
\]

holds for all \( y_1, y_2, y_3 \in \mathbb{R} \).

Proof. In order to prove this lemma we shall see that there is \( \varepsilon > 0 \) such that

\[
(8.2.1) \quad (a - \varepsilon)y_1^2 + (d - \varepsilon)y_2^2 + (f - \varepsilon)y_3^2 \geq 0
\]

holds for all \( y_1, y_2, y_3 \in \mathbb{R} \), where

\[
X := \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad A_\varepsilon := \begin{pmatrix} a - \varepsilon & \frac{b}{2} & \frac{c}{2} \\ \frac{b}{2} & d - \varepsilon & \frac{e}{2} \\ \frac{c}{2} & \frac{e}{2} & f - \varepsilon \end{pmatrix}.
\]

To confirm that there is \( \varepsilon > 0 \) such that (8.2.2) holds for all \( y_1, y_2, y_3 \in \mathbb{R} \) we put

\[
g_1(\varepsilon) := a - \varepsilon, \quad g_2(\varepsilon) := \begin{vmatrix} a - \varepsilon & \frac{b}{2} \\ \frac{b}{2} & d - \varepsilon \end{vmatrix}, \quad g_3(\varepsilon) := |A_\varepsilon|
\]

and shall show the existence of \( \varepsilon_1 > 0 \) satisfying \( g_i(\varepsilon_1) > 0 \) for \( i = 1, 2, 3 \). Now thanks to (8.2.1), we can see that

\[
g_1(0) = a > 0, \quad g_2(0) = \begin{vmatrix} a & \frac{b}{2} \\ \frac{b}{2} & d \end{vmatrix} = ad - \frac{b^2}{4} > 0
\]

and

\[
g_3(0) = \begin{vmatrix} a & \frac{b}{2} & \frac{c}{2} \\ \frac{b}{2} & d & \frac{e}{2} \\ \frac{c}{2} & \frac{e}{2} & f \end{vmatrix} = adf + \frac{bce}{4} - \frac{cd^2}{4} - \frac{b^2f}{4} - \frac{ae^2}{4} > 0.
\]

Thus a combination of the above inequalities and the continuity argument yields that there is \( \varepsilon_1 > 0 \) such that \( g_i(\varepsilon_1) > 0 \) hold for \( i = 1, 2, 3 \). Therefore aided by the Sylvester criterion, we have (8.2.2) for all \( y_1, y_2, y_3 \in \mathbb{R} \), which means the end of the proof. \( \square \)
Then we will prove the following energy estimate which leads to asymptotic behavior of solutions to (8.1.1). The proof is mainly based on the methods in [4] and Chapter 7.

**Lemma 8.2.2.** Let \( a_1, a_2 \in (0, 1) \) and let \((u, v, w)\) be a solution to (8.1.1). Then under the conditions (8.1.9)–(8.1.11), there exist a nonnegative function \( E : (0, \infty) \to \mathbb{R} \) and a constant \( \varepsilon > 0 \) such that

\[
(8.2.3) \quad \frac{d}{dt} E(t) \leq -\varepsilon \left( \int_\Omega (u - u^*)^2 + \int_\Omega (v - v^*)^2 + \int_\Omega (w - w^*)^2 + \int_\Omega |\nabla w|^2 \right)
\]

holds for all \( t > 0 \).

**Proof.** In light of (8.1.10) we can take \( q > 0 \) such that

\[
4q - (1 + q)^2a_1a_2 > 0
\]

and

\[
\frac{u^*M_1^2}{4d_1d_3a_1\mu_1} + \frac{v^*qM_2^2}{4d_2d_3a_2\mu_2} < \frac{\gamma(4q - (1 + q)^2a_1a_2)}{a_1\alpha^2q + a_2\beta^2 - a_1a_2\alpha\beta(1 + q)}
\]

hold and \( \delta > 0 \) satisfying

\[
\frac{u^*a_2\mu_2 M_1^2}{4d_1} + \frac{v^*qa_1\mu_1 M_2^2}{4d_2} < \delta < \frac{\gamma a_1a_2\mu_2(4q - (1 + q)^2a_1a_2)}{a_1\alpha^2q + a_2\beta^2 - a_1a_2\alpha\beta(1 + q)}.
\]

For all \( t > 0 \) we denote by \( A(t), B(t), C(t) \) the functions defined as

\[
A(t) := \int_\Omega \left( u(\cdot, t) - u^* - u^* \log \frac{u(\cdot, t)}{u^*} \right),
\]

\[
B(t) := \int_\Omega \left( v(\cdot, t) - v^* - v^* \log \frac{v(\cdot, t)}{v^*} \right),
\]

\[
C(t) := \frac{1}{2} \int_\Omega (w(\cdot, t) - w^*)^2,
\]

and shall confirm that the function \( E \) defined as

\[
E(t) := a_2\mu_2 A(t) + qa_1\mu_1 B(t) + \delta C(t)
\]

satisfies (8.2.3) with some \( \varepsilon > 0 \). Firstly the Taylor formula enables us to see that \( E \) is a nonnegative function for \( t > 0 \) (for more details, see [4, Lemma 3.2]). From the straightforward calculations we infer

\[
\frac{d}{dt} A(t) = -\mu_1 \int_\Omega (u - u^*)^2 - a_1\mu_1 \int_\Omega (u - u^*)(v - v^*) - d_1u^* \int_\Omega \frac{|\nabla u|^2}{u^2}
\]

\[
+ u^* \int_\Omega \frac{\chi_1(u)}{u} \nabla u \cdot \nabla w,
\]

\[
\frac{d}{dt} B(t) = -\mu_2 \int_\Omega (v - v^*)^2 - a_2\mu_2 \int_\Omega (u - u^*)(v - v^*) - d_2v^* \int_\Omega \frac{|\nabla v|^2}{v^2}
\]

\[
+ v^* \int_\Omega \frac{\chi_2(v)}{v} \nabla v \cdot \nabla w
\]

91
and
\[ \frac{d}{dt} C(t) = \alpha \int_{\Omega} (u - u^*)(w - w^*) + \beta \int_{\Omega} (v - v^*)(w - w^*) - \gamma \int_{\Omega} (w - w^*)^2 - d_3 \int_{\Omega} |\nabla w|^2. \]

Hence we have
\[ \frac{d}{dt} E(t) = I_1(t) + I_2(t), \]
where
\begin{align*}
I_1(t) &:= -a_2 \mu_1 \mu_2 \int_{\Omega} (u - u^*)^2 - a_1 a_2 \mu_1 \mu_2 (1 + q) \int_{\Omega} (u - u^*)(v - v^*) \\
&\quad + \delta \alpha \int_{\Omega} (u - u^*)(w - w^*) - a_1 \mu_1 \mu_2 q \int_{\Omega} (v - v^*)^2 \\
&\quad + \delta \beta \int_{\Omega} (v - v^*)(w - w^*) - \delta \gamma \int_{\Omega} (w - w^*)^2 \\
\end{align*}
and
\begin{align*}
I_2(t) &:= -d_1 a_2 \mu_2 u^* \int_{\Omega} \frac{|\nabla u|^2}{u^2} + a_2 \mu_2 u^* \int_{\Omega} \frac{\chi_1(w)}{u} \nabla u \cdot \nabla w - d_2 a_1 \mu_1 v^* q \int_{\Omega} \frac{|\nabla v|^2}{v^2} \\
&\quad + a_1 \mu_1 v^* q \int_{\Omega} \frac{\chi_2(w)}{v} \nabla v \cdot \nabla w - d_3 \int_{\Omega} |\nabla w|^2.
\end{align*}

In order to confirm that there is \( \varepsilon_1 > 0 \) such that
\[ (8.2.4) \quad I_1(t) \leq -\varepsilon_1 \left( \int_{\Omega} (u(\cdot, t) - u^*)^2 + \int_{\Omega} (v(\cdot, t) - v^*)^2 + \int_{\Omega} (w(\cdot, t) - w^*)^2 \right) \]
for all \( t > 0 \) we will see that the assumption of Lemma 8.2.1 is satisfied with
\[ a = a_2 \mu_1 \mu_2, \quad b = a_1 a_2 \mu_1 \mu_2 (1 + q), \quad c = -\delta \alpha, \quad d = a_1 \mu_1 \mu_2 q, \quad e = -\delta \beta, \quad f = \delta \gamma \]
and
\[ y_1 = u(\cdot, t) - u^*, \quad y_2 = v(\cdot, t) - v^*, \quad y_3 = w(\cdot, t) - w^*. \]

From the definitions of \( q, \delta > 0 \) we can see that
\[ a = a_2 \mu_1 \mu_2 > 0, \quad ad - \frac{b^2}{4} = \frac{a_1 a_2 \mu_1^2 \mu_2^2}{4} (4q - (1 + q)^2 a_1 a_2) > 0 \]
and
\begin{align*}
adf + \frac{bce}{4} - \frac{cd^2}{4} - \frac{b^2 f}{4} - \frac{ae^2}{4}
= \frac{\mu_1 \mu_2 \delta}{4} \left( (a_1 a_2 \mu_1 \mu_2 q (4q - (1 + q)^2 a_1 a_2)) - (a_1 \alpha^2 q + a_2 \beta^2 - (1 + q) a_1 a_2 \alpha \beta) \delta \right) > 0.
\end{align*}
Hence thanks to Lemma 8.2.1, there exists $\varepsilon_1 > 0$ such that (8.2.4) holds. We next verify that there is $\varepsilon_2 > 0$ such that

\[(8.2.5) \quad I_2(t) \leq -\varepsilon_2 \int_{\Omega} |\nabla w(\cdot, t)|^2 \quad \text{for all } t > 0.\]

By virtue of the Young inequality, we infer from (8.1.9) that

\[
\begin{align*}
&\int_{\Omega} \frac{\chi_1(u)}{u} \nabla u \cdot \nabla w + \frac{a_2 \mu_2 u^*}{4d_1} \int_{\Omega} |\nabla w|^2, \\
&\int_{\Omega} \frac{\chi_2(v)}{v} \nabla v \cdot \nabla w + \frac{a_1 \mu_1 v^*}{4d_2} \int_{\Omega} |\nabla w|^2, \\
\end{align*}
\]

which implies that

\[(8.2.6) \quad I_2(t) \leq -\left(\frac{a_2 \mu_2 u^* M_1^2}{4d_1} - \frac{a_1 \mu_1 v^* q M_2^2}{4d_2}\right) \int_{\Omega} |\nabla w(\cdot, t)|^2 \quad \text{for all } t > 0.\]

Therefore plugging the definition of $\delta > 0$ into (8.2.6) leads to the existence of $\varepsilon_2 > 0$ such that (8.2.5) holds, which concludes the proof of this lemma.

Then we have the following desired estimate.

**Lemma 8.2.3.** Let $a_1, a_2 \in (0, 1)$ and assume that (8.1.9)–(8.1.11) are satisfied. Then there exist $C > 0$ and $\ell > 0$ such that

\[
\|u(\cdot, t) - u^*\|_{L^\infty(\Omega)} + \|v(\cdot, t) - v^*\|_{L^\infty(\Omega)} + \|w(\cdot, t) - w^*\|_{L^\infty(\Omega)} \leq C e^{-\ell t} \quad \text{for all } t > 0.
\]

**Proof.** The same arguments as in the proofs of [4, Theorems 3.3, 3.6 and 3.7] enable us to obtain this lemma.

**Proof of Theorem 8.1.1.** Lemma 8.2.3 immediately leads to Theorem 8.1.1.

**Proof of Theorems 8.1.2 and 8.1.3.** A combination of results concerned with global existence and boundedness in (8.1.1) ([4, 115] and Chapter 7) and the standard parabolic regularity argument, along with Theorem 8.1.1 implies Theorems 8.1.2 and 8.1.3.

**8.3. Discussions**

In this section we shall confirm that the condition (8.1.10) improves the conditions assumed in the previous works aided by the view of (8.1.6) and (8.1.7). Here since

\[
\left\{(s, t) \in \mathbb{R}^2 \mid 0 \leq s < \frac{f(q)}{1+q}, \ 0 \leq t < \frac{f(q)}{1+q}\right\} \subset \left\{(s, t) \in \mathbb{R}^2 \mid s, t \geq 0, \ s + qt < f(q)\right\}
\]

holds for every $q \in I$, we can see that the condition (8.1.10) improves the conditions assumed in Chapter 7. In order to accomplish the purpose of this section, noting that

\[
\left\{(s, t) \in \mathbb{R}^2 \mid s, t \geq 0, \ s + qt < f(q)\right\} = \bigcup_{k \in [0, 1]} \left\{(s, t) \in \mathbb{R}^2 \mid 0 \leq s < kf(q), \ 0 \leq t < (1-k)\frac{f(q)}{q}\right\} =: R_q,
\]

93
we will confirm that

\[(8.3.1) \quad R_1 \subset \bigcup_{q \in I} R_q.\]

To see (8.3.1) we shall see that \(q = 1\) is not the best choice, i.e., \(q = 1\) is not a maximizer of the functions \(f(q)\) and

\[g(q) := \frac{f(q)}{q} \quad \text{for } q \in I.\]

From the straightforward calculations we infer that

\[(8.3.2) \quad \frac{df}{dq}(1) = \frac{4\gamma(1 - a_1a_2)(\beta - a_1\alpha)}{(a_1\alpha^2 + a_2\beta^2 - 2a_1a_2\alpha\beta)^2};\]

\[\frac{dg}{dq}(1) = \frac{4\gamma(1 - a_1a_2)(a_2\beta - \alpha)}{(a_1\alpha^2 + a_2\beta^2 - 2a_1a_2\alpha\beta)^2}.\]

Now we will divide into two cases and show that (8.3.1) holds for each cases.

**Case 1:** \(a_1\alpha \neq \beta\) and \(a_2\beta \neq \alpha\). In this case, by virtue of (8.3.2), we can see that

\[\frac{df}{dq}(1) \neq 0 \quad \text{and} \quad \frac{dg}{dq}(1) \neq 0,

which means that \(q = 1\) is not a maximizer of \(f\) and \(g\). Thus one of the following two properties holds:

(Case 1-1) \(\exists q_1 \in I; \quad f(q_1) > f(1), \quad g(q_1) > g(1),\)

(Case 1-2) \(\forall q \in I; \quad f(q) > f(1), \quad g(q) \leq g(1) \quad \text{or} \quad f(q) \leq f(1), \quad g(q) > h(1).\)

Therefore we obtain that (8.3.1) holds in this case.

**Case 2:** \(a_1\alpha = \beta\) or \(a_2\beta = \alpha\). We first deal with the case that \(a_1\alpha = \beta\). In this case, in view of the fact that

\[f(q) = -\frac{\gamma a_1 a_2}{\alpha(1 - a_1 a_2)} \cdot q + \frac{2\gamma(2 - a_1 a_2)}{\alpha(1 - a_1 a_2)} - \frac{\gamma a_1 a_2}{\alpha(1 - a_1 a_2)} \cdot \frac{1}{q},\]

\(q = 1\) is a maximizer of \(f\). On the other hand, thanks to (8.3.2) together with the fact that \(\alpha - a_2\beta = \alpha(1 - a_1 a_2) > 0\), we can see that \(q = 1\) is not a maximizer of \(g\); there is \(q_2 \in I\) such that

\[g(q_2) > g(1).\]

Similarly, in the case that \(a_2\beta = \alpha\), we infer that \(q = 1\) is a maximizer of \(g\) and is not a maximizer of \(f\); there exists \(q_3 \in I\) such that

\[f(q_3) > f(1).\]

Hence we derive that (8.3.1) holds also in this case.

According to the above two cases, we can obtain that (8.3.1) holds, which means the condition (8.1.10) improves the conditions assumed in [4]. Therefore we attain the purpose of this chapter.
Part II

Two-species chemotaxis systems with fluid environment
Chapter 9

Global existence and asymptotic behaviour in a two-dimensional two-species chemotaxis-Navier–Stokes system with competitive kinetics

9.1. Background and results

In this chapter we consider the following two-species chemotaxis-Navier–Stokes system with Lotka–Volterra competitive kinetics:

\[
\begin{align*}
(n_1)_t + u \cdot \nabla n_1 &= \Delta n_1 - \chi_1 \nabla \cdot (n_1 \nabla c) + \mu_1 n_1(1 - n_1 - a_1 n_2) \quad \text{in } \Omega \times (0, \infty), \\
(n_2)_t + u \cdot \nabla n_2 &= \Delta n_2 - \chi_2 \nabla \cdot (n_2 \nabla c) + \mu_2 n_2(1 - a_2 n_1 - n_2) \quad \text{in } \Omega \times (0, \infty), \\
c_t + u \cdot \nabla c &= \Delta c - (\alpha n_1 + \beta n_2)c \quad \text{in } \Omega \times (0, \infty), \\
u_t + (u \cdot \nabla)u &= \Delta u + \nabla P + (\gamma n_1 + \delta n_2)\nabla \phi, \quad \nabla \cdot u = 0 \quad \text{in } \Omega \times (0, \infty), \\
\partial_n n_1 = \partial_n n_2 = \partial_n c = 0, \quad u = 0 \quad \text{on } \partial \Omega \times (0, \infty), \\
n_1(\cdot, 0) &= n_{1,0}, \quad n_2(\cdot, 0) = n_{2,0}, \quad c(\cdot, 0) = c_0, \quad u(\cdot, 0) = u_0 \quad \text{in } \Omega,
\end{align*}
\]

where \(\Omega\) is a bounded domain in \(\mathbb{R}^2\) with smooth boundary \(\partial \Omega\) and \(\partial_n\) denotes differentiation with respect to the outward normal of \(\partial \Omega\); \(\chi_1, \chi_2, a_1, a_2 \geq 0\) and \(\mu_1, \mu_2, \alpha, \beta, \gamma, \delta > 0\) are constants; \(n_{1,0}, n_{2,0}, c_0, u_0, \phi\) are known functions satisfying

\[
\begin{align*}
0 < n_{1,0}, n_{2,0} &\in C(\overline{\Omega}), \quad 0 < c_0 \in W^{1,q}(\Omega), \quad u_0 \in D(A^\theta), \\
\phi &\in C^{1+\eta}(\overline{\Omega})
\end{align*}
\]

for some \(q > 2, \theta \in (\frac{1}{2}, 1), \eta > 0\) and \(A\) denotes the realization of the Stokes operator under homogeneous Dirichlet boundary conditions in the solenoidal subspace

\[L^2_\sigma(\Omega) := \{\varphi \in L^2(\Omega) \mid \nabla \cdot \varphi = 0 \text{ in } \Omega\}\]
of $L^2(\Omega)$.

The problem (9.1.1) is a model that describes the exercise of two species which react on a single chemoattractant in fluid, where $n_1$ and $n_2$ stand for densities of species, $c$ shows the chemical concentration, $u$ denotes the fluid velocity field and $P$ represents the pressure of the fluid. The problem (9.1.1) comes from a problem on account of the influence of chemotaxis, the Lotka–Volterra kinetics

$$(n_1)_t = \mu_1 n_1 (1 - n_1 - a_1 n_2), \quad (n_2)_t = \mu_2 n_2 (1 - a_2 n_1 - n_2)$$

and fluid. Here chemotaxis is the biased migration of cells toward more favourable environmental conditions, e.g., higher concentration of oxygen, which plays an outstanding role in a large range of biological applications ([71]). Also, multi-species chemotaxis systems would be appearing in nature and were studied by e.g., [75] and [208]. In our model we focus on the case of two-species which react on single chemoattractant. Moreover, there are some observations which reveal dynamics concerning patterns and spontaneous emergence of turbulence in populations of aerobic bacteria suspended in sessile drops of water ([179]).

From a mathematical viewpoint it is fundamental to study whether global existence and behavior of solutions to (9.1.1) can be clarified or not. Nowadays, we can find many successful studies about this basic question in some particular cases (see e.g., the recent survey [5]). In the case of single-species models reduced with $n_2 = 0$, Winkler [197] first attained global existence of solutions to (9.1.1) without Lotka–Volterra competitive kinetics in $\Omega \subset \mathbb{R}^2$ and also Winkler [201] obtained a precise stabilization result in the case that $N = 2$. In the case that $N = 3$ Winkler [204] showed global existence of weak solutions to (9.1.1). On the other hand, when $-\alpha n_1 c$ in the third equation of (9.1.1) is replaced with $n_1 - c$, Tao and Winkler [176] established global existence and asymptotic behavior of solutions to (9.1.1) with $\mu_1 > 0$ and $n_2 = 0$ in the case that $N = 2$, and Lankeit [101] established global existence of bounded solutions and their asymptotic behavior of solutions to (9.1.1) with $\mu_1 > 0$ and $n_2 = 0$ in the case that $N = 3$. As to the case of two-species, some problems on account of the influence of chemotaxis and the Lotka–Volterra kinetics, provided that $u = 0$ and the third equation is replaced with $c_t = \Delta c + \alpha n_1 + \beta n_2 - \gamma c$ in (9.1.1), were investigated (for the noncompetitive case that $a_1 = a_2 = 0$ see Negreanu and Tello [150, 151] and Chapters 3, 4; for the competitive case see Bai and Winkler [4], Stinner, Tello and Winkler [164] and Chapters 5, 7). However, the two species chemotaxis model (9.1.1) considering fluid seems not to be studied yet.

The purpose of this chapter is to establish global existence, boundedness and stabilization of solutions to the two species chemotaxis model (9.1.1) considering fluid. These are not so trivial because (9.1.1) has the following two difficult points. The first difficulty is that (9.1.1) consists of two-species chemotaxis with competitive kinetics. The second difficulty is the effect from fluid described by the Navier–Stokes equation. We will overcome these two difficulties by using recent techniques developed by [101, 176] and Chapter 7.

Now the main results read as follows. The first main result describes global existence and boundedness of solutions to (9.1.1).
Theorem 9.1.1. Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with smooth boundary and let \( \chi_1, \chi_2 \geq 0, a_1, a_2 \geq 0, \mu_1, \mu_2 > 0, \alpha, \beta, \gamma, \delta > 0 \). Suppose that \( n_{1,0}, n_{2,0}, c_0, u_0, \phi \) satisfy (9.1.2) and (9.1.3). Then there exist functions
\[
n_1, n_2 \in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)),
\]
\[
c \in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)),
\]
\[
u \in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)),
\]
\[
P \in C^{1,0}(\overline{\Omega} \times (0, \infty))
\]
which solve (9.1.1) classically in \( \Omega \times [0, \infty) \). Moreover, the solution \((n_1, n_2, c, u, P)\) of (9.1.1) is unique, up to addition of constants to \( P \), and satisfies that there exists a constant \( C_1 > 0 \) such that
\[
\|n_1(\cdot, t)\|_{L^\infty(\Omega)} + \|n_2(\cdot, t)\|_{L^\infty(\Omega)} + \|c(\cdot, t)\|_{L^\infty(\Omega)} + \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C_1 \quad (t > 0).
\]
The second main result asserts stabilization of solutions to (9.1.1).

Theorem 9.1.2. Let the assumption of Theorem 9.1.1 hold. Then the solution of (9.1.1) has the following properties:

(i) Assume that \( a_1, a_2 \in (0, 1) \). Then
\[
n_1(\cdot, t) \to N_1, \quad n_2(\cdot, t) \to N_2, \quad c(\cdot, t) \to 0, \quad u(\cdot, t) \to 0 \quad \text{in } L^\infty(\Omega) \quad \text{as } t \to \infty,
\]
where
\[
N_1 := \frac{1 - a_1}{1 - a_1 a_2}, \quad N_2 := \frac{1 - a_2}{1 - a_1 a_2}.
\]

(ii) Assume that \( a_1 \geq 1 > a_2 \). Then
\[
n_1(\cdot, t) \to 0, \quad n_2(\cdot, t) \to 1, \quad c(\cdot, t) \to 0, \quad u(\cdot, t) \to 0 \quad \text{in } L^\infty(\Omega) \quad \text{as } t \to \infty.
\]

Remark 9.1.1. The above theorem is also new even in the fluid-free problem (9.1.1) with \( u(x, t) \equiv 0 \). Indeed, in this case, stabilization holds without any upper restriction on \( \chi_1 \). This result implies “the Lotka–Volterra competitive kinetics always win the chemotactic consumption”.

The strategy for the proof of boundedness of \( n_1 \) is to establish the \( L^p \)-estimate for \( n_1 \) by using the following ordinary differential inequality:
\[
\frac{d}{dt} \int_{\Omega} n_1^p + p(p - 1) \int_{\Omega} n_1^{p-2} |\nabla n_1|^2 \leq p(p - 1) \chi_1 \int_{\Omega} n_1^{p-1} \nabla n_1 \cdot \nabla c + p\mu_1 \int_{\Omega} (n_1^p - n_1^{p+1}),
\]
and applying the method in Tao–Winkler [176]. Then we shall derive the \( L^\infty \)-estimate for \( n_1 \) via semigroup estimates and the variation of constants formula which includes the following term:
\[
I(t) := \int_s^t e^{(t-s)(\Delta^{-1})} \left[ \mu_1 n_1 (1 - n_1 - a_1 n_2) + n_1 \right] \, ds
\]
99
with some $\tau > 0$. Then, noting that
\[
(1 + \mu_1)n_1 - \mu_1 n_1^2 - \mu_1 a_1 n_1 n_2 \leq -\mu_1 \left(n_1 - \frac{1 + \mu_1}{2\mu_1}\right)^2 + \frac{(1 + \mu_1)^2}{4\mu_1},
\]
we can find an upper bound for $I(t)$, which leads to the $L^\infty$-estimate for $n_1$. On the other hand, the strategy for the proof of Theorem 9.1.2 is to modify arguments in [101] and Chapter 7. The key to the proof is to establish that
\[
\frac{d}{dt} E_1(t) \leq -\varepsilon_1 \left(\int_\Omega (n_1 - N)^2 + \int_\Omega (n_2 - \vec{N})^2\right)
\]
with some function $E_1 \geq 0$ and some $\varepsilon_1 > 0$, where $(N, \vec{N}) \in \mathbb{R}^2$ is equal to $(N_1, N_2)$ or $(0, 1)$ of the limit of $n_1, n_2$ as $t \to \infty$ as in Theorem 9.1.2 (i) or (ii). Then we obtain that there exists $C_2 > 0$ such that
\[
\int_0^\infty \int_\Omega (n_1 - N)^2 + \int_0^\infty \int_\Omega (n_2 - \vec{N})^2 \leq C_2.
\]
Combination of this integrability at infinity with a compactness argument implies $n_1 \to N$ and $n_2 \to \vec{N}$ in $L^\infty(\Omega)$ as $t \to \infty$.

This chapter is organized as follows. Section 9.2 gives some estimates for the Neumann heat semigroup and local existence in (9.1.1). Sections 9.3 and 9.4 are devoted to the proofs of Theorems 9.1.1 and 9.1.2, respectively.

**9.2. Local existence**

In this section we will recall local existence of solutions to (9.1.1).

**Lemma 9.2.1.** Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, let $\chi_1, \chi_2, a_1, a_2 \geq 0$ and let $\mu_1, \mu_2, \alpha, \beta, \gamma, \delta > 0$. For all $n_1, 0, n_2, 0, c_0, u_0, \phi$ satisfying (9.1.2) and (9.1.3), there exist $T_{\max} \in (0, \infty]$ and a classical solution $(n_1, n_2, c, u, P)$ of (9.1.1) in $\Omega \times (0, T_{\max})$ such that
\[
\begin{align*}
n_1, n_2, c, u &\in C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})), \\
n_1, n_2 > 0, \quad c > 0 &\quad \text{in } \overline{\Omega} \times (0, T_{\max}).
\end{align*}
\]
Also the solution is unique, up to addition of constants to $P$. Moreover,
\[
\lim_{t \to T_{\max}} \left(\|n_1(\cdot, t)\|_{L^\infty(\Omega)} + \|n_2(\cdot, t)\|_{L^\infty(\Omega)} + \|c(\cdot, t)\|_{W^{1,p}(\Omega)} + \|A^2 u(\cdot, t)\|_{L^2(\Omega)}\right) = \infty
\]
for all $\sigma \in (\frac{1}{2}, 1)$.

**Proof.** The proof of local existence of classical solutions to (9.1.1) is based on a standard contraction mapping argument, which can be found in [197]. Accordingly, the maximum principle is applied to yield $n_1, n_2 > 0$ and $c > 0$, in $(0, T_{\max})$. \hfill \Box
9.3. Boundedness. Proof of Theorem 9.1.1

In this section we will prove Theorem 9.1.1 through a series of lemmas.

Lemma 9.3.1. There exist constants \( C_7 > 0 \) and \( C_8 > 0 \) such that the solution of (9.1.1) satisfies

\[
\int_{\Omega} n_i(\cdot, t) \leq C_7 \quad \text{for all } t \in (0, T_{\max})
\]

and

\[
\int_{t}^{t+\tau} \int_{\Omega} n_i^2 \leq C_8 \quad \text{for all } t \in (0, T_{\max} - \tau)
\]

for \( i = 1, 2 \), where \( \tau := \min\{1, \frac{1}{16} T_{\max}\} \).

Proof. Integrating the first equation in (9.1.1) over \( \Omega \) yields

\[
\frac{d}{dt} \int_{\Omega} n_1 = \rho \int_{\Omega} n_1 - \mu_1 \int_{\Omega} n_1^2 - \mu_{11} \int_{\Omega} n_1 n_2 \quad \text{for all } t \in (0, T_{\max}),
\]

because \( \partial_t n_1 = \partial_v c = 0 \) and \( \int_{\Omega} u \cdot \nabla n_1 = -\int_{\Omega} n_1 \nabla \cdot u = 0 \) for all \( t \in (0, T_{\max}) \). Since we have from the Schwarz inequality and the positivity of \( n_1 \) and \( n_2 \) that

\[
\int_{\Omega} n_1^2 \geq \frac{1}{|\Omega|} \left( \int_{\Omega} n_1 \right)^2 \quad \text{and} \quad \int_{\Omega} n_1 n_2 \geq 0
\]

for all \( t \in (0, T_{\max}) \), we see from (9.3.3) that \( y(t) := \int_{\Omega} n_1(\cdot, t) \) satisfies

\[
y'(t) \leq \mu_1 y(t) - \frac{\mu_{11}}{|\Omega|} y^2(t) \quad \text{for all } t \in (0, T_{\max}),
\]

which by an ODE comparison shows (9.3.1):

\[
y(t) \leq C_7 := \max \left\{ \int_{\Omega} n_{1,0}, |\Omega| \right\} \quad \text{for all } t \in (0, T_{\max}).
\]

Moreover, (9.3.2) results upon an integration of (9.3.3) in time. \( \square \)

Lemma 9.3.2. There exists a constant \( C_9 > 0 \) such that

\[
\|c(\cdot, t)\|_{L^\infty(\Omega)} \leq C_9 \quad \text{for all } t \in (0, T_{\max})
\]

and \( t \mapsto \|c(\cdot, t)\|_{L^\infty(\Omega)} \) is nonincreasing on \( (0, \infty) \). Moreover, there exists a constant \( C_{10} > 0 \) such that

\[
\int_{0}^{T_{\max}} \int_{\Omega} |\nabla c|^2 \leq C_{10}.
\]
Proof. From the straightforward calculations we can see that for all \( p > 1 \),

\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} c^p = -\int_{\Omega} c^{p-1} u \cdot \nabla c + \int_{\Omega} c^{p-1} \Delta c - \int_{\Omega} (\alpha n_1 + \beta n_2) c^p
\]
\[
= -\frac{1}{p} \int_{\Omega} c^p \nabla \cdot u - (p-1) \int_{\Omega} c^{p-2} |\nabla c|^2 - \int_{\Omega} (\alpha n_1 + \beta n_2) c^p.
\]

Noting that \( \nabla \cdot u = 0 \) in \( \Omega \times (0, T_{\text{max}}) \) and \( n_1, n_2, c \) are positive, we have from (9.3.6) that

\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} c^p \leq -(p-1) \int_{\Omega} c^{p-2} |\nabla c|^2 \leq 0,
\]

which implies

\[
\|c(\cdot, t_2)\|_{L^p(\Omega)} \leq \|c(\cdot, t_1)\|_{L^p(\Omega)} \quad \text{for all } t_2 \geq t_1 \geq 0.
\]

Hence taking the limits as \( p \to \infty \), we obtain that (9.3.4) holds with \( C_9 = \|c_0\|_{L^\infty(\Omega)} \).

Moreover, now integrating (9.3.7) with \( p = 2 \) over \( (0, t) \), we obtain

\[
\frac{1}{2} \int_{\Omega} c^2 + \int_0^t \int_{\Omega} |\nabla c|^2 \leq \frac{1}{2} \int_{\Omega} c_0^2,
\]

which means that (9.3.5) holds. \( \square \)

Remark 9.3.1. Boundedness of \( c \) can be proved also by the maximum principle.

Lemma 9.3.3. There exists a constant \( C_{11} > 0 \) satisfying

\[
\|u(\cdot, t)\|_{L^2(\Omega)} \leq C_{11} \quad \text{for all } t \in (0, T_{\text{max}})
\]

and

\[
\int_t^{t+\tau} \int_{\Omega} |\nabla u|^2 \leq C_{11}, \quad \int_t^{t+\tau} \int_{\Omega} |u|^4 \leq C_{11} \quad \text{for all } t \in (0, T_{\text{max}} - \tau),
\]

where \( \tau = \min\{1, \frac{1}{6} T_{\text{max}}\} \).

Proof. Noting from the Schwarz inequality that

\[
(\gamma n_1 + \delta n_2)^2 \leq (\gamma^2 + \delta^2)(n_1^2 + n_2^2),
\]

we can show this lemma by a similar argument as in [176, Lemmas 3.4 and 3.5]. \( \square \)

Lemma 9.3.4. There exists a constant \( C_{12} > 0 \) such that

\[
\|\nabla u(\cdot, t)\|_{L^2(\Omega)} \leq C_{12} \quad \text{for all } t \in (0, T_{\text{max}}).
\]

Proof. Combination of the inequality (9.3.8) and a similar proof as in [176, Lemma 3.6] leads to this lemma. \( \square \)
Lemma 9.3.5. There exists a constant $C_{13} > 0$ such that

\begin{equation}
\| \nabla c(\cdot, t) \|_{L^2(\Omega)} \leq C_{13} \quad \text{for all } t \in (0, T_{\text{max}})
\end{equation}

and

\begin{equation}
\int_{t}^{t+\tau} \int_{\Omega} |\Delta c|^2 \leq C_{13} \quad \text{for all } t \in (0, T_{\text{max}} - \tau),
\end{equation}

where $\tau = \min\{1, \frac{1}{6} T_{\text{max}}\}$.

Proof. Multiplying the third equation in (9.1.1) by $-\Delta c$ and integrating it over $\Omega$, we obtain

\begin{equation}
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla c|^2 + \int_{\Omega} |\Delta c|^2 = -\int_{\Omega} |\nabla c|^2 - \int_{\Omega} (\alpha n_1 + \beta n_2) c \Delta c + \int_{\Omega} (u \cdot \nabla c) \Delta c.
\end{equation}

Thanks to the Young inequality and Lemma 9.3.2, we see

\begin{equation}
- \int_{\Omega} (\alpha n_1 + \beta n_2) c \Delta c \leq \frac{1}{4} \int_{\Omega} |\Delta c|^2 + (\alpha^2 + \beta^2) \|c_0\|_{L^\infty(\Omega)}^2 \int_{\Omega} (n_1^2 + n_2^2).
\end{equation}

Noting from the Gagliardo–Nirenberg inequality that there exists $C_{GN} > 0$ such that

\begin{equation}
\|\nabla c\|_{L^4(\Omega)} \leq C_{GN}\|\Delta c\|_{L^2(\Omega)}^{\frac{3}{2}}\|\nabla c\|_{L^2(\Omega)}^{\frac{1}{2}},
\end{equation}

from the Hölder inequality and the Young inequality we can find $C_{14} > 0$ satisfying

\begin{equation}
\int_{\Omega} (u \cdot \nabla c) \Delta c \leq \|u\|_{L^4(\Omega)} \|\nabla c\|_{L^4(\Omega)} \|\Delta c\|_{L^2(\Omega)} \\
\leq C_{GN}\|u\|_{L^4(\Omega)} \|\nabla c\|_{L^2(\Omega)}^{\frac{3}{2}}\|\Delta c\|_{L^2(\Omega)}^{\frac{1}{2}} \\
\leq \frac{1}{4} \|\Delta c\|_{L^2(\Omega)}^2 + C_{14}\|u\|_{L^4(\Omega)}^4 \|\nabla c\|_{L^2(\Omega)}^2.
\end{equation}

Combination of (9.3.11), (9.3.12) and (9.3.13) yields

\begin{equation}
\frac{d}{dt} \int_{\Omega} |\nabla c|^2 + \int_{\Omega} |\Delta c|^2 \leq h_1(t) + h_2(t) \int_{\Omega} |\nabla c|^2,
\end{equation}

where

\begin{align*}
\quad h_1(t) &= C_{15} \int_{\Omega} (n_1^2(\cdot, t) + n_2^2(\cdot, t)) \quad \text{and} \quad h_2(t) := C_{15} \int_{\Omega} |u(\cdot, t)|^4
\end{align*}

with some $C_{15} > 0$. Here Lemmas 9.3.1 and 9.3.3 imply that there exists $C_{16} > 0$ such that

\begin{equation}
\int_{t}^{t+\tau} \int_{\Omega} |\nabla c|^2 + \int_{t}^{t+\tau} h_1(s) \, ds + \int_{t}^{t+\tau} h_2(s) \, ds \leq C_{16},
\end{equation}

103
In particular, given \( t \in (0, T_{\text{max}}) \), we can use (9.3.15) to pick \( t_0 \in (t - \tau, t) \cap [0, \infty) \) satisfying
\[
\int_{\Omega} |\nabla c(\cdot, t_0)|^2 \leq C_{17} := \max \left\{ \int_{\Omega} |\nabla c_0|^2, \frac{C_{16}}{\tau} \right\}.
\]
Therefore we obtain from (9.3.14) that
\[
\int_{\Omega} |\nabla c(\cdot, t)|^2 \leq \left( \int_{\Omega} |\nabla c(\cdot, t_0)|^2 \right) e^{\int_{t_0}^{t} h_2(\sigma) d\sigma} + \int_{t_0}^{t} h_1(s) e^{\int_{t_0}^{s} \sigma h_2(\sigma) d\sigma} ds
\]
\[
\leq C_{17} e^{C_{16}} + (t - t_0) C_{16} e^{C_{16}}
\]
\[
\leq (C_{16} + C_{17}) e^{C_{16}} =: C_{18},
\]
since \( t - t_0 \leq \tau \leq 1 \), which implies (9.3.9). Moreover, combination of (9.3.9), (9.3.15) and integration of (9.3.14) derives (9.3.10), which means the end of the proof.

**Lemma 9.3.6.** Let \( p \geq 2 \) and \( \tau = \min\{1, \frac{1}{6} T_{\text{max}}\} \). Assume that there exists \( M > 0 \) such that
\[
\int_{t}^{t+\tau} \int_{\Omega} n_i^p \leq M \quad \text{for all } t \in (0, T_{\text{max}} - \tau) \text{ and } i = 1, 2.
\]
Then there exists a constant \( C_{19}(p, M) > 0 \) such that
\[
\|n_i(\cdot, t)\|_{L^p(\Omega)} \leq C_{19}(p, M) \quad \text{for all } t \in (0, T_{\text{max}}) \text{ and } i = 1, 2
\]
and
\[
\int_{t}^{t+\tau} \int_{\Omega} n_i^{p+1} \leq C_{19}(p, M) \quad \text{for all } t \in (0, T_{\text{max}} - \tau) \text{ and } i = 1, 2.
\]
**Proof.** Multiplying the first equation in (9.1.1) by \( n_i^{p-1} \), we see from integration by parts that
\[
\frac{d}{dt} \int_{\Omega} n_i^p + p(p - 1) \int_{\Omega} n_i^{p-2} |\nabla n_i|^2
\]
\[
= p(p - 1) \chi t \int_{\Omega} n_i^{p-1} \nabla n_i \cdot \nabla c + \mu_1 \int_{\Omega} n_i^p - \mu_1 \int_{\Omega} n_i^{p+1} - \mu a_1 \int_{\Omega} n_i^{p-1} n_2,
\]
because \( \nabla \cdot u = 0 \). Now we can remove the last term because \( n_1, n_2 \) are positive. Then the same argument as in the proof of [176, Lemma 3.8] gives that (9.3.16) leads to (9.3.17) and (9.3.18) for \( i = 1 \). The proof of the case \( i = 2 \) is similar to that of the case \( i = 1 \).

**Lemma 9.3.7.** For all \( p > 1 \) there exists a constant \( C_{20}(p) > 0 \) such that
\[
\|n_i(\cdot, t)\|_{L^p(\Omega)} \leq C_{20}(p) \quad \text{for all } t \in (0, T_{\text{max}}) \text{ and } i = 1, 2.
\]
**Proof.** Starting with (9.3.2) and using Lemma 9.3.6 inductively give this lemma.
Lemma 9.3.8. For all $\sigma_1 \in (\frac{1}{2}, 1)$ there exists a constant $C_{21}(\sigma_1) > 0$ such that

$$\|A^{\sigma_1}u(\cdot, t)\|_{L^2(\Omega)} \leq C_{21}(\sigma_1) \quad \text{for all } t \in (\tau, T_{\max}),$$

where $\tau = \min\{1, \frac{1}{6}T_{\max}\}$. In particular, there exist constants $\lambda \in (0, 1)$ and $C_{22} > 0$ such that

$$\|u(\cdot, t)\|_{C^\lambda(\Omega)} \leq C_{22} \quad \text{for all } t \in (\tau, T_{\max}).$$

Proof. By noting from the boundedness of $\nabla \phi$ that there exists $C_{23} > 0$ such that

$$\|((\gamma n_1(\cdot, t) + \delta n_2(\cdot, t))\nabla \phi\|_{L^2(\Omega)} \leq C_{23}(\|n_1(\cdot, t)\|_{L^2(\Omega)} + \|n_2(\cdot, t)\|_{L^2(\Omega)}),$$

the same argument as in the proof of [176, Lemma 3.11 with $g = 0$] implies this lemma.

Lemma 9.3.9. There exists a constant $C_{24} > 0$ such that

$$\|c(\cdot, t)\|_{W^{1, \infty}(\Omega)} \leq C_{24} \quad \text{for all } t \in (0, T_{\max}).$$

Proof. From the regularity of $c$ we see that $\nabla c$ is bounded on $\overline{\Omega} \times [0, 2\tau]$. Hence we shall confirm that there exists $C_{25} > 0$ such that

$$\|c(\cdot, t)\|_{W^{1, \infty}(\Omega)} \leq C_{25} \quad \text{for all } t \in (2\tau, T_{\max}).$$

Using the variation of constants formula for $c$, we obtain

$$c(\cdot, t) = e^{(t-\tau)\Delta}c(\cdot, \tau) + \int_\tau^t e^{(t-s)\Delta}[(\alpha n_1 + \beta n_2)c(\cdot, s) - u(\cdot, s)\nabla c(\cdot, s)] \, ds.$$

Combination of the above identity with Lemma 2.2.2 yields

$$\|\nabla c(\cdot, t)\|_{L^\infty(\Omega)} \leq C_3[1 + (t-\tau)^{-1}e^{-\lambda_1(t-\tau)}\|c(\cdot, \tau)\|_{L^2(\Omega)}$$

$$+ C_3 \int_\tau^t [1 + (t-s)^{-\frac{3}{4}}e^{-\lambda_1(t-s)}]\|n_1 + \beta n_2\|_{L^4(\Omega)} \, ds$$

$$+ C_3 \int_\tau^t [1 + (t-s)^{-\frac{3}{4}}e^{-\lambda_1(t-s)}]\|u(\cdot, s)\nabla c(\cdot, s)\|_{L^4(\Omega)} \, ds.$$}

Here, noting from Lemmas 9.3.2 and 9.3.7 that there exists $C_{26} > 0$ such that

$$\|(\alpha n_1 + \beta n_2)c\|_{L^4(\Omega)} \leq \|c\|_{L^\infty(\Omega)}(\alpha\|n_1\|_{L^4(\Omega)} + \beta\|n_2\|_{L^4(\Omega)}) \leq C_{26},$$

a similar argument as in the proof of [176, Lemma 3.9] can be applied to this lemma.

Now we will establish the $L^\infty$-estimate for $n_1$. This proof is based on the method in the proof of Theorem 3.1.1.
**Lemma 9.3.10.** There exists a constant $C_{27} > 0$ such that

$$
\| n_i(\cdot, t) \|_{L^\infty(\Omega)} \leq C_{27} \quad \text{for all } t \in (0, T_{\text{max}}) \text{ and } i = 1, 2.
$$

**Proof.** Thanks to the regularity of $n_1$, it suffices to show that there exist $\tau_1 \in (0, T_{\text{max}})$ and $C_{28}(\tau_1) > 0$ such that

$$
\| n_1(\cdot, t) \|_{L^\infty(\Omega)} \leq C_{28}(\tau_1) \quad \text{for all } t \in (\tau_1, T_{\text{max}}).
$$

Employing the variation of constants formula for $n_1$, we derive that for all $t \in (\tau, T_{\text{max}}),

\begin{equation}
(9.3.19) \quad n_1(\cdot, t) = e^{(t-\frac{\tau}{2})(\Delta-1)}n_1 \left( \cdot, \frac{\tau}{2} \right) - \int_{\frac{\tau}{2}}^{t} e^{(t-s)(\Delta-1)} \nabla \cdot (n_1 \chi_1 \nabla c + n_1 u) \, ds \\
+ \int_{\frac{\tau}{2}}^{t} e^{(t-s)(\Delta-1)} \left[ \mu_1 n_1 (1 - n_1 - a_1 n_2) + n_1 \right] \, ds \\
=: J_1 + J_2 + J_3.
\end{equation}

Letting $p > 2$, $\bar{\eta} \in \left( \frac{1}{p}, \frac{1}{2} \right)$ and $\varepsilon \in \left( 0, \frac{1}{2} - \bar{\eta} \right)$, we have that

$$
0 < 2\bar{\eta} - \frac{2}{p} \quad \text{and} \quad \bar{\eta} + \varepsilon + \frac{1}{2} < 1.
$$

Then we see from Lemma 2.2.1 that for all $t > \tau,

\begin{equation}
(9.3.20) \quad \| J_1 \|_{L^\infty(\Omega)} = \left\| e^{(t-\frac{\tau}{2})(\Delta-1)}n_1 \left( \cdot, \frac{\tau}{2} \right) \right\|_{L^\infty(\Omega)} \\
\leq C_4 \left\| (-\Delta + 1)^{\bar{\eta}} e^{(t-\frac{\tau}{2})(\Delta-1)}n_1 \left( \cdot, \frac{\tau}{2} \right) \right\|_{L^p(\Omega)} \\
\leq C_4 C_5 \left( t - \frac{\tau}{2} \right)^{-\bar{\eta}} e^{-\lambda_2 t} \left\| n_1 \left( \cdot, \frac{\tau}{2} \right) \right\|_{L^p(\Omega)} \\
\leq 2^{\bar{\eta}} C_4 C_5 C_{20}(p) \tau^{-\bar{\eta}} e^{-\lambda_2 \tau}.
\end{equation}

We next have from Lemma 2.2.1 that

$$
\| J_2 \|_{L^\infty(\Omega)} \leq \int_{\frac{\tau}{2}}^{t} \left\| e^{(t-s)(\Delta-1)} \nabla \cdot (n_1 \chi_1 \nabla c + n_1 u) \right\|_{L^\infty(\Omega)} \, ds \\
\leq C_4 \int_{\frac{\tau}{2}}^{t} \left\| (-\Delta + 1)^{\bar{\eta}} e^{(t-s)(\Delta-1)} \nabla \cdot (n_1 \chi_1 \nabla c + n_1 u) \right\|_{L^p(\Omega)} \, ds \\
\leq C_4 C_6 \int_{\frac{\tau}{2}}^{t} (t-s)^{-\bar{\eta} - \varepsilon - \frac{1}{4}} e^{-(\lambda_3 + 1)(t-s)} \left\| n_1 \chi_1 \nabla c + n_1 u \right\|_{L^p(\Omega)} \, ds.
$$

Here, noting from Lemmas 9.3.7, 9.3.8 and 9.3.9 that

$$
\| n_1 \chi_1 \nabla c + n_1 u \|_{L^p(\Omega)} \leq \chi_1 \| n_1 \|_{L^p(\Omega)} \| c \|_{W^{1, \infty}(\Omega)} + \| n_1 \|_{L^p(\Omega)} \| u \|_{L^\infty(\Omega)} \\
\leq C_{29}
$$

106
with some $C_{29} > 0$, we have

\begin{equation}
\|J_2\|_{L^\infty(\Omega)} \leq C_4 C_6 C_{29} \int_{\frac{t}{2}}^{t} (t-s)^{-\bar{\eta} + \frac{1}{2}} e^{-(\lambda_3 + 1)(t-s)} \, ds \\
\quad \leq C_4 C_6 C_{29} \int_{0}^{\infty} r^{-(\bar{\eta} + \frac{1}{2})} e^{-(\lambda_3 + 1)r} \, dr < \infty.
\end{equation}

Finally, since the identity

\begin{equation}
(1 + \mu_1)n_1 - \mu_1 n_1^2 = -\mu_1 \left( n_1 - \frac{1 + \mu_1}{2\mu_1} \right)^2 + \frac{(1 + \mu_1)^2}{4\mu_1}
\end{equation}

holds, the positivity of $n_1, n_2$ and the order preserving property give

\begin{align*}
J_3 &= \int_{\frac{t}{2}}^{t} e^{(t-s)(\Delta-1)} \left[ (1 + \mu_1)n_1 - \mu_1 n_1^2 - \mu_1 a_1 n_1 n_2 \right] \, ds \\
&\leq \int_{\frac{t}{2}}^{t} e^{(t-s)(\Delta-1)} \left[ (1 + \mu_1)n_1 - \mu_1 n_1^2 \right] \, ds \\
&= \int_{\frac{t}{2}}^{t} e^{(t-s)(\Delta-1)} \left[ -\mu_1 \left( n_1 - \frac{1 + \mu_1}{2\mu_1} \right)^2 + \frac{(1 + \mu_1)^2}{4\mu_1} \right] \, ds \\
&\leq \frac{(1 + \mu_1)^2}{4\mu_1} \int_{\frac{t}{2}}^{t} e^{(t-s)\Delta} e^{-(t-s)} \, ds.
\end{align*}

Then we establish from the maximum principle that

\begin{equation}
J_3 \leq \frac{(1 + \mu_1)^2}{4\mu_1} \int_{\frac{t}{2}}^{t} e^{-(t-s)} \, ds \\
\quad \leq \frac{(1 + \mu_1)^2}{4\mu_1} \left( 1 - e^{-\frac{t}{2}} \right).
\end{equation}

Therefore plugging (9.3.20), (9.3.21) and (9.3.22) into (9.3.19) derives that there exists $C_{30}(\tau) > 0$ such that

\begin{equation}
n_1(t) \leq \|J_1\|_{L^\infty(\Omega)} + \|J_2\|_{L^\infty(\Omega)} + J_3 \\
\quad \leq C_{30}(\tau), \quad t \in (\tau, T_{\text{max}}).
\end{equation}

Combination of (9.3.23) with the positivity of $n_1$ yields

\begin{equation}
\|n_1(\cdot, t)\|_{L^\infty(\Omega)} \leq C_{30}(\tau), \quad t \in (\tau, T_{\text{max}}).
\end{equation}

A similar argument leads to the $L^\infty$-estimate for $n_2$.

**Proof of Theorem 9.1.1.** Combination of (9.2.1) and Lemmas 9.3.8, 9.3.9 and 9.3.10 directly leads to Theorem 9.1.1.
9.4. Stabilization. Proof of Theorem 9.1.2

9.4.1. Key estimate. Case 1: $a_1, a_2 \in (0, 1)$

In this section we will show a key estimate for stabilization in (9.1.1) in the case that $a_1, a_2 \in (0, 1)$. The following lemma is a cornerstone in deriving it.

**Lemma 9.4.1.** Let $a_1, a_2 \in (0, 1)$. Under the assumption of Theorem 9.1.1, the solution of (9.1.1) has the property that there exist $k_1, \ell_1 > 0$ and $\varepsilon_1 > 0$ such that the nonnegative functions $E_1$ and $F_1$ defined by

$$E_1 := \int_{\Omega} \left( n_1 - N_1 \log \frac{n_1}{N_1} \right) + k_1 \int_{\Omega} \left( n_2 - N_2 \log \frac{n_2}{N_2} \right) + \frac{\ell_1}{2} \int_{\Omega} c^2$$

and

$$F_1 := \int_{\Omega} \left( n_1 - N_1 \right)^2 + \int_{\Omega} \left( n_2 - N_2 \right)^2$$

satisfy

$$\frac{d}{dt} E_1(t) \leq -\varepsilon_1 F_1(t) \quad \text{for all } t > 0,$$

where $N_1 = \frac{1-a_1}{1-a_1 a_2}$ and $N_2 = \frac{1-a_2}{1-a_1 a_2}$.

**Proof.** We now let $k_1, \ell_1 > 0$ be fixed later and denote by $A_1, A_2$ and $B_1$ the functions defined as

$$A_i(t) := \int_{\Omega} \left( n_i - N_i \log \frac{n_i}{N_i} \right) \quad \text{for } i = 1, 2, \quad B_1(t) := \frac{1}{2} \int_{\Omega} c^2,$$

and we write as

$$E_1(t) = A_1(t) + k_1 A_2(t) + \ell_1 B_1(t).$$

Writing as

$$A_1(t) = \int_{\Omega} \left( H(n_1) - H(N_1) + N_1 \right)$$

where $H(s) := s - N_1 \log s$ ($s > 0$), we see from the Taylor formula that $A_1(t) \geq 0$ for $t > 0$ (for more details, see [4, Lemma 3.2]). Similarly, we can have that $A_2(t) \geq 0$ for $t > 0$. Therefore $E_1$ is a nonnegative function. By the straightforward calculations we infer that

$$\frac{d}{dt} A_i(t) = -\mu_i \int_{\Omega} (n_i - N_i)^2 - a_i \mu_i \int_{\Omega} (n_1 - N_1)(n_2 - N_2) - N_i \int_{\Omega} \frac{\nabla n_i}{n_i^2}$$

$$+ N_i \chi_i \int_{\Omega} \frac{\nabla n_i \cdot \nabla c}{n_i} \quad \text{for } i = 1, 2.$$
and
\[
\frac{d}{dt} B_1(t) = - \int_{\Omega} |\nabla c|^2 - \int_{\Omega} c^2 (\alpha n_1 + \beta n_2).
\]

In light of the positivity of \(n_1, n_2, c\) and the Young inequality we have
\[
\frac{d}{dt} E_1(t) \leq -\mu_1 \int_{\Omega} (n_1 - N_1)^2 - (a_1 \mu_1 + k_1 a_2 \mu_2) \int_{\Omega} (n_1 - N_1)(n_2 - N_2)
- k_1 \mu_2 \int_{\Omega} (n_2 - N_2)^2 + \left( \frac{N_1 \chi_1^2}{4} + \frac{k_1 N_2 \chi_2^2}{4} - \ell_1 \right) \int_{\Omega} |\nabla c|^2.
\]

Noting from the Young inequality that for all \(\varepsilon \in (0, \mu_1)\),
\[
-\mu_1 (n_1 - N_1)^2 - (a_1 \mu_1 + k_1 a_2 \mu_2)(n_1 - N_1)(n_2 - N_2) - k_1 \mu_2 (n_2 - N_2)^2
\leq -\varepsilon (n_1 - N_1)^2 - \left( k_1 \mu_2 - \frac{(a_1 \mu_1 + k_1 a_2 \mu_2)^2}{4(\mu_1 - \varepsilon)} \right) (n_2 - N_2)^2,
\]
choosing \(k_1 = \frac{a_1 \mu_1}{a_2 \mu_2}\) and using the continuity argument, we can find \(\varepsilon > 0\) such that
\[
k_1 \mu_2 - \frac{(a_1 \mu_1 + k_1 a_2 \mu_2)^2}{4(\mu_1 - \varepsilon)} > 0.
\]

Therefore if we choose \(\ell_1 > \frac{N_1 \chi_1^2}{4} + \frac{k_1 N_2 \chi_2^2}{4}\), then we can see that there exists \(\varepsilon_1 > 0\) such that (9.4.1) holds. \(\square\)

Thanks to Lemma 9.4.1, we can show the key estimate, which plays an important role in the proof of Theorem 9.1.2.

**Lemma 9.4.2.** Let \(a_1, a_2 \in (0, 1)\). Under the assumption of Theorem 9.1.1, the solution of (9.1.1) satisfies that there exists a constant \(C_{31} > 0\) such that
\[
\int_0^\infty \int_{\Omega} (n_1 - N_1)^2 + \int_0^\infty \int_{\Omega} (n_2 - N_2)^2 \leq C_{31}.
\]

**Proof.** Integrating (9.4.1) over \((0, t)\), we infer
\[
E_1(t) + \varepsilon_1 \int_0^t F_1(s) \, ds \leq E_1(0).
\]

Thus combination of (9.4.3) with the nonnegativity of \(E_1\) implies (9.4.2). \(\square\)

**9.4.2. Key estimate. Case 2: \(a_1 \geq 1 > a_2\)**

This section is devoted to giving a key estimate for stabilization in (9.1.1) in the case \(a_1 \geq 1 > a_2\). The following lemma plays an important role in deriving it.
Lemma 9.4.3. Let \( a_1 \geq 1 > a_2 \). Under the assumption of Theorem 9.1.1, the solution of (9.1.1) has the property that there exist \( k_2, \ell_2 > 0 \) and \( \varepsilon_2 > 0 \) such that the nonnegative functions \( E_2 \) and \( F_2 \) defined by

\[
E_2 := \int_{\Omega} n_1 + k_2 \int_{\Omega} (n_2 - \log n_2) + \frac{\ell_2}{2} \int_{\Omega} c^2
\]

and

\[
F_2 := \int_{\Omega} n_1^2 + \int_{\Omega} (n_2 - 1)^2
\]

satisfy

\[
(9.4.4) \quad \frac{d}{dt} E_2(t) \leq -\varepsilon_2 F_2(t) \quad \text{for all } t > 0.
\]

Proof. We let \( k_2, \ell_2 > 0 \) be fixed later and denote by \( A_3, A_4 \) and \( B_2 \) the functions defined as

\[
A_3(t) := \int_{\Omega} n_1, \quad A_4(t) := \int_{\Omega} (n_2 - \log n_2), \quad B_2(t) := \frac{1}{2} \int_{\Omega} c^2,
\]

and we write as

\[
E_2(t) = A_3(t) + k_2 A_4(t) + \ell_2 B_2(t).
\]

A similar argument as in the proof of Lemma 9.4.1 leads to the nonnegativity of \( A_4 \). By the straightforward calculations we infer from the positivity of \( n_1, n_2 \) and the relation \( a_1 \geq 1 \) that

\[
\frac{d}{dt} A_3(t) = \mu_1 \int_{\Omega} n_1 (1 - n_1 - a_1 n_2)
\]

\[
\leq -\mu_1 \int_{\Omega} n_1 (n_2 - 1) - \mu_1 \int_{\Omega} n_1^2
\]

and

\[
\frac{d}{dt} A_4(t) = -\mu_2 \int_{\Omega} (n_2 - 1)^2 - a_2 \mu_2 \int_{\Omega} n_1 (n_2 - 1) - \int_{\Omega} \frac{\nabla n_2}{n_2^2} \left( \nabla n_2 \cdot \nabla c \right)
\]

\[
+ \chi_2 \int_{\Omega} \frac{\nabla n_2 \cdot \nabla c}{n_2}
\]

as well as

\[
\frac{d}{dt} B_2(t) = \int_{\Omega} |\nabla c|^2 - \int_{\Omega} c^2 (\alpha n_1 + \beta n_2).
\]

In light of the Young inequality we have

\[
\frac{d}{dt} E_2(t) \leq -\mu_1 \int_{\Omega} n_1^2 - (\mu_1 + k_2 a_2 \mu_2) \int_{\Omega} n_1 (n_2 - 1) - k_2 \mu_2 \int_{\Omega} (n_2 - 1)^2
\]

\[
+ \left( \frac{k_2 \chi_2^2}{4} - \ell_2 \right) \int_{\Omega} |\nabla c|^2.
\]

Therefore a similar argument as in the proof of Lemma 9.4.1 implies that there exist \( \varepsilon_2 > 0 \) and \( k_2 > 0 \) satisfying (9.4.4). □
By virtue of Lemma 9.4.3, we obtain the key estimate.

**Lemma 9.4.4.** Let $a_1 \geq 1 > a_2$. Under the assumption of Theorem 9.1.1, the solution of (9.1.1) satisfies that there exists a constant $C_{32} > 0$ such that

\[ \int_0^\infty \int_\Omega n_1^2 + \int_0^\infty \int_\Omega (n_2 - 1)^2 \leq C_{32}. \]  

**(Proof.** Integrating (9.4.4) over $(0,t)$, we infer

\[ E_2(t) + \varepsilon_2 \int_0^t F_2(s) \, ds \leq E_2(0). \] 

Thus combination of (9.4.6) with the nonnegativity of $E_2$ implies (9.4.5).

**9.4.3. Proof of Theorem 9.1.2**

In this section we complete the proof of Theorem 9.1.2 through some lemmas. Invoking Lemmas 9.4.2 and 9.4.4, we shall discuss convergence of $(n_i(\cdot,t))_{t>0}$ as $t \to \infty$. However, integrability around $t = \infty$ does not assure convergence as $t \to \infty$, and so we need the following lemma. The proof is based on parabolic regularity theory.

**Lemma 9.4.5.** Under the assumption of Theorem 9.1.1, there exist constants $C_{33} > 0$ and $\theta_0 > 0$ such that

\[ \|n_i\|_{C^{\theta_0, \frac{\theta_0}{2}}(\Omega \times [t,t+1])} \leq C_{33} \quad \text{for all } t \geq 1 \text{ and } i = 1, 2, \]

where $n_1, n_2$ are the first and second components of the solution.

**(Proof.** Aided by Lemmas 9.3.8, 9.3.9 and 9.3.10, standard parabolic regularity argument ([98, Theorem V.1.1], or [156, Theorem 1.3] with the argument in [100, Proof of Lemma 7.3]) yields the assertion of this lemma.

In the proof of convergence of $(n_i(\cdot,t))_{t>0}$ as $t \to \infty$ we will apply the following lemma (see Lemma 6.3.1).

**Lemma 9.4.6.** Let $n \in C^0(\overline{\Omega} \times [0,\infty))$ satisfy that there exist constants $C^* > 0$ and $\theta^* > 0$ such that

\[ \|n\|_{C^{\theta^*, \frac{\theta^*}{2}}(\Omega \times [t,t+1])} \leq C^* \quad \text{for all } t \geq 1. \]

Assume that

\[ \int_0^\infty \int_\Omega (n(x,t) - N^*)^2 \, dx \, dt < \infty \]

with some constant $N^* > 0$. Then

\[ n(\cdot,t) \to N^* \text{ in } C^0(\overline{\Omega}) \text{ as } t \to \infty. \]
Now we have already prepared all tools to prove stabilization of \( n_1, n_2 \).

**Lemma 9.4.7.** Under the assumption of Theorem 9.1.1, the solution of (9.1.1) satisfies the following.

(i) Assume that \( a_1, a_2 \in (0, 1) \). Then

\[
\|n_1(\cdot, t) - N_1\|_{L^\infty(\Omega)} \to 0, \quad \|n_2(\cdot, t) - N_2\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty,
\]

where \( N_1 = \frac{1-a_1}{1-a_1a_2}, N_2 = \frac{1-a_2}{1-a_1a_2} \).

(ii) Assume that \( a_1 \geq 1 > a_2 \). Then

\[
\|n_1(\cdot, t)\|_{L^\infty(\Omega)} \to 0, \quad \|n_2(\cdot, t) - 1\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty.
\]

**Proof.** In view of Lemmas 9.4.2, 9.4.4 and 9.4.5 we only need to apply Lemma 9.4.6. \( \square \)

We next give the following lemma to establish the decay properties of \( c \) and \( u \).

**Lemma 9.4.8.** Let \( a_2 \in (0, 1) \). Under the assumption of Theorem 9.1.1, the solution of (9.1.1) satisfies that for all \( C \in (0, |\Omega| \min\{N_2, 1\}) \) there exists \( T > 0 \) such that

\[
\int_\Omega n_2 \geq C \quad \text{for all } t > T.
\]

**Proof.** We first deal with the case that \( a_1, a_2 \in (0, 1) \). Noting from Lemma 9.4.7 that

\[
\int_\Omega n_2 \to \int_\Omega N_2 = N_2|\Omega| \quad \text{as } t \to \infty,
\]

we have that for all \( \varepsilon > 0 \) there exists \( T_1 > 0 \) such that

\[
\int_\Omega n_2 \geq N_2|\Omega| - \varepsilon \quad \text{for all } t > T_1.
\]

In the case that \( a_1 \geq 1 > a_2 \) we also see that for all \( \varepsilon > 0 \) there exists \( T_2 > 0 \) such that

\[
\int_\Omega n_2 \geq |\Omega| - \varepsilon \quad \text{for all } t > T_2,
\]

which means the end of the proof. \( \square \)

Aided by Lemma 9.4.8, similar arguments as in [201, Sections 4 and 6] lead to the following result.

**Lemma 9.4.9.** Under the assumption of Theorem 9.1.1, the solution of (9.1.1) satisfies

\[
\|c(\cdot, t)\|_{L^\infty(\Omega)} \to 0, \quad \|u(\cdot, t)\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty.
\]

**Proof of Theorem 9.1.2.** Lemmas 9.4.7 and 9.4.9 directly show Theorem 9.1.2. \( \square \)
Chapter 10

Global existence and asymptotic behaviour in a three-dimensional two-species chemotaxis-Navier–Stokes system with competitive kinetics

10.1. Problem and results

This chapter deals with the following two-species chemotaxis-Navier–Stokes system with Lotka–Volterra competitive kinetics:

\[
\begin{cases}
  (n_1)_t + u \cdot \nabla n_1 = \Delta n_1 - \chi_1 \nabla \cdot (n_1 \nabla c) + \mu_1 n_1 (1 - n_1 - a_1 n_2) & \text{in } \Omega \times (0, \infty), \\
  (n_2)_t + u \cdot \nabla n_2 = \Delta n_2 - \chi_2 \nabla \cdot (n_2 \nabla c) + \mu_2 n_2 (1 - a_2 n_1 - n_2) & \text{in } \Omega \times (0, \infty), \\
  c_t + u \cdot \nabla c = \Delta c - (\alpha n_1 + \beta n_2) c & \text{in } \Omega \times (0, \infty), \\
  u_t + \kappa (u \cdot \nabla) u = \Delta u + \nabla P + (\gamma n_1 + \delta n_2) \nabla \Phi, & \nabla \cdot u = 0 & \text{in } \Omega \times (0, \infty), \\
  \partial_{\nu} n_1 = \partial_{\nu} n_2 = \partial_{\nu} c = 0, & u = 0 & \text{on } \partial \Omega \times (0, \infty), \\
  n_1(\cdot, 0) = n_{1,0}, & n_2(\cdot, 0) = n_{2,0}, & c(\cdot, 0) = c_0, & u(\cdot, 0) = u_0 & \text{in } \Omega,
\end{cases}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \) with smooth boundary \( \partial \Omega \) and \( \partial_{\nu} \) denotes differentiation with respect to the outward normal of \( \partial \Omega \); \( \kappa = 1, \chi_1, \chi_2, a_1, a_2 \geq 0 \) and \( \mu_1, \mu_2, \alpha, \beta, \gamma, \delta > 0 \) are constants; \( n_{1,0}, n_{2,0}, c_0, u_0, \Phi \) are known functions satisfying

\[
\begin{align*}
  0 < n_{1,0}, n_{2,0} & \in C(\overline{\Omega}), & 0 < c_0 & \in W^{1,q}(\Omega), & u_0 & \in D(A^{\theta}), \\
  \Phi & \in C^{1+\lambda}(\overline{\Omega})
\end{align*}
\]

for some \( q > 3, \theta \in (\frac{3}{4}, 1), \lambda \in (0, 1) \) and \( A \) denotes the realization of the Stokes operator under homogeneous Dirichlet boundary conditions in the solenoidal subspace \( L^2_{\alpha}(\Omega) \) of \( L^2(\Omega) \).
In the mathematical point of view, difficulties of this problem are mainly caused by the chemotaxis terms $-\chi_1 \nabla \cdot (n_1 \nabla c)$, $-\chi_2 \nabla \cdot (n_2 \nabla c)$, the competitive kinetics $\mu_1 n_1 (1 - n_1 - a_1 n_2)$, $\mu_2 n_2 (1 - a_2 n_1 - n_2)$ and the Navier–Stokes equation which is the fourth equation in (10.1.1). In the case that $n_2 = 0$, global existence of weak solutions, and their eventual smoothness and stabilization were shown in [101]. On the other hand, in the case that $n_2 \neq 0$ and $\Omega \subset \mathbb{R}^2$, global existence and boundedness of classical solutions to (10.1.1) were attained (Chapter 9). Moreover, in the case that $\kappa = 0$ in (10.1.1), which namely means that the fourth equation in (10.1.1) is the Stokes equation, global existence and stabilization are in Chapter 11 (see later); in the case that $\kappa = 0$ in (10.1.1) and that $-(\alpha n_1 + \beta n_2)c$ is replaced with $+\alpha n_1 + \beta n_2 - c$, boundedness of classical solutions to the Keller–Segel-Stokes system and their asymptotic behaviour are in Chapter 12 (see later).

As we mentioned above, global classical solutions are found in (10.1.1) in the 2D setting and the case that $\kappa = 0$. However, global existence of solutions in 3-dimensional setting has not been attained. Thus the main purposes of this chapter is to obtain global existence of solutions to (10.1.1) in the case that $\Omega \subset \mathbb{R}^3$. Nevertheless, because of the difficulties of the Navier–Stokes equation, we can not expect global existence of classical solutions to (10.1.1) in the 3-dimensional case. Therefore our goal is to obtain global existence of weak solutions to (10.1.1) in the following sense.

**Definition 10.1.1.** A quadruple $(n_1, n_2, c, u)$ is called a (global) weak solution of (10.1.1) if

\[
\begin{align*}
n_1, n_2 &\in L^2_{\text{loc}}([0, \infty); L^2(\Omega)) \cap L^{\frac{4}{3}}_{\text{loc}}([0, \infty); W^{1,\frac{4}{3}}(\Omega)), \\
c &\in L^2_{\text{loc}}([0, \infty); W^{1,2}(\Omega)), \\
u &\in L^2_{\text{loc}}([0, \infty); W^{1,2}_{0,\sigma}(\Omega))
\end{align*}
\]

and for all $T > 0$ the identities

\[
\begin{align*}
- \int_0^T \int_{\Omega} n_1 \varphi_t - \int_{\Omega} n_{1,0} \varphi(\cdot, 0) &- \int_0^T \int_{\Omega} n_1 u \cdot \nabla \varphi \\
&= - \int_0^T \int_{\Omega} \nabla n_1 \cdot \nabla \varphi + \chi_1 \int_0^T \int_{\Omega} n_1 \nabla c \cdot \nabla \varphi + \mu_1 \int_0^T \int_{\Omega} n_1 (1 - n_1 - a_1 n_2) \varphi,
\end{align*}
\]

\[
\begin{align*}
- \int_0^T \int_{\Omega} n_2 \varphi_t - \int_{\Omega} n_{2,0} \varphi(\cdot, 0) &- \int_0^T \int_{\Omega} n_2 u \cdot \nabla \varphi \\
&= - \int_0^T \int_{\Omega} \nabla n_2 \cdot \nabla \varphi + \chi_2 \int_0^T \int_{\Omega} n_2 \nabla c \cdot \nabla \varphi + \mu_2 \int_0^T \int_{\Omega} n_2 (1 - a_2 n_1 - n_2) \varphi,
\end{align*}
\]

\[
\begin{align*}
- \int_0^T \int_{\Omega} c \varphi_t - \int_{\Omega} c_0 \varphi(\cdot, 0) &- \int_0^T \int_{\Omega} c u \cdot \nabla \varphi \\
&= - \int_0^T \int_{\Omega} \nabla c \cdot \nabla \varphi - \int_0^T \int_{\Omega} (\alpha n_1 + \beta n_2) c \varphi,
\end{align*}
\]

\[
\begin{align*}
- \int_0^T \int_{\Omega} u \cdot \psi_t - \int_{\Omega} u_0 \cdot \psi(\cdot, 0) &- \int_0^T \int_{\Omega} u \otimes u \cdot \nabla \psi \\
&= - \int_0^T \int_{\Omega} \nabla u \cdot \nabla \psi + \int_0^T \int_{\Omega} (\gamma n_1 + \delta n_2) \nabla \psi \cdot \nabla \Phi
\end{align*}
\]
hold for all \( \varphi \in C_0^\infty(\bar{\Omega} \times [0, \infty)) \) and all \( \psi \in C_0^{\infty}(\Omega \times [0, \infty)) \), respectively.

Now the main results read as follows. The first theorem is concerned with global existence of weak solutions to (10.1.1).

**Theorem 10.1.1.** Let \( \Omega \subset \mathbb{R}^3 \) be a bounded smooth domain and let \( \chi_1, \chi_2, a_1, a_2 \geq 0 \) and \( \mu_1, \mu_2, \alpha, \beta, \gamma, \delta > 0 \). Assume that \( n_{1,0}, n_{2,0}, c_0, u_0 \) satisfy (10.1.2) with some \( q > 3 \) and \( \theta \in (\frac{3}{2}, 1) \) and \( \Phi \in C^{1,\lambda}(\Omega) \) for some \( \lambda \in (0, 1) \). Then there is a weak solution of (10.1.1), which can be approximated by a sequence of solutions \((n_{1,\varepsilon}, n_{2,\varepsilon}, c_\varepsilon, u_\varepsilon)\) of (10.2.1) (see Section 10.2) in a pointwise manner.

The second theorem gives eventual smoothness and stabilization.

**Theorem 10.1.2.** Let the assumption of Theorem 10.1.1 be satisfied. Then there are \( T > 0 \) and \( \alpha' \in (0, 1) \) such that the solution \((n_1, n_2, c, u)\) given by Theorem 10.1.1 satisfies

\[
n_1, n_2, c \in C^{2+\alpha',1+\alpha'}(\bar{\Omega} \times [T, \infty)), \quad u \in C^{2+\alpha',1+\alpha'}(\bar{\Omega} \times [T, \infty)).
\]

Moreover, the solution of (10.1.1) has the following properties:

(i) Assume that \( a_1, a_2 \in (0, 1) \). Then

\[
n_1(\cdot, t) \to N_1, \quad n_2(\cdot, t) \to N_2, \quad c(\cdot, t) \to 0, \quad u(\cdot, t) \to 0 \quad \text{in} \quad L^\infty(\Omega)
\]

as \( t \to \infty \), where

\[
N_1 := \frac{1 - a_1}{1 - a_1 a_2}, \quad N_2 := \frac{1 - a_2}{1 - a_1 a_2}.
\]

(ii) Assume that \( a_1 \geq 1 > a_2 \). Then

\[
n_1(\cdot, t) \to 0, \quad n_2(\cdot, t) \to 1, \quad c(\cdot, t) \to 0, \quad u(\cdot, t) \to 0 \quad \text{in} \quad L^\infty(\Omega)
\]

as \( t \to \infty \).

The proofs of the main theorems are based on the arguments in [101]. The strategies for the proofs is to construct energy estimates for the solution \((n_{1,\varepsilon}, n_{2,\varepsilon}, c_\varepsilon, u_\varepsilon)\) of (10.2.1). In Section 10.2 we consider the energy function \( \mathcal{F}_\varepsilon \) defined as

\[
\mathcal{F}_\varepsilon := \int_\Omega n_{1,\varepsilon} \log n_{1,\varepsilon} + \int_\Omega n_{2,\varepsilon} \log n_{2,\varepsilon} + \frac{\chi}{2} \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + k_4 \chi \int_\Omega |u_\varepsilon|^2
\]

with some constant \( \chi > 0 \). Noting that for all \( \rho, \xi_i > 0 \) there exists \( C > 0 \) such that

\[
\int_\Omega \nabla c_\varepsilon \cdot \nabla n_{i,\varepsilon} \left( \frac{\chi_i}{1 + \varepsilon n_{i,\varepsilon}} - \frac{\chi \alpha (\text{or} \chi \beta)}{1 + \varepsilon (\alpha n_{i,\varepsilon} + \beta n_{2,\varepsilon})} \right) \\
\leq \rho \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \xi_i \int_\Omega \frac{|\nabla n_{i,\varepsilon}|^2}{n_{i,\varepsilon}} + C \int_\Omega n_{i,\varepsilon}^2 \quad (i = 1, 2),
\]
which did not appear in the previous work [101], from the estimate for the energy function $\mathcal{F}_\varepsilon$ we obtain global-in-time solvability of approximate solutions. Then we moreover see convergence as $\varepsilon \searrow 0$. Furthermore, in Section 10.3, according to an argument similar to Chapter 9, by putting

$$
G_{\varepsilon,B} := \int_\Omega \left( n_{1\varepsilon} - N_1 \log \frac{n_{1\varepsilon}}{N_1} \right) + \int_\Omega \left( n_{2\varepsilon} - N_2 \log \frac{n_{2\varepsilon}}{N_2} \right) + \frac{B}{2} \int_\Omega c_\varepsilon^2
$$

with suitable constant $B > 0$ and establishing the Hölder estimates for the solution of (10.1.1) through the estimate for the energy function $G_{\varepsilon,B},$ we can discuss convergence of $(n_1(\cdot,t), n_2(\cdot,t), c(\cdot,t), u(\cdot,t))$ as $t \to \infty$.

10.2. Proof of Theorem 10.1.1 (Global existence)

We will start by considering an approximate problem with parameter $\varepsilon > 0$, namely:

(10.2.1)

$$
\begin{aligned}
(n_{1\varepsilon})_t + u_\varepsilon \cdot \nabla n_{1\varepsilon} &= \Delta n_{1\varepsilon} - \chi_1 \nabla \cdot \left( \frac{n_{1\varepsilon}}{1 + \varepsilon n_{1\varepsilon}} \nabla c_\varepsilon \right) + \mu_1 n_{1\varepsilon} (1 - n_{1\varepsilon} - a_1 n_{2\varepsilon}), \\
(n_{2\varepsilon})_t + u_\varepsilon \cdot \nabla n_{2\varepsilon} &= \Delta n_{2\varepsilon} - \chi_2 \nabla \cdot \left( \frac{n_{2\varepsilon}}{1 + \varepsilon n_{2\varepsilon}} \nabla c_\varepsilon \right) + \mu_2 n_{2\varepsilon} (1 - a_2 n_{1\varepsilon} - n_{2\varepsilon}), \\
(c_\varepsilon)_t + u_\varepsilon \cdot \nabla c_\varepsilon &= \Delta c_\varepsilon - c_\varepsilon \frac{1}{\varepsilon} \log (1 + \varepsilon (a n_{1\varepsilon} + \beta n_{2\varepsilon})), \\
(u_\varepsilon)_t + (Y_\varepsilon u_\varepsilon \cdot \nabla) u_\varepsilon &= \Delta u_\varepsilon + \nabla P_\varepsilon + (\gamma n_{1\varepsilon} + \delta n_{2\varepsilon}) \nabla \varphi, \quad \nabla \cdot u_\varepsilon = 0, \\
\partial_\nu n_{1\varepsilon}|_{\partial \Omega} = \partial_\nu n_{2\varepsilon}|_{\partial \Omega} = \partial_\nu c_\varepsilon|_{\partial \Omega} = 0, \quad u_\varepsilon|_{\partial \Omega} = 0, \\
n_{1\varepsilon}(\cdot,0) = n_{1,0}, \quad n_{2\varepsilon}(\cdot,0) = n_{2,0}, \quad c_\varepsilon(\cdot,0) = c_0, \quad u_\varepsilon(\cdot,0) = u_0,
\end{aligned}
$$

where $Y_\varepsilon = (1 + \varepsilon A)^{-1}$, and provide estimates for its solutions. We first give the following result which states local existence in (10.1.1).

**Lemma 10.2.1.** Let $\chi_1, \chi_2, a_1, a_2 \geq 0$, $\mu_1, \mu_2, \alpha, \beta, \gamma, \delta > 0$, and $\varphi \in C^{1+\lambda}(\overline{\Omega})$ for some $\lambda \in (0,1)$ and assume that $n_{1,0}, n_{2,0}, c_0, u_0$ satisfy (10.1.2) with some $q > 3, \theta \in (\frac{3}{4},1)$. Then for all $\varepsilon > 0$ there are $T_{\max,\varepsilon}$ and uniquely determined functions:

$$
\begin{aligned}
n_{1\varepsilon}, n_{2\varepsilon} &\in C^0(\overline{\Omega} \times [0, T_{\max,\varepsilon})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max,\varepsilon})), \\
c_\varepsilon &\in C^0(\overline{\Omega} \times [0, T_{\max,\varepsilon})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max,\varepsilon})) \cap L^\infty([0, T_{\max,\varepsilon}); W^{1,q}(\Omega)), \\
u_\varepsilon &\in C^0(\overline{\Omega} \times [0, T_{\max,\varepsilon})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max,\varepsilon})),
\end{aligned}
$$

which together with some $P_\varepsilon \in C^{1,0}(\overline{\Omega} \times (0, T_{\max,\varepsilon}))$ solve (10.2.1) classically. Moreover, $n_{1\varepsilon}, n_{2\varepsilon}$ and $c_\varepsilon$ are positive and the following alternative holds: $T_{\max,\varepsilon} = \infty$ or

(10.2.2) \[ \|n_{1\varepsilon}(\cdot,t)\|_{L^\infty(\Omega)} + \|n_{2\varepsilon}(\cdot,t)\|_{L^\infty(\Omega)} + \|c_\varepsilon(\cdot,t)\|_{W^{1,q}(\Omega)} + \|A^q u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \to \infty \]

as $t \nearrow T_{\max,\varepsilon}$.

We next show the following lemma which holds a key for the proof of Theorem 10.1.1. This lemma derives the estimate for the energy function.
Lemma 10.2.2. For all $\xi_1, \xi_2 \in (0, 1)$ and $\chi > 0$ there are $C, \overline{C}, \widetilde{C}, k, \overline{k} > 0$ such that

$$F_\varepsilon := \int_\Omega n_{1,\varepsilon} \log n_{1,\varepsilon} + \int_\Omega n_{2,\varepsilon} \log n_{2,\varepsilon} + \frac{\chi}{2} \int_\Omega \frac{\nabla c_\varepsilon^2}{c_\varepsilon} + \overline{k} \int_\Omega |u_\varepsilon|^2$$

satisfies

$$\frac{d}{dt} F_\varepsilon \leq -\frac{\mu_1}{4} \int_\Omega n_{1,\varepsilon}^2 \log n_{1,\varepsilon} - \frac{\mu_2}{4} \int_\Omega n_{2,\varepsilon}^2 \log n_{2,\varepsilon} + (1 - \xi_1) \int_\Omega \frac{\nabla n_{1,\varepsilon}^2}{n_{1,\varepsilon}} + (1 - \xi_2) \int_\Omega \frac{\nabla n_{2,\varepsilon}^2}{n_{2,\varepsilon}} + C \int_\Omega n_{1,\varepsilon}^2 + \overline{C} \int_\Omega n_{2,\varepsilon}^2 + \widetilde{C}$$

$$- k \int_\Omega c_\varepsilon |D^2 \log c_\varepsilon|^2 - k \int_\Omega \frac{\nabla c_\varepsilon^4}{c_\varepsilon^2} - k \int_\Omega |\nabla u_\varepsilon|^2$$

on $(0, T_{\text{max}, \varepsilon})$ for all $\varepsilon > 0$.

Proof. Noting, the boundedness of $s(1 - s)$ and $s(1 - \frac{s}{2}) \log s$, we have that there exists $C_1 > 0$ such that

$$\frac{d}{dt} \int_\Omega n_{1,\varepsilon} \log n_{1,\varepsilon} = - \int_\Omega \frac{\nabla n_{1,\varepsilon}^2}{n_{1,\varepsilon}} + \chi \int_\Omega \frac{\nabla c_\varepsilon \cdot \nabla n_{1,\varepsilon}}{1 + \varepsilon n_{1,\varepsilon}} + \mu_1 \int_\Omega n_{1,\varepsilon} (1 - n_{1,\varepsilon} - a_1 n_{2,\varepsilon}) \log n_{1,\varepsilon} + \mu_1 \int_\Omega n_{1,\varepsilon} (1 - n_{1,\varepsilon} - a_1 n_{2,\varepsilon})$$

$$\leq - \int_\Omega \frac{\nabla n_{1,\varepsilon}^2}{n_{1,\varepsilon}} + \chi \int_\Omega \frac{\nabla c_\varepsilon \cdot \nabla n_{1,\varepsilon}}{1 + \varepsilon n_{1,\varepsilon}} - \frac{\mu_1}{2} \int_\Omega n_{1,\varepsilon}^2 \log n_{1,\varepsilon}$$

$$- \mu_1 a_1 \int_\Omega n_{1,\varepsilon} n_{2,\varepsilon} \log n_{1,\varepsilon} - \mu_1 a_1 \int_\Omega n_{1,\varepsilon} n_{2,\varepsilon} + C_1.$$  

Similarly, there is $C_2 > 0$ such that

$$\frac{d}{dt} \int_\Omega n_{2,\varepsilon} \log n_{2,\varepsilon} \leq - \int_\Omega \frac{\nabla n_{2,\varepsilon}^2}{n_{2,\varepsilon}} + \chi \int_\Omega \frac{\nabla c_\varepsilon \cdot \nabla n_{2,\varepsilon}}{1 + \varepsilon n_{2,\varepsilon}} - \frac{\mu_2}{2} \int_\Omega n_{2,\varepsilon}^2 \log n_{2,\varepsilon}$$

$$- \mu_2 a_2 \int_\Omega n_{1,\varepsilon} n_{2,\varepsilon} \log n_{2,\varepsilon} - \mu_2 a_2 \int_\Omega n_{1,\varepsilon} n_{2,\varepsilon} + C_2.$$  

According to an argument similar to that in the proof of [101, Lemma 2.8], there exist $k_1, C_3, C_4 > 0$ such that

$$\frac{d}{dt} \int_\Omega \frac{\nabla c_\varepsilon^2}{c_\varepsilon} \leq - k_1 \int_\Omega c_\varepsilon |D^2 \log c_\varepsilon|^2 - k_1 \int_\Omega \frac{\nabla c_\varepsilon^4}{c_\varepsilon^2}$$

$$+ C_3 + C_4 \int_\Omega |\nabla u_\varepsilon|^2 - 2 \int_\Omega \frac{\alpha \nabla c_\varepsilon \cdot \nabla n_{1,\varepsilon} + \beta \nabla c_\varepsilon \cdot \nabla n_{2,\varepsilon}}{1 + \varepsilon (a n_{1,\varepsilon} + \beta n_{2,\varepsilon})}.$$  

117
Now we let \( \bar{k}, \eta_1, \eta_2, k \) be constants satisfying \( \frac{C_1}{2} - \bar{k} = -\frac{k_1}{4}, \eta_1 = \frac{\mu_1}{4k}, \eta_2 = \frac{\mu_2}{4k} \) and \( k = \frac{\sqrt{2}}{4} \). Then we have

\[
\frac{d}{dt} \int_{\Omega} |u_\varepsilon|^2 \leq -2 \int_{\Omega} \nabla u_\varepsilon |^2 - 2 \int_{\Omega} u_\varepsilon \cdot (Y_\varepsilon u_\varepsilon \cdot \nabla) u_\varepsilon + 2 \int_{\Omega} u_\varepsilon \cdot (\gamma n_{1,\varepsilon} + \delta n_{2,\varepsilon}) \nabla \Phi.
\]

From the Schwarz inequality, the Poincaré inequality, the Young inequality and the fact that \( \int_{\Omega} \varphi^2 \leq a \int_{\Omega} \varphi \log \varphi + |\Omega| \varphi^\frac{2}{4} \) holds for any positive function \( \varphi \) and any \( a > 0 \), there exist \( C_5, C_\eta, C_{n_2} > 0 \) such that

\[
\gamma \int_{\Omega} |n_{1,\varepsilon} \nabla \Phi \cdot u_\varepsilon| \leq \gamma \|\nabla \Phi\|_{L^\infty} \left( \int_{\Omega} n_{1,\varepsilon} \right)^{\frac{1}{2}} \left( \int_{\Omega} |u_\varepsilon|^2 \right)^{\frac{1}{2}}
\]

\[
\leq \gamma \|\nabla \Phi\|_{L^\infty} \left( \int_{\Omega} n_{1,\varepsilon} \right)^{\frac{1}{2}} \left( C_5 \int_{\Omega} |\nabla u_\varepsilon|^2 \right)^{\frac{1}{2}}
\]

\[
\leq \gamma^2 C_5 \|\nabla \Phi\|_{L^\infty}^2 \int_{\Omega} n_{1,\varepsilon} + \frac{1}{4} \int_{\Omega} |\nabla u_\varepsilon|^2
\]

\[
\leq \frac{\eta_2}{2} \int_{\Omega} n_{1,\varepsilon} \log n_{1,\varepsilon} + \frac{C_{n_2}}{2} + \frac{1}{4} \int_{\Omega} |\nabla u_\varepsilon|^2
\]

and

\[
\delta \int_{\Omega} |n_{2,\varepsilon} \nabla \Phi \cdot u_\varepsilon| \leq \frac{\eta_2}{2} \int_{\Omega} n_{2,\varepsilon} \log n_{2,\varepsilon} + \frac{C_{n_2}}{2} + \frac{1}{4} \int_{\Omega} |\nabla u_\varepsilon|^2
\]

hold. Therefore we have

(10.2.6) \( \frac{d}{dt} \int_{\Omega} |u_\varepsilon|^2 \leq - \int_{\Omega} |\nabla u_\varepsilon|^2 + \eta_1 \int_{\Omega} n_{1,\varepsilon} \log n_{1,\varepsilon} + \eta_2 \int_{\Omega} n_{2,\varepsilon} \log n_{2,\varepsilon} + C_{n_1} + C_{n_2} \).

Thus a combination of (10.2.3), (10.2.4), (10.2.5) and (10.2.6) leads to

\[
\frac{d}{dt} \left[ \int_{\Omega} n_{1,\varepsilon} \log n_{1,\varepsilon} + \int_{\Omega} n_{2,\varepsilon} \log n_{2,\varepsilon} + \frac{\chi}{2} \int_{\Omega} |\nabla c_\varepsilon|^2 + \bar{k} \chi \int_{\Omega} |u_\varepsilon|^2 \right]
\]

\[
\leq \left( \bar{k} \chi \eta_1 - \frac{\mu_1}{2} \right) \int_{\Omega} n_{1,\varepsilon} \log n_{1,\varepsilon} + \frac{\mu_2}{2} \int_{\Omega} n_{2,\varepsilon} \log n_{2,\varepsilon}
\]

\[
- \left( \int_{\Omega} |\nabla n_{1,\varepsilon}|^2 + \int_{\Omega} |\nabla n_{2,\varepsilon}|^2 \right) + \left( \frac{\chi}{2} C_4 - \bar{k} \chi \right) \int_{\Omega} |\nabla u_\varepsilon|^2
\]

\[
+ \int_{\Omega} \nabla c_\varepsilon \cdot \nabla n_{1,\varepsilon} \left( \frac{\chi_1}{1 + \varepsilon n_{1,\varepsilon}} - \frac{\chi_2}{1 + \varepsilon (an_{1,\varepsilon} + \beta n_{2,\varepsilon})} \right)
\]

\[
+ \int_{\Omega} \nabla c_\varepsilon \cdot \nabla n_{2,\varepsilon} \left( \frac{\chi_2}{1 + \varepsilon n_{2,\varepsilon}} - \frac{\chi_3}{1 + \varepsilon (an_{1,\varepsilon} + \beta n_{2,\varepsilon})} \right)
\]

\[
- \frac{\chi}{2} k_1 \int_{\Omega} c_\varepsilon \cdot D^2 \log c_\varepsilon|^2 - \frac{\chi}{2} k_1 \int_{\Omega} \left| \nabla c_\varepsilon \right|^4 + C_1 + C_2 + \frac{\chi}{2} C_3 + \bar{k} \chi (C_{n_1} + C_{n_2})
\]

\[
- \mu_1 a_1 \int_{\Omega} n_{1,\varepsilon} n_{2,\varepsilon} (\log n_{1,\varepsilon} + 1) - \mu_2 a_2 \int_{\Omega} n_{1,\varepsilon} n_{2,\varepsilon} (\log n_{2,\varepsilon} + 1).
\]
Here, since $n_{1,e}, n_{2,e}$ are nonnegative, we can find $C_6, C_7 > 0$ such that

$$\int_{\Omega} \nabla c_{e} \cdot \nabla n_{1,e} \left( \frac{\chi_1}{1 + \varepsilon n_{1,e}} - \frac{\chi_\alpha}{1 + \varepsilon (\alpha n_{1,e} + \beta n_{2,e})} \right)$$

$$\leq (\chi_1 + \chi_\alpha) \int_{\Omega} |\nabla c_{e} \cdot \nabla n_{1,e}|$$

$$\leq \frac{\chi_1 k_1}{8 \|c_0\|_{L^\infty}^3} \int_{\Omega} |\nabla c_{e}|^2 + C_6 \int_{\Omega} |\nabla n_{1,e}|^4$$

$$\leq \frac{\chi_1 k_1}{8} \int_{\Omega} \frac{|\nabla c_{e}|^4}{c_0^3} + \xi_1 \int_{\Omega} \frac{|\nabla n_{1,e}|^2}{n_{1,e}} + C_7 \int_{\Omega} n_{1,e}^2$$

and there is $C_8 > 0$ such that

$$\int_{\Omega} \nabla c_{e} \cdot \nabla n_{2,e} \left( \frac{\chi_2}{1 + \varepsilon n_{2,e}} - \frac{\chi_\beta}{1 + \varepsilon (\alpha n_{1,e} + \beta n_{2,e})} \right)$$

$$\leq \frac{\chi_2 k_1}{8} \int_{\Omega} \frac{|\nabla c_{e}|^4}{c_0^3} + \xi_2 \int_{\Omega} \frac{|\nabla n_{2,e}|^2}{n_{2,e}} + C_8 \int_{\Omega} n_{2,e}^2,$$

which with the fact that $s \log s \geq -\frac{1}{\varepsilon} (s > 0)$ enables us to obtain

$$\left( \bar{k}\chi \eta_1 - \frac{\mu_1}{2} \right) \int_{\Omega} n_{1,e}^2 \log n_{1,e} + \left( \bar{k}\chi \eta_2 - \frac{\mu_2}{2} \right) \int_{\Omega} n_{2,e}^2 \log n_{2,e}$$

$$- \left( \int_{\Omega} \frac{|\nabla n_{1,e}|^2}{n_{1,e}} + \int_{\Omega} \frac{|\nabla n_{2,e}|^2}{n_{2,e}} \right) + \left( \frac{\chi}{2} C_1 - \bar{k}\chi \right) \int_{\Omega} |\nabla u_e|^2$$

$$+ \int_{\Omega} \nabla c_{e} \cdot \nabla n_{1,e} \left( \frac{\chi_1}{1 + \varepsilon n_{1,e}} - \frac{\chi_\alpha}{1 + \varepsilon (\alpha n_{1,e} + \beta n_{2,e})} \right)$$

$$+ \int_{\Omega} \nabla c_{e} \cdot \nabla n_{2,e} \left( \frac{\chi_2}{1 + \varepsilon n_{2,e}} - \frac{\chi_\beta}{1 + \varepsilon (\alpha n_{1,e} + \beta n_{2,e})} \right)$$

$$- \frac{\chi_1 k_1}{8} \int_{\Omega} c_0 D^2 \log c_{e}^2 - \frac{\chi_2 k_1}{8} \int_{\Omega} \frac{|\nabla c_{e}|^4}{c_0^3} + C_1 + C_2 + \frac{\chi}{2} C_3 + \bar{k}\chi (C_{n_1} + C_{n_2})$$

$$- \mu_1 a_1 \int_{\Omega} n_{1,e} n_{2,e} (\log n_{1,e} + 1) - \mu_2 a_2 \int_{\Omega} n_{1,e} n_{2,e} (\log n_{2,e} + 1)$$

$$\leq - \frac{\mu_1}{4} \int_{\Omega} n_{1,e}^2 \log n_{1,e} - \frac{\mu_2}{4} \int_{\Omega} n_{2,e}^2 \log n_{2,e}$$

$$- (1 - \xi_1) \int_{\Omega} \frac{|\nabla n_{1,e}|^2}{n_{1,e}} - (1 - \xi_2) \int_{\Omega} \frac{|\nabla n_{2,e}|^2}{n_{1,e}}$$

$$- k \int_{\Omega} |\nabla u_e|^2 - k \int_{\Omega} c_e D^2 \log c_e - k \int_{\Omega} \frac{|\nabla c_e|^4}{c_0^3} + C_7 \int_{\Omega} n_{1,e}^2 + C_8 \int_{\Omega} n_{2,e}^2 + C_9.$$

Therefore we obtain this lemma. \(\square\)

**Proof of Theorem 10.1.1.** Let \(\tau = \min\{1, \frac{1}{2} T_{\max,e}\}, \xi_1, \xi_2 \in (0, 1)\) and \(\chi > 0\). Lemma
10.2.2, the facts that \( s^2 \log s \geq s \log s - \frac{1}{2s} (s > 0) \) and \( n_{1, \varepsilon}, n_{2, \varepsilon}, c_\varepsilon > 0 \) imply
\[
\frac{d}{dt} F_\varepsilon + F_\varepsilon \leq C \int_{\Omega} n_{1, \varepsilon}^2 + C \int_{\Omega} n_{2, \varepsilon}^2 + \tilde{C}'
\]
for some \( C, \tilde{C}, \tilde{C}' > 0 \). According to \([101, \text{Lemma 2.5}]\), there exists \( C_1 > 0 \) such that
\[
\int_{t}^{t+\tau} \int_{\Omega} n_{i, \varepsilon}^2 \leq C_1
\]
for all \( t \in (0, T_{\text{max}, \varepsilon} - \tau) \) and each \( i = 1, 2 \). From the uniform Gronwall type lemma (see e.g., \([176, \text{Lemma 3.2}]\)) we can find \( C_2 > 0 \) such that
\[
\begin{align*}
(10.2.7) & \quad \int_{\Omega} n_{1, \varepsilon} \log n_{1, \varepsilon} + \int_{\Omega} n_{2, \varepsilon} \log n_{2, \varepsilon} + \frac{\chi}{2} \int_{\Omega} |\nabla c_\varepsilon|^2 + \bar{\kappa} \int_{\Omega} |u_\varepsilon|^2 \leq C_2 \\
& \quad + (1 - \xi_1) \int_{t}^{t+\tau} \int_{\Omega} |\nabla n_{1, \varepsilon}|^2 + (1 - \xi_2) \int_{t}^{t+\tau} \int_{\Omega} |\nabla n_{2, \varepsilon}|^2 \leq C_3
\end{align*}
\]
and
\[
(10.2.9) & \quad \int_{t}^{t+\tau} \int_{\Omega} \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \int_{t}^{t+\tau} \int_{\Omega} |u_\varepsilon|^2 \leq C_3
\]
as well as
\[
(10.2.10) & \quad \int_{t}^{t+\tau} \int_{\Omega} |\nabla n_{1, \varepsilon}|^\frac{4}{3} + \int_{t}^{t+\tau} \int_{\Omega} |\nabla n_{2, \varepsilon}|^\frac{4}{3} \\
& \quad + \int_{t}^{t+\tau} \int_{\Omega} |\nabla c_\varepsilon|^2 + \int_{t}^{t+\tau} \int_{\Omega} |\nabla c_\varepsilon|^4 + \int_{t}^{t+\tau} \int_{\Omega} n_{1, \varepsilon}^2 + \int_{t}^{t+\tau} \int_{\Omega} n_{2, \varepsilon}^2 \leq C_3
\]
for all \( t \in [0, T_{\text{max}, \varepsilon} - \tau) \). Now we assume \( T_{\text{max}, \varepsilon} < \infty \) for some \( \varepsilon > 0 \). From (10.2.7), (10.2.8), (10.2.9) and (10.2.10), we can see that there exists \( C_4 > 0 \) such that
\[
\|n_{1, \varepsilon}(\cdot, t)\|_{L^\infty(\Omega)} \leq C_4,
\|n_{2, \varepsilon}(\cdot, t)\|_{L^\infty(\Omega)} \leq C_4
\]
and
\[
\|c_\varepsilon(\cdot, t)\|_{W^{1,q}(\Omega)} \leq C_4
\]
as well as
\[
\|A^\varepsilon u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \leq C_4
\]
and
for all $t \in (0, T_{\max, \varepsilon})$, which is inconsistent with (10.2.2). Therefore we obtain $T_{\max, \varepsilon} = \infty$ for all $\varepsilon > 0$, which means global existence and boundedness of $(n_{1, \varepsilon}, n_{2, \varepsilon}, c_{\varepsilon}, u_{\varepsilon})$. We next verify convergence of the solution $(n_{1, \varepsilon}, n_{2, \varepsilon}, c_{\varepsilon}, u_{\varepsilon})$. Due to Lemma 10.2.2 and arguments similar to those in [101], we establish that for all $T > 0$ there is $C_{\varepsilon} > 0$ such that

\begin{align}
\|(n_{1, \varepsilon})_{t}\|_{L^1((0,T);W^2,4(\Omega))} &\leq C_{\varepsilon}, \\
\|(n_{2, \varepsilon})_{t}\|_{L^1((0,T);W^1,4(\Omega))} &\leq C_{\varepsilon}, \\
\|(c_{\varepsilon})_{t}\|_{L^2((0,T);W^1,1(\Omega))} &\leq C_{\varepsilon}, \\
\|(u_{\varepsilon})_{t}\|_{L^2((0,T);W^{1,3}(\Omega))} &\leq C_{\varepsilon}
\end{align}

for all $\varepsilon > 0$, which together with arguments in [101] implies that there exist a sequence $(\varepsilon_j)_{j \in \mathbb{N}}$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$ and functions $n_1, n_2, c, u$ such that

$$
n_1, n_2 \in L^2_{loc}([0, \infty); L^2(\Omega)) \cap C^0_{loc}([0, \infty); W^{1,4}(\Omega)), \\
c \in L^2_{loc}([0, \infty); W^{1,2}(\Omega)), \\
u \in L^2_{loc}([0, \infty); W^{1,2}_{0,\sigma}(\Omega))\$$

and that

\begin{align}
n_{1, \varepsilon} &\to n_1 \quad \text{in } L^4_{loc}([0, \infty); L^p(\Omega)) \quad \text{for all } p \in \left[1, \frac{12}{5}\right) \text{ and a.e. in } \Omega \times (0, \infty), \\
n_{2, \varepsilon} &\to n_2 \quad \text{in } L^4_{loc}([0, \infty); L^p(\Omega)) \quad \text{for all } p \in \left[1, \frac{12}{5}\right) \text{ and a.e. in } \Omega \times (0, \infty), \\
c_{\varepsilon} &\to c \quad \text{in } C^0_{loc}([0, \infty); L^p(\Omega)) \quad \text{for all } p \in [1, 6) \text{ and a.e. in } \Omega \times (0, \infty), \\
u_{\varepsilon} &\to u \quad \text{in } L^4_{loc}([0, \infty); L^p(\Omega)) \quad \text{for all } p \in [1, 6) \text{ and a.e. in } \Omega \times (0, \infty), \\
c_{\varepsilon} &\to c \quad \text{weakly* in } L^\infty(\Omega \times (t, t+1)) \quad \text{for all } t \geq 0, \\
abla n_{1, \varepsilon} &\to \nabla n_1 \quad \text{weakly in } L^4_{loc}([0, \infty); L^4(\Omega)), \\
abla n_{2, \varepsilon} &\to \nabla n_2 \quad \text{weakly in } L^4_{loc}([0, \infty); L^4(\Omega)), \\
abla c_{\varepsilon} &\to \nabla c \quad \text{weakly* in } L^4_{loc}([0, \infty); L^2(\Omega)), \\
abla u_{\varepsilon} &\to \nabla u \quad \text{weakly in } L^4_{loc}([0, \infty); L^2(\Omega)), \\
_y u_{\varepsilon} &\to u \quad \text{in } L^4_{loc}([0, \infty); L^2(\Omega)), \\
n_{1, \varepsilon} &\to n_1 \quad \text{in } L^4_{loc}([0, \infty); L^2(\Omega)), \\
n_{2, \varepsilon} &\to n_2 \quad \text{in } L^4_{loc}([0, \infty); L^2(\Omega))
\end{align}

as $\varepsilon = \varepsilon_j \searrow 0$. Thus we see that $(n_1, n_2, c, u)$ is a weak solution to (10.1.1) in the sense of Definition 10.1.1, which means the end of the proof. 

\[ \square \]

### 10.3. Proof of Theorem 10.1.2 (Eventual smoothness and stabilization)

In this section we will prove Theorem 10.1.2. The following lemma plays an important role in the proof of Theorem 10.1.2.
Lemma 10.3.1. (i) Assume that \( a_1, a_2 \in (0,1) \). Then there exists \( C > 0 \) such that for all \( \varepsilon > 0 \),

\[
\int_0^\infty \int_\Omega (n_{1,\varepsilon} - N_1)^2 \leq C, \quad \int_0^\infty \int_\Omega (n_{2,\varepsilon} - N_2)^2 \leq C,
\]

where \( N_1 = \frac{1-a_1}{1-a_1 a_2}, \ N_2 = \frac{1-a_2}{1-a_1 a_2} \).

(ii) Assume \( a_1 \geq a_2 > 0 \). Then there exists \( C > 0 \) such that for all \( \varepsilon > 0 \),

\[
\int_0^\infty \int_\Omega n_{1,\varepsilon}^2 \leq C, \quad \int_0^\infty \int_\Omega (n_{2,\varepsilon} - 1)^2 \leq C.
\]

Proof. Due to arguments similar to those in Lemmas 9.4.1–9.4.4, by using the energy functions

\[
G_{\varepsilon,B} := \int_\Omega \left( n_{1,\varepsilon} - N_1 \log \frac{n_{1,\varepsilon}}{N_1} \right) + \int_\Omega \left( n_{2,\varepsilon} - N_2 \log \frac{n_{2,\varepsilon}}{N_2} \right) + \frac{B}{2} \int_\Omega c_{\varepsilon}^2
\]

in the case that \( a_1, a_2 \in (0,1) \), and

\[
G_{\varepsilon,B} := \int_\Omega n_{1,\varepsilon} + \int_\Omega (n_{2,\varepsilon} - \log n_{2,\varepsilon}) + \frac{B}{2} \int_\Omega c_{\varepsilon}^2
\]

in the case that \( a_1 \geq 1 > a_2 > 0 \), we can see this lemma. \( \square \)

Proof of Theorem 10.1.2. According to an argument similar to that in the proof of [101, Lemmas 3.4 and 3.5], for all \( \eta > 0 \) and \( p \in (1, \infty) \) there are \( T > 0, \varepsilon_0 > 0 \) and \( C_1 > 0 \) such that for all \( t > T \) and \( \varepsilon \in (0, \varepsilon_0) \),

\[
\|c_{\varepsilon}(\cdot, t)\|_{L^\infty(\Omega)} < \eta, \quad \|n_{1,\varepsilon}(\cdot, t)\|_{L^p(\Omega)} \leq C_1, \quad \|n_{2,\varepsilon}(\cdot, t)\|_{L^p(\Omega)} \leq C_1.
\]

We next consider the estimate for \( u_{\varepsilon} \). Since \( \nabla \cdot u_{\varepsilon} = 0 \), it follows from the Young inequality, the Poincaré inequality, boundedness of \( \nabla \Phi \) and (10.2.1) that there exists \( C_2 > 0 \) such that

\[
\frac{d}{dt} \int_\Omega |u_{\varepsilon}|^2 = -2 \int_\Omega |\nabla u_{\varepsilon}|^2 - 2 \int_\Omega u_{\varepsilon} \cdot (Y_{\varepsilon} u_{\varepsilon} \cdot \nabla) u_{\varepsilon} + 2 \int_\Omega u_{\varepsilon} \cdot (\gamma n_{1,\varepsilon} + \delta n_{2,\varepsilon}) \nabla \Phi
\]

\[
= -2 \int_\Omega |\nabla u_{\varepsilon}|^2 - 2 \int_\Omega u_{\varepsilon} \cdot (Y_{\varepsilon} u_{\varepsilon} \cdot \nabla) u_{\varepsilon}
\]

\[
+ 2\gamma \int_\Omega u_{\varepsilon} \cdot (n_{1,\varepsilon} - n_{1,\infty}) \nabla \Phi + 2\delta \int_\Omega u_{\varepsilon} \cdot (n_{2,\varepsilon} - n_{2,\infty}) \nabla \Phi
\]

\[
\leq - \int_\Omega |\nabla u_{\varepsilon}|^2 - 2 \int_\Omega u_{\varepsilon} \cdot (Y_{\varepsilon} u_{\varepsilon} \cdot \nabla) u_{\varepsilon}
\]

\[
+ C_2 \int_\Omega (n_{1,\varepsilon} - n_{1,\infty})^2 + C_2 \int_\Omega (n_{2,\varepsilon} - n_{2,\infty})^2,
\]

122
where

\[
(n_{1,\infty}, n_{2,\infty}) = \begin{cases} \quad (N_1, N_2) & (a_1, a_2 \in (0, 1)), \\ (0, 1) & (a_1 \geq 1 > a_2 > 0).
\end{cases}
\]

Then, noticing from straightforward calculations that

\[
\int_{\Omega} u_\varepsilon \cdot (Y_\varepsilon u_\varepsilon \cdot \nabla) u_\varepsilon = \frac{1}{2} \int_{\Omega} \nabla \cdot (Y_\varepsilon u_\varepsilon) |u_\varepsilon|^2 = 0,
\]

thanks to Lemma 10.3.1, we obtain from integration of the above inequality over \((0, \infty)\) that there exists \(C_3 > 0\) such that

\[
\int_{0}^{\infty} \int_{\Omega} |\nabla u_\varepsilon|^2 \leq C_3.
\]

According to an argument similar to that in the proof of [101, Lemmas 3.7–3.11], there exist \(\alpha' > 0\), \(T^* > T\), \(C_4 > 0\) such that for all \(t > T^*\) there exists \(\varepsilon_1 > 0\) such that for all \(\varepsilon \in (0, \varepsilon_1)\),

\[
\|n_{1, \varepsilon}\|_{C^{1+\alpha', \frac{\alpha'}{2}}([\Omega \times [t, t+1])} \leq C_4,
\]

\[
\|n_{2, \varepsilon}\|_{C^{1+\alpha', \frac{\alpha'}{2}}([\Omega \times [t, t+1])} \leq C_4,
\]

and

\[
\|c_{\varepsilon}\|_{C^{1+\alpha', \frac{\alpha'}{2}}([\Omega \times [t, t+1])} \leq C_4,
\]

as well as

\[
\|u_\varepsilon\|_{C^{1+\alpha', \frac{\alpha'}{2}}([\Omega \times [t, t+1])} \leq C_4
\]

hold. Then aided by arguments similar to those in the proofs of [101, Corollary 3.3–Lemma 3.13], from (10.2.11) there are \(\alpha' \in (0, 1)\) and \(T_0 > 0\) as well as a subsequence \(\varepsilon_j \searrow 0\) such that for all \(t > T_0\)

\[
n_{1, \varepsilon} \rightarrow n_1, \quad n_{2, \varepsilon} \rightarrow n_2, \quad c_{\varepsilon} \rightarrow c \quad \text{in} \quad C^{1+\alpha', \frac{\alpha'}{2}}([\Omega \times [t, t+1])
\]

and

\[
u_\varepsilon \rightarrow u \quad \text{in} \quad C^{1+\alpha', \frac{\alpha'}{2}}([\Omega \times [t, t+1])
\]

as \(\varepsilon = \varepsilon_j \searrow 0\), and then

(10.3.1)

\[
\|n_1\|_{C^{1+\alpha', \frac{\alpha'}{2}}([\Omega \times [t, t+1])} \leq C_4,
\]

\[
\|n_2\|_{C^{1+\alpha', \frac{\alpha'}{2}}([\Omega \times [t, t+1])} \leq C_4
\]

123
and
\begin{equation}
\|c\|_{C^{1+a',\frac{n'}{2}}(\Omega \times [t,t+1])} \leq C_4
\end{equation}
as well as
\begin{equation}
\|u\|_{C^{1+a',\frac{n'}{2}}(\Omega \times [t,t+1])} \leq C_4.
\end{equation}

Then we obtain that
\[ n_1, n_2, c \in C^{2+a',1+\frac{n'}{2}}(\Omega \times [T_0, \infty)) \]
and
\[ u \in C^{2+a',1+\frac{n'}{2}}(\Omega \times [T_0, \infty)). \]

Finally, from (10.3.1)–(10.3.3) the solution \((n_1, n_2, c, u)\) of (10.2.1) constructed in (10.2.12) fulfills
\[ n_1(\cdot, t) \to N_1, \quad n_2(\cdot, t) \to N_2, \quad c(\cdot, t) \to 0, \quad u(\cdot, t) \to 0 \quad \text{in} \ C^1(\Omega) \quad (t \to \infty) \]
in the case that \(a_1, a_2 \in (0, 1)\), and
\[ n_1(\cdot, t) \to 0, \quad n_2(\cdot, t) \to 1, \quad c(\cdot, t) \to 0, \quad u(\cdot, t) \to 0 \quad \text{in} \ C^1(\Omega) \quad (t \to \infty) \]
in the case that \(a_1 \geq 1 > a_2 > 0\), which enable us to see Theorem 10.1.2. \qed
Chapter 11

Global existence and asymptotic behavior of classical solutions for a 3D two-species chemotaxis-Stokes system with competitive kinetics

11.1. Motivation and results

In this chapter we consider the following two-species chemotaxis-fluid system with competitive terms:

\[
\begin{aligned}
(n_1)_t + u \cdot \nabla n_1 &= \Delta n_1 - \chi_1 \nabla \cdot (n_1 \nabla c) + \mu_1 n_1 (1 - n_1 - a_1 n_2), \quad x \in \Omega, \ t > 0, \\
(n_2)_t + u \cdot \nabla n_2 &= \Delta n_2 - \chi_2 \nabla \cdot (n_2 \nabla c) + \mu_2 n_2 (1 - a_2 n_1 - n_2), \quad x \in \Omega, \ t > 0, \\
c_t + u \cdot \nabla c &= \Delta c - (\alpha n_1 + \beta n_2) c, \quad x \in \Omega, \ t > 0, \\
u_t + \kappa (u \cdot \nabla) u &= \Delta u + \nabla P + (\gamma n_1 + \delta n_2) \nabla \phi, \quad \nabla \cdot u = 0, \quad x \in \Omega, \ t > 0, \\
\partial_n n_1 = \partial_n n_2 = \partial_n c &= 0, \quad u = 0, \quad x \in \partial \Omega, \ t > 0, \\
n_i(x, 0) &= n_{i,0}(x), \ c(x, 0) = c_0(x), \ u(x, 0) = u_0(x), \quad x \in \Omega, \ i = 1, 2,
\end{aligned}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \) with smooth boundary \( \partial \Omega \) and \( \partial_n \) denotes differentiation with respect to the outward normal of \( \partial \Omega \); \( \kappa \in \{0, 1\} \) (in this chapter we will deal with the case that \( \kappa = 0 \)), \( \chi_1, \chi_2, a_1, a_2 \geq 0 \) and \( \mu_1, \mu_2, \alpha, \beta, \gamma, \delta > 0 \) are constants; \( n_{1,0}, n_{2,0}, c_0, u_0, \phi \) are known functions satisfying

\[
\begin{aligned}
0 < n_{1,0}, n_{2,0} &\in C(\overline{\Omega}), \quad 0 < c_0 \in W^{1,q}(\Omega), \quad u_0 \in D(A^\vartheta), \\
\phi &\in C^{1+\eta}(\overline{\Omega})
\end{aligned}
\]

for some \( q > 3, \ \vartheta \in (\frac{3}{2}, 1) \), \( \eta > 0 \) and \( A \) is the Stokes operator in \( L^2(\Omega) \).

The problem (11.1.1) is a generalized system to the chemotaxis-fluid system which is proposed by Tuval et al. [179]. This system describes the evolution of two competing species which react on a single chemoeattractant in a liquid surrounding environment.
Here \( n_1, n_2 \) represent the population densities of species, \( c \) stands for the concentration of chemoattractant, \( u \) shows the fluid velocity field and \( P \) represents the pressure of the fluid. The problem (11.1.1) comes from a problem on account of the influence of chemotaxis, the Lotka–Volterra competitive kinetics and the fluid. In the mathematical point of view, the chemotaxis term: \( \nabla \cdot (n_1 \nabla c) \) and the Stokes equation give difficulties in mathematical analysis.

The one-species system (11.1.1) with \( n_2 = 0 \) has been studied: it is known that there exist global classical solutions in the 2-dimensional setting; however, in the 3-dimensional setting, only global weak solutions are known to exist. In this one-species system with \( \mu_1 = 0 \), Winkler first attained global existence of classical solutions to (11.1.1), \( \kappa = 0 \) in the 3-dimensional setting and \( \kappa = 1 \) in the 2-dimensional setting ([197]), and also established asymptotic stability of solutions to (11.1.1) ([201]). Moreover, the convergence rate has been already studied ([212]). Recently, Winkler [206] attained global existence and eventual smoothness of weak solutions and their asymptotic behavior for the 3-dimensional chemotaxis-Navier–Stokes system.

In the analysis of the one-species case the logistic source can enhance the possibility of global existence of solutions. In the 3-dimensional setting, Lankeit [101] obtained global existence of weak solutions in (11.1.1) with \( n_2 = 0 \), \( \kappa = 1 \) and with additional external force \( f \) in the fourth equation, and also derived eventual smoothness and asymptotic behavior. Even for more complicated problems, Keller–Segel-fluid systems where \( -(\alpha n_1 + \beta n_2)c \) is replaced with \( -c + \alpha n_1 \) in (11.1.1) with \( n_2 = 0 \), logistic source is shown to be helpful for establishing classical bounded solutions. In the 3-dimensional setting, Tao and Winkler [174] established global existence and boundedness of classical solutions by assuming that \( \mu_1 > 23 \). In the 2-dimensional case, Tao and Winkler [176] also showed global existence of bounded classical solutions in the Keller–Segel-Navier–Stokes system with logistic source with \( +r n_1 - \mu_1 n_1^2 \) for any \( \mu_1 > 0 \), and their asymptotic behavior were obtained when \( r = 0 \). For more related works we refer to Ishida [80], Wang and the first author [187], Wang and Xiang [190], Black [15], the first author [27], the first author and Lankeit [30], Kozono, Miura and Sugiyama [93]. These results fully parallel to those for the fluid free model; we can find counterparts in [104, 152, 170].

On the other hand, the study on two-species competitive chemotaxis systems with signal consumption seems pending. We can only find related research with signal production in which the asymptotic behavior of solutions usually relies on some smallness assumption for the chemotaxis sensitivities (e.g., for the noncompetitive case \( a_1 = a_2 = 0 \), see Negreanu and Tello [150, 151] and Chapters 3, 4; for the competitive case see Tello and Winkler [178], Stinner, Tello and Winkler [164], Bai and Winkler [4] and Chapter 5, 7.

As mentioned above, the chemotaxis-fluid systems (\( n_2 = 0 \) in (11.1.1)) and the chemotaxis systems with competitive terms (\( u = 0 \) in (11.1.1)) were studied by many mathematicians. However, the problem (11.1.1), which is the combination of chemotaxis-fluid systems and chemotaxis-competition systems, had not been studied. Recently, global existence, boundedness of classical solutions and their asymptotic behavior were shown only in the 2-dimensional setting (Chapter 9). On the other hand, the 3-dimensional setting is a more realistic problem; however, in the 3-dimensional chemotaxis-Navier–Stokes setting, difficulties of the Navier–Stokes equation strongly affect: all works which dealt with the 3-dimensional chemotaxis-Navier–Stokes system could establish only weak solutions.
The purpose of this chapter is to obtain global existence and boundedness of classical solutions, and their asymptotic stability in the 3-dimensional chemotaxis-Stokes setting. The main results read as follows. The first theorem gives global existence and boundedness in (11.1.1) under smallness conditions of the chemotactic effect and largeness conditions of the logistic power. This result provides some guide to methods and results for the 3-dimensional Keller–Segel-Stokes system which is the case that $-(\alpha n_1 + \beta n_2)c$ is replaced with $-c + \alpha n_1 + \beta n_2$ in (11.1.1) (see Chapter 12 later).

**Theorem 11.1.1.** Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary and let $\kappa = 0$, $\chi_1, \chi_2, a_1, a_2 \geq 0$, $\mu_1, \mu_2, \alpha, \beta, \gamma, \delta > 0$. Suppose that (11.1.2) and (11.1.3) hold with some $q > 3$, $\theta \in (\frac{3}{4}, 1)$ and $\eta > 0$. Then there exists a constant $\xi_0 = \xi_0(\alpha, \beta) > 0$ such that whenever $\chi := \max\{\chi_1, \chi_2\}$ and $\mu := \min\{\mu_1, \mu_2\}$ satisfy $\chi\|c_0\|_{L^\infty(\Omega)} < \xi_0\mu$, the problem (11.1.1) possesses a classical solution $(n_1, n_2, c, u, P)$ such that

$$
n_1, n_2 \in C(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)),
$$

$$
c \in C(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)) \cap L^\infty((0, \infty); W^{1,q}(\Omega)),
$$

$$
u \in C(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)) \cap L^\infty((0, \infty); D(A^\theta)),
$$

$$
P \in C^{1,\theta}(\overline{\Omega} \times (0, \infty)).
$$

Also, the solution is unique in the sense that it allows up to addition of spatially constant functions to the pressure $P$. Moreover, there exists a constant $C > 0$ such that

$$
\|n_1(\cdot, t)\|_{L^\infty(\Omega)} + \|n_2(\cdot, t)\|_{L^\infty(\Omega)} + \|c(\cdot, t)\|_{W^{1,q}(\Omega)} + \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t \in (0, \infty).
$$

The second theorem is concerned with asymptotic stability in (11.1.1).

**Theorem 11.1.2.** Let the assumption of Theorem 11.1.1 hold. Then the solution of (11.1.1) has the following properties:

(i) Assume that $a_1, a_2 \in (0, 1)$. Then

$$
n_1(\cdot, t) \to N_1, \quad n_2(\cdot, t) \to N_2, \quad c(\cdot, t) \to 0, \quad u(\cdot, t) \to 0 \quad \text{in } L^\infty(\Omega) \quad \text{as } t \to \infty,
$$

where

$$
N_1 := \frac{1 - a_1}{1 - a_1 a_2}, \quad N_2 := \frac{1 - a_2}{1 - a_1 a_2}.
$$

(ii) Assume that $a_1 \geq 1 > a_2$. Then

$$
n_1(\cdot, t) \to 0, \quad n_2(\cdot, t) \to 1, \quad c(\cdot, t) \to 0, \quad u(\cdot, t) \to 0 \quad \text{in } L^\infty(\Omega) \quad \text{as } t \to \infty.
$$

(iii) Assume that $a_2 \geq 1 > a_1$. Then

$$
n_1(\cdot, t) \to 1, \quad n_2(\cdot, t) \to 0, \quad c(\cdot, t) \to 0, \quad u(\cdot, t) \to 0 \quad \text{in } L^\infty(\Omega) \quad \text{as } t \to \infty.
$$

**Remark 11.1.1.** This theorem is concerned with asymptotic behaviour in the cases that $a_1, a_2 < 1$, $a_2 < 1 \leq a_1$ and $a_1 < 1 \leq a_2$. However, the case that $a_1, a_2 \geq 1$ is still open: Even in not only the fluid-free case but also the Lotka–Volterra competition system asymptotic behaviour in the case that $a_1, a_2 \geq 1$ is a difficult problem (see sentences after [4, Theorem 1.1]).
The strategy for the proof of Theorem 11.1.1 is to derive an $L^p$-estimate for $n_i$ with $p > \frac{3}{2}$. By using a differential inequality we can see
\[
\int_{\Omega} n_1^p + \int_{\Omega} n_2^p \leq C \int_{s_0}^t e^{-(p+1)(t-s)} \int_{\Omega} |\Delta c|^{p+1} - C \int_{s_0}^t e^{-(p+1)(t-s)} \left( \int_{\Omega} n_1^{p+1} + \int_{\Omega} n_2^{p+1} \right)
\]
with some $C > 0$ and $s_0 > 0$. The maximal Sobolev regularity (see Lemma 11.2.2) will be used to control $\int_{s_0}^t e^{-(p+1)(t-s)} \int_{\Omega} |\Delta c|^{p+1}$. Combining the maximal Sobolev regularity with some estimate for $|Au|^2$, we can obtain the $L^p$-estimate for $n_i$. On the other hand, the strategy for the proof of Theorem 11.1.2 is to derive the following inequality:
\[
\int_0^\infty \int_{\Omega} (n_1 - N_1)^2 + \int_0^\infty \int_{\Omega} (n_2 - N_2)^2 + \int_0^\infty \int_{\Omega} (\alpha n_1 + \beta n_2)^2 \leq C
\]
with some $C > 0$, where $(N_1, N_2, 0, 0)$ is a constant solution to (11.1.1). This together with
\[
\inf_{(x,t) \in \Omega \times (0,\infty)} n_2(x,t) > 0
\]
and
\[
y'(t) + b_1 y(t) \leq h(t), \quad t \in (0, \infty)
\]
with some constant $b_1 > 0$, where
\[
y(t) := \int_{\Omega} |u(\cdot,t)|^2 \quad \text{and} \quad h(t) := b_2 \int_{\Omega} (n_1(\cdot,t) - N_1)^2 + b_2 \int_{\Omega} (n_2(\cdot,t) - N_2)^2
\]
with some constant $b_2 > 0$, enables us to see stabilization in (11.1.1).

This chapter is organized as follows. In Section 11.2 we collect basic facts which will be used later. In Section 11.3 we prove global existence and boundedness (Theorem 11.1.1). Section 11.4 is devoted to showing asymptotic stability (Theorem 11.1.2).

\section{11.2. Local existence and basic inequality}

In this section we will provide some results which will be used later. The following lemma gives local existence of solutions to (11.1.1).

\begin{lemma}
Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. Suppose that (11.1.2) and (11.1.3) hold with some $q > 3$, $\vartheta \in \left(\frac{3}{q}, 1\right)$ and $\eta > 0$. Then there exists $T_{\max} \in (0, \infty]$ such that the problem (11.1.1) possesses a classical solution $(n_1, n_2, c, u, P)$ fulfilling
\[
n_1, n_2 \in C(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})),
\]
\[
c \in C(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})) \cap L^\infty_{\text{loc}}([0, T_{\max}); W^{1,q}(\Omega)),
\]
\[
u \in C(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})) \cap L^\infty_{\text{loc}}([0, T_{\max}); D(A^{\vartheta})),
\]
\[
n_1, n_2 > 0, \quad c > 0 \quad \text{in} \ \Omega \times (0, T_{\max}).
\]
Also, the solution is unique up to addition of spatially constant functions to the pressure $P$. Moreover, either $T_{\max} = \infty$ or
\[
\limsup_{t \nearrow T_{\max}} (\|n_1(\cdot,t)\|_{L^\infty(\Omega)} + \|n_2(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,q}(\Omega)} + \|A^{\vartheta} u(\cdot,t)\|_{L^2(\Omega)}) = \infty.
\]

128
Proof. The proof of local existence of classical solutions to (11.1.1) is based on a standard
contraction mapping argument, which can be found in [197]. Moreover, the maximum
principle is applied to yield \( n_1, n_2 > 0 \) and \( c > 0 \) in \( \Omega \times (0, T_{\max}) \).

In the following, by \((n_1, n_2, c, u, P)\) we will denote the corresponding solution to
(11.1.1) given by Lemma 11.2.1 and by \( T_{\max} \) its maximal existence time. Now we fix
\( s_0 \in (0, T_{\max}) \). Then from the regularity properties the solution \((n_1, n_2, c, u, P)\) of (11.1.1)
provided by Lemma 11.2.1 satisfies that
\[
  c(\cdot, s_0) \in C^2(\Omega) \quad \text{with} \quad \partial_\nu c(\cdot, s_0) = 0 \quad \text{on} \partial\Omega.
\]
In particular, we can put
\[
  M = M(s_0, n_{1,0}, n_{2,0}, c_0, u_0) > 0 \quad \text{defined as}
\]
(11.2.1)
\[
  M := \|c(\cdot, s_0)\|_{W^{2,p}(\Omega)} < \infty
\]
(see e.g., [210]). The following lemma is referred to as a variation of the maximal Sobolev
regularity (see [70, Theorem 3.1]), which is important to prove Theorem 11.1.1.

**Lemma 11.2.2.** Let \( s_0 \in (0, T_{\max}) \). Then for all \( p > 1 \) there exists a constant \( C_1 = C_1(p) \) \( > 0 \) such that the solution \((n_1, n_2, c, u, P)\) of (11.1.1) satisfies
\[
  \int_{s_0}^{t} \int_{\Omega} e^{ps} |\Delta c|^p \leq C_1 \int_{s_0}^{t} \int_{\Omega} e^{ps} |2c - (\alpha n_1 + \beta n_2) c - u \cdot \nabla c|^p + C_1 |\Omega| M^p e^{p s_0}
\]
for all \( t \in (s_0, T_{\max}) \).

**Proof.** Let \( s_0 \in (0, T_{\max}) \) and let \( t \in (s_0, T_{\max}) \). We rewrite the third equation as
\[
  c_t = (\Delta - 1)c - c + (2c - (\alpha n_1 + \beta n_2) c - u \cdot \nabla c),
\]
and use the transformation \( \tilde{c}(\cdot, s) = e^{s} c(\cdot, s), s \in (s_0, t) \). Then \( \tilde{c} \) satisfies
\[
\begin{align*}
  &\tilde{c}_t = (\Delta - 1)\tilde{c} + f, \quad x \in \Omega, \ s \in (s_0, t), \\
  &\partial_\nu \tilde{c} = 0, \quad x \in \partial\Omega, \ s \in (s_0, t), \\
  &\tilde{c}(\cdot, s_0) = e^{s_0} c(\cdot, s_0) \in W^{2,p}(\Omega), \ x \in \Omega,
\end{align*}
\]
where
\[
  f := e^{s} (2c - (\alpha n_1 + \beta n_2) c - u \cdot \nabla c) \in L^p(s_0, t; L^p(\Omega)).
\]
Therefore an application of the maximal Sobolev regularity [70, Theorem 3.1] to \( \tilde{c} \) implies
this lemma. \( \square \)
11.3. Boundedness. Proof of Theorem 11.1.1

In this section we will prove Theorem 11.1.1 by preparing a series of lemmas. We first recall elementary inequalities which are useful in the proof of Theorem 11.1.1.

**Lemma 11.3.1.** There exists a constant \( C_2 = C_2(\|n_{1,0}\|_{L^1(\Omega)}, \|n_{2,0}\|_{L^1(\Omega)}, |\Omega|) > 0 \) such that
\[
\int_{\Omega} n_i(\cdot, t) \leq C_2 \quad \text{for all } t \in (0, T_{\text{max}}) \text{ and } i = 1, 2.
\]
Moreover,
\[
\int_{t}^{t+\tau} \int_{\Omega} n_i^2 \leq \left( \tau + \frac{1}{\mu_i} \right) C_2
\]
holds for all \( t \in (0, T_{\text{max}} - \tau) \), all \( \tau \in (0, T_{\text{max}}) \) and \( i = 1, 2 \).

**Proof.** Integrating the first and second equations in (11.1.1) implies this lemma. \( \square \)

**Lemma 11.3.2.** The function \( t \mapsto \|c(\cdot, t)\|_{L^1(\Omega)} \) is nonincreasing. In particular,
\[
\|c(\cdot, t)\|_{L^1(\Omega)} \leq \|c_0\|_{L^1(\Omega)}
\]
holds for all \( t \in (0, T_{\text{max}}) \). Moreover, we have
\[
\int_{0}^{T_{\text{max}}} \int_{\Omega} |\nabla c|^2 \leq \frac{1}{2} \int_{\Omega} |c_0|^2.
\]

**Proof.** We can show the \( L^1 \)-estimate for \( c \) by applying the maximum principle to the third equation in (11.1.1) (see e.g., [201, Lemma 2.1]). Moreover, multiplying the third equation in (11.1.1) by \( c \) and integrating it over \( \Omega \times (0, T_{\text{max}}) \), we have this lemma. \( \square \)

**Lemma 11.3.3.** For all \( r \in (1, 3) \),
\[
\|u(\cdot, t)\|_{L^r(\Omega)} \leq C_3 \quad \text{for all } t \in (0, T_{\text{max}})
\]
with some \( C_3 = C_3(r, \gamma, \delta, \|n_{1,0}\|_{L^1(\Omega)}, \|n_{2,0}\|_{L^1(\Omega)}, \|u_0\|_{D(A^\gamma)}, |\Omega|) > 0 \).

**Proof.** From well-known Neumann heat semigroup estimates together with Lemma 11.3.1 we can obtain the \( L^r \)-estimate for \( u \) with \( r \in (1, 3) \) (for more details, see [202, Corollary 3.4]). \( \square \)

The proofs of the following two lemmas are based on the methods in [210, Lemma 3.1].

**Lemma 11.3.4.** For all \( p > 1, \varepsilon > 0 \) and \( \ell > 0 \) there exist constants \( C_4 = C_4(p) > 0 \) and \( C_5 = C_5(\mu_1, \mu_2, \varepsilon, p, |\Omega|) > 0 \) such that
\[
\frac{1}{p} \int_{\Omega} n_1^{p}(\cdot, t) + \frac{1}{p} \int_{\Omega} n_2^{p}(\cdot, t)
\leq -(\mu - \varepsilon - \ell) e^{-(p-1)t} \int_{s_0}^{t} e^{(p-1)s} \left( \int_{\Omega} n_1^{p+1}(\cdot, s) + \int_{\Omega} n_2^{p+1}(\cdot, s) \right) ds
+ C_4 \ell^{p} \chi^{p+1} e^{-(p-1)t} \int_{s_0}^{t} e^{(p-1)s} \int_{\Omega} |\Delta c(\cdot, s)|^{p+1} ds + C_5
\]
for all \( t \in (s_0, T_{\text{max}}) \), where \( \chi := \max\{\chi_1, \chi_2\} \) and \( \mu := \min\{\mu_1, \mu_2\} \).
Proof. Let $p > 1$. Multiplying the first equation in (11.1.1) by $n_1^{p-1}$ and integrating it over $\Omega$, we see that

$$
\frac{1}{p} \frac{d}{dt} \int_{\Omega} n_1^p = -\frac{1}{p} \int_{\Omega} u \cdot \nabla n_1^p + \int_{\Omega} n_1^{p-1} \Delta n_1 + \chi_1 \frac{p-1}{p} \int_{\Omega} \nabla n_1^p \cdot \nabla c + \mu_1 \int_{\Omega} n_1^p - \mu_1 \int_{\Omega} n_1^{p+1} - a_1 \mu_1 \int_{\Omega} n_1^p n_2
$$

for all $t \in (0, T_{\text{max}})$. Noting from $\nabla \cdot u = 0$ in $\Omega \times (0, T_{\text{max}})$ that $\int_{\Omega} u \cdot \nabla n_1^p = -\int_{\Omega} (\nabla \cdot u)n_1^p = 0$, we obtain from integration by parts and the nonnegativity of $n_1, n_2$ that

\begin{equation}
\frac{1}{p} \frac{d}{dt} \int_{\Omega} n_1^p = -(p-1) \int_{\Omega} n_1^{p-2} |\nabla n_1|^2 - \chi_1 \frac{p-1}{p} \int_{\Omega} n_1^{p} \Delta c + \mu_1 \int_{\Omega} n_1^p
- \mu_1 \int_{\Omega} n_1^{p+1} - a_1 \mu_1 \int_{\Omega} n_1^p n_2
\end{equation}

$$
\leq -\chi_1 \frac{p-1}{p} \int_{\Omega} n_1^{p} \Delta c + \mu_1 \int_{\Omega} n_1^p - \mu_1 \int_{\Omega} n_1^{p+1}
= -\frac{p+1}{p} \int_{\Omega} n_1^{p} - \chi_1 \frac{p-1}{p} \int_{\Omega} n_1^{p} \Delta c + (\mu_1 + \frac{p+1}{p}) \int_{\Omega} n_1^p
- \mu_1 \int_{\Omega} n_1^{p+1}
$$

for all $t \in (s_0, T_{\text{max}})$. Now we let $\varepsilon > 0$ and $\ell > 0$. By the Young inequality there exists a constant $C_6 = C_6(\mu_1, \varepsilon, p, |\Omega|) > 0$ such that

\begin{equation}
(\mu_1 + \frac{p+1}{p}) \int_{\Omega} n_1^{p} \leq \varepsilon \int_{\Omega} n_1^{p+1} + C_6.
\end{equation}

Moreover, the second term on the right-hand side of (11.3.1) can be estimated as

\begin{equation}
-\chi_1 \frac{p-1}{p} \int_{\Omega} n_1^{p} \Delta c \leq \chi_1 \int_{\Omega} n_1^{p} |\Delta c| \leq \ell \int_{\Omega} n_1^{p+1} + C_7 \ell^{-p} \chi_1^{p+1} \int_{\Omega} |\Delta c|^{p+1}
\end{equation}

with some $C_7 = C_7(p) > 0$. Hence we derive from (11.3.1), (11.3.2) and (11.3.3) that

$$
\frac{1}{p} \frac{d}{dt} \int_{\Omega} n_1^p + \frac{p+1}{p} \int_{\Omega} n_1^p \leq -(\mu_1 - \varepsilon - \ell) \int_{\Omega} n_1^{p+1}
+ C_7 \ell^{-p} \chi_1^{p+1} \int_{\Omega} |\Delta c|^{p+1} + C_6
$$

for all $t \in (s_0, T_{\text{max}})$. Therefore there exists $C_8 = C_8(\mu_1, \varepsilon, p, |\Omega|) > 0$ such that

\begin{equation}
\frac{1}{p} \int_{\Omega} n_1^p \leq e^{-(p+1)(t-s_0)} \frac{1}{p} \int_{\Omega} n_1^p' (\xi, s_0) - (\mu_1 - \varepsilon - \ell) \int_{s_0}^{t} e^{-(p+1)(t-s)} \int_{\Omega} n_1^{p+1}
+ C_7 \ell^{-p} \chi_1^{p+1} \int_{s_0}^{t} e^{-(p+1)(t-s)} \int_{\Omega} |\Delta c|^{p+1} + C_6 \int_{s_0}^{t} e^{-(p+1)(t-s)}
\leq -(\mu_1 - \varepsilon - \ell)e^{-(p+1)t} \int_{s_0}^{t} e^{(p+1)s} \int_{\Omega} n_1^{p+1}
+ C_7 \ell^{-p} \chi_1^{p+1} e^{-(p+1)t} \int_{s_0}^{t} e^{(p+1)s} \int_{\Omega} |\Delta c|^{p+1} + C_8
$$

131
for each \( t \in (s_0, T_{\text{max}}) \). Similarly, we see that

\[
(11.3.5) \quad \frac{1}{p} \int_\Omega n_2^p \leq -(\mu_2 - \varepsilon - \ell) e^{-(p+1)t} \int_{s_0}^t e^{(p+1)s} \int_\Omega n_2^{p+1} + C_{10} \]

for all \( t \in (s_0, T_{\text{max}}) \) with some \( C_9 = C_9(\mu_2, \varepsilon, p, |\Omega|) > 0 \). Thus from (11.3.4) and (11.3.5) we have that there exists \( C_{10} = C_{10}(\mu_1, \mu_2, \varepsilon, p, |\Omega|) > 0 \) such that

\[
\frac{1}{p} \int_\Omega n_1^p + \frac{1}{p} \int_\Omega n_2^p \leq -(\mu - \varepsilon - \ell) e^{-(p+1)t} \int_{s_0}^t e^{(p+1)s} \left( \int_\Omega n_1^{p+1} + \int_\Omega n_2^{p+1} \right) + 2C_{10} \]

holds for all \( t \in (s_0, T_{\text{max}}) \), where \( \chi = \max\{\chi_1, \chi_2\} \) and \( \mu = \min\{\mu_1, \mu_2\} \).

In order to control \( \int_{s_0}^t e^{(p+1)s} \int_\Omega |\nabla c|^{p+1} \) we provide the following two lemmas.

**Lemma 11.3.5.** For all \( \varepsilon > 0 \) and \( p \in (1, 2) \) there is \( C_{11} = C_{11}(p, \alpha, \beta) > 0 \) and \( C_{12} = C_{12}(\varepsilon, p, \alpha, \beta, \|n_1\|_{L^1(\Omega)}, \|n_2\|_{L^1(\Omega)}, \|c_0\|_{L^\infty(\Omega)}, \|u_0\|_{D(A^*)}, M, |\Omega|) > 0 \), where \( M \) is defined as (11.2.1), such that

\[
\int_{s_0}^t e^{(p+1)s} \int_\Omega |\nabla c|^{p+1} \leq C_{11} \int_{s_0}^t e^{(p+1)s} \left( \int_\Omega n_1^{p+1} + \int_\Omega n_2^{p+1} \right) + \varepsilon \int_{s_0}^t e^{(p+1)s} \int_\Omega |Au|^2 + C_{12} e^{(p+1)t} + C_{12}
\]

for all \( t \in (s_0, T_{\text{max}}) \).

**Proof.** Fix \( \theta \in (1, 2) \) and put \( \theta' = \frac{\theta}{\theta - 1} \). We then derive from Lemma 11.2.2 that

\[
(11.3.6) \quad \int_{s_0}^t e^{(p+1)s} \int_\Omega |\nabla c|^{p+1} \leq C_{13} \int_{s_0}^t e^{(p+1)s} \int_\Omega \left| 2c - (\alpha n_1 + \beta n_2) c - u \cdot \nabla c \right|^{p+1} + |\Omega|M^{p+1} C_{13} e^{(p+1)s_0}
\]

holds for all \( t \in (s_0, T_{\text{max}}) \) with some \( C_{13} = C_{13}(p) > 0 \). Lemma 11.3.2 and the Hölder inequality imply

\[
(11.3.7) \quad \int_{s_0}^t e^{(p+1)s} \int_\Omega \left| 2c - (\alpha n_1 + \beta n_2) c - u \cdot \nabla c \right|^{p+1} \leq C_{14} \int_{s_0}^t e^{(p+1)s} \left( \int_\Omega n_1^{p+1} + n_2^{p+1} \right) + C_{14} e^{(p+1)t}
\]

\[
\leq C_{14} \int_{s_0}^t e^{(p+1)s} \left( \int_\Omega n_1^{p+1} + n_2^{p+1} \right) + C_{14} e^{(p+1)t}
\]

132
for all \( t \in (s_0, T_{\max}) \) with some \( C_{14} = C_{14}(p, \alpha, \beta) > 0 \). Here we see from the Gagliardo–Nirenberg inequality and Lemma 11.3.2 that there exist constants \( C_{15} = C_{15}(p, |\Omega|) \), \( C_{16} = C_{16}(p, \|c_0\|_{L^\infty(\Omega)}, |\Omega|) > 0 \) such that

\[
(11.3.8) \quad \|\nabla c\|_{L^{s(p+1)\eta}(\Omega)}^{(p+1)} \leq C_{15}\|\Delta c\|_{L^{s(p+1)\eta}(\Omega)}^{d(p+1)}\|c\|_{L^\infty(\Omega)}^{(1-d)(p+1)} + C_{15}\|c\|_{L^1(\Omega)}^{p+1} \leq C_{16}\|\Delta c\|_{L^{s(p+1)\eta}(\Omega)}^{d(p+1)} + C_{16}
\]

with \( a := \frac{1 - \frac{3}{(p+1)\eta}}{\frac{2}{p+1}} \in (\frac{1}{2}, 1) \). By (11.3.6), (11.3.7), (11.3.8) and the Young inequality it holds that

\[
(11.3.9) \quad \int_{s_0}^t e^{(p+1)s} \int_0^t \|\Delta c\|^{p+1} \leq \frac{C_{14}\|c_0\|_{L^\infty(\Omega)}^{p+1}}{1 - a} \int_{s_0}^t e^{(p+1)s} \int_\Omega (n_1^{p+1} + n_2^{p+1})
\]

\[
+ (C_{14}C_{16})^{\frac{1}{1 - a}} \int_{s_0}^t e^{(p+1)s} \|u\|_{L^{s(p+1)\eta}(\Omega)}^{p+1} \int_{s_0}^t e^{(p+1)s} \|u\|_{L^{s(p+1)\eta}(\Omega)}^{p+1}
\]

\[
+ \frac{C_{14}C_{16}}{1 - a} \int_{s_0}^t e^{(p+1)s} \|u\|_{L^{s(p+1)\eta}(\Omega)}^{p+1} + \frac{C_{14}}{1 - a} e^{(p+1)t} + \frac{|\Omega|M^{p+1}}{C_1} 1 - ae^{(p+1)s_0}
\]

for all \( t \in (s_0, T_{\max}) \). Here we use \( p < 2 \), which namely enables us to pick \( r \in (1, 3) \) such that

\[
(11.3.10) \quad \frac{2 - \frac{3}{(p+1)\eta}}{1 - \frac{3}{(p+1)\eta}} \cdot (p + 1) \cdot \frac{3}{2} \cdot \frac{\frac{3}{p + 1} - \frac{p + 1}{(p+1)\eta}}{\frac{2}{2} + \frac{3}{r}} < 2
\]

holds. Then, by the Gagliardo–Nirenberg inequality, Lemma 11.3.3 and the Young inequality, for any \( \tilde{\varepsilon} > 0 \) we find \( C_{17} = C_{17}(\tilde{\varepsilon}, p, \|n_{1,0}\|_{L^1(\Omega)}, \|n_{2,0}\|_{L^1(\Omega)}, \|u_0\|_{D(A^\theta)}, |\Omega|) > 0 \) such that

\[
(11.3.11) \quad \|u(\cdot, s)\|_{L^{s(p+1)\eta}(\Omega)}^{p+1} \leq \|Au(\cdot, s)\|_{L^1(\Omega)}^{p+1} \|u(\cdot, s)\|_{L^1(\Omega)}^{p+1} \leq C_{17} + \tilde{\varepsilon}\|Au(\cdot, s)\|_{L^2(\Omega)}^2
\]

for all \( s \in (s_0, T_{\max}) \) with \( b := \frac{3}{2} \cdot \frac{3}{p + 1} - \frac{p + 1}{(p+1)\eta} \in (0, 1) \), since \( \frac{p+1}{1 - a} b < 2 \) from (11.3.10). Similarly, there exists a constant \( C_{18} = C_{18}(\tilde{\varepsilon}, p, \|n_{1,0}\|_{L^1(\Omega)}, \|n_{2,0}\|_{L^1(\Omega)}, \|u_0\|_{D(A^\theta)}, |\Omega|) > 0 \) such that

\[
(11.3.12) \quad \|u(\cdot, s)\|_{L^{s(p+1)\eta}(\Omega)}^{p+1} \leq C_{18} + \tilde{\varepsilon}\|Au(\cdot, s)\|_{L^2(\Omega)}^2
\]

for all \( s \in (s_0, T_{\max}) \). Thus combination of (11.3.9) with (11.3.11) and (11.3.12) yields that there is \( C_{19} = C_{19}(\tilde{\varepsilon}, p, \alpha, \beta, \|n_{1,0}\|_{L^1(\Omega)}, \|n_{2,0}\|_{L^1(\Omega)}, \|c_0\|_{L^\infty(\Omega)}, \|u_0\|_{D(A^\theta)}, M, |\Omega|) > 0 \) such that

\[
\int_{s_0}^t e^{(p+1)s} \int_\Omega |\Delta c(\cdot, s)|^{p+1} ds \leq \frac{C_{14}\|c_0\|_{L^\infty(\Omega)}^{p+1}}{1 - a} \int_{s_0}^t e^{(p+1)s} \int_\Omega (n_1^{p+1}(\cdot, s) + n_2^{p+1}(\cdot, s)) ds
\]

\[
+ \left( (C_{14}C_{16})^{\frac{1}{1 - a}} + \frac{C_{14}C_{16}^2}{1 - a} \right) \tilde{\varepsilon} \int_{s_0}^t e^{(p+1)s}\|Au(\cdot, s)\|_{L^2(\Omega)}^2 ds + C_{19}e^{(p+1)t} + C_{19}
\]

133
for all \( t \in (s_0, T_{\text{max}}) \), with which \( \varepsilon := (((C_{14}C_{16})^{\frac{1}{p-1}} + C_{14}C_{16})^\frac{1}{p-1}) \) implies this lemma.

\[ \]

**Lemma 11.3.6.** For all \( p > 1 \) there exist constants \( C_{20} = C_{20}(p, \gamma, \delta) > 0 \) and \( C_{21} = C_{21}(p, \gamma, \delta, \|n_{1,0}\|_{L^1(\Omega)}, \|n_{2,0}\|_{L^1(\Omega)}, |\Omega|) > 0 \) such that

\[
\int_{s_0}^{t} e^{(p+1)s} \int_{\Omega} |Au|^2 \leq 2e^{(p+1)s_0} \int_{\Omega} |A^{\frac{1}{2}} u(\cdot, s_0)|^2 + C_{20} \int_{s_0}^{t} e^{(p+1)s} \left( \int_{\Omega} n_1^{p+1} + \int_{\Omega} n_2^{p+1} \right) + C_{21} e^{(p+1)t}
\]

for all \( t \in (s_0, T_{\text{max}}) \).

**Proof.** It follows from the fourth equation in (11.1.1), the Young inequality and the continuity of the Helmholtz projection on \( L^2(\Omega; \mathbb{R}^3) \) ([59, Theorem 1]) that

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |A^{\frac{1}{2}} u|^2 + \int_{\Omega} |Au|^2 = \int_{\Omega} Au \cdot \mathcal{P}[(\gamma n_1 + \delta n_2) \nabla \phi] \leq \frac{1}{4} \int_{\Omega} |Au|^2 + \int_{\Omega} |(\gamma n_1 + \delta n_2) \nabla \phi|^2 \\
\leq \frac{1}{4} \int_{\Omega} |Au|^2 + (\gamma^2 + \delta^2) \|\nabla \phi\|_{L^\infty(\Omega)} \int_{\Omega} (n_1^2 + n_2^2) \\
\leq \frac{1}{4} \int_{\Omega} |Au|^2 + (\gamma^2 + \delta^2) \|\nabla \phi\|_{L^\infty(\Omega)}^2 \int_{\Omega} (n_1^2 + n_2^2),
\]

and hence there exists a constant \( C_{22} = C_{22}(\gamma, \delta) > 0 \) such that

\[
(11.3.13) \quad \frac{1}{2} \frac{d}{dt} \int_{\Omega} |A^{\frac{1}{2}} u|^2 + \frac{3}{4} \int_{\Omega} |Au|^2 \leq |\Omega| C_{22} + C_{22} \left( \int_{\Omega} n_1^{p+1} + \int_{\Omega} n_2^{p+1} \right)
\]

and we derive from [48, Part2, Theorem 14.1], Lemma 11.3.3 and the Young inequality that

\[
(11.3.14) \quad \int_{\Omega} |A^{\frac{1}{2}} u|^2 \leq C_{23} \|Au\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \leq C_{24} + \frac{1}{p+1} \|Au\|_{L^2(\Omega)}^2
\]

with some constants \( C_{23}, C_{24} = C_{24}(p, \|n_{1,0}\|_{L^1(\Omega)}, \|n_{2,0}\|_{L^1(\Omega)}, |\Omega|) > 0 \). By virtue of (11.3.13) and (11.3.14) we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |A^{\frac{1}{2}} u|^2 + \frac{p+1}{2} \int_{\Omega} |A^{\frac{1}{2}} u|^2 + \frac{1}{4} \int_{\Omega} |Au|^2 \\
\leq |\Omega| C_{22} + \frac{(p+1)C_{24}}{2} + C_{22} \left( \int_{\Omega} n_1^{p+1} + \int_{\Omega} n_2^{p+1} \right),
\]

and hence we have

\[
\int_{s_0}^{t} e^{(p+1)s} \int_{\Omega} |Au|^2 \leq 2e^{(p+1)s_0} \int_{\Omega} |A^{\frac{1}{2}} u(\cdot, s_0)|^2 + 4C_{22} \int_{s_0}^{t} e^{(p+1)s} \left( \int_{\Omega} n_1^{p+1} + \int_{\Omega} n_2^{p+1} \right) + C_{25} e^{(p+1)t}
\]

for all \( t \in (s_0, T_{\text{max}}) \) with some constant \( C_{25} = C_{25}(p, \gamma, \delta, \|n_{1,0}\|_{L^1(\Omega)}, \|n_{2,0}\|_{L^1(\Omega)}, |\Omega|) > 0 \), which concludes the proof. \( \square \)
Lemma 11.3.7. For all \( p \in (1, 2) \) and for all \( \ell > 0 \) there exist positive constants \( K = K(p, \alpha, \beta) > 0 \) and \( C_{26} = C_{26}(p, \ell, \mu_1, \mu_2, \chi_1, \chi_2, \alpha, \beta, \gamma, \delta, n_{1,0}, n_{2,0}, c_0, u_0) > 0 \) such that if \( \mu > \mu_{p,\ell} := \ell + K\ell^{-p}\|c_0\|_{L^\infty(\Omega)}^{p+1} \), then

\[
\|n_i(\cdot, t)\|_{L^p(\Omega)} \leq C_{26} \quad \text{for all } t \in (s_0, T_{\max})
\]

and for \( i = 1, 2 \).

Proof. Lemmas 11.3.4, 11.3.6 and Lemma 11.3.5 with \( \varepsilon = C_4^{-1}C_{20}^{-1}\ell^p\chi^{-(p+1)}\varepsilon \) enable us to find constants \( K = K(p, \alpha, \beta) > 0 \) and \( L = L(p, \mu_1, \mu_2, \alpha, \beta, \gamma, \delta, n_{1,0}, n_{2,0}, c_0, u_0) > 0 \) such that

\[
\frac{1}{p} \int_{\Omega} n_1^p(\cdot, t) + \frac{1}{p} \int_{\Omega} n_2^p(\cdot, t) \\
\leq - (\mu - 2\varepsilon - \ell - K\ell^{-p}\|c_0\|_{L^\infty(\Omega)}^{p+1}) e^{-(p+1)t} \int_{s_0}^t e^{(p+1)s} \left( \int_{\Omega} n_{1}^{p+1}(\cdot, s) + \int_{\Omega} n_{2}^{p+1}(\cdot, s) \right) ds \\
+ L\ell^{-p}\chi^{p+1} + L
\]

for all \( t \in (s_0, T_{\max}) \). Assume that \( \mu > \mu_{p,\ell} \). Then there exists \( \varepsilon \in (0, \frac{\mu - \mu_{p,\ell}}{2}) \) such that

\[
\mu - 2\varepsilon - \ell - K\ell^{-p}\|c_0\|_{L^\infty(\Omega)}^{p+1} \geq 0.
\]

Thus we derive that

\[
\frac{1}{p} \int_{\Omega} n_1^p(\cdot, t) + \frac{1}{p} \int_{\Omega} n_2^p(\cdot, t) \leq L\ell^{-p}\chi^{p+1} + L
\]

holds for all \( t \in (s_0, T_{\max}) \), which enables us to obtain this lemma.

The proof of the following lemma is based on the method in [210, Proof of Theorem 1].

Lemma 11.3.8. There is a constant \( \xi_0 = \xi_0(\alpha, \beta) > 0 \) such that if \( \chi\|c_0\|_{L^\infty(\Omega)} < \xi_0\mu \), then for all \( p \in \left( \frac{3}{2}, 2 \right) \),

\[
\|n_i(\cdot, t)\|_{L^p(\Omega)} \leq C_{27}
\]

holds for all \( t \in (0, T_{\max}) \) and \( i = 1, 2 \) with some \( C_{27} = C_{27}(p, \mu_1, \mu_2, \chi_1, \chi_2, \alpha, \beta, \gamma, \delta, n_{1,0}, n_{2,0}, c_0, u_0) > 0 \).

Proof. Let \( p \in \left( \frac{3}{2}, 2 \right) \) and let \( \xi = \xi(p, \alpha, \beta) > 0 \) satisfy

\[
\inf_{\ell > 0} \mu_{p,\ell} = \inf_{\ell > 0} (\ell + K\ell^{-p}\|c_0\|_{L^\infty(\Omega)}^{p+1}) = \frac{1}{\xi}\chi\|c_0\|_{L^\infty(\Omega)}.
\]

Then we can see that \( \chi\|c_0\|_{L^\infty(\Omega)} < \xi \mu \) implies \( \mu > \mu_{p,\ell} \) with some \( \ell > 0 \). Therefore Lemma 11.3.7 implies that there exists a constant \( C_{28} = C_{28}(p, \mu_1, \mu_2, \chi_1, \chi_2, \alpha, \beta, \gamma, \delta, n_{1,0}, n_{2,0}, c_0, u_0) > 0 \) such that

\[
\|n_i(\cdot, t)\|_{L^p(\Omega)} \leq C_{28} \quad \text{for all } t \in (0, T_{\max})
\]

for each \( i = 1, 2 \). Now if we put \( \xi_0 = \xi_0(\alpha, \beta) := \inf_{p \in \left( \frac{3}{2}, 2 \right)} \xi(p, \alpha, \beta) \) and assume that \( \chi\|c_0\|_{L^\infty(\Omega)} < \xi_0\mu \), then we can see this lemma.
Lemma 11.3.9. Assume \( \|c_0\|_{L^\infty(\Omega)} < \xi_0\mu \), where \( \xi_0 \) is the constant defined in Lemma 11.3.8. Then there is a constant \( C_{29} = C_{29}(p, \mu_1, \mu_2, \chi_1, \chi_2, \alpha, \beta, \gamma, \delta, n_{1,0}, n_{2,0}, c_0, u_0) > 0 \) such that

\[
\|A^q u(\cdot, t)\|_{L^2(\Omega)} \leq C_{29} \quad \text{and} \quad \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C_{29}
\]

for all \( t \in (0, T_{\text{max}}) \).

**Proof.** Noting that \( \frac{1}{2} + \frac{\theta}{3} (1 - \vartheta) \in \left( \frac{1}{2}, \frac{3}{2} \right) \), we can pick \( p \in \left( \frac{1}{\vartheta + \frac{3}{2} \left( \frac{1}{p} - \frac{1}{2} \right)}, 2 \right) \). It follows from Lemma 11.3.8, well-known regularization estimates for Stokes semigroup \([62, 162]\) and the continuity of the Helmholtz projection on \( L^r(\Omega; \mathbb{R}^3) \) (see e.g., \([59, \text{Theorem 1}]\)) that there exist constants \( C_{30} = C_{30}(\|\cdot\|_\Omega), C_{31} = C_{31}(\|\cdot\|_\Omega), C_{32} = C_{32}(p, \mu_1, \mu_2, \chi_1, \chi_2, \alpha, \beta, \gamma, \delta, n_{1,0}, n_{2,0}, c_0, u_0, |\Omega|), C_{33} = C_{33}(p, \mu_1, \mu_2, \chi_1, \chi_2, \alpha, \beta, \gamma, \delta, n_{1,0}, n_{2,0}, c_0, u_0, |\Omega|) > 0 \) and \( \lambda > 0 \) such that

\[
\|A^q u(\cdot, t)\|_{L^2(\Omega)} \leq \|A^q e^{-tA} u_0\|_{L^2(\Omega)} + \int_0^t \|A^q e^{-tA} A P[(\gamma n_1(\cdot, s) + \delta n_2(\cdot, s))\nabla \varphi]\|_{L^2(\Omega)} \, ds
\]

\[
\leq \|A^q u_0\|_{L^2(\Omega)} + C_{30} \int_0^t (t - s)^{-\vartheta - \frac{3}{2} (\frac{1}{p} - \frac{1}{2})} e^{-\lambda(t-s)} \|P[(\gamma n_1(\cdot, s) + \delta n_2(\cdot, s))\nabla \varphi]\|_{L^p(\Omega)} \, ds
\]

\[
\leq \|A^q u_0\|_{L^2(\Omega)} + C_{31} \int_0^t (t - s)^{-\vartheta - \frac{3}{2} (\frac{1}{p} - \frac{1}{2})} e^{-\lambda(t-s)} \|\gamma n_1(\cdot, s) + \delta n_2(\cdot, s)\|_{L^p(\Omega)} \, ds
\]

\[
\leq \|A^q u_0\|_{L^2(\Omega)} + C_{32} \int_0^\infty \sigma^{-\vartheta - \frac{3}{2} (\frac{1}{p} - \frac{1}{2})} e^{-\lambda \sigma} \, d\sigma \leq C_{33}
\]

for all \( t \in (0, T_{\text{max}}) \) since \( \vartheta + \frac{3}{2} (\frac{1}{p} - \frac{1}{2}) < 1 \). Moreover, the properties of \( D(A^q) \) (\([61, \text{Theorem 3}]\) and \([68, \text{Theorem 1.6.1}]\)) imply that there exists \( C_{34} = C_{34}(\|\cdot\|_\Omega) > 0 \) such that

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C_{34} \|A^q u(\cdot, t)\|_{L^2(\Omega)}
\]

for all \( t \in (0, T_{\text{max}}) \), which concludes the proof. \( \square \)

Lemma 11.3.10. Assume \( \|c_0\|_{L^\infty(\Omega)} < \xi_0\mu \), where \( \xi_0 \) is the constant defined in Lemma 11.3.8. Then for all \( r \in [1, 6) \cap [1, q] \) there exists \( C_{35} > 0 \) such that

\[
\|\nabla c(\cdot, t)\|_{L^r(\Omega)} \leq C_{35} \quad \text{for all} \quad t \in (0, T_{\text{max}}).
\]

**Proof.** We first verify the \( L^2 \)-estimate for \( \nabla c \). Multiplying the third equation by \( -\Delta c \) and integrating it over \( \Omega \), we obtain from integration by parts and the Young inequality that

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla c|^2 + \int_\Omega |
\Delta c|^2 \leq (\alpha^2 + \beta^2) \|c_0\|_{L^\infty(\Omega)} \int_\Omega (n_1^2 + n_2^2) + C_{29} \int_\Omega |\nabla c|^2
\]

on \( (0, T_{\text{max}}) \). Using the Hölder inequality and the Gagliardo–Nirenberg inequality, we can find \( C_{36} > 0 \) such that

\[
(11.3.15) \quad \|\nabla c\|_{L^2(\Omega)}^2 \leq |\Omega| \|\nabla c\|_{L^2(\Omega)}^2 \leq C_{36} \|\Delta c\|_{L^2(\Omega)} \|c\|_{L^\infty(\Omega)} + \|c\|_{L^\infty(\Omega)},
\]

136
which leads to

\[(11.3.16)\quad (2C_{29} + 1) \int_\Omega |\nabla c|^2 \leq 2 \int_\Omega |\Delta c|^2 + \left( \frac{(2C_{29} + 1)^2C_{36}^2}{8} + (2C_{29} + 1)C_{36} \right) ||c_0||_{L^\infty(\Omega)}^2.\]

Therefore, since

\[
\frac{d}{dt} \int_\Omega |\nabla c|^2 + \int_\Omega |\nabla c|^2 \leq 2(\alpha^2 + \beta^2)||c_0||_{L^\infty(\Omega)} \int_\Omega (n_1^2 + n_2^2) + \left( \frac{(2C_{29} + 1)^2C_{36}^2}{8} + (2C_{29} + 1)C_{36} \right) ||c_0||_{L^\infty(\Omega)}^2
\]

holds and \( \int_t^{t+\tau} \int_\Omega (n_1^2 + n_2^2) \) is bounded on \((0, T_{\text{max}} - \tau)\) with \( \tau = \min\{1, T_{\text{max}}\}\), a Gronwall-type lemma (see e.g., [174, Lemma 2.3]) yields the \(L^2\)-estimate for \(\nabla c\). Next we let \( r \in (3, 6) \cap (1, q] \) and fix \( p \in (\frac{3p}{q+3}, 2) \), and will see the \(L^r\)-estimate for \(\nabla c\). An application of the variation of constants formula for \(c\) leads to

\[(11.3.17)\quad ||\nabla c(\cdot, t)||_{L^r(\Omega)} \leq ||\nabla e^{t(\Delta-1)}c_0||_{L^r(\Omega)}
+ \int_0^t ||\nabla e^{(t-s)(\Delta-1)}(\alpha n_1(\cdot, s) + \beta n_2(\cdot, s) + 1)c(\cdot, s)||_{L^r(\Omega)} ds
+ \int_0^t ||\nabla e^{(t-s)(\Delta-1)}u(\cdot, s) \cdot \nabla c(\cdot, s)||_{L^r(\Omega)} ds
\]

for all \( t \in (s_0, T_{\text{max}}) \). Noting that \( q > r \), we derive from the Hölder inequality and Lemma 2.2.2 that there exist constants \(C_{37}, C_{38} > 0\) such that

\[(11.3.18)\quad ||\nabla e^{t(\Delta-1)}c_0||_{L^r(\Omega)} \leq C_{37} ||\nabla e^{t(\Delta-1)}c_0||_{L^q(\Omega)} \leq C_{38} ||\nabla c_0||_{L^q(\Omega)}.
\]

We next establish the estimate for the second term on the right-hand side of (11.3.17). Lemmas 11.3.2 and 11.3.8 yield that there exist constants \(C_{39}, C_{40} > 0\) such that

\[(11.3.19)\quad \int_0^t ||\nabla e^{(t-s)(\Delta-1)}(\alpha n_1(\cdot, s) + \beta n_2(\cdot, s) + 1)c(\cdot, s)||_{L^r(\Omega)} ds
\leq C_{39} \int_0^t [1 + (t-s)^{-\frac{1}{2} - \frac{3}{2}(\frac{1}{p} - \frac{1}{r})}]e^{-(t-s)}(||n_1(\cdot, s)||_{L^p(\Omega)} + ||n_2(\cdot, s)||_{L^p(\Omega)} + ||\Omega||_{L^r}^\frac{1}{r}) ds
\leq C_{40} \int_0^t [1 + (t-s)^{-\frac{1}{2} - \frac{3}{2}(\frac{1}{p} - \frac{1}{r})}]e^{-(t-s)} ds
\]

for all \( t \in (s_0, T_{\text{max}}) \). Here, since \( \frac{1}{2} + \frac{3}{2}(\frac{1}{p} - \frac{1}{r}) < 1 \) holds from \( r \leq q < \frac{3p}{3p - 2} \), we have

\[(11.3.20)\quad \int_0^t [1 + (t-s)^{-\frac{1}{2} - \frac{3}{2}(\frac{1}{p} - \frac{1}{r})}]e^{-(t-s)} ds \leq \int_0^\infty [1 + \sigma^{-\frac{1}{2} - \frac{3}{2}(\frac{1}{p} - \frac{1}{r})}]e^{-\sigma} d\sigma < \infty.
\]
Combination of (11.3.19) and (11.3.20) derives that

\( (11.3.21) \quad \int_0^t \| \nabla e(t-s)(\Delta^{-1})(\alpha n_1(\cdot, s) + \beta n_2(\cdot, s) + 1)c(\cdot, s) \|_{L^r(\Omega)} \, ds \leq C_{41} \)

with some constant \( C_{41} > 0 \). Finally we will deal with the third term on the right-hand side of (11.3.17). Noticing that \( \frac{1}{2} + \frac{3}{2}(\frac{1}{2} - \frac{1}{r}) < 1 \) and

\[ \| u(\cdot, s) \cdot \nabla c(\cdot, s) \|_{L^2(\Omega)} \leq C_{42} \]

for all \( s \in (0, T_{\text{max}}) \) with some \( C_{42} > 0 \), we find \( C_{43}, C_{44}, C_{45} > 0 \) such that

\( (11.3.22) \quad \int_0^t \| \nabla e(t-s)(\Delta^{-1})u(\cdot, s) \cdot \nabla c(\cdot, s) \|_{L^r(\Omega)} \, ds \)

\[ \leq C_{43} \int_0^t [1 + (t-s)^{-\frac{1}{2} - \frac{3}{2}(\frac{1}{2} - \frac{1}{r})}]e^{-(t-s)}\| u(\cdot, s) \cdot \nabla c(\cdot, s) \|_{L^2(\Omega)} \, ds \]

\[ \leq C_{44} \int_0^t [1 + (t-s)^{-\frac{1}{2} - \frac{3}{2}(\frac{1}{2} - \frac{1}{r})}]e^{-(t-s)} \, ds \leq C_{45} \]

for all \( t \in (0, T_{\text{max}}) \). Therefore in light of (11.3.17), (11.3.18), (11.3.21) and (11.3.22) there exists a constant \( C_{46} > 0 \) that

\[ \| \nabla c(\cdot, t) \|_{L^r(\Omega)} \leq C_{46} \]

for all \( t \in (0, T_{\text{max}}) \).

Then we will derive the \( L^\infty \)-estimate for \( n_i \) by using well-known semigroup estimates.

**Lemma 11.3.11.** Assume \( \chi \| c_0 \|_{L^\infty(\Omega)} < \xi_0 \mu \), where \( \xi_0 \) is the constant defined in Lemma 11.3.8. Then there exists a constant \( C_{47} > 0 \) such that

\[ \| n_i(\cdot, t) \|_{L^\infty(\Omega)} \leq C_{47} \quad \text{for all } t \in (0, T_{\text{max}}) \]

for \( i = 1, 2 \).

**Proof.** The proof is based on an argument in the proof of [5, Lemma 3.2]. We let \( p \in (\frac{3}{2}, 2) \) with \( \frac{3p}{3-p} < q \). Then, thanks to Lemma 11.3.8, we obtain

\[ \| n_1(\cdot, t) \|_{L^p(\Omega)} \leq C_{48} \]

for all \( t \in (0, T_{\text{max}}) \) with some \( C_{48} > 0 \). Now we can choose \( r \in (3, q) \) such that \( p > \frac{3p}{3+r} \) and \( \theta > 1 \) satisfying \( \frac{1}{\theta} < \min\left\{ 1 - \frac{r(3-p)}{3p}, \frac{q-r}{q} \right\} \), and put \( \theta' := \frac{\theta}{\theta - 1} \), and then

\[ r\theta' < \frac{3p}{3-p} \quad \text{and} \quad r\theta' < q \]

hold. Now for all \( T' \in (0, T_{\text{max}}) \) we note that

\[ B(T') := \sup_{t \in (0, T')} \| n_1(\cdot, t) \|_{L^\infty(\Omega)} \]

138
is finite. In order to obtain the estimate for $B(T')$ for all $t \in (0, T')$ we put $t_0 := (t - 1)^+$ and represent $n_1$ according to

$$n_1(\cdot, t) = e^{(t-t_0)\Delta}n_1(\cdot, t_0) - \int_{t_0}^{t} e^{(t-s)\Delta} \nabla \cdot (\chi n_1(\cdot, s) \nabla c(\cdot, s) + n_1(\cdot, s)u(\cdot, s)) \, ds$$

$$+ \mu_1 \int_{t_0}^{t} e^{(t-s)\Delta}n_1(\cdot, s)(1 - n_1(\cdot, s) - a_1n_2(\cdot, s)) \, ds$$

$$=: I_1(\cdot, t) + I_2(\cdot, t) + I_3(\cdot, t).$$

In the case that $t \leq 1$, from the order preserving property of the Neumann heat semigroup we know that

$$\|I_1(\cdot, t)\|_{L^\infty(\Omega)} \leq \|n_{1,0}\|_{L^\infty(\Omega)} \quad \text{for all} \quad t \in (0, \min\{1, T'\}).$$

In the case that $t > 1$, by using the $L^p$-$L^q$ estimate for $(e^{s\Delta})_{s \geq 0}$ (see Lemma 2.2.2) and Lemma 11.3.8 we can see that there exists a constant $C_{49} > 0$ such that

$$\|I_1(\cdot, t)\|_{L^\infty(\Omega)} \leq \|n_1(\cdot, t_0)\|_{L^p(\Omega)} \leq C_{49} \quad \text{for all} \quad t \in (1, T').$$

Thanks to the elementary inequality

$$\mu_1 n_1(1 - n_1 - a_1n_2) \leq -\mu_1 \left( n_1 - \frac{1 + \mu_1}{2\mu_1} \right)^2 + \frac{(1 + \mu_1)^2}{4\mu_1} \leq \frac{(1 + \mu_1)^2}{4\mu_1}$$

together with the maximum principle, we see that there exists a constant $C_{50} > 0$ such that

$$I_3(\cdot, t) \leq C_{50} \quad \text{for all} \quad t \in (1, T').$$

Next we obtain from the known smoothing property of $(e^{s\Delta})_{s \geq 0}$ (see [52]) that

$$\int_{t_0}^{t} \left\| e^{(t-s)\Delta} \nabla \cdot (\chi n_1(\cdot, s) \nabla c(\cdot, s) + n_1(\cdot, s)u(\cdot, s)) \right\|_{L^\infty(\Omega)} \, ds$$

$$\leq C_{51} \sup_{s \in (0, T')} \left( \chi_1 \|n_1(\cdot, s)\nabla c(\cdot, s)\|_{L^\infty(\Omega)} + \|n_1(\cdot, s)u(\cdot, s)\|_{L^\infty(\Omega)} \right) \int_{0}^{1} \sigma^{-\frac{1}{2} - \frac{3}{p'}} \, d\sigma$$

for all $t \in (0, T')$ with some $C_{51} > 0$. Here we note from $\frac{1}{2} + \frac{3}{2p'} < 1$ that $\int_{0}^{1} \sigma^{-\frac{1}{2} - \frac{3}{p'}} \, d\sigma$ is finite. Then we can obtain that

$$\|n_1(\cdot, s)\nabla c(\cdot, s)\|_{L^\infty(\Omega)} \leq \|n_1(\cdot, s)\|_{L^p(\Omega)} \|\nabla c(\cdot, s)\|_{L^{q}(\Omega)}$$

$$\leq B(T')^{1 - \frac{3}{p}} \|n_1(\cdot, s)\|_{L^{q}(\Omega)} \|\nabla c(\cdot, s)\|_{L^{q}(\Omega)}$$

for all $s \in (0, T')$. Noting from $r\theta' < \frac{3p}{3-p}$ that

$$r\theta' \in (3, 6) \cap (1, q],$$

139
we have from Lemma 11.3.10 that there exists $C_{52} > 0$ such that
\[ \| \nabla c(\cdot; s) \|_{L^{r'}(\Omega)} \leq C_{52} \quad \text{for all } s \in (0, T'). \]
Therefore we can find $C_{53} > 0$ satisfying
\[ \| n_1(\cdot; s) \nabla c(\cdot; s) \|_{L^r(\Omega)} \leq C_{53} \quad \text{for all } s \in (0, T'). \]
Similarly, from Lemma 11.3.9 there exists a constant $C_{54} > 0$ such that
\[ \| n_1(\cdot; s) u(\cdot; s) \|_{L^r(\Omega)} \leq C_{54} B(T')^{1 - \frac{1}{r}} \| n_1(\cdot; s) \|_{L^1(\Omega)}. \]
Therefore Lemma 11.3.1 leads to the existence of $C_{55}, C_{56} > 0$ such that
\[ n_1(\cdot; t) \leq \| I_1(\cdot; t) \|_{L^\infty(\Omega)} + \| I_2(\cdot; t) \|_{L^\infty(\Omega)} + I_3(\cdot; t) \leq C_{55} + C_{56} B(T')^{1 - \frac{1}{r}} \]
for all $t \in (0, T')$, which implies from the positivity of $n_1$ that
\[ B(T') \leq C_{55} + C_{56} B(T')^{1 - \frac{1}{r}}. \]
Noticing that $r\theta > 1$, we derive that there exists $C_{57} > 0$ such that
\[ B(T') = \sup_{t \in (0, T')} \| n_1(\cdot; t) \|_{L^\infty(\Omega)} \leq C_{57} \quad \text{for all } T' \in (0, T_{\text{max}}). \]
Similarly we prove that there exists a constant $C_{58} > 0$ such that $\| n_2(\cdot; t) \|_{L^\infty(\Omega)} \leq C_{58}$ for all $t \in (0, T_{\text{max}})$. Therefore we can attain the conclusion of the proof. \(\square\)

**Proof of Theorem 11.1.1.** Combination of Lemmas 11.2.1, 11.3.9, 11.3.10 and 11.3.11 directly leads to Theorem 11.1.1. \(\square\)

### 11.4. Stabilization. Proof of Theorem 11.1.2

Now we assume that $\| c_0 \|_{L^\infty(\Omega)} < \xi_0 \mu$ holds, where $\xi_0$ is the constant defined in Lemma 11.3.8. In this section we will show stabilization in (11.1.1). Here asymptotic stability in the case that $a_1 < 1 \leq a_2$ can be shown by arguments similar to those in the case that $a_2 < 1 \leq a_1$. Thus we only show convergence in the cases that $a_1, a_2 < 1$ and $a_2 < 1 \leq a_1$. The same arguments as in the proofs of Theorem 9.1.2 yield the following two lemmas; therefore we write brief proofs.

**Lemma 11.4.1.** Let $a_1, a_2 \in (0, 1)$ and let $(n_1, n_2, c, u, P)$ be a solution to (11.1.1). Under the assumption of Theorem 11.1.1, there exist $k_1, \ell_1 > 0$ and $\varepsilon_1 > 0$ such that the nonnegative functions $E_1$ and $F_1$ defined by
\[
E_1 := \int_{\Omega} \left( n_1 - N_1 \log \frac{n_1}{N_1} \right) + k_1 \int_{\Omega} \left( n_2 - N_2 \log \frac{n_2}{N_2} \right) + \ell_1 \int_{\Omega} c^2
\]
and
\[
F_1 := \int_{\Omega} (n_1 - N_1)^2 + \int_{\Omega} (n_2 - N_2)^2 + \int_{\Omega} (\alpha n_1 + \beta n_2) c^2
\]
satisfy

\[ \frac{d}{dt} E_1(t) \leq -\varepsilon_1 F_1(t) \quad \text{for all } t > 0, \]

where

\[ N_1 := \frac{1 - a_1}{1 - a_1 a_2}, \quad N_2 := \frac{1 - a_2}{1 - a_1 a_2}, \]

which implies that the solution of (11.1.1) satisfies the following properties:

\[ \|n_1(\cdot, t) - N_1\|_{L^\infty(\Omega)} \to 0, \quad \|n_2(\cdot, t) - N_2\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty. \]

**Proof.** We put \( k_1 = a_1 a_2 \) and \( \ell_1 > \frac{N_1 x_1^2}{4} + \frac{k_1 N_2 x_2^2}{4} \). Then straightforward calculations and the Young inequality derive that the functions

\[ E_1 := \int_\Omega \left( n_1 - N_1 \log \frac{n_1}{N_1} \right) + k_1 \int_\Omega \left( n_2 - N_2 \log \frac{n_2}{N_2} \right) + \ell_1 \int_\Omega c^2 \]

and

\[ F_1 := \int_\Omega (n_1 - N_1)^2 + \int_\Omega (n_2 - N_2)^2 + \int_\Omega (\alpha n_1 + \beta n_2) c^2 \]

satisfy (11.4.1) with some \( \varepsilon_1 > 0 \). Moreover, noting from integrating (11.4.1) over \((0, \infty)\) and nonnegativity of \( E_1 \) that

\[ \varepsilon_1 \int_0^\infty \int_\Omega (n_1 - N_1)^2 + \varepsilon_1 \int_0^\infty \int_\Omega (n_2 - N_2)^2 \leq E_1(0), \]

we can see from standard compactness arguments (see, e.g., Lemma 6.3.1) that (11.4.2) holds.

\[ \square \]

**Lemma 11.4.2.** Let \( a_1 \geq 1 > a_2 \) and let \((n_1, n_2, c, u, P)\) be a solution to (11.1.1). Under the assumption of Theorem 11.1.1, there exist \( k_2, \ell_2 > 0 \) and \( \varepsilon_2 > 0 \) such that the nonnegative functions \( E_2 \) and \( F_2 \) defined by

\[ E_2 := \int_\Omega n_1 + k_2 \int_\Omega (n_2 - \log n_2) + \ell_2 c^2 \]

and

\[ F_2 := \int_\Omega n_1^2 + \int_\Omega (n_2 - 1)^2 + \int_\Omega (\alpha n_1 + \beta n_2) c^2 \]

satisfy

\[ \frac{d}{dt} E_2(t) \leq -\varepsilon_2 F_2(t) \quad \text{for all } t > 0, \]

which implies that

\[ \|n_1(\cdot, t)\|_{L^\infty(\Omega)} \to 0, \quad \|n_2(\cdot, t) - 1\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty. \]
Proof. We put \( k_2 = \frac{m_1}{a_2 \mu_2} \) and \( \ell_2 > \frac{k_2 \chi_2^2}{4} \). Then we can obtain from straightforward calculations and the Young inequality that the functions

\[
E_2 := \int_{\Omega} n_1 + k_2 \int_{\Omega} (n_2 - \log n_2) + \frac{\ell_2}{2} \int_{\Omega} c^2
\]

and

\[
F_2 := \int_{\Omega} n_1^2 + \int_{\Omega} (n_2 - 1)^2 + \int_{\Omega} (\alpha n_1 + \beta n_2) c^2
\]

satisfy (11.4.3) with some \( \varepsilon_2 > 0 \). Furthermore, since integrating (11.4.3) over \((0, \infty)\) and nonnegativity of \( E_2 \) yield that

\[
\varepsilon_2 \int_0^\infty \int_{\Omega} n_1^2 + \varepsilon_2 \int_0^\infty \int_{\Omega} (n_2 - 1)^2 \leq E_2(0),
\]

the standard compactness arguments enable us to obtain (11.4.4).

We next show the lower estimate for \( n_2 \), which will be used later.

**Lemma 11.4.3.** Let \( a_2 \in (0, 1) \). Under the assumption of Theorem 11.1.1, there exist constants \( C_{59} > 0 \) and \( T^* > 0 \) such that

\[
n_2(x, t) \geq C_{59} \quad \text{for all } x \in \Omega \text{ and all } t > T^*.
\]

**Proof.** We first deal with the case that \( a_1, a_2 \in (0, 1) \). Now thanks to Lemma 11.4.1, there is \( T^* > 0 \) such that

\[
\| n_2(\cdot, t) - N_2 \|_{L^\infty(\Omega)} \leq \frac{N_2}{2} \quad \text{for all } t > T^*,
\]

which means that

\[
n_2(x, t) \geq \frac{N_2}{2} \quad \text{for all } x \in \Omega \text{ and all } t > T^*.
\]

In the case that \( a_1 \geq 1 > a_2 \) a similar argument leads to the lower estimate for \( n_2 \). Therefore we can conclude the proof.

Finally we give the following lemma to establish the decay properties of \( c \) and \( u \).

**Lemma 11.4.4.** Under the assumption of Theorem 11.1.1, the solution of (11.1.1) satisfies

\[
\| c(\cdot, t) \|_{L^\infty(\Omega)} \to 0, \quad \| u(\cdot, t) \|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty.
\]

**Proof.** Noting from Lemmas 11.4.1 and 11.4.2 that

\[
\int_{T^*}^\infty \int_{\Omega} (\alpha n_1 + \beta n_2) c^2 < \infty
\]

and using Lemma 11.4.3, we can establish that

\[
\int_{T^*}^\infty \int_{\Omega} c^2 < \infty.
\]
which entails
\[ \| c(\cdot, t) \|_{L^\infty(\Omega)} \to 0 \]
as \( t \to \infty \) by using the same argument as in Lemma 6.3.1. Next we will show that
\[ \| u(\cdot, t) \|_{L^\infty(\Omega)} \to 0 \]
as \( t \to \infty \). Let \( \theta \in (\frac{3}{4}, 1) \) and \( a = \frac{\theta}{\pi} \in (0, 1) \). It follows from combination of [61, Theorem 3], [68, Theorem 1.6.1], [48, Part 2, Theorem 14.1] and Lemma 11.3.9 that there exist constants \( C_{60}, C_{61}, C_{62} > 0 \) such that
\begin{equation}
|u|_{L^2(\Omega)} \leq C_{60} \| A^\theta u \|_{L^2(\Omega)}
\leq C_{61} \| A^\theta u \|_{L^2(\Omega)} \| A^0 u \|_{L^2(\Omega)}^{1-a}
\leq C_{62} \| u \|_{L^2(\Omega)}^{1-a},
\end{equation}
which means that it is sufficient to show that
\[ \| u(\cdot, t) \|_{L^2(\Omega)} \to 0 \quad \text{as} \quad t \to \infty. \]
We first note from the Poincaré inequality that there exists a constant \( C_{63} > 0 \) such that
\[ \| u(\cdot, s) \|_{L^2(\Omega)} \leq C_{63} \| \nabla u(\cdot, s) \|_{L^2(\Omega)} \]
for all \( s \in (0, \infty) \). Put
\[ (n_{1,\infty}, n_{2,\infty}) = \begin{cases} (N_1, N_2) & (a_1, a_2 \in (0, 1)), \\
(0, 1) & (a_1 \geq 1 > a_2 > 0). \end{cases} \]
We infer from the fourth equation in (11.1.1) and the Young inequality that
\[ \frac{1}{2} \frac{d}{dt} \int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 
\leq \int_\Omega (\gamma(n_1 - n_{1,\infty}) + \delta(n_2 - n_{2,\infty})) \nabla \phi \cdot u + (\gamma n_{1,\infty} + \delta n_{2,\infty}) \int_\Omega \nabla \phi \cdot u
\leq \frac{1}{4C_{63}} \int_\Omega |u|^2 + C_{64} \left( \int_\Omega (n_1 - n_{1,\infty})^2 + \int_\Omega (n_2 - n_{2,\infty})^2 \right) + (\gamma n_{1,\infty} + \delta n_{2,\infty}) \int_\Omega \nabla \phi \cdot u \]
for all \( t \in (0, \infty) \) with some constant \( C_{64} > 0 \). Since \( \nabla \cdot u = 0 \) in \( \Omega \times (0, \infty) \), the functions
\[ y(t) := \int_\Omega |u(\cdot, t)|^2 \]
and
\[ h(t) := 2C_{64} \left( \int_\Omega (n_1 - n_{1,\infty})^2 + \int_\Omega (n_2 - n_{2,\infty})^2 \right) \]
satisfy
\[ y'(t) + C_{65} y(t) \leq h(t) \]
with some $C_{65} > 0$. Hence it holds that
\begin{equation}
(11.4.6) \quad y(t) \leq y(0)e^{-C_{65}t} + \int_0^t e^{-C_{65}(t-s)}h(s) \, ds
\leq y(0)e^{-C_{65}t} + \int_0^t \frac{1}{2} e^{-C_{65}(t-s)}h(s) \, ds + \int_0^t e^{-C_{65}(t-s)}h(s) \, ds.
\end{equation}

Here we see from Lemma 11.3.11 that there exists a constant $C_{66} > 0$ such that $h(s) \leq C_{66}$ for all $s > 0$, and hence we have
\begin{equation}
(11.4.7) \quad \int_0^t \frac{1}{2} e^{-C_{65}(t-s)}h(s) \, ds \leq C_{66}e^{-C_{65}t} \int_0^t \frac{1}{2} e^{C_{65}s} \, ds \leq C_{67}e^{-\frac{C_{65}}{2}t}
\end{equation}
with some $C_{67} > 0$. On the other hand, noting from (11.4.1) and (11.4.3) that
\[ \int_0^\infty h(s) \, ds < \infty, \]
we can see that
\begin{equation}
(11.4.8) \quad 0 \leq \int_0^t e^{-C_{65}(t-s)}h(s) \, ds \leq \int_0^t h(s) \, ds \to 0 \quad \text{as } t \to \infty.
\end{equation}

Therefore combination of (11.4.6) with (11.4.7) and (11.4.8) leads to
\[ \|u(\cdot, t)\|_{L^2(\Omega)}^2 = y(t) \to 0 \quad \text{as } t \to \infty, \]
which with (11.4.5) means the end of the proof.

\section*{Proof of Theorem 11.1.2.} Lemmas 11.4.1, 11.4.2, and 11.4.4 directly show Theorem 11.1.2.
Chapter 12

Global existence and asymptotic behavior of classical solutions for a 3D two-species Keller–Segel-Stokes system with competitive kinetics

12.1. Background and results

Let \( n \) denote the density of the cells and \( c \) present the concentration of the chemical signal. The mathematical model describing the above mechanism which was first proposed by Keller–Segel \([89]\) reads as

\[
\begin{align*}
n_t &= \Delta n - \nabla \cdot (n \nabla c), \\
c_t &= \Delta c - c + n.
\end{align*}
\]

In the above system which is called Keller–Segel system, it is known that the size of initial data determine whether classical solutions of the above system exist globally or not \( ([25, 76, 144, 194]) \). In the 2-dimensional setting global classical solutions of the above system exist when the mass of an initial data \( n_0 \) is sufficiently small \( ([144]) \). On the other hand, there exist initial data such that the mass of the initial data is large enough and a solution blows up in finite time in a 2-dimensional bounded domain \( ([76]) \).

In the higher dimensional case it is known that, if an initial data \( (n_0, c_0) \) is sufficiently small with respect to suitable Lebesgue norm, then global existence and boundedness of classical solutions hold \( ([25, 194]) \). Then we expect that existence of blow-up solutions hold under some largeness condition for initial data also in the higher-dimensional case; however, Winkler \([194]\) showed that for all \( m > 0 \) there exist initial data \( n_0 \) such that \( \|n_0\|_{L^1(\Omega)} = m \) and a solution blows up in finite time in the case that the domain is a ball in \( \mathbb{R}^N \) with \( N \geq 3 \).

As we mentioned above, we can see that whether solutions of the Keller–Segel system blow up or not depends on initial data. On the other hand, it is known that the logistic term can prevent solutions from blowing up even though a initial data is large enough \( ([100, 152, 195]) \). Osaki–Tsujikawa–Yagi–Mimura \([152]\) obtained that, in the
2-dimensional case, the chemotaxis system with logistic source

\[
\begin{aligned}
n_t &= \Delta n - \chi \nabla \cdot (n \nabla c) + rn - \mu n^2, \\
c_t &= \Delta c - c + n,
\end{aligned}
\]

where \(\chi, r, \mu > 0\), possesses a unique global solution for all \(\chi, r, \mu > 0\) and all initial data. In the higher dimensional case, Winkler [195, 199] established existence of global classical solutions under the condition that \(\mu > 0\) is sufficiently large. Moreover, asymptotic behavior of the solutions was obtained: \(n(\cdot, t) \to \frac{r}{\mu}, c(\cdot, t) \to \frac{c^*}{\mu}\) in \(L^\infty(\Omega)\) as \(t \to \infty\) ([67]).

Recently, Lankeit [100] obtained global existence of weak solutions to the chemotaxis system with logistic source for all \(r \in \mathbb{R}\) and any \(\mu > 0\) and the eventual smoothness of the solutions was derived if \(r \in \mathbb{R}\) is small. More related works which deal with the Keller–Segel system and the chemotaxis system with logistic source can be found in [5, Section 3].

The chemotaxis system with logistic source is a single-species case, and has a classical solution which converges to the constant steady state \((\frac{r}{\mu}, \frac{c^*}{\mu})\) under certain conditions on the parameters. On the other hand, the two-species chemotaxis-competition system

\[
\begin{aligned}
(n_1)_t &= \Delta n_1 - \chi_1 \nabla \cdot (n_1 \nabla c) + \mu_1 n_1 (1 - n_1 - a_1 n_2), \\
(n_2)_t &= \Delta n_2 - \chi_2 \nabla \cdot (n_2 \nabla c) + \mu_2 n_2 (1 - a_2 n_1 - n_2), \\
c_t &= \Delta c - c + \alpha n_1 + \beta n_2,
\end{aligned}
\]

where \(\chi_1, \chi_2, \mu_1, \mu_2, a_1, a_2, \alpha, \beta > 0\), which describes the evolution of two competing species which react on a single chemoattractant, has different dynamics depending on \(a_1\) and \(a_2\) ([4, 115, 164, 178] and Chapters 5, 6, 7, 8). In the 2-dimensional case Bai–Winkler [4] obtained global existence of solutions to the above system for all parameters. In the higher-dimensional case global existence and boundedness in the above system were established in [115] and Chapter 7. Moreover, it was shown that the solutions of the above system have the same asymptotic behavior as the solutions of Lotka–Volterra competition model: \(n_1(\cdot, t) \to \frac{1-a_1}{1-a_1 a_2}, n_2(\cdot, t) \to \frac{1-a_2}{1-a_1 a_2}, c(\cdot, t) \to \frac{\alpha(1-a_1)+\beta(1-a_2)}{1-a_1 a_2}\) in \(L^\infty(\Omega)\) as \(t \to \infty\) in the case that \(a_1, a_2 \in (0, 1), \) and \(n_1(\cdot, t) \to 0, n_2(\cdot, t) \to 1, c(\cdot, t) \to \beta\) in \(L^\infty(\Omega)\) as \(t \to \infty\) in the case that \(a_1 \geq 1 > a_2 > 0\) ([4] and Chapters 7, 8). More related works can be found in [115, 164, 178] and Chapters 5, 6.

Recently, the chemotaxis-fluid system

\[
\begin{aligned}
n_t + u \cdot \nabla n &= \Delta n - \chi \nabla \cdot (n \nabla c) + rn - \mu n^2, \\
c_t + u \cdot \nabla c &= \Delta c + g(n, c), \\
u_t + \kappa (u \cdot \nabla)u &= \Delta u + \nabla P + n \nabla \phi, \quad \nabla \cdot u = 0,
\end{aligned}
\]

where \(\chi > 0, r, \mu \geq 0, \kappa = 0\) (the Stokes case) or \(\kappa = 1\) (the Navier–Stokes case), which is the chemotaxis system with the fluid influence according to the Navier–Stokes equation, was intensively studied ([111, 174, 176, 197]). In the case that \(g(n, c) = -nc\) and \(r = \mu = 0\) (chemotaxis-Navier–Stokes system) Winkler [197] first overcame the difficulties of the chemotactic effect and the fluid influence, and showed existence of global classical solutions in the 2-dimensional case and global existence of weak solutions to the
above system with $\kappa = 0$ in the 3-dimensional setting; in the case that $\kappa = 1$, global weak solutions for the three-dimensional case were also constructed ([204, 206]); however, even though in the 3-dimensional chemotaxis-Stokes case, classical solutions were not found. In the case that $q(n, c) = -c + n$ and $\kappa = 0$ (Keller–Segel-Stokes system) Li–Xiao [111] showed global existence and boundedness under the smallness condition for the mass of initial data only in the 2-dimensional case. On the other hand, also in the chemotaxis-fluid system, the logistic source can be helpful for obtaining classical bounded solutions; global classical bounded solutions were established in the 2-dimensional Keller–Segel-Navier–Stokes system ([176]) and in the 3-dimensional Keller–Segel-Stokes system under the condition that $\mu$ is large enough ([174]); on the other hand, global weak solutions were obtained for all $\mu > 0$ by Lankeit [101]. More related works which deal with stabilization of solutions to the above system can be found in [101, 174, 176, 201, 206]: the case that $r = \mu = 0$ is in [201, 206]; the case that $\mu > 0$ is found in [101, 174, 176].

As we discussed before, the chemotaxis-competition system and the chemotaxis-fluid system were intensively studied. However, there are not many results on a coupled two-species chemotaxis-fluid system which have difficulties of the chemotaxis effect, the competitive kinetics and the fluid influence. Recently, the problem which is a combination of the chemotaxis-Navier–Stokes system and the chemotaxis-competition system was studied in the 2-dimensional case and the 3-dimensional case (Chapters 9, 11); in the 2-dimensional case global existence and asymptotic behavior of classical bounded solutions to the two-species chemotaxis-Navier–Stokes system were established (Chapter 9), and in the 3-dimensional case existence and stabilization of global classical bounded solutions to the two-species chemotaxis-Stokes system hold under the condition that $\mu_1, \mu_2$ are sufficiently large (Chapter 11). However, the two-species Keller–Segel-Stokes system with competitive kinetics

(12.1.1)

\[
\begin{align*}
(n_1)_t + u \cdot \nabla n_1 &= \Delta n_1 - \chi_1 \nabla \cdot (n_1 \nabla c) + \mu_1 n_1(1 - n_1 - a_1 n_2), \quad x \in \Omega, \ t > 0, \\
(n_2)_t + u \cdot \nabla n_2 &= \Delta n_2 - \chi_2 \nabla \cdot (n_2 \nabla c) + \mu_2 n_2(1 - a_2 n_1 - n_2), \quad x \in \Omega, \ t > 0, \\
c_t + u \cdot \nabla c &= \Delta c - c + \alpha n_1 + \beta n_2, \quad x \in \Omega, \ t > 0, \\
u_t &= \Delta u + \nabla P + (\gamma n_1 + \delta n_2) \nabla \phi, \quad \nabla \cdot u = 0, \quad x \in \Omega, \ t > 0, \\
\partial \nu n_1 &= \partial \nu n_2 = \partial \nu c = 0, \quad u = 0, \quad x \in \partial \Omega, \ t > 0, \\
n_i(x, 0) &= n_{i,0}(x), \ c(x, 0) = c_0(x), \ u(x, 0) = u_0(x), \quad x \in \Omega, \ i = 1, 2,
\end{align*}
\]

where $\Omega$ is a bounded domain, $\chi_1, \chi_2, a_1, a_2 \geq 0$ and $\mu_1, \mu_2, \alpha, \beta, \gamma, \delta > 0$ are constants, $n_{1,0}, n_{2,0}, c_0, u_0, \phi$ are known functions, has not been studied yet; we cannot apply the same argument as in Chapter 9 because of lacking the $L^\infty$-information and only having $L^1$-estimate for $c$. The purpose of this chapter is to obtain global existence and asymptotic behavior in (12.1.1) in a 3-dimensional domain. Throughout this chapter, we assume that the known functions $n_{1,0}, n_{2,0}, c_0, u_0, \phi$ satisfy

(12.1.2) \quad \begin{align*}
0 < n_{1,0}, n_{2,0} &\in C^{0}(\overline{\Omega}), \quad 0 < c_0 \in W^{1,q}(\Omega), \quad u_0 \in D(A^\theta), \\
\phi &\in C^{1+\eta}(\overline{\Omega})
\end{align*}

(12.1.3)

for some $q > 3$, $\theta \in (\frac{2}{3}, 1)$, $\eta > 0$ and $A$ is the Stokes operator in $L^2(\Omega)$ (see [162]). Then
the main results read as follows. The first theorem gives global existence and boundedness in (12.1.1).

**Theorem 12.1.1.** Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary and let $\chi_1, \chi_2 > 0$, $a_1, a_2 \geq 0$, $\mu_1, \mu_2, \alpha, \beta, \gamma, \delta > 0$. Assume that (12.1.2) and (12.1.3) are satisfied. Then there is a constant $\xi_0 = \xi_0(\alpha, \beta) > 0$ such that whenever $\chi := \max\{\chi_1, \chi_2\}$ and $\mu := \min\{\mu_1, \mu_2\}$ satisfy $\frac{\chi}{\mu} < \xi_0$, the problem (12.1.1) possesses a classical solution $(n_1, n_2, c, u, P)$ such that

$$n_1, n_2 \in C^0(\Omega \times [0, \infty)) \cap C^{2,1}(\Omega \times (0, \infty)),$$

$$c \in C^0(\Omega \times [0, \infty)) \cap C^{2,1}(\Omega \times (0, \infty)) \cap L_{\text{loc}}^\infty([0, \infty); W^{1,q}(\Omega)),$$

$$u \in C^0(\Omega \times [0, \infty)) \cap C^{2,1}(\Omega \times (0, \infty)) \cap L_{\text{loc}}^\infty([0, \infty); D(A^0)),$$

$$P \in C^{1,0}(\Omega \times (0, \infty)).$$

Also, the above solution is unique up to addition of spatial constants to the pressure $P$. Moreover, the above solution is bounded in the following sense:

$$\sup_{t > 0}(\|n_1(\cdot, t)\|_{L^\infty(\Omega)} + \|n_2(\cdot, t)\|_{L^\infty(\Omega)} + \|c(\cdot, t)\|_{W^{1,q}(\Omega)} + \|u(\cdot, t)\|_{D(A^0)}) < \infty.$$

The second theorem asserts asymptotic behavior of solutions to (12.1.1).

**Theorem 12.1.2.** Assume that the assumption of Theorem 12.1.1 is satisfied. Then the following properties hold:

(i) In the case that $a_1, a_2 \in (0, 1)$, under the condition that there exists $\delta_1 > 0$ such that

$$4\delta_1 - (1 + \delta_1)^2 a_1 a_2 > 0$$

and

$$\frac{\chi^2_1 (1 - a_1)}{4 a_1 \mu_1 (1 - a_1 a_2)} + \frac{\delta_1 \chi^2_2 (1 - a_2)}{4 a_2 \mu_2 (1 - a_1 a_2)} < \frac{4\delta_1 - (1 + \delta_1)^2 a_1 a_2}{a_1 \alpha \delta \delta + a_2 \beta^2 - a_1 a_2 \alpha \beta (1 + \delta_1)},$$

the solution of the problem (12.1.1) converges to a constant stationary solution of (12.1.1) as follows:

$$n_1(\cdot, t) \to N_1, \quad n_2(\cdot, t) \to N_2, \quad c(\cdot, t) \to C^\ast, \quad u(\cdot, t) \to 0 \quad \text{in} \ L^\infty(\Omega) \quad \text{as} \ t \to \infty,$$

where

$$N_1 := \frac{1 - a_1}{1 - a_1 a_2}, \quad N_2 := \frac{1 - a_2}{1 - a_1 a_2}, \quad C^\ast := \alpha N_1 + \beta N_2.$$

(ii) In the case that $a_1 \geq 1 > a_2$, under the condition that there exist $\delta'_1 > 0$ and $a'_1 \in [1, a_1]$ such that

$$4\delta'_1 - a'_1 a_2 (1 + \delta'_1)^2 > 0$$

and

$$\mu_2 > \frac{\chi^2_2 \delta'_1 (\alpha^2 a'_1 \delta'_1 + \beta^2 a_2 - \alpha \beta a'_1 a_2 (1 + \delta'_1))}{4 a_2 (4\delta'_1 - a'_1 a_2 (1 + \delta'_1)^2)},$$

the solution of the problem (12.1.1) converges to a constant stationary solution of (12.1.1) as follows:

$$n_1(\cdot, t) \to 0, \quad n_2(\cdot, t) \to 1, \quad c(\cdot, t) \to \beta, \quad u(\cdot, t) \to 0 \quad \text{in} \ L^\infty(\Omega) \quad \text{as} \ t \to \infty.$$
Remark 12.1.1. Here we note that it follows from the inequality $4\delta_1 - (1 + \delta_1)^2a_1a_2 > 0$ that
$$a_1\alpha^2\delta_1 + a_2\beta^2 - a_1a_2\alpha\beta(1 + \delta_1) > 0$$
holds. Indeed, we can verify that
$$0 < a_1\alpha^2(4\delta_1 - (1 + \delta_1)^2a_1a_2) + a_2(2\beta - a_1\alpha(1 + \delta_1))^2$$
$$= 4(a_1\alpha^2\delta_1 + a_2\beta^2 - a_1a_2\alpha\beta(1 + \delta_1)).$$

The strategy for the proof of Theorem 12.1.1 is to confirm the $L^p$-estimates for $n_1$ and $n_2$ with $p > \frac{3}{2}$. By using the differential inequality we can obtain
$$\int_{\Omega} n_1^p(\cdot, t) + \int_{\Omega} n_2^p(\cdot, t) \leq C + C\int_{s_0}^{t} e^{-(p+1)(t-s)} \int_{\Omega} |\Delta c(\cdot, s)|^{p+1} ds$$
$$- C\int_{s_0}^{t} e^{-(p+1)(t-s)} \left( \int_{\Omega} n_1^{p+1}(\cdot, s) + \int_{\Omega} n_2^{p+1}(\cdot, s) \right) ds$$
with some $C > 0$ and $s_0 > 0$. The technical innovation of this strategy is to control $\int_{s_0}^{t} e^{-(p+1)(t-s)} \int_{\Omega} |\Delta c(\cdot, s)|^{p+1} ds$ by applying variations of the maximal Sobolev regularity for $-\Delta$ associated with homogeneous Neumann boundary conditions and the Stokes operator $A$. Then we can obtain the $L^p$-estimates for $n_1$ and $n_2$. Here the keys of this strategy are the $L^2$-estimate for $\nabla c$ (Lemma 12.3.2) and the maximal Sobolev regularity for the Stokes equation (Lemma 12.2.3); these enable us to overcome difficulties of applying an argument similar to that in Chapter 11. On the other hand, the strategy for the proof of Theorem 12.1.2 is to confirm the following inequality:

$$\int_{0}^{\infty} \int_{\Omega} (n_1 - N_1)^2 + \int_{0}^{\infty} \int_{\Omega} (n_2 - N_2)^2 + \int_{0}^{\infty} \int_{\Omega} (c - C^*)^2 < \infty,$$
where $(N_1, N_2, C^*, 0)$ is a constant stationary solution of (12.1.1). In order to obtain this estimate we will find a nonnegative function $E$ satisfying
$$\frac{d}{dt} E(t) \leq -\varepsilon \int_{\Omega} \left[ (n_1 - N_1)^2 + (n_2 - N_2)^2 + (c - C^*)^2 \right]$$
with some $\varepsilon > 0$. The above inequality and the nonnegativity of $E(t)$ enable us to attain the desired estimate (12.1.4).

The plan of this chapter is as follows. In Section 12.2 we collect basic facts which will be used later. Section 12.3 is devoted to proving global existence and boundedness (Theorem 12.1.1). In Section 12.4 we show asymptotic stability (Theorem 12.1.2).

12.2. Local existence and basic inequality

In this section we will give some results which will be used later. We can prove the following lemma which gives local existence of classical solutions to (12.1.1) by a straightforward adaptation of the reasoning in [197, Lemma 2.1].
Lemma 12.2.1. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. Assume that (12.1.2) and (12.1.3) are satisfied. Then there exists $T_{\text{max}} \in (0, \infty)$ such that the problem (12.1.1) possesses a classical solution $(n_1, n_2, c, u, P)$ satisfying

$$
n_1, n_2 \in C^0(\bar{\Omega} \times [0, T_{\text{max}})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\text{max}}));$$
$$
c \in C^0(\bar{\Omega} \times [0, T_{\text{max}})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\text{max}})) \cap L^\infty_{\text{loc}}([0, T_{\text{max}}); W^{1,q}(\Omega));$$
$$
u \in C^0(\bar{\Omega} \times [0, T_{\text{max}})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\text{max}})) \cap L^\infty_{\text{loc}}([0, T_{\text{max}}); D(A^q));$$
$$
P \in C^{1,0}(\bar{\Omega} \times (0, T_{\text{max}}));$$
$$
n_1, n_2 > 0, \quad c > 0 \quad \text{in} \quad \Omega \times (0, T_{\text{max}}).$$

Also, the above solution is unique up to addition of spatial constants to the pressure $P$.
Moreover, either $T_{\text{max}} = \infty$ or

$$
\|n_1(\cdot, t)\|_{L^\infty(\Omega)} + \|n_2(\cdot, t)\|_{L^\infty(\Omega)} + \|c(\cdot, t)\|_{W^{1,q}(\Omega)} + \|A^q u(\cdot, t)\|_{L^2(\Omega)} \to \infty \quad \text{as} \quad t \to T_{\text{max}}.
$$

Given any $s_0 \in (0, T_{\text{max}})$, we can derive from the regularity properties that

$$
c(\cdot, s_0) \in C^2(\bar{\Omega}) \quad \text{with} \quad \partial_c c(\cdot, s_0) = 0 \quad \text{on} \quad \partial \Omega.
$$

In particular, there exists $M = M(s_0) > 0$ such that

$$
\|c(\cdot, s_0)\|_{W^{2,\infty}(\Omega)} \leq M
$$

(see e.g., [210]).

The following two lemmas provided as variations of the maximal Sobolev regularity hold keys for global existence and boundedness of solutions to (12.1.1).

Lemma 12.2.2. Let $s_0 \in (0, T_{\text{max}})$. Then for all $p > 1$ there exists a positive constant $C = C(p, n_1, n_2, c, u_0)$ such that

$$
\int_{s_0}^t \int_\Omega e^{ps} |\nabla c(\cdot, s)|^p \, ds \leq C \left( \int_{s_0}^t \int_\Omega e^{ps} \left( |c + \alpha n_1 + \beta n_2 - u \cdot \nabla c(\cdot, s)|^p \, ds + 1 \right) \right)
$$

holds for all $t \in (s_0, T_{\text{max}})$.

Proof. The proof is similar to that of Lemma 11.2.2. Let $s_0 \in (0, T_{\text{max}})$ and $t \in (s_0, T_{\text{max}})$. By using the transformation $\tilde{c}(\cdot, s) = e^{s} c(\cdot, s)$, $s \in (s_0, t)$, and the maximal Sobolev regularity [70, Theorem 3.1] for $\tilde{c}$ we obtain this lemma.

Lemma 12.2.3. Let $s_0 \in (0, T_{\text{max}})$. Then for all $p > 1$ there exists a constant $C = C(p, \gamma, \delta, n_1, n_2, c, u_0) > 0$ such that

$$
\int_{s_0}^t e^{ps} \int_\Omega |Au(\cdot, s)|^p \, ds \leq C \left( \int_{s_0}^t e^{ps} \int_\Omega |u(\cdot, s)|^p \, ds + \int_{s_0}^t e^{ps} \int_\Omega (n_1^p(\cdot, s) + n_2^p(\cdot, s)) \, ds + 1 \right)
$$

for all $t \in (s_0, T_{\text{max}})$. 

150
Proof. Letting $s_0 \in (0, T_{\text{max}})$ and $t \in (s_0, T_{\text{max}})$ and putting $\tilde{u}(\cdot, s) := e^s u(\cdot, s)$, $s \in (s_0, t)$, we obtain from the forth equation in (12.1.1) that

$$\tilde{u}_s = \Delta \tilde{u} + \tilde{u} + e^s (\gamma n_1 + \delta n_2) \nabla \phi + e^s \nabla P,$$

which derives

$$\tilde{u}_s + A \tilde{u} = \mathcal{P} [\tilde{u} + e^s (\gamma n_1 + \delta n_2) \nabla \phi],$$

where $\mathcal{P}$ denotes the Helmholtz projection mapping $L^2(\Omega)$ onto its subspace $L^2_s(\Omega)$ of all solenoidal vector field. Thus we derive from [63, Theorem 2.7] that there exist positive constants $C_1 = C_1(p)$ and $C_2 = C_2(p, \gamma, \delta)$ such that

$$\int_{s_0}^{t} \| \tilde{u}_s(\cdot, s) \|_{L^p(\Omega)}^p \, ds + \int_{s_0}^{t} \| A \tilde{u}(\cdot, s) \|_{L^p(\Omega)}^p \, ds$$

$$\leq C_1 \left( \int_{s_0}^{t} \| \tilde{u}_s(\cdot, s) + e^s (\gamma n_1(\cdot, s) + \delta n_2(\cdot, s)) \nabla \phi \|_{L^p(\Omega)}^p \, ds \right) + 1$$

$$\leq C_2 \left( \int_{s_0}^{t} e^{ps} \| u(\cdot, s) \|_{L^p(\Omega)}^p \, ds + \int_{s_0}^{t} e^{ps} (\| n_1(\cdot, s) \|_{L^p(\Omega)}^p + \| n_2(\cdot, s) \|_{L^p(\Omega)}^p) \, ds \right) + 1$$

for all $t \in (s_0, T_{\text{max}})$. Hence we can prove this lemma.

\[\square\]

12.3. Boundedness. Proof of Theorem 12.1.1

We will prove Theorem 12.1.1 by preparing a series of lemmas in this section.

Lemma 12.3.1. There exists a constant $C = C(n_{1,0}, n_{2,0}, c_0, u_0, |\Omega|) > 0$ such that for $i = 1, 2$,

$$\int_{\Omega} n_i(\cdot, t) \leq C \quad \text{for all } t \in (0, T_{\text{max}})$$

and

$$\int_{t}^{t+\tau} \int_{\Omega} n_i^2 \leq C \quad \text{for all } t \in (0, T_{\text{max}} - \tau),$$

where $\tau := \min\{1, \frac{1}{2} T_{\text{max}}\}$.

Proof. The above lemma can be proved by the same argument as in the proof of Lemma 9.3.1.

\[\square\]

Lemma 12.3.2. We have

$$\| \nabla c(\cdot, t) \|_{L^2(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\text{max}})$$

with some $C = C(\alpha, \beta, \gamma, \delta, n_{1,0}, n_{2,0}, c_0, u_0, |\Omega|) > 0$. Moreover, for all $p \in [1, 6)$,

$$\| c(\cdot, t) \|_{L^p(\Omega)} \leq \tilde{C} \quad \text{for all } t \in (0, T_{\text{max}})$$

with some $\tilde{C} = \tilde{C}(p, \alpha, \beta, \gamma, \delta, n_{1,0}, n_{2,0}, c_0, u_0, |\Omega|) > 0$. 

151
Proof. Integrating the third equation in (12.1.1) over $\Omega$ together with the $L^1$-estimates for $n_1$ and $n_2$ provided by Lemma 12.3.1 implies that for all $t \in (0, T_{\max})$,

(12.3.1) \[ \|c(\cdot, t)\|_{L^1(\Omega)} \leq C_1 \]

holds with some constant $C_1 = C_1(\alpha, \beta, n_{1,0}, n_{2,0}, c_0, u_0, |\Omega|) > 0$. We next see from an argument similar to that in the proofs of [174, Lemmas 2.5 and 2.6] that there is

(12.3.2) \[ \|\nabla c(\cdot, t)\|_{L^2(\Omega)} \leq C_2 \quad \text{for all } t > 0. \]

Thanks to (12.3.1) and (12.3.2), we have from the Gagliardo–Nirenberg inequality (see Lemma 2.1.1) that

\[ \|c(\cdot, t)\|_{L^p(\Omega)} \leq C_3\|\nabla c(\cdot, t)\|_{L^2(\Omega)}^{\frac{2-p}{p}} \|c(\cdot, t)\|_{L^1(\Omega)}^{\frac{p}{2}} + C_3\|c(\cdot, t)\|_{L^1(\Omega)} \leq C_3C_2^\alpha C_1^{1-a} + C_3C_1 \]

for all $t \in (0, T_{\max})$ with some positive constant $C_3 = C_3(|\Omega|)$, where $a = \frac{6}{5}(1 - \frac{1}{p}) \in (0, 1)$. Thus we can prove this lemma.

Lemma 12.3.3. Let $r \in (1, 3)$. Then we have

\[ \|u(\cdot, t)\|_{L^r(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\max}) \]

with some constant $C = C(r, \gamma, \delta, n_{1,0}, n_{2,0}, c_0, u_0, |\Omega|) > 0$.

Proof. The $L^r$-boundedness of $u$ for $r \in (1, 3)$ can be obtained from the well-known Neumann heat semigroup estimates together with Lemma 12.3.1 (for more details, see [202, Corollary 3.4]).

Now we fix $s_0 \in (0, T_{\max}) \cap (0, 1]$. We will obtain the $L^p$-estimates for $n_1$ and $n_2$ by preparing a series of lemmas.

Lemma 12.3.4. For all $p > 1$, $\varepsilon > 0$ and $\ell > 0$ there exist constants $C = C(p) > 0$ and $\tilde{C} = \tilde{C}(\mu_1, \mu_2, \varepsilon, p, |\Omega|) > 0$ such that

\[ \frac{1}{p} \int_{\Omega} n_1^p(\cdot, t) + \frac{1}{p} \int_{\Omega} n_2^p(\cdot, t) \leq - (\mu - \varepsilon - \ell) e^{-(p+1)t} \int_{s_0}^t e^{(p+1)s} \int_{\Omega} (n_1^{p+1}(\cdot, s) + n_2^{p+1}(\cdot, s)) ds \]

\[ + C\ell^{-p} \chi^{p+1} e^{-(p+1)t} \int_{s_0}^t e^{(p+1)s} \int_{\Omega} |\Delta c(\cdot, s)|^{p+1} ds + \tilde{C} \]

for all $t \in (s_0, T_{\max})$, where $\chi = \max\{\chi_1, \chi_2\}$ and $\mu = \min\{\mu_1, \mu_2\}$.

Proof. We can prove this lemma by using the same argument as in the proof of Lemma 11.3.4.

The following lemma and its proof provide the main part of this chapter.
Lemma 12.3.5. For all $\varepsilon > 0$ and all $p \in \left(\frac{3}{2}, 2\right)$, there exist positive constants $C = C(p, \alpha, \beta)$ and $\tilde{C} = \tilde{C}(\varepsilon, p, \alpha, \beta, \gamma, \delta, n_{1,0}, n_{2,0}, c_0, u_0, |\Omega|)$ such that

$$\int_{s_0}^{t} e^{(p+1)s} \int_{\Omega} |\Delta c(\cdot, s)|^{p+1} ds \leq C \int_{s_0}^{t} e^{(p+1)s} \left( n_{1,0}^{p+1}(\cdot, s) + n_{2,0}^{p+1}(\cdot, s) \right) ds$$

$$+ \varepsilon \int_{s_0}^{t} e^{(p+1)s} \int_{\Omega} |Au(\cdot, s)|^{p+1} ds + \tilde{C} e^{(p+1)t} + \tilde{C}$$

for all $t \in (s_0, T_{\text{max}})$.

Proof. Fix $\theta \in \left(\frac{3}{2}, 2\right)$ and put $\theta' = \frac{\theta}{\theta-1} \in (2, 3)$. From Lemma 12.2.2 we have

$$\int_{s_0}^{t} e^{(p+1)s} \int_{\Omega} |\Delta c(\cdot, s)|^{p+1} ds \leq C_1 \int_{s_0}^{t} e^{(p+1)s} \int_{\Omega} |(c + \alpha n_1 + \beta n_2 - u \cdot \nabla c)(\cdot, s)|^{p+1} ds + C_1$$

with some positive constant $C_1 = C_1(p, n_{1,0}, n_{2,0}, c_0, u_0)$. Now Lemma 12.3.2 and the H"older inequality imply that there is a constant $C_2 = C_2(p, \alpha, \beta, n_{1,0}, n_{2,0}, c_0, u_0) > 0$ such that

$$\int_{s_0}^{t} e^{(p+1)s} \int_{\Omega} |(c + \alpha n_1 + \beta n_2 - u \cdot \nabla c)(\cdot, s)|^{p+1} ds \leq C_2 \int_{s_0}^{t} e^{(p+1)s} \int_{\Omega} \left( n_{1,0}^{p+1}(\cdot, s) + n_{2,0}^{p+1}(\cdot, s) + |(u \cdot \nabla c)(\cdot, s)|^{p+1} \right) ds + C_3 e^{(p+1)t}$$

$$\leq C_2 \int_{s_0}^{t} e^{(p+1)s} \left( \int_{s_0}^{t} n_{1,0}^{p+1}(\cdot, s) ds + \int_{s_0}^{t} n_{2,0}^{p+1}(\cdot, s) ds \right) ds$$

$$+ C_2 \int_{s_0}^{t} e^{(p+1)s} \left( \int_{\Omega} |u(\cdot, s)|^{p+1} ds \right)^{\frac{p+1}{p}} \left( \int_{\Omega} |\nabla c(\cdot, s)|^{p+1} ds \right)^{\frac{p}{p+1}} ds + C_3 e^{(p+1)t}$$

holds for all $t \in (s_0, T_{\text{max}})$ with some $C_3 = C_3(p, \alpha, \beta, \gamma, \delta, n_{1,0}, n_{2,0}, c_0, u_0, |\Omega|) > 0$. Here the Gagliardo–Nirenberg inequality and Lemma 12.3.2 imply that there exists a constant $C_4 = C_4(p, |\Omega|) > 0$ such that

$$\int_{s_0}^{t} e^{(p+1)s} \int_{\Omega} |\nabla c|^{p+1} ds \leq C_4 \int_{s_0}^{t} e^{(p+1)s} \int_{\Omega} |\Delta c|^{a(p+1)} ds + C_4 \int_{s_0}^{t} e^{(p+1)s} \int_{\Omega} |\nabla c|^{p+1} ds + C_5$$

$$\leq C_5 \int_{s_0}^{t} e^{(p+1)s} \int_{\Omega} |\Delta c|^{a(p+1)} ds + C_5$$

with some $C_5 = C_5(p, \alpha, \beta, \gamma, \delta, n_{1,0}, n_{2,0}, c_0, u_0, |\Omega|) > 0$ and

$$a := \frac{3 - \frac{6}{(p+1)\theta'}}{5 - \frac{6}{p+1}} \in (0, 1).$$

We derive from (12.3.3), (12.3.4), (12.3.5) and the Young inequality that there exists a
positive constant $C_6 = C_6(p, \alpha, \beta, \gamma, \delta, n_{1,0}, n_{2,0}, c_0, u_0, |\Omega|)$ such that

$$
\int_{s_0}^{t} e^{(p+1)s} \int_{\Omega} |\Delta c(\cdot, s)|^{p+1} ds \\
\leq C_2 \int_{s_0}^{t} e^{(p+1)s} \int_{\Omega} (n_1^{p+1}(\cdot, s) + n_2^{p+1}(\cdot, s)) ds + a \int_{s_0}^{t} e^{(p+1)s} \int_{\Omega} |\Delta c(\cdot, s)|^{p+1} ds \\
+ C_6 \int_{s_0}^{t} e^{(p+1)s} \left( \|u(\cdot, s)\|_{L^{p+1}(\Omega)}^{p+1} + \|u(\cdot, s)\|_{L^{p+1}(\Omega)}^{p+1} \right) ds + C_6 e^{(p+1)t} + C_6.
$$

Hence we can obtain that there exists a constant $C_7 = C_7(p, \alpha, \beta) > 0$ such that

$$(12.3.6) \quad \int_{s_0}^{t} e^{(p+1)s} \int_{\Omega} |\Delta c(\cdot, s)|^{p+1} ds \leq C_7 \int_{s_0}^{t} e^{(p+1)s} \int_{\Omega} (n_1^{p+1}(\cdot, s) + n_2^{p+1}(\cdot, s)) ds \\
+ C_8 \int_{s_0}^{t} e^{(p+1)s} \left( \|u(\cdot, s)\|_{L^{p+1}(\Omega)}^{p+1} + \|u(\cdot, s)\|_{L^{p+1}(\Omega)}^{p+1} \right) ds \\
+ C_8 e^{(p+1)t} + C_8$$

with some $C_8 = C_8(p, \alpha, \beta, \gamma, \delta, n_{1,0}, n_{2,0}, c_0, u_0, |\Omega|)$. Here we can choose $r \in (1, 3)$ such that

$$(12.3.7) \quad \frac{5 - \frac{6}{p+1}}{2 - \frac{6}{(p+1)\theta}} = \frac{\frac{1}{r} - \frac{1}{(p+1)\theta}}{\frac{1}{r} - \frac{1}{p+1}} < 1$$

holds. Now it follows from the Gagliardo–Nirenberg inequality, Lemma 12.3.3 and the Young inequality that

$$(12.3.8) \quad \|u(\cdot, s)\|_{L^{p+1}(\Omega)}^{p+1} \leq \|Au(\cdot, s)\|_{L^{p+1}(\Omega)}^{p+1} \|u(\cdot, s)\|_{L^{p+1}(\Omega)}^{p+1} (1-b) \\
\leq C_9 + \frac{\bar{\gamma}}{2C_8} \|Au(\cdot, s)\|_{L^{p+1}(\Omega)}^{p+1}$$

holds with

$$b := \frac{1}{r} - \frac{1}{(p+1)\theta} \in (0, 1)$$

and some constant $C_9 = C_9(\bar{\gamma}, p, \gamma, \delta, n_{1,0}, n_{2,0}, c_0, u_0, |\Omega|) > 0$, because

$$\frac{p+1}{1-a} b = (p+1) \cdot \frac{5 - \frac{6}{p+1}}{2 - \frac{6}{(p+1)\theta}} \cdot \frac{\frac{1}{r} - \frac{1}{(p+1)\theta}}{\frac{1}{r} - \frac{1}{p+1}} < p+1$$

holds from (12.3.7). We moreover have from the fact $\frac{1}{r-a} > 1$, the Young inequality and (12.3.8) that there is a constant $C_{10} = C_{10}(\bar{\gamma}, p, \gamma, \delta, n_{1,0}, n_{2,0}, c_0, u_0, |\Omega|) > 0$ such that

$$(12.3.9) \quad \|u(\cdot, s)\|_{L^{p+1}(\Omega)}^{p+1} \leq C_{10} + \frac{\bar{\gamma}}{2C_8} \|Au(\cdot, s)\|_{L^{p+1}(\Omega)}^{p+1}$$
for all \( s \in (s_0, T_{\max}) \). Thus, combining (12.3.6), (12.3.8) and (12.3.9), we can establish that there exists \( C_{11} = C_{11}(\xi, p, \alpha, \beta, \gamma, \delta, n_{1,0}, n_{2,0}, c_0, u_0, |\Omega|) > 0 \) such that

\[
\int_{s_0}^{t} e^{(p+1)s} \int_{\Omega} |\Delta c(\cdot, s)|^{p+1} \, ds \leq C_7 \int_{s_0}^{t} e^{(p+1)s} \int_{\Omega} (n_1^{p+1}(\cdot, s) + n_2^{p+1}(\cdot, s)) \, ds + \int_{s_0}^{t} e^{(p+1)s} \int_{\Omega} |Au(\cdot, s)|^{p+1} \, ds + C_{11} \epsilon^{(p+1)t} + C_{11}
\]

for all \( t \in (s_0, T_{\max}) \), which concludes the proof of this lemma.

We give the following lemma to control \( \int_{s_0}^{t} e^{(p+1)s} \int_{\Omega} |Au(\cdot, s)|^{p+1} \, ds \).

**Lemma 12.3.6.** Let \( p \in \left(\frac{3}{2}, 2\right) \). Then there is \( C = C(p, \gamma, \delta, n_{1,0}, n_{2,0}, c_0, u_0) > 0 \) such that

\[
\int_{s_0}^{t} e^{(p+1)s} \int_{\Omega} |Au(\cdot, s)|^{p+1} \, ds \leq C \int_{s_0}^{t} e^{(p+1)s} \int_{\Omega} (n_1^{p+1}(\cdot, s) + n_2^{p+1}(\cdot, s)) \, ds + C \epsilon^{(p+1)t} + C
\]

for all \( t \in (s_0, T_{\max}) \).

**Proof.** A combination of Lemmas 12.2.3 and 12.3.3 implies this lemma.

The following lemma is concerned with the \( L^p \)-estimates for \( n_1 \) and \( n_2 \).

**Lemma 12.3.7.** There exists a constant \( \xi_0 = \xi_0(\alpha, \beta) > 0 \) (independent of \( p \)) such that if \( \frac{\lambda}{p} < \xi_0 \), then for all \( p \in \left(\frac{3}{2}, 2\right) \) there exists a constant \( C > 0 \) such that

\[
\|n_i(\cdot, t)\|_{L^p(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\max}) \text{ and for } i = 1, 2.
\]

**Proof.** By Lemma 12.3.6, for all \( t \in (s_0, T_{\max}) \) the inequality

\[
(12.3.10) \quad \int_{s_0}^{t} e^{(p+1)s} \int_{\Omega} |Au(\cdot, s)|^{p+1} \, ds \leq C_1 \int_{s_0}^{t} e^{(p+1)s} \int_{\Omega} (n_1^{p+1}(\cdot, s) + n_2^{p+1}(\cdot, s)) \, ds + C_1 \epsilon^{(p+1)t} + C_1
\]

holds with some positive constant \( C_1 = C_1(p, \gamma, \delta, \|n_{1,0}\|_{L^1(\Omega)}, \|n_{2,0}\|_{L^1(\Omega)}, \|u_0\|_{D(A^\ell)}, |\Omega|) \). Choosing \( \xi = \frac{1}{C_1} \) in Lemma 12.3.5, we can obtain from Lemma 12.3.4 and (12.3.10) that there are positive constants \( K = K(p, \alpha, \beta) \) and \( L > 0 \) depending on all parameters \( p, \alpha, \beta, \gamma, \delta \) and all initial data \( n_{1,0}, n_{2,0}, c_0, u_0 \) such that

\[
(12.3.11) \quad \frac{1}{p} \int_{\Omega} n_1^p(\cdot, t) + \frac{1}{p} \int_{\Omega} n_2^p(\cdot, t)
\]

\[
\leq - (\mu - \varepsilon - \ell - K\ell^p\chi^{p+1}) \int_{s_0}^{t} e^{(p+1)s} \int_{\Omega} (n_1^{p+1}(\cdot, s) + n_2^{p+1}(\cdot, s)) \, ds + L\ell^p\chi^{p+1} + L
\]

155
holds for all \( t \in (s_0, T_{\text{max}}) \), where \( \varepsilon > 0 \) and \( \ell > 0 \). Here there exists a constant \( \xi = \xi(p, \alpha, \beta) > 0 \) such that

\[
\inf_{\ell > 0} (\ell + K \ell^{-p} \chi^{p+1}) = \frac{1}{\xi} \chi.
\]

Now we put \( \xi_0 = \xi_0(\alpha, \beta) := \inf_{p \in (\frac{3}{2}, 2)} \xi(p, \alpha, \beta) \). If \( \frac{\xi}{\mu} < \xi_0 \), then we see from the fact \( \frac{\xi}{\mu} \leq \frac{\xi}{\ell_0} < \mu \) and (12.3.12) that

\[
\inf_{\ell > 0} (\ell + K \ell^{-p} \chi^{p+1}) < \mu,
\]

and hence there exists a constant \( \ell > 0 \) such that

\[
\mu > \ell + K \ell^{-p} \chi^{p+1}.
\]

Therefore, under the condition that \( \frac{\xi}{\mu} < \xi_0 \), we can find \( \varepsilon > 0 \) satisfying

\[
\mu - \varepsilon - \ell - K \ell^{-p} \chi^{p+1} \geq 0,
\]

which enables us to obtain from (12.3.11) that

\[
\frac{1}{p} \int \Omega n_1^p(\cdot, t) + \frac{1}{p} \int \Omega n_2^p(\cdot, t) \leq L \ell^{-p} \chi^{p+1} + L
\]

holds for all \( t \in (s_0, T_{\text{max}}) \). Thus we can obtain this lemma.

\[\square\]

Remark 12.3.1. In this proof we can choose arbitrary \( \ell > 0 \), when we do not care about the condition for \( \mu > 0 \). If we choose \( \ell = 1 \), then \( \mu > 0 \) have to satisfy that \( \mu > 1 + K \chi^{p+1} \), which namely means that \( \mu \leq 1 \) does not derive the \( L^p \)-estimate for \( n_i \). Thus, we utilize \( \ell > 0 \) to derive the best condition for \( \mu > 0 \).

Lemma 12.3.8. Assume \( \frac{\xi}{\mu} < \xi_0 \), where \( \xi_0 \) is the constant defined in Lemma 12.3.7. Then there exists a constant \( C > 0 \) such that

\[
\|A^0 u(\cdot, t)\|_{L^2(\Omega)} \leq C \quad \text{and} \quad \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C
\]

for all \( t \in (0, T_{\text{max}}) \).

Proof. Thanks to Lemma 12.3.7, we can show this lemma by the same argument as in the proof of Lemma 11.3.9.

\[\square\]

Lemma 12.3.9. Assume \( \frac{\xi}{\mu} < \xi_0 \), where \( \xi_0 \) is the constant defined in Lemma 12.3.7. Then for all \( r \in [1, 6) \cap [1, q] \) there exists \( C > 0 \) such that

\[
\|\nabla c(\cdot, t)\|_{L^r(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\text{max}}).
\]

Proof. This proof is based on that of Lemma 11.3.10. We first note from Lemma 12.3.7 that for all \( p \in (\frac{3}{2}, 2) \),

\[
\|\alpha n_1(\cdot, s) + \beta n_2(\cdot, s)\|_{L^p(\Omega)} \leq C_1
\]

with some constant \( C_1 > 0 \). Moreover, we see from Lemmas 12.3.2 and 12.3.8 that

\[
\|u(\cdot, s) \cdot \nabla c(\cdot, s)\|_{L^2(\Omega)} \leq C_2
\]

with some constant \( C_2 > 0 \). Therefore an argument similar to that in the proof of Lemma 11.3.10 implies this lemma.

\[\square\]
Lemma 12.3.10. Assume $\frac{a_2}{a_1} < \xi_0$, where $\xi_0$ is the constant defined in Lemma 12.3.7. Then there exists a constant $C > 0$ such that
\[ \|n_i(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\max}) \text{ and for } i = 1, 2. \]

Proof. We can prove this lemma in the same way as in the proof of Lemma 11.3.11. 

Proof of Theorem 12.1.1. Lemmas 12.2.1, 12.3.8, 12.3.9 and 12.3.10 directly drive Theorem 12.1.1.

12.4. Asymptotic behavior. Proof of Theorem 12.1.2

We first recall the following lemma which will give stabilization in (12.1.1).

Lemma 12.4.1 (Lemma 6.3.1). Let $n \in C^0(\Omega \times [0, \infty))$ satisfy that there exist constants $C > 0$ and $\alpha_0 \in (0, 1)$ such that
\[ \|n\|_{C^\alpha_0, \frac{\alpha_0}{2} (\Omega \times [t, t+1])} \leq C \quad \text{for all } t \geq 1. \]
Assume that
\[ \int_1^\infty \int_\Omega (n - N^*)^2 < \infty \]
with some constant $N^*$. Then
\[ n(\cdot, t) \to N^* \quad \text{in } L^\infty(\Omega) \quad \text{as } t \to \infty. \]

12.4.1. Case 1: $a_1, a_2 \in (0, 1)$

Now we assume that $\frac{a_2}{a_1} < \xi_0$ holds, where $\xi_0$ is the constant defined in Lemma 12.3.7, and will prove asymptotic behavior of solutions to (12.1.1) in the case $a_1, a_2 \in (0, 1)$. In this case we also suppose that there exists $\delta_1 > 0$ such that
\begin{align*}
4\delta_1 - (1 + \delta_1)^2 a_1 a_2 > 0 \quad & \text{and} \\
\frac{\chi_a^2(1 - a_1)}{4a_1 \mu_1 (1 - a_1 a_2)} + \frac{\delta_1 \chi_a^2(1 - a_2)}{4a_2 \mu_2 (1 - a_1 a_2)} < & \frac{4\delta_1 - (1 + \delta_1)^2 a_1 a_2}{a_1 \alpha^2 \beta + a_2 \beta^2 - a_1 a_2 \alpha \beta (1 + \delta_1)}. \quad (12.4.2)
\end{align*}

The following lemma asserts that the assumption of Lemma 12.4.1 is satisfied in the case that $a_1, a_2 < 1$.

Lemma 12.4.2. Let $(n_1, n_2, c, u, P)$ be a solution to (12.1.1). If $a_1, a_2 \in (0, 1)$, then there exist constants $C > 0$ and $\alpha_0 \in (0, 1)$ such that
\[ \|n_1\|_{C^\alpha_0, \frac{\alpha_0}{2} (\Omega \times [t, t+1])} + \|n_2\|_{C^\alpha_0, \frac{\alpha_0}{2} (\Omega \times [t, t+1])} + \||c||_{C^\alpha_0, \frac{\alpha_0}{2} (\Omega \times [t, t+1])} \leq C \]
for all $t \geq 1$. Moreover, $n_1, n_2$ and $c$ satisfy
\[ \int_{1}^{\infty} \int_{\Omega} (n_1 - N_1)^2 + \int_{1}^{\infty} \int_{\Omega} (n_2 - N_2)^2 + \int_{1}^{\infty} \int_{\Omega} (c - C^*)^2 < \infty, \]
where
\[ N_1 := \frac{1 - a_1}{1 - a_1 a_2}, \quad N_2 := \frac{1 - a_2}{1 - a_1 a_2}, \quad C^* := \alpha N_1 + \beta N_2. \]
Proof. We can first obtain from Lemmas 12.3.8, 12.3.9, 12.3.10 and [98] that (12.4.3) holds. Next we will confirm (12.4.4). We put
\[ E_1 := \int_\Omega \left( n_1 - N_1 - N_1 \log \frac{n_1}{N_1} \right) + \delta_1 a_1 \mu_1 a_3 \mu_2 \int_\Omega \left( n_2 - N_2 - N_2 \log \frac{n_2}{N_2} \right) + \frac{\delta_2}{2} \int_\Omega (c - C^*)^2, \]
where \( \delta_1 > 0 \) is a constant defined as in (12.4.1)–(12.4.2) and \( \delta_2 > 0 \) is a constant satisfying
\[ \frac{a_2 \mu_2 \chi_1^2 N_1}{4} + \frac{\delta_1 a_1 \mu_1 \chi_2^2 N_2}{4} < \delta_2 < \frac{a_1 a_2 \mu_1 \mu_2 (4 \delta_1 - (1 + \delta_1)^2 a_1 a_2)}{a_1 \alpha_2 \delta_1 + a_2 \beta^2 - a_1 a_2 \alpha \beta (1 + \delta_1)}. \]
Then noting from \( \nabla \cdot u = 0 \) that
\[ \int_\Omega u \cdot \nabla (\log n_i) = 0 \]
for all \( i = 1, 2 \) and
\[ \int_\Omega u \cdot \nabla (c^2) = 0, \]
we see from an argument similar to that in the proof of Lemma 8.2.2 that there exists a constant \( \varepsilon_1 > 0 \) such that
\[ \frac{d}{dt} E_1(t) \leq -\varepsilon_1 \left( \int_\Omega (n_1 - N_1)^2 + \int_\Omega (n_2 - N_2)^2 + \int_\Omega (c - C^*)^2 \right) \text{ for all } t > 0. \]
Thus we have from the nonnegativity of \( E_1 \) that
\[ \int_1^\infty \int_\Omega (n_1 - N_1)^2 + \int_1^\infty \int_\Omega (n_2 - N_2)^2 + \int_1^\infty \int_\Omega (c - C^*)^2 \leq \frac{1}{\varepsilon_1} E_1(1) < \infty, \]
which leads to (12.4.4). \( \square \)

12.4.2. Case 2: \( a_1 \geq 1 > a_2 \)

In this section we assume that \( \frac{\alpha}{\mu} < \xi_0 \) holds, where \( \xi_0 \) is the constant defined in Lemma 12.3.7, and will obtain stabilization in (12.1.1) in the case \( a_1 \geq 1 > a_2 \). In this case we also suppose that there exist constants \( \delta_1' > 0 \) and \( a_1' \in [1, a_1] \) such that
\begin{align*}
(12.4.5) \quad & 4 \delta_1' - a_1 a_2 (1 + \delta_1')^2 > 0 \quad \text{and} \\
(12.4.6) \quad & \mu_2 > \frac{\chi_2^2 \delta_1' (\alpha^2 a_1' \delta_1' + \beta^2 a_2 - \alpha \beta a_1' a_2 (1 + \delta_1'))}{4 a_2 (4 \delta_1' - a_1' a_2 (1 + \delta_1')^2)}. 
\end{align*}
We shall show the following lemma to verify that the assumption of Lemma 12.4.1 is satisfied in the case that \( a_1 \geq 1 > a_2 \).

Lemma 12.4.3. Let \( (n_1, n_2, c, u, P) \) be a solution to (12.1.1). If \( a_1 \geq 1 > a_2 \), then there exist constants \( C > 0 \) and \( \alpha_0 \in (0, 1) \) such that
\[ \|n_1\|_{\mathcal{C}^{\alpha_0} \mathcal{N}_1 (\Omega \times [t, t+1])} + \|n_2\|_{\mathcal{C}^{\alpha_0} \mathcal{N}_2 (\Omega \times [t, t+1])} + \|c\|_{\mathcal{C}^{\alpha_0} \mathcal{N}_1 (\Omega \times [t, t+1])} \leq C \]
for all \( t \geq 1 \). Moreover, we have
\[ \int_1^\infty \int_\Omega n_1^2 + \int_1^\infty \int_\Omega (n_2 - 1)^2 + \int_1^\infty \int_\Omega (c - \beta)^2 < \infty. \]
Proof. We first see from Lemmas 12.3.8, 12.3.9, 12.3.10 and [98] that (12.4.7) holds. Next we will show (12.4.8). We put

\[ E_2 := \int_{\Omega} n_1 + \delta'_1 a_1 \mu_1 \int_{\Omega} (n_2 - 1 - \log n_2) + \frac{\delta'_2}{2} \int_{\Omega} (c - \beta)^2, \]

where \( \delta'_1 > 0 \) and \( a'_1 \in [1, a_1] \) are constants defined as in (12.4.5)–(12.4.6) and \( \delta'_2 > 0 \) is a constant satisfying

\[ a'_1 \mu_1 \chi^2 \delta'_1 < \delta'_2 < \frac{a'_1 \mu_1 (4 \delta'_1 - a'_1 a_2 (1 + \delta'_1)^2)}{a^2 a'_1 \delta'_1 + \beta^2 a_2 - \alpha \beta a'_1 a_2 (1 + \delta'_1)}. \]

Then noting from \( \nabla \cdot u = 0 \) that

\[ \int_{\Omega} u \cdot \nabla (\log n_i) = 0 \]

for \( i = 1, 2 \) and

\[ \int_{\Omega} u \cdot \nabla (c^2) = 0, \]

we derive from an argument similar to that in the proof of Lemma 7.4.1 that there exists a constant \( \varepsilon_2 > 0 \) such that

\[ \frac{d}{dt} E_2(t) \leq -\varepsilon_2 \left( \int_{\Omega} n_1^2 + \int_{\Omega} (n_2 - 1)^2 + \int_{\Omega} (c - \beta)^2 \right) \]

for all \( t > 0 \). Hence we obtain from the nonnegativity of \( E_1 \) that

\[ \int_{1}^{\infty} \int_{\Omega} n_1^2 + \int_{1}^{\infty} \int_{\Omega} (n_2 - 1)^2 + \int_{1}^{\infty} \int_{\Omega} (c - \beta)^2 \leq \frac{1}{\varepsilon_2} E_2(1) < \infty, \]

which means that the desired estimate (12.4.8) holds.

12.4.3. Convergence for \( u \)

Finally we provide the following lemma with respect to the decay properties of \( u \). The proof is same as that of Lemma 11.4.4. Here we write a sketch of the proof.

Lemma 12.4.4. Under the assumptions of Theorems 12.1.1 and 12.1.2, the solution of (12.1.1) has the following property:

\[ \| u(\cdot, t) \|_{L^\infty(\Omega)} \to 0 \quad \text{as} \quad t \to \infty. \]

Proof. A combination of [61, Theorem 3], [68, Theorem 1.6.1], [48, Part 2, Theorem 14.1] and Lemma 12.3.8 implies that there exist constants \( C_1, C_2, C_3 > 0 \) such that

\[ \| u \|_{L^\infty(\Omega)} \leq C_1 \| A^0 u \|_{L^2(\Omega)} \leq C_2 \| A^0 u \|_{L^2(\Omega)}^{\frac{1}{2}} \| A^0 u \|_{L^2(\Omega)}^{\frac{1}{2} - \alpha} \leq C_3 \| u \|_{L^2(\Omega)}^{\frac{1}{2} - \alpha}, \]

which means that it is sufficient to confirm that

\[ \| u(\cdot, t) \|_{L^2(\Omega)} \to 0 \quad \text{as} \quad t \to \infty. \]
Put

\[(n_{1,\infty}, n_{2,\infty}) = \begin{cases} (N_1, N_2) & (a_1, a_2 \in (0, 1)), \\ (0, 1) & (a_1 \geq 1 > a_2 > 0). \end{cases}\]

Aided by the Poincaré inequality, we infer from the fourth equation in (12.1.1) and the Young inequality that

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u|^2 + \int_{\Omega} |\nabla u|^2 \leq \frac{1}{4} \int_{\Omega} |\nabla u|^2 + C_4 \left( \int_{\Omega} (n_1 - n_{1,\infty})^2 + \int_{\Omega} (n_2 - n_{2,\infty})^2 \right) + (\gamma n_{1,\infty} + \delta n_{2,\infty}) \int_{\Omega} \nabla \phi \cdot u
\]

for all \( t \in (0, \infty) \) with some constant \( C_1 > 0 \). Since \( \nabla \cdot u = 0 \) in \( \Omega \times (0, \infty) \), the functions

\[ y(t) := \int_{\Omega} |u(\cdot, t)|^2 \quad \text{and} \quad h(t) := 2C_4 \left( \int_{\Omega} (n_1 - n_{1,\infty})^2 + \int_{\Omega} (n_2 - n_{2,\infty})^2 \right) \]

satisfy

\[ y'(t) + C_5 y(t) \leq h(t) \]

with some \( C_5 > 0 \). Hence it holds that

\[ y(t) \leq y(0) e^{-C_5t} + \int_0^t e^{-C_5(t-s)} h(s) \, ds. \]  

(12.4.9)

Here we obtain from Lemma 12.3.10 that there exists a constant \( C_6 > 0 \) such that \( h(s) \leq C_6 \) for all \( s > 0 \), and hence we have

\[ \int_0^{\frac{t}{2}} e^{-C_5(t-s)} h(s) \, ds \leq C_6 e^{-C_5t} \int_0^{\frac{t}{2}} e^{C_5s} \, ds \leq C_7 e^{-\frac{C_5t}{2}} \]

with some \( C_7 > 0 \). On the other hand, noticing from (12.4.4) and (12.4.8) that

\[ \int_0^\infty h(s) \, ds < \infty, \]

we can see that

\[ 0 \leq \int_{\frac{t}{2}}^t e^{-C_5(t-s)} h(s) \, ds \leq \int_{\frac{t}{2}}^t h(s) \, ds \to 0 \quad \text{as} \quad t \to \infty. \]  

(12.4.11)

Thus a combination of (12.4.9) with (12.4.10) and (12.4.11) implies to

\[ ||u(\cdot, t)||_{L^2(\Omega)}^2 = y(t) \to 0 \quad \text{as} \quad t \to \infty, \]

which means the end of the proof.

\[ \square \]

**12.4.4. Proof of Theorem 12.1.2**

**Proof of Theorem 12.1.2.** A combination of Lemmas 12.4.1, 12.4.2, 12.4.3 and 12.4.4 directly leads to Theorem 12.1.2. 

\[ \square \]
Part III

Related works: Single-species chemotaxis systems
Chapter 13

A unified method for boundedness in Keller–Segel systems with signal-dependent sensitivity

13.1. Problem and results

In this chapter we consider the Keller–Segel system with signal-dependent sensitivity

\[
\begin{align*}
\begin{cases}
    u_t &= \Delta u - \nabla \cdot (u\chi(v)\nabla v), & x \in \Omega, \ t > 0, \\
    v_t &= \Delta v + u - v, & x \in \Omega, \ t > 0, \\
    \nabla u \cdot \nu = \nabla v \cdot \nu = 0, & x \in \partial \Omega, \ t > 0, \\
    u(x, 0) = u_0(x), & v(x, 0) = v_0(x), \ x \in \Omega,
\end{cases}
\end{align*}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) \((n \geq 2)\) with smooth boundary \( \partial \Omega \) and \( \nu \) is the outward normal vector to \( \partial \Omega \). The initial data \( u_0 \) and \( v_0 \) are assumed to be nonnegative functions. The unknown function \( u(x, t) \) represents the population density of species and \( v(x, t) \) shows the concentration of the substance at place \( x \) and time \( t \). As to the sensitivity function \( \chi \), we are interested in functions generalizing

\[
\chi(s) = \frac{K}{s} \quad \text{and} \quad \chi(s) = \frac{K}{(1 + s)^2} \quad (s > 0),
\]

where \( K > 0 \) is a constant. In a mathematical view, the difficulty caused by the sensitivity function is to deal with the additional term \( u\chi'(v)|\nabla v|^2 \) which does not appear in the case that \( \chi \) is a constant. Moreover, when \( \chi \) is the singular sensitivity function, it is delicate to derive the estimate for \( \frac{1}{v} \). In the case that

\[
\chi(v) = \frac{K}{v} \quad (K > 0),
\]

Winkler [196] first attained global existence of classical solutions when \( K < \sqrt{\frac{2}{n}} \) and global existence of weak solutions when \( K < \sqrt{\frac{n+2}{3n-4}} \). However, the result in [196] could
not arrive at boundedness of solutions to (13.1.1). To overcome the difficulty in a singular sensitivity, Fujie [50] established the uniform-in-time lower estimate for $v$, and showed global existence of classical bounded solutions to (13.1.1) when $\chi(v) = \frac{K}{v}$ with $K > 0$ satisfying

$$K < \sqrt{\frac{2}{n}}.$$  

Moreover, Stinner–Winkler [165] proved global existence of weak power-$\lambda$ solutions to (13.1.1) for all $K > 0$ in the radial setting. As to the problem (13.1.1) with $\chi(v) = \frac{K}{v}$, (13.1.2) $K < \sqrt{\frac{2}{n}}$: furthermore Fujie–Senba [54] dealt with the 2-dimensional problem which was replaced $v_t$ with $v$ in (13.1.1) and showed global existence and boundedness of radially symmetric solutions under the condition that $s \to 0 (s \to \infty)$ and $\tau > 0$ is sufficiently small. On the other hand, in the case that

$$\chi(v) \leq \frac{K}{(1 + \alpha v)^k} \quad (k > 1, \ \alpha > 0, \ K > 0),$$

Winkler [193] established global existence and boundedness in (13.1.1) without any restriction on $K > 0$. The result in [193] affected the result in [58] which mentioned global existence of classical bounded solutions to (13.1.1) when $\chi$ is the strong singular sensitivity function such that $\chi(v) = \frac{K}{v}$ for $k > 1$, and the method in [193] was used in Zhang–Li [213] and Zheng–Mu [220]. However, one cannot verify the assertion in [193] that global existence and boundedness hold for all $K > 0$, because there is a gap in the proof (see [193, p. 1670], the calculation for seeing the derivative of $\psi(s) := (1 + \alpha s)^{-2k}(1 + \beta s)^{\kappa + 2}$ with $\kappa \in (0, 2k - 2)$ and $\beta > \alpha > 0$, which claims that

$$\psi'(s) = (1 + \alpha s)^{-2k-1}(1 + \beta s)^{\kappa + 1}\{ -2k(1 + \beta s) + (\kappa + 2)(1 + \alpha s) \}$$

$$\leq (1 + \alpha s)^{-2k-1}(1 + \beta s)^{\kappa + 1}\{ -(2k - \kappa - 2) - [2k\beta - (\kappa + 2)\alpha]s \}$$

$$\leq 0$$

for all $s \geq 0$ by letting $\beta$ be sufficiently large. The largeness of $\beta$ plays an essential role in the proof of boundedness of solutions for all $K > 0$. However, the above calculation is not correct. Indeed, since

$$\psi'(s) = (1 + \alpha s)^{-2k-1}(1 + \beta s)^{\kappa + 1}\{ -2k\alpha(1 + \beta s) + \beta(\kappa + 2)(1 + \alpha s) \}$$

$$= (1 + \alpha s)^{-2k-1}(1 + \beta s)^{\kappa + 1}\{ [\kappa(\kappa + 2)\beta - 2k\alpha] - \alpha\beta(2k - \kappa - 2)s \},$$

we cannot let $\beta$ be large for $\psi'(s) \leq 0$, which means that the proof is not correct). Recently, Fujie tried modifying it; nevertheless it also has a gap (cf. [51]). In general, it does not seem to be easy to show global existence of bounded solutions for arbitrary $K > 0$. Moreover, if $k \to 1$, then the condition “arbitrary $K > 0$” cannot connect with (13.1.2).
The purpose of this chapter is to obtain global existence and boundedness in (13.1.1) under a more natural and proper condition for $\chi$ and to build a mathematical bridge between the cases $k = 1$ and $k > 1$. We shall suppose that $\chi$ satisfies that

$$\chi \in C^{1+\lambda}((0, \infty)) \quad \text{and} \quad 0 \leq \chi(s) \leq \frac{K}{(a+s)^k} \quad (s > 0)$$

(13.1.3)

with some $\lambda > 0$, $k \geq 1$, $a \geq 0$ and $K > 0$ fulfilling

$$K < k(a+\eta)^{k-1}\sqrt{\frac{2}{n}}.$$  

(13.1.4)

Here

$$\eta := \sup_{\tau > 0} \left( \min \left\{ e^{-2\tau} \min_{x \in \Omega} v_0(x), \ c_0 \|u_0\|_{L^1(\Omega)} (1 - e^{-\tau}) \right\} \right),$$

(13.1.5)

where $c_0 > 0$ is a lower bound for the fundamental solution of $w_t = \Delta w - w$ with Neumann boundary condition (for more detail, see Remark 13.2.1). We suppose that

$$0 \leq u_0 \in C^1(\overline{\Omega}) \setminus \{0\} \quad \text{and} \quad \begin{cases} 0 < v_0 \in W^{1,q}(\Omega) \ (\exists q > n) \quad (a = 0), \\ 0 \leq v_0 \in W^{1,q}(\Omega) \setminus \{0\} \ (\exists q > n) \quad (a > 0). \end{cases}$$

(13.1.6)

Now the main results read as follows.

**Theorem 13.1.1.** Let $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Assume that $\chi$ satisfies (13.1.3) with some $\lambda > 0$, $k \geq 1$, $a \geq 0$, $K > 0$ fulfilling (13.1.4). Then for any $u_0, v_0$ satisfying (13.1.6) with some $q > n$, there exists an exactly one pair $(u, v)$ of functions

$$u, \ v \in C^1(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty))$$

which solves (13.1.1). Moreover, the solution $(u, v)$ is uniformly bounded, i.e., there exists a constant $C > 0$ such that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C \quad \text{for all } t \geq 0.$$  

**Remark 13.1.1.** The unified condition (13.1.3) with $K > 0$ satisfying (13.1.4) can connect with the condition in [50]. Indeed, when $k = 1$, (13.1.4) becomes (13.1.2). However, this condition is not optimal; Lankeit [102] and Fujie–Senba [54] extended the condition for $K$ in the 2-dimensional setting.

The main theorem tells us the result in the typical case of singular sensitivity.

**Corollary 13.1.2.** Let $\chi(s) = \frac{K}{s}$ with $K < \sqrt{\frac{2}{n}}$. Then for any $u_0, v_0$ satisfying (13.1.6) with some $q > n$, (13.1.1) has a unique global bounded classical solution.
The strategy for the proof of Theorem 13.1.1 is to construct the estimate for \( \int_\Omega u^p \) with some \( p > \frac{n}{2} \). One of the keys for this strategy is to derive the unified inequality
\[
\frac{d}{dt} \int_\Omega u^p \varphi(v) + \varepsilon p(p - 1) \int_\Omega u^{p-2} \varphi(v) |\nabla u|^2 \leq c \int_\Omega u^p \varphi(v) - r \int_\Omega u^{p+1} \frac{\varphi(v)}{(a + v)^k}
\]
for some \( \varepsilon \in [0, 1) \) and \( c > 0 \), where
\[
\varphi(s) := \exp \left\{ -r \int_s^\eta \frac{1}{(a + \tau)^k} d\tau \right\} \quad (s \geq \eta)
\]
with \( r > 0 \). This function \( \varphi \) constructed in [150] and Chapter 3 unifies mathematical structures in the cases \( k > 1 \) and \( k = 1 \).

This chapter is organized as follows. In Section 13.2 we collect basic facts which will be used later. In Section 13.3 we give a unified viewpoint in energy estimates. Section 13.4 is devoted to the proof of global existence and boundedness (Theorem 13.1.1).

### 13.2. Local existence and basic inequalities

In this section we will collect elementary results. We first recall the uniform-in-time lower estimate for \( v \) established by Fujie [50, 51].

**Lemma 13.2.1.** Let \( u \in C(\overline{\Omega} \times [0,T)) \) be a nonnegative function such that, with some \( m > 0 \), \( \int_\Omega u(\cdot, t) = m \) for every \( t \in [0,T) \). If \( v_0 \in C(\overline{\Omega}) \) is a positive function in \( \Omega \) and \( v \in C^{2, 1}(\overline{\Omega} \times (0,T)) \cap C(\overline{\Omega} \times [0,T)) \) is a classical solution of
\[
\begin{align*}
  v_t &= \Delta v - v + u, \quad x \in \Omega, \ t \in (0, T), \\
  \nabla v \cdot \nu &= 0 \quad \quad x \in \partial\Omega, \ t \in (0, T), \\
  v(x, 0) &= v_0(x) \quad x \in \Omega,
\end{align*}
\]
then for all \( t \in (0, T) \),
\[
\inf_{x \in \Omega} v(x, t) \geq \tilde{\eta} > 0,
\]
where
\[
\tilde{\eta} := \sup_{\tau > 0} \left( \min \left\{ e^{-2\tau} \min_{x \in \overline{\Omega}} v_0(x), \ c_0 m (1 - e^{-\gamma}) \right\} \right).
\]

**Remark 13.2.1.** When \( \Omega \) is a convex bounded domain, the proof of this lemma is given in [50, Lemma 2.2] and the constant \( \tilde{\eta} \) can be explicitly represented as
\[
\tilde{\eta} := \sup_{\tau > 0} \left\{ \min_{x \in \overline{\Omega}} e^{-\tau} \min_{x \in \overline{\Omega}} v_0(x), \ m \cdot \int_0^\tau \frac{1}{(4\pi r)^\frac{n}{2}} e^{-r \left( \frac{\max_{x \in \overline{\Omega}} |x-y|^2}{4r} \right)} dr \right\}.
\]

On the other hand, if we do not assume the convexity of \( \Omega \), then using the positivity of the fundamental solution \( U(t, x; s, y) \) to \( w_t = \Delta w - w \) in \( \Omega \times (0, T) \) with \( \nabla w \cdot \nu = 0 \) on \( \partial \Omega \times (0, T) \) (see e.g., [83]), we have that there exists \( c_0 > 0 \) such that for all \( \tau \in (0, \frac{T}{2}) \),
\[
U(s + \tau, x; s, y) = U(\tau, x; 0, y) \geq c_0 > 0 \quad \text{for all } x, y \in \overline{\Omega}, \ s > 0.
\]
Then we can see the conclusion of Lemma 13.2.1 by a similar argument as in [50, Lemma 2.2].

We next recall the well-known result about local existence of solutions to (13.1.1) (see [102, Theorem 2.3], [54, Proposition 2.2] and [193, Lemma 2.1]).

**Lemma 13.2.2.** Assume that \( \chi \) satisfies (13.1.3) with some \( \lambda > 0, k \geq 1, a \geq 0, K > 0 \) and the initial data \( u_0, v_0 \) fulfil (13.1.6) for some \( q > n \). Then there exist \( T_{\text{max}} \in (0, \infty] \) and exactly one pair \( (u,v) \) of nonnegative functions

\[
\begin{aligned}
    u &\in C(\overline{\Omega} \times [0, T_{\text{max}})) \cap C^2(\overline{\Omega} \times (0, T_{\text{max}})), \\
v &\in C(\overline{\Omega} \times [0, T_{\text{max}})) \cap C^2(\overline{\Omega} \times (0, T_{\text{max}})) \cap L^\infty_{\text{loc}}([0, T_{\text{max}}); W^{1,q}(\Omega))
\end{aligned}
\]

which solves (13.1.1) in the classical sense and satisfies the mass conservation

\[
\int_{\Omega} u(\cdot, t) = \int_{\Omega} u_0 \quad \text{for all } t \in (0, T_{\text{max}})
\]

and the lower estimate

\[
\inf_{x \in \Omega} v(x, t) \geq \eta \quad \text{for all } t \in (0, T_{\text{max}}),
\]

where \( \eta > 0 \) is defined as (13.1.5). Moreover, if \( T_{\text{max}} < \infty \), then

\[
\lim_{t \nearrow T_{\text{max}}} \left( \| u(\cdot, t) \|_{L^\infty(\Omega)} + \| v(\cdot, t) \|_{W^{1,\infty}(\Omega)} \right) = \infty.
\]

At the end of this section we shall recall the result about the estimate for \( v \) in dependence on boundedness features of \( u \) derived by a straightforward application of well-known smoothing estimates for the heat semigroup under homogeneous Neumann boundary conditions (see [196, Lemma 2.4], [50, Lemma 2.4]).

**Lemma 13.2.3.** Let \( T > 0 \) and \( 1 \leq \theta, \mu \leq \infty \). If \( \frac{\theta}{2}(\frac{1}{\theta} - \frac{1}{\mu}) < 1 \), then there exists \( C_1(\mu, \theta) > 0 \) such that

\[
\| v(\cdot, t) \|_{L^\mu(\Omega)} \leq C_1(\mu, \theta) \left( 1 + \sup_{s \in (0, T)} \| u(\cdot, s) \|_{L^\theta(\Omega)} \right) \quad \text{for all } t \in (0, T).
\]

### 13.3. A unified viewpoint in energy estimates

Let \( (u, v) \) be the solution of (13.1.1) on \([0, T_{\text{max}}]\) as in Lemma 13.2.2. For the proof of Theorem 13.1.1 we will recall an useful fact to derive the \( L^\infty \)-estimate for \( u \).

**Lemma 13.3.1.** Assume that the solution \( u \) of (13.1.1) given in Lemma 13.2.2 satisfies

\[
\| u(\cdot, t) \|_{L^p(\Omega)} \leq C(p) \quad \text{for all } t \in (0, T_{\text{max}})
\]

with some \( p > \frac{n}{2} \) and \( C(p) > 0 \). Then there exists a constant \( C' > 0 \) such that

\[
\| u(\cdot, t) \|_{L^\infty(\Omega)} + \| v(\cdot, t) \|_{W^{1,\infty}(\Omega)} \leq C' \quad \text{for all } t \in (0, T_{\text{max}}).
\]

167
Proof. Combination of (13.2.2) and the same argument as in [196, Lemma 3.4] leads to the $L^\infty$-estimate for $u$. Moreover, the same argument as in the proof of [193, Theorem 3.2] implies the conclusion of this lemma.

**Unified test function.** Thanks to Lemmas 13.2.2 and 13.3.1 we will only make sure that the $L^p$-estimate for $u$ with some $p > \frac{n}{2}$ to show global existence and boundedness of solutions to (13.1.1). To establish (13.3.1) we introduce the functions $g$ and $\varphi$ by

\[(13.3.2)\quad g(s) := -r \int_0^s \frac{1}{(a + \tau)^k} d\tau, \quad \varphi(s) := \exp\{g(s)\} \quad (s \geq \eta),\]

where $r > 0$ is a constant fixed later and $\eta$ is defined as (13.1.5). When $k > 1$, by straightforward calculations we have

\[\varphi(s) = C_\varphi \exp\left\{ \frac{r}{(k-1)(a+s)^{k-1}} \right\}\]

with $C_\varphi = \exp\{-r(k-1)^{-1}(a+\eta)^{-k+1}\} > 0$, which is a similar function used in [193]. On the other hand, when $k = 1$, it follows that

\[\varphi(s) = \frac{C_\varphi}{(a+s)^r}\]

with some constant $C_\varphi = (a+\eta)^r > 0$, which is a similar function used in [50]. Now we shall prove the following unified inequality by using the test function $\varphi(v)$.

**Lemma 13.3.2.** Assume that $\chi$ satisfies (13.1.3) with some $\lambda > 0$, $k \geq 1$, $a \geq 0$, $K > 0$. Then for all $\varepsilon \in [0,1)$ there exists $c > 0$ such that

\[(13.3.3)\quad \frac{d}{dt} \int_\Omega u^p \varphi(v) + \varepsilon p(p-1) \int_\Omega u^{p-2} \varphi(v)|\nabla u|^2
\leq \int_\Omega u^p H_{\varepsilon,r}(v) \varphi(v)|\nabla v|^2 + c \int_\Omega u^p \varphi(v) - r \int_\Omega u^{p+1} \frac{\varphi(v)}{(a+v)^k},\]

where $H_{\varepsilon,r}$ is the function defined for $s \geq \eta$ by

\[(13.3.4)\quad H_{\varepsilon,r}(s) := -\frac{K r}{(a+s)^{k+1}} + \left( \frac{\varepsilon p K r}{1-\varepsilon} + \frac{p(p-1)K^2}{4(1-\varepsilon)} + \frac{(\varepsilon p + 1 - \varepsilon)r^2}{(1-\varepsilon)(p-1)} \right) \frac{1}{(a+s)^{2k}}\]

**Proof.** We let $p \geq 1$. From (13.1.1) we have

\[(13.3.5)\quad \frac{d}{dt} \int_\Omega u^p \varphi(v) = p \int_\Omega u^{p-1} \varphi(v) \nabla \cdot (\nabla u - u \chi(v) \nabla v) + \int_\Omega u^p \varphi'(v)(\Delta v - v + u).\]

Integration by parts yields

\[(13.3.6)\quad p \int_\Omega u^{p-1} \varphi(v) \nabla \cdot (\nabla u - u \chi(v) \nabla v) + \int_\Omega u^p \varphi'(v) \Delta v
= -p \int_\Omega \nabla (u^{p-1} \varphi(v)) \cdot (\nabla u - u \chi(v) \nabla v) - \int_\Omega (u^p \varphi'(v)) \cdot \nabla v
= -p(p-1) \int_\Omega u^{p-2} \varphi(v) |\nabla u|^2 + \int_\Omega u^{p-1} (p(p-1) \varphi(v) \chi(v) - 2p \varphi'(v)) \nabla u \cdot \nabla v
+ \int_\Omega u^p (-\varphi''(v) + p \varphi'(v) \chi(v)) |\nabla v|^2.\]
Invoking the Young inequality, we infer that for all $\varepsilon \in [0, 1)$,
\[
(13.3.7) \quad \int_{\Omega} u^{p-1} (p(p-1)\varphi(v)\chi(v) - 2p\varphi'(v)) \nabla u \cdot \nabla v \\
\leq (1 - \varepsilon)p(p-1) \int_{\Omega} u^{p-2} \varphi(v) |\nabla u|^2 \\
+ \int_{\Omega} u^{p} \frac{p(p-1)\varphi(v)\chi(v) - 2p\varphi'(v))^2}{4(1 - \varepsilon)p(p-1)\varphi(v)} |\nabla v|^2.
\]
Thus combination of (13.3.5), (13.3.6) and (13.3.7) yields that
\[
(13.3.8) \quad \frac{d}{dt} \int_{\Omega} u^p \varphi(v) + \varepsilon p(p-1) \int_{\Omega} u^{p-2} \varphi(v) |\nabla u|^2 \\
\leq \int_{\Omega} u^p F_{\varphi}(v) |\nabla v|^2 + \int_{\Omega} u^p \varphi'(v)(-v + u),
\]
where
\[
F_{\varphi}(s) := -\varphi''(s) - \frac{\varepsilon}{1-\varepsilon} p\varphi'(s)\chi(s) + \frac{p(p-1)}{4(1 - \varepsilon)} \chi(s)^2 \varphi(s) + \frac{p\varphi'(s)^2}{(1 - \varepsilon)(p-1)\varphi(s)} \quad (s \geq \eta).
\]
Noting that
\[
\varphi'(s) = g'(s)\varphi(s) \quad \text{and} \quad \varphi''(s) = g''(s)\varphi(s) + g'(s)^2\varphi(s) \quad (s \geq \eta),
\]
we can rewrite the function $F_{\varphi}(s)$ as
\[
F_{\varphi}(s) = \left(-g''(s) - \frac{\varepsilon}{1-\varepsilon} pg'(s)\chi(s) + \frac{p(p-1)}{4(1 - \varepsilon)} \chi(s)^2 \varphi(s) + \frac{\varepsilon p - 1 - \varepsilon)g'(s)^2}{(1 - \varepsilon)(p-1)}\right) \varphi(s)
\]
for all $s \geq \eta$. Recalling by (13.3.2) that
\[
g'(s) = \frac{-r}{(a+s)^k} \quad \text{and} \quad g''(s) = \frac{rk}{(a+s)^{k+1}} \quad (s \geq \eta),
\]
we see from (13.1.3) that
\[
(13.3.9) \quad F_{\varphi}(s) \leq H_{\varepsilon,r}(s)\varphi(s)
\]
for all $s \geq \eta$, where $H_{\varepsilon,r}$ is defined as (13.3.4). Therefore we obtain from (13.3.8) together with (13.3.9) that
\[
\frac{d}{dt} \int_{\Omega} u^p \varphi(v) + \varepsilon p(p-1) \int_{\Omega} u^{p-2} \varphi(v) |\nabla u|^2 \\
\leq \int_{\Omega} u^p H_{\varepsilon,r}(v)\varphi(v) |\nabla v|^2 - r \int_{\Omega} u^p \varphi(v) \frac{(-v + u)}{(a+v)^k}.
\]
We finally confirm from the boundedness of the function $s \mapsto \frac{s}{(a+s)^k}$ on $[\eta, \infty)$ ($k \geq 1$) that there exists $c > 0$ satisfying
\[
\int_{\Omega} u^p \varphi(v) \frac{rv}{(a+v)^k} \leq c \int_{\Omega} u^p \varphi(v),
\]
and thus we obtain (13.3.3). \qed
13.4. Global existence and boundedness

In this section we will show the $L^p$-estimate for $u$ with $p > \frac{n}{2}$ by using Lemma 13.3.2.

13.4.1. Energy estimate in the case $k > 1$

We first derive the energy estimate in the case $k > 1$. In this subsection we assume that $\chi$ satisfies (13.1.3) with some $k > 1$. Now we shall show the following inequality by modifying the method in Chapter 3.

**Lemma 13.4.1.** Assume that (13.1.3) and (13.1.4) are satisfied with some $\lambda > 0$, $k > 1$, $a \geq 0$ and $K > 0$. Then there exist $p > \frac{n}{2}$, $r > 0$ and $\varepsilon \in (0,1)$ such that

$$H_{\varepsilon,r}(s) \leq 0 \quad \text{for all } s \geq \eta,$$

where $H_{\varepsilon,r}$ is defined as (13.3.4) and $\eta > 0$ is defined as (13.1.5), which implies that

$$\frac{d}{dt} \int_{\Omega} u^p \varphi(v) + \varepsilon p(p-1) \int_{\Omega} u^{p-2} \varphi(v) |\nabla u|^2 \leq c \int_{\Omega} u^p \varphi(v) - r \int_{\Omega} u^{p+1} \frac{\varphi(v)}{(a+v)^k}.$$

**Proof.** We take $p \geq 1$, $r > 0$ and $\varepsilon \in [0,1)$ which will be fixed later. Due to the definition of $H_{\varepsilon,r}$ we write as

$$H_{\varepsilon,r}(s) = a_1(s)r^2 + a_2(s)r + a_3(s),$$

where

$$a_1(s) := \frac{\varepsilon p + 1 - \varepsilon}{(1 - \varepsilon)(p-1)(a+s)^{2k}},$$

$$a_2(s) := \frac{\varepsilon pK}{(1 - \varepsilon)(a+s)^{2k}} - \frac{k}{(a+s)^{k+1}},$$

$$a_3(s) := \frac{p(p-1)K^2}{4(1 - \varepsilon)(a+s)^{2k}}.$$

Noting from the condition (13.1.4) that there exist $p > \frac{n}{2}$ and $\varepsilon \in (0,1)$ satisfying

$$K < \frac{(1 - \varepsilon)k(a+\eta)(k-1)}{\varepsilon p + \sqrt{p(\varepsilon p + 1 - \varepsilon)}}$$

we have that

$$\frac{\varepsilon p + \sqrt{p(\varepsilon p + 1 - \varepsilon)}}{1 - \varepsilon} K < k(a+s)^{k-1} \quad \text{for all } s \geq \eta.$$

This implies that the discriminant

$$D_r(s) = a_2(s)^2 - 4a_1(s)a_3(s)$$

$$= \left( \frac{\varepsilon pK}{(1 - \varepsilon)(a+s)^{2k}} - \frac{k}{(a+s)^{k+1}} \right)^2 - \frac{p(\varepsilon p + 1 - \varepsilon)K^2}{(1 - \varepsilon)^2(a+s)^{4k}}$$

$$= \frac{1}{(a+s)^{4k}} \left( \frac{\varepsilon pK}{1 - \varepsilon} - k(a+s)^{k-1} \right)^2 - \frac{p(\varepsilon p + 1 - \varepsilon)K^2}{(1 - \varepsilon)^2}.$$
is positive for all \( s \geq \eta \). Finally, we show that there exists \( r > 0 \) such that

\[
H_{\varepsilon,r}(s) = a_1(s)r^2 + a_2(s)r + a_3(s) \leq 0
\]

for all \( s \geq \eta \). Because \( D_r(s) \) is positive, we can define

\[
r_{\pm}(s) := \frac{-a_2(s) \pm \sqrt{D_r(s)}}{2a_1(s)} = \frac{(1 - \varepsilon)(p - 1)}{2(\varepsilon p + 1 - \varepsilon)} \times \left( -\frac{\varepsilon p K}{1 - \varepsilon} + k(a + s)^{k-1} \pm \sqrt{\left( \frac{\varepsilon p K}{1 - \varepsilon} - k(a + s)^{k-1} \right)^2 - \frac{p(\varepsilon p + 1 - \varepsilon)K^2}{(1 - \varepsilon)^2}} \right).
\]

Then we see that \( H_{\varepsilon,r}(s) \leq 0 \) for each \( s \geq \eta \) and all \( r \in (r_-, r_+(s)) \). Setting the functions

\[
\tilde{r}_\pm(\tau) := \tau \pm \sqrt{\tau^2 - \frac{p(\varepsilon p + 1 - \varepsilon)K^2}{(1 - \varepsilon)^2}} \quad \left( \tau \geq \frac{K\sqrt{p(\varepsilon p + 1 - \varepsilon)}}{1 - \varepsilon} \right),
\]

we note that

\[
(13.4.4) \quad r_{\pm}(s) = \frac{(1 - \varepsilon)(p - 1)}{2(\varepsilon p + 1 - \varepsilon)} \tilde{r}_\pm \left( -\frac{\varepsilon p K}{1 - \varepsilon} + k(a + s)^{k-1} \right).
\]

Since the functions \( \tilde{r}_\pm \) satisfy that

\[
\frac{d\tilde{r}_+}{d\tau}(\tau) > 0 \quad \text{and} \quad \frac{d\tilde{r}_-}{d\tau}(\tau) < 0
\]

for all \( \tau \geq \frac{K\sqrt{p(\varepsilon p + 1 - \varepsilon)}}{1 - \varepsilon} \), we obtain from (13.4.2) that

\[
\tilde{r}_- \left( -\frac{\varepsilon p K}{1 - \varepsilon} + k(a + s)^{k-1} \right) < \tilde{r}_- \left( \frac{K\sqrt{p(\varepsilon p + 1 - \varepsilon)}}{1 - \varepsilon} \right)
\]

\[
= \tilde{r}_+ \left( \frac{K\sqrt{p(\varepsilon p + 1 - \varepsilon)}}{1 - \varepsilon} \right) < \tilde{r}_+ \left( -\frac{\varepsilon p K}{1 - \varepsilon} + k(a + s)^{k-1} \right)
\]

holds for all \( s \geq \eta \). Therefore putting

\[
r_0 := \frac{(1 - \varepsilon)(p - 1)}{2(\varepsilon p + 1 - \varepsilon)} \tilde{r}_- \left( \frac{K\sqrt{p(\varepsilon p + 1 - \varepsilon)}}{1 - \varepsilon} \right) = \frac{(p - 1)K}{2} \sqrt{\frac{p}{\varepsilon p + 1 - \varepsilon}} > 0
\]

and recalling (13.4.4), we have \( r = r_0 \in (r_-(s), r_+(s)) \) for all \( s \geq \eta \), which means that (13.4.3) holds for all \( s \geq \eta \). This implies the end of the proof.

Now we are ready to show the \( L^p \)-estimate in the case \( k > 1 \).
Lemma 13.4.2. Assume that (13.1.3) and (13.1.4) are satisfied with some \( \lambda > 0, k > 1, a \geq 0 \) and \( K > 0 \). Then there exist \( p > \frac{n}{2} \) and \( C > 0 \) such that

\[
\| u(\cdot, t) \|_{L^p(\Omega)} \leq C \quad \text{for all} \quad t > 0.
\]

Proof. The proof is similar as in [193]. From Lemmas 13.3.2 and 13.4.1 we obtain (13.4.1) with some \( p > \frac{n}{2}, r > 0 \) and \( \varepsilon \in (0, 1) \). We shall show the \( L^p \)-estimate for \( u \) by using (13.4.1). From the positivity of \( u, v \) and \( \varphi \) we have that

\[
-r \int_\Omega u^{p+1} \frac{\varphi'(v)}{(a + v)^k} \leq 0.
\]

We next deal with the term \( \int_\Omega u^{p-1} \varphi(v) |\nabla u|^2 \). Noting the boundedness of \( \varphi \):

\[
C_\varphi \leq \varphi(s) \leq 1 \quad (s \geq \eta)
\]

and combining the Gagliardo–Nirenberg inequality (see Lemma 2.1.1) with the mass conservation property (13.2.1), we deduce that there exists \( c_1 > 0 \) such that

\[
\int_\Omega u^p \varphi(v) \leq \int_\Omega u^p = \| u^\frac{p}{2} \|_{L^2(\Omega)}^2 \leq c_1 \left( \| \nabla u^\frac{p}{2} \|_{L^2(\Omega)} + \| u^\frac{p}{2} \|_{L^p(\Omega)} \right)^{2a} \| u^\frac{p}{2} \|_{L^p(\Omega)}^{2(1-a)}
\]

\[
= c_1 \left( \| \nabla u^\frac{p}{2} \|_{L^2(\Omega)} + \| u^\frac{p}{2} \|_{L^p(\Omega)} \right)^{2a} \| u^\frac{p}{2} \|_{L^p(\Omega)}^{p(1-a)}
\]

with \( a = \frac{pn - 2}{p^2 + 1} \in (0, 1) \). From (13.4.1), (13.4.5) and combination of (13.4.7) and

\[
C_\varphi \int_\Omega |\nabla u^\frac{p}{2}|^2 \leq \int_\Omega u^{p-2} \varphi(v) |\nabla u|^2
\]

we see that there exist \( c_2, c_3 > 0 \) such that

\[
\frac{d}{dt} \int_\Omega u^p \varphi(v) \leq c \int_\Omega u^p \varphi(v) - c_2 \left( \int_\Omega u^p \varphi(v) \right)^{\frac{1}{2}} + c_3,
\]

which implies that there exist \( p > \frac{n}{2}, r > 0 \) (determined in Lemma 13.4.1) and \( C > 0 \) satisfying

\[
\int_\Omega u^p \varphi(v) \leq C.
\]

Therefore we obtain from (13.4.6) that

\[
\int_\Omega u^p \leq C C_\varphi^{-1}.
\]

Thus we attain the goal of the proof. \( \square \)
13.4.2. Energy estimates in the case \( k = 1 \)

In this section we assume that the sensitivity function \( \chi \) satisfies (13.1.3) with \( k = 1 \). We first show the following estimate for \( H_{0,r}(s) \).

**Lemma 13.4.3.** Assume that (13.1.3) and (13.1.4) are satisfied with \( k = 1 \) and with some \( \lambda > 0 \), \( a \geq 0 \) and \( K > 0 \). Then for each \( p \in (1, \frac{1}{K^2}) \),

\[
H_{0,r}(s) \leq 0 \quad \text{for all } r \in I_p \text{ and all } s \geq \eta,
\]

where \( H_{0,r} \) is defined as (13.3.4) with \( \varepsilon = 0 \) and

\[
I_p := \left( \frac{p-1}{2} \left( 1 - \sqrt{1 - pK^2} \right), \frac{p-1}{2} \left( 1 + \sqrt{1 - pK^2} \right) \right),
\]

which means

\[
\frac{d}{dt} \int_{\Omega} u^p \varphi(v) \leq c \int_{\Omega} u^p \varphi(v) - r \int_{\Omega} u^{p+1} \frac{\varphi(v)}{a + v}.
\]

**Proof.** We pick \( p \geq 1 \) and \( r > 0 \) which will be fixed later. Due to the definition of \( H_{0,r}(s) \) we write as

\[
H_{0,r}(s) = \frac{1}{(a + s)^2} \left( \frac{1}{p-1} r^2 - r + \frac{p(p-1)K^2}{4} \right)
\]

\[
= \frac{b_1 r^2 + b_2 r + b_3}{(a + s)^2}.
\]

From (13.1.4) we note that \( (1, \frac{1}{K^2}) \) is not an empty set. If we choose \( p \in (1, \frac{1}{K^2}) \), then the discriminant of \( b_1 r^2 + b_2 r + b_3 \) is nonnegative:

\[
D_r = 1 - pK^2 \geq 0.
\]

Thus we can define the interval \( I_p \) as (13.4.8) for each \( p \in (1, \frac{1}{K^2}) \). Then we have from straightforward calculations that for each \( p \in (1, \frac{1}{K^2}) \) and all \( r \in I_p \),

\[
b_1 r^2 + b_2 r + b_3 \leq 0
\]

holds, which yields that for all \( p \in (1, \frac{1}{K^2}) \) there exists the interval \( I_p \) such that

\[
H_{0,r}(s) \leq 0
\]

for all \( r \in I_p \) and all \( s \geq \eta \). This implies the end of the proof. \( \square \)

We next show the estimate for \( \int_{\Omega} u^p \varphi(v) \). However we cannot easily obtain the estimate for \( \int_{\Omega} u^p \) because \( \varphi \) is not a bounded function. Therefore we will show the following lemma, which has an important role for obtaining the \( L^p \)-estimate.
Lemma 13.4.4. Assume that (13.1.3) and (13.1.4) are satisfied with $k = 1$ and with some $\lambda > 0$, $a \geq 0$ and $K > 0$. Suppose that $p \in (1, \frac{1}{K^2})$, $r \in I_p$ such that $p - r \geq 1$. If there exists $C > 0$ such that

$$\|v(\cdot, t)\|_{L^{p-r}(\Omega)} \leq C \quad \text{for all } t > 0,$$

then there exists $C_2(p, r) > 0$ satisfying

$$\int_{\Omega} u^p \varphi(v) \leq C_2(p, r) \quad \text{for all } t > 0.$$

Proof. The proof is similar as in [50]. We let $p \in (1, \frac{1}{K^2})$. We denote by $I_p$ the interval defined as (13.4.8), and choose $r \in I_p$. We shall attain the conclusion from (13.4.9). By virtue of the Hölder inequality, we infer that

$$\int_{\Omega} u^p \varphi(v) \leq C_{\varphi} \left( \int_{\Omega} u^{p+1} \frac{\varphi(v)}{a + v} \right)^{\frac{p}{p+1}} \left( \int_{\Omega} (a + v)^{p-r} \right)^{\frac{1}{p-r}}.$$

Noting from (13.4.10) and the fact $p - r \geq 1$ that

$$\|a + v(\cdot, t)\|_{L^{p-r}(\Omega)} \leq a|\Omega|^{\frac{1}{p-r}} + C \quad \text{for all } t > 0,$$

we obtain that there exists $c_1 > 0$ such that

$$\int_{\Omega} u^p \varphi(v) \leq c_1 \left( \int_{\Omega} u^{p+1} \frac{\varphi(v)}{a + v} \right)^{\frac{p}{p+1}}.$$

Plugging this inequality into (13.4.9), we have the conclusion from a standard ODE comparison argument. 

Lemma 13.4.5. Assume that (13.1.3) and (13.1.4) are satisfied with $k = 1$ and with some $\lambda > 0$, $a \geq 0$ and $K > 0$. Then there exist $p > \frac{n}{2}$ and $C > 0$ such that

$$\|u(\cdot, t)\|_{L^p(\Omega)} \leq C.$$

Proof. We use a similar argument as in [50, Proof of Main Theorem (Step 1)]. First we choose a pair $(p_0, r_0)$ such that

$$\begin{cases} p_0 \in \left( 1, \min \left\{ \frac{1}{K^2}, n + 1, \frac{n + 2}{n - 2} \right\} \right), \\ r_0 := \frac{p_0 - 1}{2}. \end{cases}$$

Then $p_0 > r_0$, $r_0 < \frac{n}{2}$, $r_0 \in I_{p_0}$, $p_0 - r_0 \geq 1$ and $p_0 - r_0 < \frac{n}{n - 2}$. Since $\frac{n}{2} \left( 1 - \frac{1}{p_0 - r_0} \right) < 1$ from the inequality $p_0 - r_0 < \frac{n}{n - 2}$, it follows from Lemma 13.2.3 and (13.2.1) that there exists $b_0 > 0$ such that

$$\|v(\cdot, t)\|_{L^{p_0-r_0}(\Omega)} \leq C_1(p_0 - r_0, 1) \left( 1 + \sup_{s \in (0, T_{\max})} \|u(\cdot, s)\|_{L^1(\Omega)} \right) \leq b_0$$

174
for all $t \in (0, T_{\max})$. Thus we have from the fact $p_0 - r_0 \geq 1$ and Lemma 13.4.4 that
\[
\int_{\Omega} u^{p_0}(a + v)^{-r_0} \leq C_{p_0} C_2(p_0, r_0).
\]

We will show that for all $q_0 \in (\frac{2p_0}{p_0+1}, \min\{p_0, \frac{n(p_0-r_0)}{n-2r_0}\})$ there exists $b'_0 > 0$ such that
\[
(13.4.11) \quad \int_{\Omega} u^{q_0} \leq b'_0.
\]

We first note from the fact $p_0 \geq 1$ that
\[
\frac{2p_0}{p_0+1} \leq p_0 \quad \text{and} \quad \frac{2p_0}{p_0+1} \leq \frac{n(p_0+1)}{2(n - p_0+1)} = \frac{n(p_0 - r_0)}{n - 2r_0}.
\]

Now we fix $q_0 \in (\frac{2p_0}{p_0+1}, \min\{p_0, \frac{n(p_0-r_0)}{n-2r_0}\})$. Applying the Hölder inequality, we infer that
\[
(13.4.12) \quad \int_{\Omega} u^{q_0} \leq \left( \int_{\Omega} u^{p_0}(a + v)^{-r_0} \right)^{\frac{q_0}{p_0}} \left( \int_{\Omega} (a + v)^{\frac{q_0 r_0}{p_0 - q_0}} \right)^{\frac{p_0 - q_0}{p_0}} \leq \left( C_{p_0} C_2(p_0, r_0) \right)^{\frac{q_0}{p_0}} \|a + v(\cdot, t)\|_{L^{\frac{p_0}{p_0-r_0}}(\Omega)}^\frac{q_0 r_0}{p_0 - q_0}.
\]

We can confirm from the fact $\frac{n}{2} (\frac{1}{q_0} - \frac{p_0-q_0}{q_0 r_0}) < 1$, $\frac{q_0 r_0}{p_0 - q_0} \geq 1$ (from the fact $q_0 \geq \frac{2p_0}{p_0+1}$) and Lemma 13.2.3 that
\[
(13.4.13) \quad \|v(\cdot, t)\|_{L^{\frac{p_0}{p_0-r_0}}(\Omega)} \leq C_1 \left( \frac{q_0 r_0}{p_0 - q_0}, q_0 \right) \left( 1 + \sup_{s \in (0, T_{\max})} \|u(\cdot, s)\|_{L^{p_0}(\Omega)} \right)
\]
for all $t \in (0, T_{\max})$. Therefore plugging (13.4.13) into (13.4.12) derives that there exists $b'_0 > 0$ satisfying
\[
\|u(\cdot, t)\|_{L^{p_0}(\Omega)} \leq b'_0 \left( 1 + \sup_{s \in (0, T_{\max})} \|u(\cdot, s)\|_{L^{p_0}(\Omega)} \right)^\frac{r_0}{p_0}
\]
for all $t \in (0, T_{\max})$. Noting that $\frac{r_0}{p_0} < 1$, we obtain (13.4.11). In the above argument, since $\frac{n}{2} < \min\{\frac{1}{K^2}, n+1\}$, if $\frac{n+2}{n-2} > \frac{n}{2}$, then we can fix $p_0 > \frac{n}{2}$. Thus we can find $q_0 > \frac{n}{2}$ due to $p_0 < \frac{n(p_0+1)}{2(n-p_0+1)} = \frac{n(p_0-r_0)}{n-2r_0}$ if $\frac{n+2}{n-2} > \frac{n}{2}$. On the other hand, if $\frac{n+2}{n-2} \leq \frac{n}{2}$, then we proceed the iteration argument. For all $i \in \mathbb{N}$, we fix a pair $(p_i, r_i)$ defined inductively such that
\[
\begin{align*}
(p_i &\in \left( p_{i-1}, \min\left\{ \frac{1}{K^2}, n+1, \frac{p_{i-1}(n+2)}{n-2p_{i-1}} \right\} \right), \\
(r_i &:= \frac{p_i - 1}{2},
\end{align*}
\]
175
then we can see that \( p_i > r_i, r_i < \frac{n}{2} \) and \( r_i \in I_{p_i} \). Moreover, since the relation \( p_i < \frac{p_{i-1}(n+2)}{n-2p_{i-1}} \) implies that

\[
p_i - r_i < \frac{n \cdot \frac{n(p_{i-1}+1)}{2(n-p_{i-1}+1)}}{n - 2 \cdot \frac{n(p_{i-1}+1)}{2(n-p_{i-1}+1)}},
\]

we can find some \( q_{i-1} \in \left( \frac{2p_{i-1}}{p_{i-1}+1}, \min\{p_{i-1}, \frac{n(p_{i-1}+1)}{2(n-p_{i-1}+1)} \} \right) \) satisfying

\[
p_i - r_i < \frac{nq_{i-1}}{n - 2q_{i-1}},
\]

which means that \( \frac{n}{2} \left( \frac{1}{q_{i-1}} - \frac{1}{p_i - r_i} \right) < 1 \). Therefore we obtain

(13.4.14) \[ \|v(\cdot, t)\|_{L^{p_i-r_i}(\Omega)} \leq C_1(p_i - r_i, q_{i-1}) \left( 1 + \sup_{s \in (0, T_{\text{max}})} \|u(\cdot, s)\|_{L^{q_i-1}(\Omega)} \right) \]

for all \( t \in (0, T_{\text{max}}) \). According to (13.4.14) and \( p_i - r_i \geq 1 \) into Lemma 13.4.4, we have

\[
\int_{\Omega} u^{p_i}(a + v)^{-r_i} \leq C_\varphi r_i C_2(p_i, r_i).
\]

Due to a similar argument to the first iteration, we can show that for all \( t \in (0, T_{\text{max}}) \),

\[
\|u(\cdot, t)\|_{L^{q_i}(\Omega)} \leq b'_i(q_i) \quad \text{for all } q_i \in \left( \frac{2p_i}{p_i + 1}, \min\left\{ \frac{n(p_i+1)}{2(n-p_i+1)} \right\} \right)
\]

with some constant \( b'_i(q_i) > 0 \). Because the increasing function \( f(x) := \frac{x(n+2)}{n-2x} \) satisfies \( f(x) > 1 \) and \( f(x) \to \infty \) as \( x \to \frac{n}{2} \), we can find some \( i_0 \in \mathbb{N} \) such that \( p_{i_0} > \frac{n}{2} \) and \( p_{i_0-1} \leq \frac{n}{2} \), and hence \( q_{i_0} > \frac{n}{2} \). Therefore we verify

\[
\|u(\cdot, t)\|_{L^p(\Omega)} \leq C
\]

with some \( p > \frac{n}{2} \) and some \( C > 0 \), which completes the proof.

13.4.3. Proof of Theorem 13.1.1

Combination of the \( L^p \)-estimate for \( u \) (see Lemma 13.4.2 or Lemma 13.4.5) and Lemma 13.3.1 directly leads to the conclusion of Theorem 13.1.1. \( \square \)
Chapter 14

Global existence and boundedness in a fully parabolic chemotaxis system with signal-dependent sensitivity and logistic term

14.1. Problem and results

The following problem which describes the movement of species with chemotaxis

\[ u_t = \Delta u - \nabla \cdot (u \chi(v) \nabla v) + \mu u(1-u), \]
\[ v_t = \Delta v + u - v, \]

where \( \chi \) is a function and \( \mu \geq 0 \) is a constant, is called a Keller-Segel system or a chemotaxis system, and is studied intensively. The function \( \chi \) appearing in the above problem is called signal-dependent sensitivity, and examples of this function \( \chi \) are as follows:

- \( \chi(s) = K \) (linear)
- \( \chi(s) = \frac{K}{s} \) (singular)
- \( \chi(s) = \frac{K}{(1+s)^2} \) (regular)

for \( s > 0 \) with some constant \( K > 0 \). Previous works which deal with the constant sensitivity can be found in [25, 67, 76, 144, 194, 199]; the singular sensitivity is treated in [50, 54, 55, 102, 105]; we can find works related to the regular sensitivity in [54, 55, 129, 137, 138, 150, 151, 211]; variation of chemotaxis systems are in [5]. Here we focus on the case that \( \chi \) is a function generalizing the regular sensitivity:

\[ \chi(s) = \frac{K}{(a+s)^k} \quad (s > 0) \tag{14.1.1} \]

with some constants \( a \geq 0, k > 1 \) and \( K > 0 \). In a mathematical view, one of difficulties caused by the sensitivity function \( \chi \) is to deal with the additional term \( u\chi'(v)|\nabla v|^2 \) which does not appear in the case that \( \chi \) is a constant. In the case that \( \mu = 0 \), by using an
energy estimate to overcome the difficulties of the sensitivity function, under the condition that $\chi$ fulfills (14.1.1) with some constants $a \geq 0$, $k > 1$ and $K > 0$ satisfying
\[
K < k(a + \eta)^{k-1} \sqrt{\frac{2}{n}},
\]
where $\eta$ is a constant defined as
\[
\eta := \sup_{\tau > 0} \left( \min_{x \in \Omega} \left\{ e^{-2\tau} \min_{x \in \Omega} v_0(x), \: c_0 \| u_0 \|_{L^1(\Omega)} (1 - e^{-\tau}) \right\} \right) \geq 0
\]
(see [51] and Chapter 13), global existence and boundedness were established (Chapter 13). Recently, Fujie–Senba [54, 55] established conditions for global existence and boundedness in a problem generalizing the chemotaxis system with $\mu = 0$. More related works which deal with a two-species chemotaxis system with competitive kinetics can be found in [150, 151, 211] and Chapters 3, 7, 8; global existence and boundedness are in [150, 151, 211] and Chapters 3, 7; asymptotic behavior is shown in Chapters 7, 8.

In summary, the conditions (14.1.1)–(14.1.2) lead to global existence and boundedness in the chemotaxis system with $\mu = 0$. However, the case that $\mu > 0$ has not been studied. The purpose of this work is to derive conditions for global existence and boundedness in the chemotaxis system.

In this chapter we consider the chemotaxis system with signal-dependent sensitivity and logistic term
\[
\begin{aligned}
    u_t &= \Delta u - \nabla \cdot (u \chi(v) \nabla v) + \mu u(1 - u), \quad x \in \Omega, \: t > 0, \\
    v_t &= \Delta v + u - v, \quad x \in \Omega, \: t > 0, \\
    \nabla u \cdot \nu = \nabla v \cdot \nu &= 0, \quad x \in \partial \Omega, \: t > 0, \\
    u(x, 0) &= u_0(x), \: v(x, 0) = v_0(x), \quad x \in \Omega,
\end{aligned}
\]
where $\Omega$ is a bounded domain in $\mathbb{R}^n$ ($n \geq 2$) with smooth boundary $\partial \Omega$ and $\nu$ is the outward normal vector to $\partial \Omega$; $\mu > 0$ is a constant; the initial data $u_0$ and $v_0$ are assumed to be nonnegative functions. The unknown function $u(x, t)$ represents the population density of species and $v(x, t)$ shows the concentration of the substance at place $x$ and time $t$. As to the sensitivity function $\chi$, we are interested in functions generalizing
\[
\chi(s) = \frac{K}{(1 + s)^2} \quad (s > 0),
\]
where $K > 0$ is a constant.

In order to achieve our purpose we shall suppose that $\chi$ satisfies that
\[
\chi \in C^{1+\lambda}((0, \infty)) \quad \text{and} \quad 0 \leq \chi(s) \leq \frac{K}{(a + s)^k} \quad (s > 0)
\]
with some $\lambda > 0$, $k > 1$, $a > 0$ and $K > 0$ fulfilling
\[
K < ka^{k-1} \sqrt{\frac{2}{n}}.
\]
Now the main result reads as follows.
Theorem 14.1.1. Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded domain with smooth boundary and let $\mu > 0$. Assume that $\chi$ satisfies (14.1.4) with some $\lambda > 0$, $k > 1$, $a > 0$, $K > 0$ fulfilling (14.1.5). Then for any $u_0, v_0$ satisfying
\begin{equation}
0 \leq u_0 \in C(\overline{\Omega}) \setminus \{0\} \quad \text{and} \quad 0 \leq v_0 \in W^{1,q}(\Omega) \setminus \{0\}
\end{equation}
with some $q > n$, there exists an exactly one pair $(u, v)$ of positive functions

$u, v \in C(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty))$

which solves (14.1.3). Moreover, the solution $(u, v)$ is uniformly bounded, i.e., there exists a constant $C > 0$ such that

$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,q}(\Omega)} \leq C$

for all $t > 0$.

Here we give one remark: The condition (14.1.5) is more restricted condition than (14.1.2) except the case that $\eta = 0$ (which is the case that $\min_{x \in \Omega} v_0(x) = 0$). The reason is that it is difficult to see the uniform-in-time lower estimate for $v$ because of lacking information about the lower estimate for $u$. Moreover, the condition (14.1.5) is independent of $\mu > 0$: The question “can the logistic term relax conditions for global existence and boundedness?” is still open problem in (14.1.3).

The strategy for the proof of Theorem 14.1.1 is to construct the $L^p$-estimate for $u$ with some $p > \frac{n}{2}$. One of keys for this strategy is to derive the inequality

\[
\frac{d}{dt} \int_{\Omega} u^p \varphi(v) \leq c \int_{\Omega} u^p \varphi(v) - \mu p \int_{\Omega} u^{p+1} \varphi(v)
\]

for some constant $c > 0$, where

$\varphi(s) := \exp \left\{ -r \int_0^s \frac{1}{(a + \tau)^{\frac{p}{p-1}}} d\tau \right\} \quad (s \geq 0)$

with some $r > 0$. Thanks to this strategy, we obtain

$\int_{\Omega} u^p \varphi(v) \leq C$

with some $C > 0$, which together with the lower estimate for $\varphi$ implies the $L^p$-estimate for $u$. Thus in light of the well-known semigroup estimates, we can attain the $L^\infty$-estimate for $u$.

14.2. Proof of the main result

In this section we will prove Theorem 14.1.1. We first recall the well-known result about local existence of solutions to (14.1.3) (see e.g., [5, Lemma 3.1]).
Lemma 14.2.1. Assume that \( \chi \) satisfies (14.1.4) with some \( \lambda > 0, k > 1, a > 0, K > 0 \) and the initial data \( u_0, v_0 \) fulfil (14.1.6) for some \( q > n \). Then there exist \( T_{\text{max}} \in (0, \infty] \) and exactly one pair \((u, v)\) of positive functions

\[
\begin{align*}
  u &\in C(\bar{\Omega} \times [0, T_{\text{max}}]) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\text{max}})), \\
v &\in C(\bar{\Omega} \times [0, T_{\text{max}}]) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\text{max}})) \cap L^\infty_{\text{loc}}([0, T_{\text{max}}); W^{1,q}(\Omega))
\end{align*}
\]

which solves (14.1.3) in the classical sense. Moreover, if \( T_{\text{max}} < \infty \), then

\[
\lim_{t \to T_{\text{max}}^-} (\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,q}(\Omega)}) = \infty.
\]

In the following, we let \((u, v)\) be the solution of (14.1.3) on \([0, T_{\text{max}}]\) as in Lemma 14.2.1.

For the proof of Theorem 14.1.1 we will recall a useful fact to derive the \( L^1 \)-estimate for \( u \).

Lemma 14.2.2. Assume that the solution \((u, v)\) of (14.1.3) satisfies

\[
\|u(\cdot, t)\|_{L^p(\Omega)} \leq C(p)
\]

for all \( t \in (0, T_{\text{max}}) \) with some \( p > \frac{n}{2} \) and \( C(p) > 0 \). Then there exists a constant \( C' > 0 \) such that

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,q}(\Omega)} \leq C'
\]

for all \( t \in (0, T_{\text{max}}) \).

Proof. The same argument as in the proof of [5, Lemma 3.2] yields this result. \(\square\)

Thanks to Lemmas 14.2.1 and 14.2.2 we will only make sure that the \( L^p \)-estimate for \( u \) holds with some \( p > \frac{n}{2} \) to show global existence and boundedness of solutions to (14.1.3). To establish (14.2.1) we introduce the functions \( g \) and \( \varphi \) by

\[
\begin{align*}
g(s) &:= -r \int_0^s \frac{1}{(a + \tau)^k} d\tau, \\
\varphi(s) &:= \exp\{g(s)\} \quad (s \geq 0),
\end{align*}
\]

where \( r > 0 \) is a constant fixed later. Here we note from straightforward calculations that

\[
\varphi(s) = C_\varphi \exp\left\{ \frac{r}{(k-1)(a+s)^{k-1}} \right\}
\]

with

\[
C_\varphi = \exp\{-r(k-1)^{-1}a^{-k+1}\} > 0.
\]

Now we shall prove the following inequality by using the test function \( \varphi(v) \).
Lemma 14.2.3. Assume that $\chi$ satisfies (14.1.4) with some $\lambda > 0$, $k > 1$, $a > 0$, $K > 0$. Then there exists $c > 0$ such that

$$\frac{d}{dt} \int_\Omega u^p \phi(v) \leq \int_\Omega u^p H_r(v) \phi(v) |\nabla v|^2 + c \int_\Omega u^p \phi(v) - \mu p \int_\Omega u^{p+1} \phi(v),$$

where $H_r$ is the function defined by

$$H_r(s) := -\frac{kr}{(a+s)^{k+1}} + \left(\frac{p(p-1)K^2}{4} + \frac{r^2}{p-1}\right) \frac{1}{(a+s)^{2k}}$$

for $s \geq 0$.

Proof. Let $p \geq 1$. From (14.1.3) we have

$$\frac{d}{dt} \int_\Omega u^p \phi(v) = p \int_\Omega u^{p-1} \phi(v) \nabla \cdot (\nabla u - u \chi(v) \nabla v) + \mu p \int_\Omega u^p \phi(v)(1 - u)$$

$$+ \int_\Omega u^p \phi'(v)(\Delta v - v + u).$$

Then integration by parts derives

$$\int_\Omega u^{p-1} \phi(v) \nabla \cdot (\nabla u - u \chi(v) \nabla v) + \int_\Omega u^p \phi'(v) \Delta v$$

$$= -p \int_\Omega \nabla (u^{p-1} \phi(v)) \cdot (\nabla u - u \chi(v) \nabla v) - \int_\Omega \nabla (u^p \phi'(v)) \cdot \nabla v$$

$$= -p(p-1) \int_\Omega u^{p-2} \phi(v)|\nabla u|^2 + \int_\Omega u^{p-1}(p(p-1) \phi(v) \chi(v) - 2p \phi'(v)) \nabla u \cdot \nabla v$$

$$+ \int_\Omega u^p (-\phi''(v) + p \phi'(v) \chi(v)) |\nabla v|^2.$$ 

Due to the Young inequality, we infer that

$$\int_\Omega u^{p-1} (p(p-1) \phi(v) \chi(v) - 2p \phi'(v)) \nabla u \cdot \nabla v$$

$$\leq p(p-1) \int_\Omega u^{p-2} \phi(v) |\nabla u|^2 + \int_\Omega \frac{u^p (p(p-1) \phi(v) \chi(v) - 2p \phi'(v))^2}{4p(p-1) \phi(v)} |\nabla v|^2.$$ 

Thus a combination of (14.2.6), (14.2.7) and (14.2.8) yields that

$$\frac{d}{dt} \int_\Omega u^p \phi(v) \leq \int_\Omega u^p F_\phi(v) |\nabla v|^2 + \mu p \int_\Omega u^p \phi(v)(1 - u) + \int_\Omega u^p \phi'(v)(-v + u),$$

where

$$F_\phi(s) := -\phi''(s) + \frac{p(p-1)}{4} \chi(s)^2 \phi(s) + \frac{p \phi'(s)^2}{(p-1) \phi(s)} (s \geq 0).$$

Noting that

$$\phi'(s) = g'(s) \phi(s) \quad (s \geq 0)$$

181
and
\[ \varphi''(s) = g''(s)\varphi(s) + g'(s)^2\varphi(s) \quad (s \geq 0), \]
we can rewrite the function \( F_\varphi(s) \) as
\[ F_\varphi(s) = \left( -g''(s) + \frac{p(p-1)}{4}\varphi(s) \right) \varphi(s) \quad (s \geq 0). \]

Recalling by (14.2.2) that
\[ g'(s) = \frac{-r}{(a+s)^k} \quad (s \geq 0) \]
and
\[ g''(s) = \frac{rk}{(a+s)^{k+1}} \quad (s \geq 0), \]
we obtain from (14.1.4) that
\[ F_\varphi(s) \leq H_r(s)\varphi(s) \quad \text{for all } s \geq 0, \]
where \( H_r \) is defined as (14.2.5). Therefore we see from (14.2.9) together with (14.2.10) that
\[ \frac{d}{dt} \int_\Omega u^p \varphi(v) \leq \int_\Omega u^p H_r(v)\varphi(v)|\nabla v|^2 + \mu \int_\Omega u^p \varphi(v)(1-u) - r \int_\Omega u^p \varphi(v)(-v + u) \]
and thus we obtain (14.2.4). \( \square \)

Now we shall confirm the following inequality which enables us to see the \( L^p \)-estimate for \( u \).

**Lemma 14.2.4.** Assume that (14.1.4) and (14.1.5) are satisfied with some \( \lambda > 0 \), \( k > 1 \), \( a > 0 \) and \( K > 0 \). Then there exist \( p > \frac{n}{2} \) and \( r > 0 \) such that
\[ H_r(s) \leq 0 \quad \text{for all } s \geq 0, \]
where \( H_r \) is defined as (14.2.5), which implies that
\[ \frac{d}{dt} \int_\Omega u^p \varphi(v) \leq c \int_\Omega u^p \varphi(v) - \mu \int_\Omega u^{p+1} \varphi(v) \]
holds.
Proof. We take \( p \geq 1, r > 0 \) which will be fixed later. Due to the definition of \( H_r \) we write as

\[
H_r(s) = a_1(s)r^2 + a_2(s)r + a_3(s),
\]

where

\[
a_1(s) := \frac{1}{(p - 1)(a + s)^{2k}},
\]

and

\[
a_2(s) := -\frac{k}{(a + s)^{k+1}},
\]

as well as

\[
a_3(s) := \frac{p(p - 1)K^2}{4(a + s)^{2k}}.
\]

Then the same argument as in the proof of Lemma 13.4.1 with \( \varepsilon = 0 \) enables us to find some \( p > \frac{n}{2} \) such that

\[
H_r(s) = a_1(s)r^2 + a_2(s)r + a_3(s) \leq 0
\]

holds for all \( s \geq 0 \) with

\[
r := \frac{(p - 1)K\sqrt{p}}{2} > 0,
\]

which means that (14.2.11) holds. Moreover, from a combination of Lemma 14.2.3 and (14.2.11) we obtain that

\[
\frac{d}{dt} \int_{\Omega} u^p \varphi(v) \leq \int_{\Omega} u^p H_r(v)\varphi(v) |\nabla v|^2 + c \int_{\Omega} u^p \varphi(v) - \mu p \int_{\Omega} u^{p+1} \varphi(v),
\]

\[
\leq c \int_{\Omega} u^p \varphi(v) - \mu p \int_{\Omega} u^{p+1} \varphi(v),
\]

which completes the proof. \( \square \)

Now we are ready to show the \( L^p \)-estimate for \( u \). By using an argument similar to that in the proof of Lemma 3.3.2 we can verify the following lemma.

**Lemma 14.2.5.** Assume that (14.1.4) and (14.1.5) are satisfied with some \( \lambda > 0, k > 1, a > 0 \) and \( K > 0 \). Then there exist \( p > \frac{n}{2} \) and \( C > 0 \) such that

\[
\|u(\cdot, t)\|_{L^p(\Omega)} \leq C
\]

for all \( t \in (0, T_{\text{max}}) \).
Proof. From Lemma 14.2.4 we obtain (14.2.12) with some $p > \frac{n}{2}$ and $r > 0$. We shall show the $L^p$-estimate for $u$ by using (14.2.12). We first note from the definition of $\varphi$ (see (14.2.2)–(14.2.3)) that

$$C_\varphi \leq \varphi(s) \leq 1 \quad (s \geq 0).$$

Noticing from the Hölder inequality and (14.2.13) that

$$\int_\Omega u^p \varphi(v) \leq \left( \int_\Omega \varphi(v) \right)^{\frac{1}{p+1}} \left( \int_\Omega u^{p+1} \varphi(v) \right)^{\frac{p}{p+1}} \leq |\Omega|^{\frac{1}{p+1}} \left( \int_\Omega u^{p+1} \varphi(v) \right)^{\frac{p}{p+1}},$$

we infer from (14.2.12) that

$$\frac{d}{dt} \int_\Omega u^p \varphi(v) \leq c \int_\Omega u^p \varphi(v) - \mu p |\Omega|^{-\frac{1}{p+1}} \left( \int_\Omega u^p \varphi(v) \right)^{\frac{p+1}{p}},$$

which implies that there exists $C > 0$ satisfying

$$\int_\Omega u^p \varphi(v) \leq C.$$

Therefore we obtain from (14.2.13) that

$$\int_\Omega u^p \leq CC_\varphi^{-1},$$

which entails this lemma.

Proof of Theorem 14.1.1. Lemmas 14.2.2 and 14.2.5 directly lead to the conclusion of Theorem 14.1.1.
Chapter 15

The fast signal diffusion limit in a chemotaxis system with strong signal sensitivity

15.1. Motivation and results

Partial differential equation is one of topics of mathematical analysis, and many mathematicians study variations of partial differential equations intensively. Here there are several types of partial differential equations, e.g., parabolic partial differential equation and elliptic partial differential equation, and these often describe many phenomena which appear in natural science, especially, physics, chemistry and biology. Therefore partial differential equations frequently play an important role in analysis of some phenomenon, moreover, in some case elliptic partial differential equations help analysis of parabolic partial differential equations, e.g., analysis of steady states or simplified equations of parabolic partial differential equations. Here we focus on biological phenomena, especially chemotaxis which is one of important properties and is related to e.g., the movement of sperm, the migration of neurons or lymphocytes and the tumor invasion. Chemotaxis is the property such that species move towards higher concentration of a chemical substance when they plunge into hunger. One of examples of species which have chemotaxis is Dictyostelium discoideum. Keller–Segel [89] studied the aggregation of Dictyostelium discoideum due to an attractive chemical substance, and proposed the following system of partial differential equations

\[ u_t = \Delta u - \chi \nabla \cdot (u \nabla v), \quad \lambda v_t = \Delta v - v + u \]

in \( \Omega \times (0, \infty) \), where \( \Omega \subset \mathbb{R}^n \) is a bounded domain, \( \chi > 0, \lambda = 0 \) (parabolic-elliptic system) or \( \lambda > 0 \) (parabolic-parabolic system). The above system is called as Keller–Segel system or chemotaxis system. In this problem the parabolic-elliptic system was first investigated, and then the result for the parabolic-elliptic system provided us some conjecture of results for the parabolic-parabolic system and the interaction between the parabolic-elliptic system and the parabolic-parabolic system made progress on researches of the Keller–Segel system. In the two-dimensional parabolic-elliptic case Nagai [140] showed a condition for existence of global bounded solutions or blow-up solutions in the
radially symmetric situation, which tells us that the size of initial data will determine whether classical solutions of the above system exist globally or not. After the above pioneering work, the Keller–Segel system was studied intensively; conditions for global existence or blow-up in the parabolic-elliptic system were studied in \cite{141,144} and in the parabolic-parabolic system were investigated in \cite{10,25,60,69,76,144,194}; blow-up asymptotics of solutions for the parabolic-elliptic system is in \cite{158,160} and for the parabolic-parabolic system is in \cite{69,126,143}. More related works can be found in \cite{4,115,164,177,178,195} and Chapters 5, 6, 7; a chemotaxis system with logistic term in the parabolic-elliptic case is in \cite{177} and in the parabolic-parabolic case is in \cite{195}; global existence and stabilization in a two-species chemotaxis-competition system were shown in the parabolic-parabolic-elliptic case (\cite{164,178} and Chapters 5, 6) and in the parabolic-parabolic-parabolic case (\cite{4,115} and Chapter 7).

Moreover, the chemotaxis system with signal-dependent sensitivity
\[
\begin{align*}
  u_t &= \Delta u - \nabla \cdot (u\chi(v)\nabla v), \\
  \lambda v_t &= \Delta v - v + u
\end{align*}
\]
in $\Omega \times (0, \infty)$, where $\Omega \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) is a bounded domain, $\lambda \geq 0$ is a constant and $\chi$ is a nonnegative function generalizing
\[
\chi(v) = \frac{\chi_0}{v} \quad \text{and} \quad \chi(v) = \frac{\chi_0}{(1 + v)^k} \quad (\chi_0 > 0, \ k > 1),
\]
was also studied in the parabolic-elliptic case firstly, and then researches of the above system were developed by the interaction between the parabolic-elliptic system and the parabolic-parabolic system. In the parabolic-elliptic system with $\chi(v) = \frac{\chi_0}{v}$ ($\chi_0 > 0$) Nagai-Senba \cite{142} showed that if $n = 2$, or $n \geq 3$ and $\chi_0 < \frac{2}{n-2}$ then a radial solution is global and bounded, and if $n \geq 3$ and $\chi_0 > \frac{2n}{n-2}$ then there exists some initial data such that a radial solution blows up in finite time. In the nonradial case Biler \cite{11} obtained global existence of solutions to the parabolic-elliptic system with $\chi(v) = \frac{\chi_0}{v}$ ($\chi_0 > 0$) under the conditions that $n = 2$ and $\chi_0 \leq 1$, or $n \geq 3$ and $\chi_0 < \frac{2}{n}$. Thanks to these results, we can expect that conditions for global existence in the above system were determined by a dimension of a domain and a smallness of $\chi$ in some sense. Indeed, global existence and boundedness of solutions to the parabolic-elliptic system with $\chi(v) = \frac{\chi_0}{v}$ ($\chi_0 > 0, \ k \geq 1$) were derived under some smallness conditions for $\chi_0$ (\cite{57}). Moreover, Fujie–Senba \cite{53} established global existence and boundedness in the two-dimensional parabolic-elliptic system with more general sensitivity function. On the other hand, also in the parabolic-parabolic case, it was shown that some smallness condition for $\chi$ implies global existence and boundedness; in the case that $\chi(v) = \frac{\chi_0}{v}$ ($\chi_0 > 0$) Winkler \cite{196} obtained global existence of classical solutions under the condition that $\chi_0 < \sqrt{\frac{2}{n}}$ and Fujie \cite{50} established boundedness of these solutions; in the case that $\chi(v) \leq \frac{\chi_0}{(a + v)^k}$ ($\chi_0 > 0, \ a \geq 0, \ k > 1$) some smallness condition for $\chi_0$ leads to global existence and boundedness (Chapter 3); recently, Fujie–Senba \cite{54} showed global existence and boundedness of radially symmetric solutions to the parabolic-parabolic system with more general sensitivity function and small $\lambda$ in a two-dimensional ball.

In summary, in the setting that $\Omega$ is a bounded domain, parabolic-elliptic chemotaxis systems often provided us some guide to how we could deal with parabolic-parabolic
chemotaxis systems; however, the relation between the both systems has not been studied. Namely, in the setting that $\Omega$ is a bounded domain, it still remains to analyze on the following question:

\[ \text{Does a solution of the parabolic-parabolic system converge to that of the parabolic-elliptic problem as } \lambda \searrow 0? \]

If we can obtain some positive answer to this question, then we can see that solutions of both systems have some similar properties; thus an answer will enable us to establish approaches to obtain properties for solutions of the chemotaxis systems. Here, in the setting that $\Omega$ is the whole space $\mathbb{R}^n$, there are some positive answers to this question in 2-dimensional case ([157]) and $n$-dimensional case ([109]). Therefore we can expect a positive answer to this question also in the setting that $\Omega$ is a bounded domain. In order to obtain an answer to this question we first deal with the chemotaxis system with strong signal sensitivity, because we have already provided all tools to establish the $L^1(\Omega \times [0,\infty))$-estimate for solutions via a priori estimate in Chapter 13 (This is likely to enable us to see a uniform-in-$\lambda$ estimate). The purpose of this work is to obtain some positive answer to this question in the chemotaxis system with strong signal sensitivity.

In this chapter we consider convergence of a solution for the parabolic-parabolic chemotaxis system with signal-dependent sensitivity

\[
\begin{align*}
(u_\lambda)_t &= \Delta u_\lambda - \nabla \cdot (u_\lambda \chi(v_\lambda) \nabla v_\lambda), & x &\in \Omega, \ t > 0, \\
\lambda(v_\lambda)_t &= \Delta v_\lambda - v_\lambda + u_\lambda, & x &\in \Omega, \ t > 0, \\
\nabla u_\lambda \cdot \nu &= \nabla v_\lambda \cdot \nu = 0, & x &\in \partial \Omega, \ t > 0, \\
u_\lambda(x,0) &= u_{\text{init}}(x), \ v_\lambda(x,0) = v_{\text{init}}(x), & x &\in \Omega
\end{align*}
\]

to that of the parabolic-elliptic chemotaxis system

\[
\begin{align*}
u_t &= \Delta u - \nabla \cdot (u \chi(v) \nabla v), & x &\in \Omega, \ t > 0, \\
0 &= \Delta v - v + u, & x &\in \Omega, \ t > 0, \\
\nabla u \cdot \nu &= \nabla v \cdot \nu = 0, & x &\in \partial \Omega, \ t > 0, \\
u(x,0) &= u_{\text{init}}(x), & x &\in \Omega,
\end{align*}
\]

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ ($n \geq 2$) with smooth boundary $\partial \Omega$ and $\nu$ is the outward normal vector to $\partial \Omega$; $\lambda > 0$ is a constant; the initial functions $u_{\text{init}}, v_{\text{init}}$ are assumed to be nonnegative functions; the sensitivity function $\chi$ is assumed to be generalization of the regular function:

\[ \chi(s) = \frac{\chi_0}{(1+s)^k} \quad (s > 0), \]

where $\chi_0 > 0$ and $k > 1$ are constants. The unknown functions $u_\lambda(x,t)$ and $u(x,t)$ represent the population density of the species and $v_\lambda(x,t)$ and $v(x,t)$ show the concentration of the chemical substance at place $x$ and time $t$.

Now the main results read as follows. We suppose that the sensitivity function $\chi$ satisfies that

\[
\chi \in C^{1+\theta}(0,\infty) \quad \text{and} \quad 0 \leq \chi(s) \leq \frac{\chi_0}{(a+s)^k} \quad (s > 0)
\]

187
with some $\vartheta \in (0, 1)$, $\alpha \geq 0$, $k > 1$ and $\chi_0 > 0$. The first theorem is concerned with global existence and boundedness in (15.1.1) under a condition depending on $\lambda$.

**Theorem 15.1.1.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ ($n \geq 2$) with smooth boundary, and let $\lambda > 0$. Assume that $\chi$ satisfies (15.1.3) with some $\vartheta \in (0, 1)$, $\alpha \geq 0$, $k > 1$ and $\chi_0 > 0$ satisfying

\begin{equation}
\chi_0 < \frac{4k(a + \eta)^{k-1}}{(1 - \lambda)^{n+1} + \sqrt{n(n\lambda^2 - 2n\lambda + n + 8\lambda)}},
\end{equation}

where

$$
\eta := \sup_{\tau > 0} \left( \min \left\{ \frac{e^{-2\tau}}{\min_{x \in \Omega} \|\nu_{\text{init}}\|_{L^1(\Omega)}(1 - e^{-\tau})} \right\} \right)
$$

and a constant $c_0 > 0$ is a lower bound for the fundamental solution of $w_t = \Delta w - w$ with Neumann boundary condition. Then for all $u_{\text{init}}, v_{\text{init}}$ satisfying

\begin{equation}
0 \leq u_{\text{init}} \in C(\overline{\Omega}) \setminus \{0\}, \quad \begin{cases} 
0 < v_{\text{init}} \in W^{1,q}(\Omega) \ (\exists q > n) \ (a = 0), \\
0 \leq v_{\text{init}} \in W^{1,q}(\Omega) \setminus \{0\} \ (\exists q > n) \ (a \neq 0),
\end{cases}
\end{equation}

the problem (15.1.1) possesses a unique global solution

$$
u_{\lambda}, v_{\lambda} \in C(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty))
$$

satisfying that there exists $C > 0$ such that

$$
\|u_{\lambda}(\cdot, t)\|_{L^\infty(\Omega)} + \|v_{\lambda}(\cdot, t)\|_{W^{1,q}(\Omega)} \leq C
$$

for all $t > 0$.

The next corollary gives existence of global solutions satisfying a uniform-in-$\lambda$ estimate under a condition independent of $\lambda$.

**Corollary 15.1.2.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ ($n \geq 2$) with smooth boundary, and assume that $\chi$ satisfies (15.1.3) with some $\vartheta \in (0, 1)$, $\alpha \geq 0$, $k > 1$ and $\chi_0 > 0$ satisfying

\begin{equation}
\chi_0 < \frac{2k(a + \eta)^{k-1}}{n}.
\end{equation}

Then for all $u_{\text{init}}, v_{\text{init}}$ satisfying (15.1.5), there exists $\lambda_0 \in (0, 1)$ such that for all $\lambda \in (0, \lambda_0)$ the problem (15.1.1) possesses a unique global classical solution $(u_{\lambda}, v_{\lambda})$ satisfying that there exists $C > 0$ independent of $\lambda \in (0, \lambda_0)$ such that

\begin{equation}
\|u_{\lambda}(\cdot, t)\|_{L^\infty(\Omega)} + \|v_{\lambda}(\cdot, t)\|_{W^{1,q}(\Omega)} \leq C
\end{equation}

for all $t > 0$ and any $\lambda \in (0, \lambda_0)$.

Then the uniform-in-$\lambda$ estimate for the solution obtained in Corollary 15.1.2 leads to the following result.
Theorem 15.1.3. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ ($n \geq 2$) with smooth boundary, and assume that $\chi$ satisfies (15.1.3) and (15.1.6). Then for all $u_{\text{init}}, v_{\text{init}}$ satisfying (15.1.5), there exist unique functions

$$u \in C(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)) \text{ and } v \in C^{2,0}(\overline{\Omega} \times (0, \infty)) \cap L^\infty([0, \infty); W^{1,2}(\Omega))$$

such that the solution $(u_\lambda, v_\lambda)$ of (15.1.1) satisfies

$$u_\lambda \to u \text{ in } C_{\text{loc}}(\overline{\Omega} \times [0, \infty)), \quad v_\lambda \to v \text{ in } C_{\text{loc}}(\overline{\Omega} \times (0, \infty)) \cap L^\infty_{\text{loc}}((0, \infty); W^{1,2}(\Omega))$$

as $\lambda \searrow 0$. Moreover, the pair of the functions $(u, v)$ solves (15.1.2) classically.

Difficulties are caused by the facts that $v_\lambda$ satisfies a parabolic equation and $v$ satisfies an elliptic equation. Thus we cannot use methods only for parabolic equations and only for elliptic equations when we would like to obtain some error estimate for solutions of (15.1.1) and those of (15.1.2), and it seems to be difficult to combine these methods. Therefore we rely on a compactness method some uniform-in-$\lambda$ estimate for the solution is required. The strategy of seeing an estimate independent of $\lambda$ is to modify the methods in Chapter 13. One of keys for this strategy is to derive the differential inequality

$$\frac{d}{dt} \int_\Omega u_\lambda^b \varphi(v_\lambda) \leq -c_1 \left( \int_\Omega u_\lambda^b \varphi(v_\lambda) \right) + c_2 \int_\Omega u_\lambda^3 \varphi(v_\lambda) + c_3,$$

where $\varphi$ is some function, and $b > 1$ and $c_1, c_2, c_3 > 0$ are some constants. This together with a smoothing property of $(e^{r\Delta})_{r \geq 0}$ enables us to establish the desired estimate. Then we can see convergence of the solutions $(u_\lambda, v_\lambda)$ as $\lambda \searrow 0$ by using the uniform-in-$\lambda$ estimate and the Arzelà–Ascoli theorem.

This chapter is organized as follows. In Section 15.2 we collect basic facts which will be used later. In Section 15.3 we prove global existence and boundedness in (15.1.1) for all $\lambda > 0$, and establish a uniform-in-$\lambda$ estimate (Theorem 15.1.1 and Corollary 15.1.2). Section 15.4 is devoted to the proof of Theorem 15.1.3 according to arguments in [188]; we show convergence of the solution $(u_\lambda, v_\lambda)$ for (15.1.1) as $\lambda \searrow 0$ by using the uniform-in-$\lambda$ estimate established in Corollary 15.1.2.

15.2. Local existence and basic estimates

In this section we collect results which will be used later. We first introduce the uniform-in-time lower estimate for $v_\lambda$ which is independent of $\lambda > 0$.

Lemma 15.2.1. Let $\lambda > 0$ and let $u \in C(\overline{\Omega} \times [0, T))$ be a nonnegative function such that, with some $m > 0$, $\int_\Omega u(\cdot, t) = m$ for every $t \in [0, T)$. If $v_{\text{init}} \in C(\overline{\Omega})$ is a nonnegative function in $\overline{\Omega}$ and $v_\lambda \in C^{2,1}(\overline{\Omega} \times (0, T)) \cap C(\overline{\Omega} \times [0, T))$ is a classical solution of

$$\begin{cases}
\lambda(v_\lambda)_t = \Delta v_\lambda - v_\lambda + u, & x \in \Omega, \ t \in (0, T), \\
\nabla v_\lambda \cdot \nu = 0, & x \in \partial \Omega, \ t \in (0, T), \\
v_\lambda(x, 0) = v_{\text{init}}(x), & x \in \Omega,
\end{cases}$$

189
then
\[
\inf_{x \in \Omega} v_\lambda(x, t) \geq \tilde{\eta}
\]
holds for all \( t \in (0, T) \), where
\[
(15.2.1) \quad \tilde{\eta} := \sup_{\tau > 0} \left( \min_{x \in \Omega} \left\{ e^{-2\tau} \min_{x \in \Omega} v_{\text{init}}(x), \ c_0 m(1 - e^{-\tau}) \right\} \right).
\]

Proof. For a function \( f : \Omega \times [0, T) \rightarrow \mathbb{R} \) putting \( \tilde{f}(x, t) := f(x, \lambda t) \) for \((x, t) \in \Omega \times (0, \frac{T}{\lambda})\), we see that \( \tilde{v}_\lambda \) satisfies
\[
\begin{aligned}
(\tilde{v}_\lambda)_t &= \Delta \tilde{v}_\lambda - \tilde{v}_\lambda + \tilde{u}, \quad x \in \Omega, \ t \in (0, \frac{T}{\lambda}), \\
\nabla \tilde{v}_\lambda \cdot \nu &= 0, \quad x \in \partial \Omega, \ t \in (0, \frac{T}{\lambda}), \\
\tilde{v}_\lambda(x, 0) &= v_{\text{init}}(x), \quad x \in \Omega.
\end{aligned}
\]
Thus since the mass conservation yields that \( \int_\Omega \tilde{u}(\cdot, \lambda t) = m \) for all \( t \in (0, \frac{T}{\lambda}) \) and any \( \lambda > 0 \), we infer from [50, 51] (see also Lemma 13.2.1) that \( \tilde{\eta} \) defined as (15.2.1) satisfies
\[
\inf_{x \in \Omega} \tilde{v}_\lambda(x, t) \geq \tilde{\eta}
\]
for all \( t \in (0, \frac{T}{\lambda}) \), which implies this lemma. \( \square \)

We next recall the result which is concerned with local existence of solutions (see e.g., [193, Lemma 2.1]).

Lemma 15.2.2. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) \((n \geq 2)\) with smooth boundary, and assume that (15.1.3) and (15.1.5) are satisfied. Then for all \( \lambda > 0 \) there exists \( T_{\text{max}, \lambda} \in (0, \infty] \) such that the problem (15.1.1) possesses a unique solution \((u_\lambda, v_\lambda)\) fulfilling
\[
\begin{aligned}
u_\lambda, v_\lambda \in C(\overline{\Omega} \times [0, T_{\text{max}, \lambda}]), \quad C^{2,1}(\overline{\Omega} \times (0, T_{\text{max}, \lambda})), \\
u_\lambda(x, t) &\geq 0 \quad \text{for all } x \in \Omega \text{ and all } t > 0, \\
\int_\Omega \nu_\lambda(x, t) &= \int_\Omega v_{\text{init}} \quad \text{and} \quad \inf_{x \in \Omega} v_\lambda(x, t) \geq \eta \quad \text{for all } t \in (0, T_{\text{max}, \lambda}) \text{ and } \lambda > 0.
\end{aligned}
\]
Moreover, either \( T_{\text{max}, \lambda} = \infty \) or
\[
\limsup_{t \to T_{\text{max}, \lambda}} (\|\nu_\lambda(\cdot, t)\|_{L^\infty(\Omega)} + \|v_\lambda(\cdot, t)\|_{W^{1, q}(\Omega)}) = \infty.
\]

Then we shall provide the following two lemmas which hold keys to derive important estimates for the proofs of main results.

Lemma 15.2.3. Let \( 1 \leq \theta, \mu \leq \infty \). Then we have the following properties.

(i) If \( \frac{\mu}{2} \left( \frac{1}{\theta} - \frac{1}{\mu} \right) < 1 \), then there exists \( C = C(\theta, \mu) > 0 \) such that
\[
(15.2.2) \quad \|v_\lambda(\cdot, t)\|_{L^\mu(\Omega)} \leq C \left( 1 + \sup_{s \in (0, T_{\text{max}, \lambda})} \|u_\lambda(\cdot, s)\|_{L^\theta(\Omega)} \right)
\]
for all \( t \in (0, T_{\text{max}, \lambda}) \) and any \( \lambda > 0 \).
(ii) If \( \frac{1}{2} + \frac{n}{2} \left( \frac{1}{\theta} - \frac{1}{\mu} \right) < 1 \), then there exists \( C = C(\theta, \mu) > 0 \) such that

\[
\| \nabla v_\lambda(\cdot, t) \|_{L^p(\Omega)} \leq C \left( 1 + \sup_{s \in (0, T_{\text{max}, \lambda})} \| u_\lambda(\cdot, s) \|_{L^p(\Omega)} \right)
\]

for all \( t \in (0, T_{\text{max}, \lambda}) \) and any \( \lambda > 0 \).

**Proof.** In the case that \( \frac{n}{2} (\frac{1}{\theta} - \frac{1}{\mu}) < 1 \) by using the transformation \( \tilde{v}_\lambda(x, t) := v_\lambda(x, \lambda t) \) for \( (x, t) \in \Omega \times (0, T_{\text{max}, \lambda}) \) and a straightforward application of well-known smoothing estimates for the heat semigroup under homogeneous Neumann boundary conditions (see Lemma 2.2.2) we have that

\[
\| \tilde{v}_\lambda(\cdot, t) \|_{L^p(\Omega)} \leq C_1 \left( 1 + \sup_{s \in (0, T_{\text{max}, \lambda})} \| u_\lambda(\cdot, \lambda s) \|_{L^p(\Omega)} \right)
\]

for all \( t \in (0, T_{\text{max}, \lambda}) \) and any \( \lambda > 0 \) with some \( C_1 = C_1(\theta, \mu) > 0 \), which implies that (15.2.2) holds. Similarly, in the case that \( \frac{1}{2} + \frac{n}{2} (\frac{1}{\theta} - \frac{1}{\mu}) < 1 \) the same transformation and Lemma 2.2.2 derive (15.2.3) with some \( C_2 = C_2(\theta, \mu) > 0 \) independent of \( \lambda > 0 \).

---

**Lemma 15.2.4.** Let \( \lambda > 0 \). If there exist \( p > \frac{n}{2} \) and \( M > 0 \) such that

\[
\| u_\lambda(\cdot, t) \|_{L^p(\Omega)} \leq M \quad \text{for all } t \in (0, T_{\text{max}, \lambda}),
\]

then there exists \( C = C(p, M) > 0 \) such that

\[
\| u_\lambda(\cdot, t) \|_{L^\infty(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\text{max}, \lambda}).
\]

Moreover, if \( p \) and \( M \) are independent of \( \lambda \in (0, \lambda_0) \) with some \( \lambda_0 \in (0, 1) \), then \( C \) is also independent of \( \lambda \in (0, \lambda_0) \).

**Proof.** Thanks to assumption, there exist \( p > \frac{n}{2} \) and \( C_1 > 0 \) such that

\[
u(\cdot, t) \|_{L^p(\Omega)} \leq C_1 \quad \text{for all } t \in (0, T_{\text{max}, \lambda}).
\]

Then we can find \( r = r(p) \geq 1 \) and \( \mu = \mu(p) \geq 1 \) such that

\[
n < r < \mu < \frac{np}{(n-p)^+},
\]

because \( p > \frac{n}{2} \). Therefore Lemma 15.2.3 and (15.2.4) enable us to obtain that

\[
\| \nabla v_\lambda(\cdot, t) \|_{L^p(\Omega)} \leq C_2 \quad \text{for all } t \in (0, T_{\text{max}, \lambda})
\]

with some \( C_2 = C_2(p, C_1) > 0 \). Now we put

\[
A(T') = \sup_{t \in (0, T')} \| u_\lambda(\cdot, t) \|_{L^\infty(\Omega)} < \infty
\]

191
for $T' \in (0, T_{\text{max}, \lambda})$ and will show that $A(T') \leq C$ for all $T' \in (0, T_{\text{max}, \lambda})$ with some $C > 0$. In order to obtain the estimate for $A(T')$ we set $t_0 := (t - 1)_+$ for $t \in (0, T')$ and represent $u_\lambda$ according to

\begin{equation}
(15.2.5) \quad u_\lambda(\cdot, t) = e^{(t-t_0)\Delta}u_\lambda(\cdot, t_0) - \int_{t_0}^{t} e^{(t-s)\Delta} \nabla \cdot (u_\lambda(\cdot, s)\chi(v_\lambda(\cdot, s))) \nabla v_\lambda(\cdot, s) \, ds
= I_1(\cdot, t) + I_2(\cdot, t).
\end{equation}

In the case that $t \leq 1$ the order preserving property of the Neumann heat semigroup implies that

\begin{equation}
(15.2.6) \quad \|I_1(\cdot, t)\|_{L^\infty(\Omega)} \leq \|u_{\text{init}}\|_{L^\infty(\Omega)} \quad \text{for all } t \in (0, T').
\end{equation}

In the case that $t > 1$ by the $L^p$-$L^q$ estimates for $(e^{r\Delta})_{r \geq 0}$ (see Lemma 2.2.2) and (15.2.4) we can see that there exists $C_3 = C_3(p) > 0$ such that

\begin{equation}
(15.2.7) \quad \|I_1(\cdot, t)\|_{L^\infty(\Omega)} \leq C_3\|u_\lambda(\cdot, t_0)\|_{L^p(\Omega)} \leq C_3 C_1 \quad \text{for all } t \in (0, T').
\end{equation}

On the other hand, noting that

\[ \|u_\lambda(\cdot, t)\nabla v_\lambda(\cdot, t)\|_{L^p(\Omega)} \leq \|u_\lambda(\cdot, t)\|_{L^{n}(\Omega)} \|\nabla v_\lambda(\cdot, t)\|_{L^q(\Omega)} \leq A(T')^{1-\frac{n-r}{r\sigma}} \|u_{\text{init}}\|_{L^p(\Omega)} C_2 \]

holds for all $t \in (0, T')$, we obtain from a known smoothing property of $(e^{r\Delta})_{r \geq 0}$ (see [52, Lemma 3.3]) that

\[ \|I_2(\cdot, t)\|_{L^\infty(\Omega)} \leq \frac{\chi_0}{(a + \eta)^k} \int_{t_0}^{t} (t - s)^{-\frac{1}{2} - \frac{n}{r\sigma}} \|u_\lambda(\cdot, s)\nabla v_\lambda(\cdot, s)\|_{L^p(\Omega)} \, ds \]

\[ \leq \frac{\chi_0 A(T')^{1-\frac{n-r}{r\sigma}} \|u_{\text{init}}\|_{L^p(\Omega)} C_2}{(a + \eta)^k} \int_{0}^{1} \sigma^{-\frac{1}{2} - \frac{n}{r\sigma}} \, d\sigma \]

for all $t \in (0, T')$. Since $\int_{0}^{1} \sigma^{-\frac{1}{2} - \frac{n}{r\sigma}} \, d\sigma$ is finite from $\frac{n}{2r} < \frac{1}{2}$, there exists $C_4 = C_4(p, C_1) > 0$ such that

\begin{equation}
(15.2.8) \quad \|I_2(\cdot, t)\|_{L^\infty(\Omega)} \leq C_4 A(T')^{1-\frac{n-r}{r\sigma}} \quad \text{for all } t \in (0, T').
\end{equation}

Therefore a combination of (15.2.6) and (15.2.7), along with (15.2.8) derives that

\[ \|u_\lambda(\cdot, t)\|_{L^\infty(\Omega)} \leq \|I_1(\cdot, t)\|_{L^\infty(\Omega)} + \|I_2(\cdot, t)\|_{L^\infty(\Omega)} \leq C_5 + C_4 A(T')^{1-\frac{n-r}{r\sigma}} \]

holds for all $t \in (0, T')$ with some $C_5 = C_5(p, C_1) > 0$, which together with $\frac{n-r}{r\sigma} < 1$ implies that there exists $C_6 = C_6(p, C_1) > 0$ such that

\[ A(T') = \sup_{t \in (0, T')} \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C_6 \quad \text{for all } T' \in (0, T_{\text{max}, \lambda}). \]

Therefore we can attain the $L^\infty$-estimate for $u_\lambda$. Moreover, in the case that $p$ and $C_1$ are independent of $\lambda \in (0, \lambda_0)$ with some $\lambda_0 \in (0, 1)$, aided by Lemma 15.2.3, we can see that the constants appearing in this proof are independent of $\lambda \in (0, \lambda_0)$. \qed
15.3. Global existence

In this section we will show global existence and boundedness in (15.1.1) (Theorem 15.1.1) and the uniform-in-\(\lambda\) estimate for the solution (Corollary 15.1.2). Thanks to Lemma 15.2.4, our aim is to derive the \(L^p\)-estimate for \(u_\lambda\) with \(p > \frac{n}{2}\). We first prove the following lemma which plays an important role in the proofs of the main results.

**Lemma 15.3.1.** For all \(\varepsilon \in (0, \frac{1}{2})\), \(p > 1\) and \(\lambda \geq 0\),

\[
p\lambda^2 + p - 2p\lambda + 4\varepsilon p\lambda + 4\lambda - 4\varepsilon \lambda > 0
\]

holds.

**Proof.** Let \(\varepsilon \in (0, \frac{1}{2})\). We shall see that the discriminant of \(f_\varepsilon(\lambda) := p\lambda^2 + (-2p + 4\varepsilon p + 4 - 4\varepsilon)\lambda + p\) is negative:

\[
D_\lambda := 4(-1 - (2\varepsilon - 1)^2)p^2 - 4(1 - \varepsilon)(1 - 2\varepsilon)p + 4(1 - \varepsilon)^2 < 0
\]

for all \(p > 1\). Now we put \(g(p) := -(1 - (2\varepsilon - 1)^2)p^2 - 4(1 - \varepsilon)(1 - 2\varepsilon)p + 4(1 - \varepsilon)^2\). Then since

\[
g'(p) = -2(1 - (2\varepsilon - 1)^2)p - 4(1 - \varepsilon)(1 - 2\varepsilon) < 0
\]

from \(\varepsilon \in (0, \frac{1}{2})\) and \(g(1) = 0\) hold, we have \(g(p) < 0\) for all \(p > 1\), which means that (15.3.1) holds. Therefore we obtain

\[
f_\varepsilon(\lambda) > 0 \quad \text{for all } \lambda \geq 0 \text{ and all } p > 1,
\]

which entails this lemma. \(\square\)

In order to establish the \(L^p\)-estimate for \(u_\lambda\) with \(p > \frac{n}{2}\) we put

\[
\varphi_r(s) := \exp \left\{ -r \int_{\eta}^{s} \frac{1}{(a + \sigma)^k} \, d\sigma \right\} \quad \text{for } s \geq \eta
\]

with some \(r > 0\) and will show the following two lemmas which derive a differential inequality for \(\int_{\Omega} u_\lambda^p \varphi_r(v_\lambda)\).

**Lemma 15.3.2.** Assume that (15.1.3) is satisfied with some \(\theta \in (0, 1)\), \(a \geq 0\), \(k > 1\) and \(\chi_0 > 0\) satisfying

\[
(1 - \lambda + 2\lambda\varepsilon_0) + p + \sqrt{p(\lambda^2 + p - 2p\lambda + 4\varepsilon p\lambda + 4\lambda - 4\varepsilon \lambda)} \chi_0 \leq \frac{k}{\chi}(a + \eta)^k
\]

with some \(\varepsilon_0 \in (0, \frac{1}{2})\) and \(p > 1\). Then

\[
\frac{d}{dt} \int_{\Omega} u_\lambda^p \varphi_r(v_\lambda) \leq -\varepsilon p(p - 1) \int_{\Omega} u_\lambda^{p-2} \varphi_r(v_\lambda) |\nabla u_\lambda|^2 + r \int_{\Omega} u_\lambda^p \varphi_r(v_\lambda) \frac{v_\lambda}{(a + v_\lambda)^k}
\]

holds, where

\[
r := \lambda(p - 1)\chi_0 \sqrt{\frac{p}{p\lambda^2 - 2p\lambda + p + 4p\varepsilon \lambda + 4\lambda - 4\varepsilon \lambda}}.
\]
Proof. In light of integration by parts and the Young inequality, we obtain from straightforward calculations that
\[
\frac{d}{dt} \int_{\Omega} u_{\lambda}^p \varphi_r(v_\lambda) = -p(p-1) \int_{\Omega} u_{\lambda}^{p-2} \varphi_r(v_\lambda) |\nabla u_\lambda|^2 \\
+ \int_{\Omega} u_{\lambda}^{p-1} \left( p(p-1) \chi(v_\lambda) \varphi_r(v_\lambda) - \left( 1 + \frac{1}{\lambda} \right) p \varphi_r''(v_\lambda) \right) \nabla u_\lambda \cdot \nabla v_\lambda \\
+ \int_{\Omega} u_{\lambda}^p \left( p \chi(v_\lambda) \varphi_r'(v_\lambda) - \frac{1}{\lambda} \varphi_r''(v_\lambda) \right) |\nabla v_\lambda|^2 + \frac{1}{\lambda} \int_{\Omega} u_{\lambda}^p \varphi_r'(v_\lambda)(u_\lambda - v_\lambda) \\
\leq -\varepsilon p(p-1) \int_{\Omega} u_{\lambda}^{p-2} |\nabla u_\lambda|^2 + \int_{\Omega} u_{\lambda}^p \varphi_r(v_\lambda) H_{r,\varepsilon}(v_\lambda) |\nabla v_\lambda|^2 \\
+ \frac{r}{\lambda} \int_{\Omega} u_{\lambda}^p \varphi_r(v_\lambda) \frac{v_\lambda}{(a + v_\lambda)^k}
\]
holds for all \( \varepsilon \in (0, \frac{1}{2}) \), where
\[
H_{r,\varepsilon}(s) := \frac{p \lambda^2 - 2p \lambda + p + 4p \varepsilon \lambda + 4 \lambda - 4 \varepsilon \lambda}{4 \lambda^2 (1 - \varepsilon)(p - 1)(a + s)^{2k}} r^2 \\
+ \left( \frac{1 - \lambda + 2 \varepsilon}{2 \lambda (1 - \varepsilon)(a + s)^{2k}} - \frac{k}{\lambda(a + s)^{k+1}} \right) r + \frac{p(p-1) \lambda^2 \chi_0}{4(1 - \varepsilon)(a + s)^{2k}}.
\]
Now since (15.3.2) holds with some \( \varepsilon \in (0, \frac{1}{2}) \) and \( p > 1 \), an argument similar to that in the proof of Lemma 13.4.1 implies that
\[
H_{r,\varepsilon}(s) \leq 0 \quad \text{for all } s \geq \eta
\]
with \( r > 0 \) defined as (15.3.4), which leads to the end of the proof. \( \square \)

**Lemma 15.3.3.** Assume that (15.1.3) and (15.3.2) are satisfied with \( \varepsilon \in (0, \frac{1}{2}) \) and \( p > 1 \).
Then there exist \( b > 1 \) and \( c_1, c_2, c_3 > 0 \) such that
\[
\frac{d}{dt} \int_{\Omega} u_{\lambda}^p \varphi_r(v_\lambda) \leq -c_1 \left( \int_{\Omega} u_{\lambda}^p \varphi_r(v_\lambda) \right)^b + c_2 \int_{\Omega} u_{\lambda}^p \varphi_r(v_\lambda) + c_3.
\]
Moreover, if there are \( \lambda_0 > 0 \), \( p > 1 \) and \( \varepsilon \in (0, \frac{1}{2}) \) such that (15.3.2) holds for all \( \lambda \in (0, \lambda_0) \), then the constants \( b, c_1, c_2, c_3 \) are independent of \( \lambda \in (0, \lambda_0) \).

**Proof.** By virtue of Lemma 15.3.2, we have that (15.3.3) holds with \( r > 0 \) defined as (15.3.4). We first obtain from the boundedness of the function \( s \mapsto \frac{s}{(a+s)^k} \) on \([\eta, \infty)\) \((k > 1)\) that there is a constant \( C_1 > 0 \) satisfying
\[
\int_{\Omega} u_{\lambda}^p \varphi_r(v_\lambda) \frac{v_\lambda}{(a + v_\lambda)^k} \leq C_1 \int_{\Omega} u_{\lambda}^p \varphi_r(v_\lambda).
\]
Then the fact
\[
\exp \left\{ \frac{-r}{(k - 1)(a + \eta)^{k-1}} \right\} \leq \varphi_r(s) \leq 1 \quad \text{for all } s \geq \eta
\]

\[194\]
and the Gagliardo–Nirenberg inequality
\[ \| u_\lambda^p \|_{L^2(\Omega)} \leq C_2 \left( \| \nabla u_\lambda^p \|_{L^2(\Omega)} + \| u_\lambda^p \|_{L^{\frac{p}{2}}(\Omega)} \right)^\alpha \| u_\lambda^p \|_{L^{\frac{p}{2}}(\Omega)}^{(1-\alpha)} \]

with \( \alpha := \frac{p^n - \frac{n}{2}}{p^n + 1 - \frac{n}{2}} \in (0,1) \) and some constant \( C_2 > 0 \) (see Lemma 2.1.1) imply that there exists \( C_3 > 0 \) such that
\[ (15.3.7) \quad \int_\Omega u_\lambda^p \varphi_r(v_\lambda) \leq C_3 \left( \exp \left( \frac{r}{(k-1)(a + \eta)^{k-1}} \right) \int_\Omega u_\lambda^{p-2} \varphi_r(v_\lambda) |\nabla u_\lambda|^2 + 1 \right)^\alpha. \]

Therefore a combination of (15.3.3), (15.3.5) and (15.3.7) yields
\[ (15.3.8) \quad \frac{d}{dt} \int_\Omega u_\lambda^p \varphi_r(v_\lambda) \leq -\varepsilon p(p-1)C_3^{-\frac{1}{\alpha}} \exp \left( \frac{-r}{(k-1)(a + \eta)^{k-1}} \right) \left( \int_\Omega u_\lambda^p \varphi_r(v_\lambda) \right)^{\frac{1}{\alpha}} + \frac{rC_1}{\lambda} \int_\Omega u_\lambda^p \varphi_r(v_\lambda) + \exp \left( \frac{-r}{(k-1)(a + \eta)^{k-1}} \right). \]

Moreover, if there are \( \lambda_0 > 0, p > 1 \) and \( \varepsilon \in (0, \frac{1}{2}) \) such that (15.3.2) holds for all \( \lambda \in (0, \lambda_0) \), then noting from the definition of \( r \) (see (15.3.4)) and the existence of \( C_4 > 0 \) satisfying \( p\lambda^2 - 2p\lambda + p + 4p\varepsilon\lambda + 4\lambda - 4\varepsilon\lambda \geq C_4 \) for all \( \lambda \in [0, \lambda_0] \) (from Lemma 15.3.1) that
\[ (15.3.9) \quad 1 \leq \exp \left( \frac{r}{(k-1)(a + \eta)^{k-1}} \right) \leq \exp \left( \frac{\lambda_0(p-1)\chi_0}{(k-1)(a + \eta)^{k-1}} \sqrt{\frac{p}{C_4}} \right) =: C_5 \]

and
\[ \frac{rC_1}{\lambda} \leq C_1(p-1)\chi_0 \sqrt{\frac{p}{C_4}} =: C_6, \]

we can see from (15.3.8) that
\[ \frac{d}{dt} \int_\Omega u_\lambda^p \varphi_r(v_\lambda) \leq -\varepsilon p(p-1)C_3^{-\frac{1}{\alpha}}C_5^{-1} \left( \int_\Omega u_\lambda^p \varphi_r(v_\lambda) \right)^{\frac{1}{\alpha}} + C_6 \int_\Omega u_\lambda^p \varphi_r(v_\lambda) + 1. \]

Thus we can show this lemma. \( \square \)

Now we have already provided all tools to establish the \( L^p \)-estimate for \( u_\lambda \) under the condition (15.3.2).

**Lemma 15.3.4.** Assume that (15.1.3) and (15.3.2) are satisfied with some \( \varepsilon \in (0, \frac{1}{2}) \) and \( p > 1 \). Then there exists \( C > 0 \) such that
\[ \| u_\lambda(\cdot,t) \|_{L^p(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\text{max},\lambda}). \]

Moreover, if there are \( \lambda_0 > 0, p > 1 \) and \( \varepsilon \in (0, \frac{1}{2}) \) such that (15.3.2) holds for all \( \lambda \in (0, \lambda_0) \), then \( C \) is independent of \( \lambda \in (0, \lambda_0) \).
Proof. Since (15.3.6) holds, Lemma 15.3.3 and the standard ODE comparison argument lead to the $L^p$-estimate for $u_\lambda$. Moreover, if there are $\lambda_0 > 0$, $p > 1$ and $\varepsilon \in (0, \frac{1}{2})$ such that (15.3.2) holds for all $\lambda \in (0, \lambda_0)$, then a combination of Lemma 15.3.3 and (15.3.6), along with (15.3.9) implies the desired uniform-in-$\lambda$ estimate.

Now we are ready to attain the $L^p$-estimate for $u_\lambda$ under the condition (15.1.4) or (15.1.6). Here we note that (15.1.4) is the case of (15.3.2) with $p = \frac{n}{2}$ and $\varepsilon = 0$, and (15.1.6) is the case of (15.3.2) with $p = \frac{n}{2}$, $\varepsilon = 0$ and $\lambda = 0$.

**Corollary 15.3.5.** Assume that (15.1.3) and (15.1.4) are satisfied. Then there exist $p > \frac{n}{2}$ and $C > 0$ such that

$$\|u_\lambda(\cdot, t)\|_{L^p(\Omega)} \leq C \text{ for all } t \in (0, T_{\text{max}, \lambda}).$$

**Proof.** Invoking to (15.1.4), we obtain from the continuity argument that there are $p > \frac{n}{2}$ and $\varepsilon \in (0, \frac{1}{2})$ such that (15.3.2) holds. Thus Lemma 15.3.4 enables us to see the $L^p$-estimate for $u_\lambda$.

**Corollary 15.3.6.** Assume that (15.1.3) and (15.1.6) are satisfied. Then there exist $\lambda_0 \in (0, 1), p > \frac{n}{2}$ and $C > 0$ such that

$$\|u_\lambda(\cdot, t)\|_{L^p(\Omega)} \leq C \text{ for all } t \in (0, T_{\text{max}, \lambda}) \text{ and any } \lambda \in (0, \lambda_0).$$

**Proof.** In light of (15.1.6), we can find $\lambda_0 \in (0, 1), p > \frac{n}{2}$ and $\varepsilon \in (0, \frac{1}{2})$ such that (15.3.2) holds for all $\lambda \in (0, \lambda_0)$. Therefore from Lemma 15.3.4 we obtain this lemma.

A combination of Lemma 15.2.3 and Corollaries 15.3.5, 15.3.6 implies the following lemma.

**Lemma 15.3.7.** Assume that (15.1.3) and (15.1.4), or (15.1.3) and (15.1.6) are satisfied. Then there exists $C > 0$ such that

$$\|v_\lambda(\cdot, t)\|_{W^{1, p}(\Omega)} \leq C \text{ for all } t \in (0, T_{\text{max}, \lambda}).$$

Moreover, if (15.1.3) and (15.1.6) are satisfied, then $C$ is independent of $\lambda \in (0, \lambda_0)$ with some $\lambda_0 \in (0, 1)$.

**Proof of Theorem 15.1.1.** Lemmas 15.2.4, 15.3.7 and Corollary 15.3.5 derive Theorem 15.1.1.

**Proof of Corollary 15.1.2.** Lemmas 15.2.4, 15.3.7 and Corollary 15.3.6 lead to Corollary 15.1.2.
15.4. Convergence

In this section we will show that solutions of (15.1.1) converge to those of (15.1.2) (Theorem 15.1.3). Here we note that Arguments in this section are based on those in the proof of [188, Theorem 1.1]; thus I shall only show brief proofs. Here we assume that (15.1.3) and (15.1.6) are satisfied. Then thanks to Corollary 15.1.2, there exists $\lambda_0 \in (0, 1)$ such that for all $\lambda \in (0, \lambda_0)$ the problem (15.1.1) possesses a unique global classical solution $(u_{\lambda}, v_{\lambda})$ satisfying the uniform-in-$\lambda$ estimate (15.1.7). We first confirm the following lemma which is a cornerstone of this work.

Lemma 15.4.1. For all sequences of numbers $\{\lambda_n\}_{n \in \mathbb{N}} \subset (0, \lambda_0)$ satisfying $\lambda_n \searrow 0$ as $n \to \infty$ there exist a subsequence $\lambda_{n_j} \searrow 0$ and functions

$$u \in C(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)) \quad \text{and} \quad v \in C^{2,0}(\overline{\Omega} \times (0, \infty)) \cap L^\infty([0, \infty); W^{1,q}(\Omega))$$

such that for all $T > 0$,

$$u_{\lambda_{n_j}} \to u \quad \text{in} \quad C_{\text{loc}}(\overline{\Omega} \times [0, \infty)),
$$

$$v_{\lambda_{n_j}} \to v \quad \text{in} \quad C_{\text{loc}}(\overline{\Omega} \times (0, \infty)) \cap L^2((0, \infty); W^{1,2}(\Omega))$$

as $j \to \infty$. Moreover, $(u, v)$ solves (15.1.2) classically.

Remark 15.4.1. This lemma also implies that if $\chi$ satisfies (15.1.3) and (15.1.6) then global existence and boundedness in (15.1.2) hold.

Proof. From the assumption in this section and the standard parabolic regularity argument [156, Theorem 1.3] we see that there is $\alpha \in (0, 1)$ such that

$$\{u_{\lambda}\}_{\lambda \in (0, \lambda_0)} \text{ is bounded in } C^{\alpha, \frac{q}{2}}_{\text{loc}}(\overline{\Omega} \times [0, \infty)).$$

Thus the Arzelà–Ascoli theorem and the boundedness of $\|\nabla v_{\lambda}\|_{L^\infty(0, \infty; W^{1,q}(\Omega))}$ yields that we can find a subsequence $\lambda_{n_j} \searrow 0$ and functions

$$u \in C^{\alpha, \frac{q}{2}}_{\text{loc}}(\overline{\Omega} \times [0, \infty)) \quad \text{and} \quad v \in L^\infty(0, \infty; W^{1,q}(\Omega))$$

satisfying

$$u_{\lambda_{n_j}} \to u \quad \text{in} \quad C_{\text{loc}}(\overline{\Omega} \times [0, \infty))$$

and

$$v_{\lambda_{n_j}} \rightharpoonup v \quad \text{in} \quad L^\infty(0, \infty; W^{1,q}(\Omega))$$

as $j \to \infty$. Then arguments similar to those in the proof of [188, Theorem 1.1] enable us to attain this lemma.

We next verify the following lemma which implies that the pair of functions $(u, v)$ provided by Lemma 15.4.1 is independent of a choice of a sequence $\lambda_n \searrow 0$. 

197
Lemma 15.4.2. A solution \((\overline{u}, \overline{v})\) of (15.1.2) satisfying
\[
\overline{u} \in C(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)) \quad \text{and} \quad \overline{v} \in C^{2,0}(\overline{\Omega} \times (0, \infty)) \cap L^\infty([0, \infty); W^{1,q}(\Omega))
\]
is unique.

Proof. Let \((\overline{u}_1, \overline{v}_1)\) and \((\overline{u}_2, \overline{v}_2)\) be solutions to (15.1.2) and put
\[
y(x, t) := \overline{u}_1(x, t) - \overline{u}_2(x, t) \quad \text{for} \quad (x, t) \in \Omega \times (0, \infty).
\]
Then an argument similar to that in the proof of [164, Lemma 2.1] implies that
\[
y(x, t) = 0 \quad \text{for all} \quad (x, t) \in \Omega \times (0, \infty).
\]
Thus we can obtain this lemma.

Finally we shall establish convergence of the solution \((u_\lambda, v_\lambda)\) for (15.1.1) as \(\lambda \searrow 0\).

Lemma 15.4.3. The solution \((u_\lambda, v_\lambda)\) of (15.1.1) with \(\lambda \in (0, \lambda_0)\) satisfies that for all \(T > 0\)
\[
u_\lambda \to v \quad \text{in} \quad C^{0,2}(\overline{\Omega} \times (0, \infty)) \cap L^2_{\text{loc}}((0, \infty); W^{1,2}(\Omega))
\]
as \(\lambda \searrow 0\), where \((u, v)\) is the solution of (15.1.2) provided by Lemma 15.4.1.

Proof. Lemmas 15.4.1 and 15.4.2 yield that there exists the pair of the functions \((u, v)\) such that for any sequences \(\{\lambda_n\}_{n \in \mathbb{N}} \subset (0, \lambda_0)\) satisfying \(\lambda_n \searrow 0\) as \(n \to \infty\) there is a subsequence \(\lambda_{n_j} \searrow 0\) such that for all \(T > 0\),
\[
\begin{align*}
u_{n_j} &\to v \quad \text{in} \quad C^{0,2}(\overline{\Omega} \times (0, \infty)) \cap L^2_{\text{loc}}((0, \infty); W^{1,2}(\Omega))
\end{align*}
\]
as \(j \to \infty\), which enables us to see this lemma.

Proof of Theorem 15.1.3. Lemma 15.4.3 directly shows Theorem 15.1.3.
Chapter 16

The fast signal diffusion limit in a Keller–Segel system

16.1. Motivation and results

The subject of this work is to construct a new approach to a parabolic-elliptic Keller–Segel system from its parabolic-parabolic case, and to use the parabolic-parabolic case as a step to establish new results in the parabolic-elliptic case. In this chapter our aim is, by considering that the parabolic-elliptic system is as a limit of its parabolic-parabolic case, to establish a result such that only dealing with the parabolic-parabolic Keller–Segel system is enough to obtain new properties for solutions of its parabolic-elliptic case.

Before an introduction of a problem in this chapter, we will recall some related works on the chemotaxis system. Here chemotaxis is the property such that species move towards higher concentration of a chemical substance when they plunge into hunger. Keller–Segel [89, 90] studied the migration of the species which have chemotaxis, and proposed the following problem:

\[ u_t = \Delta u - \nabla \cdot (u \chi (v) \nabla v), \quad \lambda v_t = \Delta v - v + u \quad \text{in } \Omega \times (0, \infty), \]

where \( \Omega \subseteq \mathbb{R}^n \) (\( n \in \mathbb{N} \)) is a bounded domain, \( \lambda = 0 \) (the parabolic-elliptic system) or \( \lambda > 0 \) (the parabolic-parabolic system) is a constant and \( \chi \) is a function. This problem is called a chemotaxis system, and especially, is called a (minimal) Keller–Segel system in the case that \( \chi \) is a constant function. About the Keller–Segel system, Nanjundiah [148] first asserted that we could expect existence of a blow-up solution to the Keller–Segel system. Moreover, Childress–Percus [33] claimed the following conjecture:

- In the 1-dimensional setting, global existence holds.
- In the 2-dimensional setting, there is a critical number \( c \) such that if an initial data \( u_{\text{init}} \) satisfies \( \| u_{\text{init}} \|_{L^1(\Omega)} < c \) then global existence holds, and for any \( m > c \) there are initial data \( u_{\text{init}}, v_{\text{init}} \) such that \( \| u_{\text{init}} \|_{L^1(\Omega)} = m \) and the corresponding solution blows up in finite time.
- In the higher-dimensional setting, there are many blow-up solutions.

Here we first focus on the 2-dimensional setting. The study of the 2-dimensional Keller–Segel system is supported by the interaction between the parabolic-elliptic case and the
In order to verify the Childress–Percus conjecture Nagai [140] tried to deal with the parabolic-elliptic case which is a simplified problem of the parabolic-parabolic Keller–Segel system, and shown that, in the radial setting, $8\pi$ is the critical value in the Childress–Percus conjecture. Subsequently, Nagai–Senba–Yoshida [144] established global existence and boundedness of radial solutions in the parabolic-parabolic Keller–Segel system under the condition that $\|u_{\text{init}}\|_{L^1(\Omega)} < 8\pi$, and also obtained existence of global bounded nonradial solutions to the parabolic-parabolic system under the condition that $\|u_{\text{init}}\|_{L^1(\Omega)} < 4\pi$. Here Senba–Suzuki [159] asserted that arguments in proofs of these results could also be applied to the parabolic-elliptic case, which meant that global existence and boundedness of solutions to the parabolic-elliptic system were shown under the condition that $\|u_{\text{init}}\|_{L^1(\Omega)} < 4\pi$. Therefore in the both cases of the parabolic-elliptic system and the parabolic-parabolic system, $8\pi$ is the critical value in the Childress–Percus conjecture in the radial case, and $4\pi$ is the critical value in the nonradial case. Indeed, existence of blow-up solutions such that $\|u_{\text{init}}\|_{L^1(\Omega)}$ is larger than the critical value was shown ([69, 76, 127, 140, 141]): The radial parabolic-elliptic case was treated by a combination of the results in [69, 140]; the nonradial parabolic-elliptic case is in [141]; the radial parabolic-parabolic case can be found in [127]; the nonradial parabolic-parabolic case is in [76]. Moreover, related works which deal with blow-up asymptotics of solutions to the parabolic-elliptic case can be found in [69, 158, 160] and to the parabolic-parabolic case are in [126, 143]. In summary, in the 2-dimensional setting, the study of the Keller–Segel system was developed by the interaction between the parabolic-elliptic system and the parabolic-parabolic system, and it is shown that the Childress–Percus conjecture is true. On the other hand, the other dimensional cases have also been studied only in the parabolic-parabolic system, and it is shown that the Childress–Percus conjecture is valid also in the other dimensional cases; in the 1-dimensional setting Osaki–Yagi [153] showed global existence and boundedness of classical solutions; in the higher-dimensional case Winkler [198] obtained that for all $m > 0$ there are initial data $u_{\text{init}}, v_{\text{init}}$ such that $\|u_{\text{init}}\|_{L^1(\Omega)} = m$ and the corresponding solution blows up in finite time. Here global existence of bounded solutions to the higher-dimensional parabolic-parabolic Keller–Segel system also holds under some smallness condition for initial data $u_{\text{init}}, v_{\text{init}}$ with respect to some Lebesgue norm; Winkler [194] first established global existence and boundedness in the higher-dimensional parabolic-parabolic Keller–Segel system under the condition that $\|u_{\text{init}}\|_{L^p(\Omega)}$ and $\|\nabla v_{\text{init}}\|_{L^q(\Omega)}$ are sufficiently small with some $p > \frac{n}{2}$ and $q > n$; Cao [25] obtained global existence of bounded solutions to the parabolic-parabolic system under the smallness conditions for initial data in optimal spaces: $\|u_{\text{init}}\|_{L^{\frac{n}{2}}(\Omega)}$ and $\|\nabla v_{\text{init}}\|_{L^{n}(\Omega)}$ are small enough. Recently, in a further simplification of the parabolic-elliptic system (see (16.1.2) below) and in the case that $\Omega = \mathbb{R}^n$, Biler [12] showed that a critical mass phenomenon happens and also that a global radial solution satisfying some condition converges to a homogeneous steady state which blows up at $x = 0$ under some conditions; this result implies that a global solution of a parabolic-elliptic problem converges to an unbounded solution of its elliptic-elliptic case as $t \to \infty$ under some conditions, which means that a global solution of a parabolic-parabolic problem blows up in infinite time.

As we mentioned before, the interaction between the parabolic-elliptic system and the parabolic-parabolic system made progress on researches of the Keller–Segel system. The similar things occurred in the study of the chemotaxis system with signal-dependent...
sensitivity which is the case that \( \chi \) is a function. In the parabolic-elliptic system with 
\[ \chi(v) = \frac{\lambda_0}{v} \quad (\lambda_0 > 0) \]
Nagai–Senba [142] first showed that if \( n = 2 \), or \( n \geq 3 \) and \( \lambda_0 < \frac{2}{n-2} \)
then a radial solution is global and bounded, and if \( n \geq 3 \) and \( \lambda_0 > \frac{2\lambda_0}{n-2} \) then there exists 
some initial data such that a radial solution blows up in finite time. In the nonradial case Biler [11] obtained 
global existence of solutions to the parabolic-elliptic system with 
\[ \chi(v) = \frac{\lambda_0}{v} \quad (\lambda_0 > 0) \]
First showed that if \( n = 2 \), or \( n \geq 3 \) and \( \lambda_0 < \frac{2}{n-2} \)
Thanks to these results, we can expect that conditions for global existence in the above 
system were determined by a dimension of a domain and a smallness of \( \lambda \) in some sense. Indeed, 
global existence and boundedness of solutions to the parabolic-elliptic system 
with 
\[ \chi(v) = \frac{\lambda_0}{v^k} \quad (\lambda_0 > 0, \ k \geq 1) \]
were derived under some smallness conditions for \( \lambda_0 \) ([57]). On the other hand, also in the parabolic-parabolic case, it was shown that some 
smallness condition for \( \lambda \) leads to global existence and boundedness; in the case that 
\[ \chi(v) = \frac{\lambda_0}{v^k} \quad (\lambda_0 > 0) \]
Winkler [196] obtained global existence of classical solutions under 
the condition that \( \lambda_0 < \sqrt{\frac{2}{n}} \) and Fujie [50] established boundedness of these solutions; 
moreover, Lankeit [102] improved these results in the 2-dimensional setting; in the case that 
\[ \chi(v) \leq \frac{\lambda_0}{(a+v)^k} \quad (\lambda_0 > 0, \ a \geq 0, \ k \geq 1) \]
some smallness condition for \( \lambda_0 \) yields 
global existence and boundedness (see Chapter 13). In the case that \( \lambda \) is a more general 
sensitivity, Fujie–Senba [53] first established global existence and boundedness in the 
two-dimensional parabolic-elliptic system, and then they also showed existence of radially 
symmetric bounded solutions to the parabolic-parabolic system in a two-dimensional ball 
under the condition that \( \lambda \) is sufficiently small ([54]). Recently, in the nonradial setting, 
a sufficient condition of sensitivity functions for global existence and boundedness in the 
parabolic-parabolic system was studied by Fujie–Senba [55]. More related works which 
deal with problems with singular sensitivity \( \chi(v) = \frac{\lambda_0}{v} \) for some \( \lambda_0 > 0 \) can be found 
in [21, 105, 165, 196, 207] and Chapter 19; Winkler–Yokota [207] derived asymptotic 
behavior of classical solutions under some additional smallness condition for \( \lambda_0 \): Winkler 
[196] showed global existence of solutions in the sense of weak solution concepts under 
the condition that \( \lambda_0 < \sqrt{\frac{2}{n-2}} \); in the radial setting, under the weaker condition such 
that \( \lambda_0 < \sqrt{\frac{2}{n-2}} \), Stinner–Winkler [165] obtained existence of global solutions in some 
sense; furthermore, Lankeit–Winkler [105] constructed some weak solution concept, and 
established global existence of those solutions without radial setting under some condition 
for \( \lambda_0 \) which is the weakest condition to the best of our knowledge; some existence results 
in a Keller–Segel–(Navier–)Stokes system with singular sensitivity, that is a combination 
of the Keller–Segel system with a fluid environment, are in [21] and Chapter 19.

In summary parabolic-elliptic chemotaxis systems often gave us some guide to how 
we could deal with parabolic-parabolic chemotaxis systems; however, there have not been 
rich results on the relation between the both systems. Namely, it still remains to analyze 
on the following question except some cases:

> Does a solution of the parabolic-parabolic system converge to 
> that of the parabolic-elliptic problem as \( \lambda \searrow 0 \)?

If we can obtain some positive answer to this question, then we can see that solutions 
of both systems have some similar properties; thus an answer will enable us to establish
approaches to obtain properties for solutions of the chemotaxis systems. Here, in the case that \( \Omega \) is the whole space \( \mathbb{R}^n \), there are some positive answers to this question in 2-dimensional case ([157]) and \( n \)-dimensional case ([109]). Moreover, in the case that \( \Omega \) is a bounded domain and \( \chi(v) \leq \frac{x_0}{(a+v)^2} \) (\( x_0 > 0, a \geq 0, k > 1 \)), a positive answer to this question is also shown under the condition that \( x_0 \) is small (see Chapter 15). Therefore we can expect a positive answer to this question also in the Keller–Segel system in a bounded domain \( \Omega \) in some case. The purpose of this chapter is to give some positive answer to this question.

In order to attain this purpose, this chapter investigates the fast signal diffusion limit, which namely is convergence of a solution for the parabolic-parabolic Keller–Segel system

\[
(u_\lambda)_t = \Delta u_\lambda - \chi \nabla \cdot (u_\lambda \nabla v_\lambda), \quad x \in \Omega, \ t > 0, \\
\lambda(v_\lambda)_t = \Delta v_\lambda - v_\lambda + u_\lambda, \quad x \in \Omega, \ t > 0, \\
\nabla u_\lambda \cdot \nu = \nabla v_\lambda \cdot \nu = 0, \quad x \in \partial \Omega, \ t > 0, \\
u_\lambda(x, 0) = u_{\text{init}}(x), \ v_\lambda(x, 0) = v_{\text{init}}(x), \ x \in \Omega
\]

(16.1.1)

to that of the parabolic-elliptic Keller–Segel system

\[
u_t = \Delta u - \chi \nabla \cdot (u \nabla v), \quad x \in \Omega, \ t > 0, \\
0 = \Delta v - v + u, \quad x \in \Omega, \ t > 0, \\
\nabla u \cdot \nu = \nabla v \cdot \nu = 0, \quad x \in \partial \Omega, \ t > 0, \\
u(x, 0) = u_{\text{init}}(x), \ x \in \Omega
\]

(16.1.2)

as \( \lambda \searrow 0 \), where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) (\( n \geq 2 \)) with smooth boundary \( \partial \Omega \) and \( \nu \) is the outward normal vector to \( \partial \Omega \); \( \chi, \lambda > 0 \) is a constant; the initial functions \( u_{\text{init}}, v_{\text{init}} \) are assumed to be nonnegative functions. The unknown functions \( u_\lambda \) and \( u \) represent the population density of the species and \( v_\lambda \) and \( v \) show the concentration of the chemical substance at place \( x \) and time \( t \).

Now the main results read as follows. The first theorem is concerned with global existence and the fast signal diffusion limit of solutions for the higher-dimensional Keller–Segel system under smallness conditions for the initial data.

**Theorem 16.1.1.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) (\( n \geq 3 \)) with smooth boundary and let \( \chi > 0 \) be a constant. Assume that \( u_{\text{init}} \) and \( v_{\text{init}} \) satisfy

\[
0 \leq u_{\text{init}} \in C(\overline{\Omega}), \quad 0 \leq v_{\text{init}} \in W^{1,q}(\Omega)
\]

(16.1.3)

with some \( q > n \). Then for all \( p > \frac{n}{2} \) there exists \( \varepsilon_0 = \varepsilon_0(p, q, \chi, |\Omega|) > 0 \) such that, if \( u_{\text{init}} \) and \( v_{\text{init}} \) satisfy

\[
\|u_{\text{init}}\|_{L^p(\Omega)} < \varepsilon_0 \quad \text{and} \quad \|\nabla v_{\text{init}}\|_{L^q(\Omega)} < \varepsilon_0,
\]

(16.1.4)

then for all \( \lambda > 0 \) the problem (16.1.1) possesses a unique global bounded solution \( (u_\lambda, v_\lambda) \) which is a pair of nonnegative functions

\[
\ u_\lambda, v_\lambda \in C(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)).
\]
Moreover, if $u_{\text{init}}$ and $v_{\text{init}}$ satisfy (16.1.4), then there are unique functions
\[ u \in C(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\Omega \times (0, \infty)) \quad \text{and} \quad v \in C^{2,0}(\overline{\Omega} \times (0, \infty)) \cap L^{\infty}(0, \infty; W^{1,q}(\Omega)) \]
such that the solution $(u_\lambda, v_\lambda)$ of (16.1.1) satisfies
\[
\begin{align*}
    u_\lambda &\to u \quad \text{in} \quad C_{\text{loc}}(\overline{\Omega} \times [0, \infty)), \\
v_\lambda &\to v \quad \text{in} \quad C_{\text{loc}}(\overline{\Omega} \times (0, \infty)) \cap L^2_{\text{loc}}((0, \infty); W^{1,2}(\Omega))
\end{align*}
\]
as $\lambda \searrow 0$, and the pair of the functions $(u, v)$ solves (16.1.2) classically.

As an application of this result, we can establish a new result which provides global existence and boundedness in the higher-dimensional parabolic-elliptic Keller–Segel system (16.1.2) under some smallness condition for initial data $u_{\text{init}}$.

**Corollary 16.1.2.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ ($n \geq 3$) with smooth boundary and let $\chi > 0$ be a constant. Then for all $p > \frac{n}{2}$ there exists $\varepsilon_1 = \varepsilon(p, \chi, |\Omega|) > 0$ such that, if $u_{\text{init}} \in C(\overline{\Omega})$ satisfies that
\[
\|u_{\text{init}}\|_{L^p(\Omega)} < \varepsilon_1
\]
holds, then the problem (16.1.2) possesses a unique global bounded classical solution.

**Remark 16.1.1.** In these results we assume the smallness conditions for $\|u_{\text{init}}\|_{L^p(\Omega)}$ and $\|\nabla v_{\text{init}}\|_{L^q(\Omega)}$ with some $p > \frac{n}{2}$ and $q > n$, instead of $p = \frac{n}{2}$ and $q = n$ which are the conditions assumed in [25]; we could not attain fast signal diffusion limit under the smallness conditions in optimal spaces.

In the 2-dimensional setting, it is known that global existence and boundedness in (16.1.1) hold under the condition that $\|u_{\text{init}}\|_{L^1(\Omega)} \leq \frac{4\pi}{\chi}$ ([144]). Thanks to this previous work, we attain the fast signal diffusion limit in the 2-dimensional Keller–Segel system under the smallness conditions for the initial data in the optimal space.

**Theorem 16.1.3.** Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with smooth boundary and let $\chi > 0$ be a constant. Assume that $u_{\text{init}} \in C(\overline{\Omega})$ satisfies $\|u_{\text{init}}\|_{L^1(\Omega)} < \frac{4\pi}{\chi}$. Then there exist unique functions
\[ u \in C(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\Omega \times (0, \infty)) \quad \text{and} \quad v \in C^{2,0}(\overline{\Omega} \times (0, \infty)) \cap L^{\infty}(0, \infty; W^{1,q}(\Omega)) \]
such that for all $v_{\text{init}} \in W^{1,q}(\Omega)$ ($q > 2$) the global bounded classical solution $(u_\lambda, v_\lambda)$ of (16.1.1) satisfies
\[
\begin{align*}
    u_\lambda &\to u \quad \text{in} \quad C_{\text{loc}}(\overline{\Omega} \times [0, \infty)), \\
v_\lambda &\to v \quad \text{in} \quad C_{\text{loc}}(\overline{\Omega} \times (0, \infty)) \cap L^2_{\text{loc}}((0, \infty); W^{1,2}(\Omega))
\end{align*}
\]
as $\lambda \searrow 0$, and the pair of the functions $(u, v)$ solves (16.1.2) classically.

This result tells us a new method to obtain global existence and boundedness in the 2-dimensional parabolic-elliptic Keller–Segel system (16.1.2).
Corollary 16.1.4. Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with smooth boundary and let $\chi > 0$ be a constant. If $u_{\text{init}} \in C(\overline{\Omega})$ satisfies $\|u_{\text{init}}\|_{L^p(\Omega)} < \frac{25}{\chi}$, then the problem (16.1.2) possesses a unique global bounded classical solution.

In the proof of these main results difficulties are caused by the facts that $v_\lambda$ satisfies a parabolic equation and $v$ satisfies an elliptic equation. Thus we cannot use methods only for parabolic equations and only for elliptic equations when we would like to obtain some error estimate for solutions of (16.1.1) and those of (16.1.2), and it seems to be difficult to combine these methods. Therefore we rely on a compactness method to obtain convergence of a solution $(u_\lambda, v_\lambda)$ as $\lambda \searrow 0$, which is the same strategy as that of the proof of Theorem 15.1.3. In order to use a compactness method some estimate for the solution uniformly in time and $\lambda$ is required. In the chemotaxis system with signal dependent sensitivity the boundedness of $\int_\Omega u^{\lambda}_v(\cdot,t) \exp \{-\int_0^{\lambda-v(t)} \chi(s) \, ds\}$ with some $r > 0$ and the fact $\int_0^\infty \chi(s) \, ds < \infty$ lead to the desired estimate (see Chapter 15). Nevertheless, in the Keller–Segel setting, it is difficult to obtain the boundedness of $\int_0^{\lambda-v(\cdot,t)} \chi \, ds = \chi v_\lambda(\cdot,t)$. Thus we should give the other method to obtain the desired estimate in the Keller–Segel setting. However, in the higher-dimensional case, a construction of some estimate for the solution uniformly in time and $\lambda$ is a challenging problem: Indeed, in the previous works [25, 194] the following inequality was obtained:

$$\|u_\lambda(\cdot,t) - e^{t\Delta} u_{\text{init}}\|_{L^\infty(\Omega)} \leq C(1 + t^{-\alpha}) e^{-\beta t} \quad \text{for all } t > 0$$

with some $C, \alpha, \beta > 0$, which could not lead to the uniform-in-time estimate for the solution. This is one of the reason why we could not attain fast signal diffusion limit under the smallness conditions in optimal spaces in the higher-dimensional setting. To establish the $L^\infty$-estimate for $u_\lambda$ uniformly in time and $\lambda$ we modified the method in [194]. Let $\varepsilon > 0$ be a constant fixed later and put

$$T_\lambda := \sup \left\{ \hat{T} > 0 \mid \|u_\lambda(\cdot,t) - e^{t\Delta} u_{\text{init}}\|_{L^p(\Omega)} < \varepsilon \quad \text{for all } t \in (0, \hat{T}) \text{ and all } \lambda > 0 \right\} \leq \infty$$

with some $\theta = \frac{a}{p}$, which is different from a setting in [194]. Then, under the conditions that $\|u_{\text{init}}\|_{L^p(\Omega)} \leq \varepsilon$ and $\|\nabla v_{\text{init}}\|_{L^q(\Omega)} \leq \varepsilon$, we can see that

$$\|u_\lambda(\cdot,t) - e^{t\Delta} u_{\text{init}}\|_{L^p(\Omega)} \leq C(\varepsilon) \varepsilon \quad \text{for all } t \in (0, T_\lambda) \text{ and all } \lambda > 0,$$

where $C(\varepsilon) > 0$ is a constant such that $C(\varepsilon) \searrow 0$ as $\varepsilon \searrow 0$. Thus by choosing $\varepsilon > 0$ satisfying $C(\varepsilon) < 1$, we can obtain the $L^\theta$-estimate for $u_\lambda$ uniformly in time and $\lambda$, which with the standard $L^p$-$L^q$ estimate for the Neumann heat semigroup on bounded domains implies the desired estimate for $u_\lambda$. This strategy enables us to pass to the fast signal diffusion limit; however, it also lets us assume that $\|u_{\text{init}}\|_{L^p(\Omega)}$ and $\|\nabla v_{\text{init}}\|_{L^q(\Omega)}$ are small with some $p > \frac{a}{2}$ and $q > n$ in Theorem 16.1.1. On the other hand, in the 2-dimensional setting, by using a combination of an argument in the proof of [144, Theorem 1.1] and a compactness method we can show fast signal diffusion limit under the smallness conditions for the initial data in optimal spaces.

This chapter is organized as follows. In Section 16.2 we collect basic facts which will be used later. In Section 16.3 we prove global existence and uniform-in-$\lambda$ boundedness.
in (16.1.1); we divide the section into Sections 16.3.1 and 16.3.2 according to the higher-dimensional setting and the 2-dimensional setting, respectively. Section 16.4 is devoted to the proofs of the main results according to arguments in [188]; we show convergence of the solution \((u_\lambda, v_\lambda)\) for (16.1.1) as \(\lambda \searrow 0\) by using the uniform-in-\(\lambda\) estimate established in Section 16.3.

### 16.2. Local existence and basic property

In this section we collect results which will be used later. We first recall the well-known result concerned with local existence of solutions to (16.1.1) (see e.g., [5, Lemma 3.1]).

**Lemma 16.2.1.** Let \(\Omega\) be a bounded domain in \(\mathbb{R}^n (n \geq 2)\) with smooth boundary, and let \(\chi > 0\) be a constant. Then for all \(\lambda > 0\) and any \(u_{\text{init}}, v_{\text{init}}\) satisfying (16.1.3) there exists \(T_{\text{max}, \lambda} \in (0, \infty]\) such that the problem (16.1.1) possesses a unique solution \((u_\lambda, v_\lambda)\) fulfilling

\[
\begin{align*}
  u_\lambda &\in C(\overline{\Omega} \times [0, T_{\text{max}, \lambda}) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\text{max}, \lambda})), \\
  v_\lambda &\in C(\overline{\Omega} \times [0, T_{\text{max}, \lambda}) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\text{max}, \lambda})) \cap L^\infty([0, T_{\text{max}, \lambda}); W^{1,q}(\Omega))
\end{align*}
\]

and

\[
  u_\lambda(x, t) \geq 0 \quad \text{and} \quad v_\lambda(x, t) \geq 0 \quad \text{for all } x \in \Omega, \ t > 0 \text{ and all } \lambda > 0
\]

as well as

\[
  \int_\Omega u_\lambda(\cdot, t) = \int_\Omega u_{\text{init}} \quad \text{for all } t \in (0, T_{\text{max}, \lambda}) \text{ and all } \lambda > 0.
\]

Moreover, either \(T_{\text{max}, \lambda} = \infty\) or

\[
  \limsup_{t \to T_{\text{max}, \lambda}} (\|u_\lambda(\cdot, t)\|_{L^\infty(\Omega)} + \|v_\lambda(\cdot, t)\|_{W^{1,q}(\Omega)}) = \infty.
\]

We then give the following result which plays an important role in obtaining uniform-in-\(\lambda\) boundedness of solutions to (16.1.1).

**Lemma 16.2.2.** Let \(\lambda > 0\). If there exist \(p > \frac{n}{2}\) and \(M > 0\) such that

\[
  \|u_\lambda(\cdot, t)\|_{L^p(\Omega)} \leq M \quad \text{for all } t \in (0, T_{\text{max}, \lambda}),
\]

then there exists \(C = C(p, M) > 0\) such that

\[
  \|u_\lambda(\cdot, t)\|_{L^\infty(\Omega)} + \|v_\lambda(\cdot, t)\|_{W^{1,q}(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\text{max}, \lambda}).
\]

Moreover, if \(p\) and \(M\) are independent of \(\lambda \in (0, \lambda_0)\) with some \(\lambda_0 > 0\), then \(C\) is also independent of \(\lambda \in (0, \lambda_0)\).

**Proof.** The proof is a combination of Lemmas 15.2.3 and 15.2.4 (the proof is based on an application of the \(L^p-L^q\) estimates for the Neumann heat semigroup in the proof of [5, Lemma 3.2]). \(\Box\)
16.3. Uniform-in-\( \lambda \) boundedness

In this section we establish global existence of solutions to (16.1.1) and their uniform-in-\( \lambda \) boundedness.

16.3.1. The higher-dimensional setting

In this subsection we will deal with the higher-dimensional Keller–Segel system (16.1.1). Aided by Lemma 16.2.2, we shall only verify the \( L^p \)-estimate for \( u_\lambda \) with some \( p_0 > \frac{n}{2} \). We first prove the following lemma which enables us to pick appropriate constants in the proof of Lemma 16.3.2.

**Lemma 16.3.1.** Let \( p > \frac{n}{2} \) and \( q > n \). Then there are constants \( \theta, q_0, \mu > 0 \) such that

\[
\theta \in I_1 := \left(p, \min \left\{ \frac{np}{(np + nq - pq)_+}, \frac{np}{2(n - p)_+} \right\} \right), \\
q_0 \in I_2 := \left(\max \left\{1, \frac{np\theta}{p\theta + np - n\theta}\right\}, \min \left\{q, \frac{np}{(n - p)_+}\right\}\right), \\
\mu \in I_3 := \left(\max \left\{1, \frac{n\theta}{n + \theta}, \frac{np\theta}{p\theta + 2np - n\theta}\right\}, \min \left\{q_0, \frac{q_0\theta}{q_0 + \theta}\right\}\right).
\]

**Proof.** Since we have from the conditions \( q > n \) and \( p > \frac{n}{2} \) that

\[
(np + nq - pq)p < npq \quad \text{and} \quad 2(n - p) < n,
\]

we can verify that

\[
I_1 = \left(p, \min \left\{ \frac{np}{(np + nq - pq)_+}, \frac{np}{2(n - p)_+} \right\} \right) \neq \emptyset.
\]

Thus we can take \( \theta \in I_1 \). We next see that

\[
I_2 = \left(\max \left\{1, \frac{np\theta}{p\theta + np - n\theta}\right\}, \min \left\{q, \frac{np}{(n - p)_+}\right\}\right) \neq \emptyset.
\]

Noticing from the fact \( \theta \in I_1 \subset (p, \frac{np}{(n - p)_+}) \) that

\[
p\theta + np - n\theta > 0 \quad \text{and} \quad \frac{np\theta}{p\theta + np - n\theta} \geq 1,
\]

we will only confirm that

\[
(16.3.1) \quad \frac{np\theta}{p\theta + np - n\theta} < \min \left\{q, \frac{np}{(n - p)_+}\right\}
\]

holds. Here since the fact

\[
\theta < \min \left\{ \frac{np}{(np + nq - pq)_+}, \frac{np}{2(n - p)_+} \right\}
\]

...
implies
\[ np\theta < q(p\theta + np - n\theta) \quad \text{and} \quad \theta(n - p) < p\theta + np - n\theta, \]
we can verify that (16.3.1) is true, which tells us that \( I_2 \neq \emptyset \). Therefore we can choose \( q_0 \in I_2 \). We finally confirm that
\[
I_3 = \left( \max \left\{ 1, \frac{n\theta}{n + \theta}, \frac{np\theta}{p\theta + 2np - n\theta} \right\}, \min \left\{ q_0, \frac{q_0\theta}{q_0 + \theta} \right\} \right) \neq \emptyset.
\]
Here we note that \( p\theta + 2np - n\theta > p\theta + np - n\theta > 0 \) and \( \frac{q_0\theta}{q_0 + \theta} < q_0 \). Since the facts \( \theta > p > \frac{n}{2} \geq \frac{n}{n+1} \) \((n \geq 3)\) and \( p\theta + 2np - n\theta < p(n + \theta) \) ensure that
\[
1 \leq \frac{n\theta}{n + \theta} < \frac{np\theta}{p\theta + 2np - n\theta},
\]
we shall only see that
\[
\text{(16.3.2)} \quad \frac{np\theta}{p\theta + 2np - n\theta} < \frac{q_0\theta}{q_0 + \theta}.
\]
Now aided by the relation \( \frac{np\theta}{p\theta + np - n\theta} < q_0 \), we establish that
\[
np(\theta + q_0) < q_0(p\theta + 2np - n\theta)
\]
holds. Therefore we have (16.3.2), which enables us to find a constant \( \mu \in I_3 \). This completes the proof. \( \square \)

Then we can show the following lemma which entails the desired estimate for \( u_\lambda \).

**Lemma 16.3.2.** Let \( p > \frac{n}{2} \). Then there exists a positive constant \( \varepsilon_0 = \varepsilon_0(p, q, \chi, |\Omega|) \) such that, if \( u_{\text{init}} \) and \( v_{\text{init}} \) satisfy
\[
\| u_{\text{init}} \|_{L^p(\Omega)} < \varepsilon_0 \quad \text{and} \quad \| \nabla v_{\text{init}} \|_{L^q(\Omega)} < \varepsilon_0,
\]
then there exist \( p_0 > \frac{n}{2} \) and \( C > 0 \) which are independent of \( \lambda > 0 \) such that
\[
\| u_\lambda(\cdot, t) \|_{L^{p_0}(\Omega)} \leq C
\]
for all \( t \in (0, T_{\text{max}}, \lambda) \) and all \( \lambda > 0 \).

**Proof.** Let \( \varepsilon > 0 \) be a small constant fixed later (see (16.3.10)), and assume that \( u_{\text{init}} \) and \( v_{\text{init}} \) satisfy
\[
\text{(16.3.3)} \quad \| u_{\text{init}} \|_{L^p(\Omega)} \leq \varepsilon \quad \text{and} \quad \| \nabla v_{\text{init}} \|_{L^q(\Omega)} \leq \varepsilon
\]
with some \( p > \frac{n}{2} \). Then invoking to Lemma 16.3.1, we can take \( \theta, q_0, \mu \geq 1 \) such that
\[
\theta \in I_1, \quad q_0 \in I_2 \quad \text{and} \quad \mu \in I_3,
\]
\[ 207 \]
where $I_1, I_2, I_3$ are intervals defined in Lemma 16.3.1. Now we put

$$T_\lambda := \sup \left\{ \hat{T} \in (0, T_{\text{max}, \lambda}) \mid \| u_\lambda(\cdot, t) - e^{t\Delta} u_{\text{init}} \|_{L^p(\Omega)} < \varepsilon \quad \text{for all} \ t \in (0, \hat{T}) \right\}.$$  

Then because

$$u_\lambda(\cdot, 0) - e^{0\Delta} u_{\text{init}} = 0$$

and the function $t \mapsto u_\lambda(\cdot, t) - e^{t\Delta} u_{\text{init}}$ is continuous on $[0, T_{\text{max}, \lambda})$, $T_\lambda$ is well-defined and positive with $T_\lambda \leq T_{\text{max}, \lambda}$. We first note from the standard $L^p$-$L^q$ estimate for the Neumann heat semigroup that there is $C_1 = C_1(|\Omega|) > 0$ such that for all $r \in [1, \theta)$,

$$\| u_\lambda(\cdot, t) \|_{L^r(\Omega)} \leq \| u_\lambda(\cdot, t) - e^{t\Delta} u_{\text{init}} \|_{L^r(\Omega)} + \| e^{t\Delta} u_{\text{init}} \|_{L^r(\Omega)}$$

for all $t \in (0, T_\lambda)$, which with the relation $p < \theta$ and (16.3.3) tells us that

$$\| u_\lambda(\cdot, t) \|_{L^p(\Omega)} \leq \left( |\Omega|^{\frac{1}{p} - \frac{1}{q}} + C_1 \right) \varepsilon$$

for all $t \in (0, T_\lambda)$. We then obtain from the variation-of-constants representation for $v_\lambda$, the fact $q_0 < q$ and Lemma 2.2.2 (ii), (iii) that

$$\| \nabla v_\lambda(\cdot, t) \|_{L^{q_0}(\Omega)} \leq \| \nabla e^{t\frac{1}{\lambda}(\Delta - 1)} v_{\text{init}} \|_{L^{q_0}(\Omega)} + \frac{1}{\lambda} \int_0^t \| \nabla e^{\frac{1}{\lambda}(s - \Delta)} v_\lambda(\cdot, s) \|_{L^{q_0}(\Omega)} ds$$

$$\leq C_2 \| \nabla v_{\text{init}} \|_{L^{q_0}(\Omega)} + \frac{C_3 \varepsilon}{\lambda} \int_0^t \left( 1 + \left( \frac{t - s}{\lambda} \right)^{-\frac{1}{2} - \frac{n}{2} \left( \frac{1}{p} - \frac{1}{q_0} \right)} \right) e^{-\alpha \left( \frac{t - s}{\lambda} \right)} ds$$

$$\leq C_2 |\Omega|^{\frac{1}{n} - \frac{1}{q}} \| \nabla v_{\text{init}} \|_{L^{q_0}(\Omega)} + C_3 \varepsilon \int_0^t \left( 1 + \sigma^{-\frac{1}{2} - \frac{n}{2} \left( \frac{1}{p} - \frac{1}{q_0} \right)} \right) e^{-\alpha \sigma} d\sigma$$

for all $t \in (0, T_\lambda)$ with some $C_2 = C_2(|\Omega|) > 0$ and $C_3 = C_3(p, q, |\Omega|) > 0$. Since the fact $q_0 < \frac{np}{(n-p)\lambda}$ implies $\frac{1}{2} + \frac{n}{2} \left( \frac{1}{p} - \frac{1}{q_0} \right) < 1$, from (16.3.3) we infer that

$$C_2 |\Omega|^{\frac{1}{n} - \frac{1}{q}} \| \nabla v_{\text{init}} \|_{L^{q_0}(\Omega)} + C_3 \varepsilon \int_0^t \left( 1 + \sigma^{-\frac{1}{2} - \frac{n}{2} \left( \frac{1}{p} - \frac{1}{q_0} \right)} \right) e^{-\alpha \sigma} d\sigma \leq C_4 \varepsilon$$

holds with

$$C_4 := C_2 |\Omega|^{\frac{1}{n} - \frac{1}{q}} + C_3 \int_0^\infty \left( 1 + \sigma^{-\frac{1}{2} - \frac{n}{2} \left( \frac{1}{p} - \frac{1}{q_0} \right)} \right) e^{-\alpha \sigma} d\sigma < \infty,$$

which means that

$$\| \nabla v_\lambda(\cdot, t) \|_{L^{q_0}(\Omega)} \leq C_4 \varepsilon$$

for all $t \in (0, T_\lambda)$. Finally, in order to show $T_\lambda = T_{\text{max}, \lambda}$, we will show that

$$\| u_\lambda(\cdot, t) - e^{t\Delta} u_{\text{init}} \|_{L^p(\Omega)} \leq C \varepsilon \quad \text{for all} \ t \in (0, T_\lambda)$$
with some \(C < 1\). Employing the variation-of-constant formula for \(u_\lambda\) and Lemma 2.2.2 (iv), we see that

\[
\begin{align*}
(16.3.6) \quad & \|u_\lambda(\cdot, t) - e^{t \Delta} u_{\text{init}}\|_{L^p(\Omega)} \\
& \leq \chi \int_0^t (1 + (t - s)^{-\frac{\mu}{2} - \frac{\gamma}{2}}) e^{-\alpha(t - s)} \|u_\lambda(\cdot, s)\|_{L^p(\Omega)} ds.
\end{align*}
\]

Now thanks to the facts \(\mu < q_0\), \(1 < \frac{q_0 \mu}{q_0 - \mu} < \theta\) and (16.3.4)–(16.3.5), we obtain from the H"older inequality and the interpolation inequality that

\[
\begin{align*}
(16.3.7) \quad & \left\| u_\lambda(\cdot, s) \nabla v_\lambda(\cdot, s) \right\|_{L^q(\Omega)} \\
& \leq \left\| u_\lambda(\cdot, s) \right\|_{L^{\frac{q_0 \mu}{q_0 - \mu}}(\Omega)} \left\| \nabla v_\lambda(\cdot, s) \right\|_{L^{\frac{q_0 \mu}{q_0 - \mu}}(\Omega)} \\
& \leq C_5 e^{1-a} \left( \|u_\lambda(\cdot, s) - e^{t \Delta} u_{\text{init}}\|_{L^p(\Omega)}^{1-a} + \|e^{s \Delta}(u_{\text{init}} - \overline{u_{\text{init}}})\|_{L^p(\Omega)}^{1-a} \right)
\end{align*}
\]

with some \(C_5 = C_5(p, q, |\Omega|) > 0\), where \(a = \frac{\theta, q_0 \mu}{q_0 \mu, q_0 \mu - \theta} \in (0, 1)\) and

\[
\overline{u_{\text{init}}} := \frac{1}{|\Omega|} \int_\Omega u_{\text{init}}.
\]

Here from the H"older inequality, the Young inequality and Lemma 2.2.2 (i) we can find \(C_6 = C_6(p, q, |\Omega|) > 0\) and \(C_7 = C_7(p, q, |\Omega|) > 0\) such that

\[
(16.3.8) \quad \|e^{s \Delta} \overline{u_{\text{init}}}\|_{L^p(\Omega)}^{1-a} \leq C_6 \|u_{\text{init}}\|_{L^p(\Omega)}^{1-a} \leq C_6 e^{1-a}
\]

and

\[
(16.3.9) \quad \|e^{s \Delta}(u_{\text{init}} - \overline{u_{\text{init}}})\|_{L^p(\Omega)}^{1-a} \leq (1 - a) \|e^{s \Delta}(u_{\text{init}} - \overline{u_{\text{init}}})\|_{L^p(\Omega)} + a
\]

\[
\leq C_7 (1 + s^{-\frac{\mu}{2} \left( \frac{1}{p} - \frac{1}{q} \right) \frac{1}{\theta}}) e^{-\alpha s} \|u_{\text{init}} - \overline{u_{\text{init}}}\|_{L^p(\Omega)} + a
\]

\[
\leq C_7 (1 + |\Omega|^{-1+a}) \|e^{(1 + s^{-\frac{\mu}{2} \left( \frac{1}{p} - \frac{1}{q} \right) \frac{1}{\theta}})} e^{-\alpha s} + a.
\]

Since the fact \(\frac{1}{2} + \frac{n}{2} \left( \frac{1}{\mu} - \frac{1}{\theta} \right) < 1\) leads to

\[
\int_0^\infty (1 + \sigma^{-1} \frac{1}{2} \left( \frac{1}{\mu} - \frac{1}{\theta} \right)) e^{-\alpha \sigma} d\sigma < \infty
\]

and the relation

\[
1 - \frac{1}{2} - \frac{n}{2} \left( \frac{1}{\mu} - \frac{1}{\theta} \right) - \frac{n}{2} \left( \frac{1}{p} - \frac{1}{\theta} \right) > 0
\]

yields from [194, Lemma 1.2] that

\[
\begin{align*}
& \int_0^t (1 + (t - s)^{-\frac{n}{2} \left( \frac{1}{\mu} - \frac{1}{\theta} \right)}) e^{-\alpha(t - s)} \left( 1 + s^{-\frac{n}{2} \left( \frac{1}{p} - \frac{1}{q} \right) \frac{1}{\theta}} \right) e^{-\alpha s} ds \\
& \leq C_8 (1 + t^{\min(0,1-\frac{n}{2} \left( \frac{1}{\mu} - \frac{1}{\theta} \right), \frac{n}{2} \left( \frac{1}{p} - \frac{1}{q} \right) \frac{1}{\theta}))} e^{-\alpha t} \leq 2 C_8
\end{align*}
\]
for all $t > 0$ with some $C_8 = C_8(p, q) > 0$, plugging (16.3.8) and (16.3.9) into (16.3.6) and (16.3.7) implies that

$$
\|u_\lambda(\cdot, t) - e^{t\Delta}u_{\text{init}}\|_{L^p(\Omega)} \leq C_9 \varepsilon_1^{1+a} \left( \sup_{s \in (0, T_\lambda)} \|u_\lambda(\cdot, s) - e^{s\Delta}u_{\text{init}}\|_{L^p(\Omega)} + \varepsilon + 1 + \varepsilon^{1-a} \right)
$$

\[ \leq C_9 \varepsilon_1^{1+a}(2\varepsilon^{1-a} + \varepsilon + 1) \]

for all $t \in (0, T_\lambda)$ with some $C_9 = C_9(p, q, \chi, |\Omega|) > 0$. Thus if we take $\varepsilon > 0$ satisfying

(16.3.10) \[ C_9 \varepsilon_1^{1+a}(2\varepsilon^{1-a} + \varepsilon + 1) < 1, \]

then the continuity of the function $t \mapsto \|u_\lambda(\cdot, t) - e^{t\Delta}u_{\text{init}}\|_{L^p(\Omega)}$ concludes that

$$
T_\lambda = T_{\text{max}, \lambda},
$$

which namely means that

(16.3.11) \[ \|u_\lambda(\cdot, t) - e^{t\Delta}u_{\text{init}}\|_{L^p(\Omega)} \leq \varepsilon \]

for all $t \in (0, T_{\text{max}, \lambda})$. Here, since $\varepsilon > 0$ is independent of $\lambda > 0$, we note that (16.3.11) holds for all $t \in (0, T_{\text{max}, \lambda})$ and all $\lambda > 0$, which together with the maximum principle

$$
\|e^{t\Delta}u_{\text{init}}\|_{L^\infty(\Omega)} \leq \|u_{\text{init}}\|_{L^\infty(\Omega)} \quad \text{for all } t > 0
$$

enables us to see that

(16.3.12) \[ \|u_\lambda(\cdot, t)\|_{L^p(\Omega)} \leq \varepsilon + |\Omega|^\frac{1}{p} \|u_{\text{init}}\|_{L^\infty(\Omega)} \]

for all $t \in (0, T_{\text{max}, \lambda})$ and all $\lambda > 0$. Noticing that $\theta > p > \frac{n}{2}$ holds and $\theta, \varepsilon$ are independent of $\lambda$, from (16.3.12) we can attain the goal of the proof. \( \square \)

Here we are in the position to prove global existence and uniform-in-$\lambda$ boundedness in the higher-dimensional Keller–Segel system (16.1.1).

**Lemma 16.3.3.** Let $p > \frac{n}{2}$. Assume that $u_{\text{init}}$ and $v_{\text{init}}$ satisfy

$$
\|u_{\text{init}}\|_{L^p(\Omega)} < \varepsilon_0 \quad \text{and} \quad \|\nabla v_{\text{init}}\|_{L^q(\Omega)} < \varepsilon_0,
$$

where $\varepsilon_0$ is the constant defined in Lemma 16.3.2. Then $T_{\text{max}, \lambda} = \infty$ holds, and there exists $C > 0$ independent of $\lambda > 0$ such that

$$
\|u_\lambda(\cdot, t)\|_{L^\infty(\Omega)} \leq C
$$

for all $t \in (0, \infty)$ and all $\lambda > 0$.

**Proof.** A combination of Lemmas 16.2.2 and 16.3.2, along with the extensibility criterion directly leads to this lemma. \( \square \)

210
16.3.2. The 2-dimensional setting

In this subsection we will show uniform-in-\( \lambda \) boundedness in the 2-dimensional Keller–Segel system. The proof is mainly based on arguments in the proof of \([144, \text{Theorem 1.1}]\). Thus we will only give short proofs.

**Lemma 16.3.4.** Assume that \( u_{\text{init}} \) satisfies

\[
\|u_{\text{init}}\|_{L^1(\Omega)} < \frac{4\pi}{\chi}.
\]

Then for all \( \lambda_0 > 0 \) there exists \( C > 0 \) such that

\[
\|u_\lambda(\cdot, t)\|_{L^2(\Omega)} \leq C
\]

for all \( t \in (0, T_{\text{max}, \lambda}) \) and all \( \lambda \in (0, \lambda_0) \).

**Proof.** Let \( \lambda_0 > 0 \) be an arbitrary constant. From straightforward calculations we can verify that the function

\[
W_\lambda := \int_\Omega \left( u_\lambda \log u_\lambda - \chi u_\lambda v_\lambda + \frac{\chi}{2} (|\nabla v_\lambda|^2 + v_\lambda^2) \right)
\]

satisfies

\[
\frac{dW_\lambda}{dt} + \chi \lambda \int_\Omega |(v_\lambda)_t|^2 + \int_\Omega u_\lambda |\nabla \cdot (\log u_\lambda - \chi v_\lambda)| = 0 \tag{16.3.13}
\]

for all \( \lambda \in (0, \lambda_0) \). Then by virtue of the Jensen inequality and the Trudinger–Moser inequality, the same argument as in the proof of \([144, \text{Lemma 3.4}]\) ensures that there is \( C_1 > 0 \) such that

\[
\int_\Omega u_\lambda(\cdot, t) v_\lambda(\cdot, t) \leq C_1 \quad \text{and} \quad |W_\lambda(t)| \leq C_1 \tag{16.3.14}
\]

for all \( t \in (0, T_{\text{max}, \lambda}) \) and \( \lambda \in (0, \lambda_0) \) under the condition that \( \|u_{\text{init}}\|_{L^1(\Omega)} < \frac{4\pi}{\chi} \). Thanks to (16.3.14), the relation (16.3.13) implies that

\[
\int_\Omega |u_\lambda(\cdot, t) \log u_\lambda(\cdot, t)| \leq \max \left\{ W_\lambda(0) + C_1, \frac{1}{e} \right\} \tag{16.3.15}
\]

and

\[
\lambda \int_0^t \|v_\lambda(\cdot, s)\|_{L^2(\Omega)}^2 \, ds \leq \frac{1}{\chi} (|W_\lambda(0)| + C_1) \tag{16.3.16}
\]

for all \( t \in (0, T_{\text{max}, \lambda}) \) and \( \lambda \in (0, \lambda_0) \). Now we shall show the \( L^2 \)-boundedness of \( u_\lambda \). Multiplying the first equation in (16.1.1) by \( \frac{1}{2} u_\lambda \) and integrating it over \( \Omega \), we infer from integration by parts that

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega u_\lambda^2 = -\int_\Omega |\nabla u_\lambda|^2 + \chi \int_\Omega u_\lambda \nabla u_\lambda \cdot \nabla v_\lambda
\]

\[
= -\int_\Omega |\nabla u_\lambda|^2 - \frac{\chi \lambda}{2} \int_\Omega (v_\lambda)_t^2 + \frac{\chi}{2} \int_\Omega u_\lambda^3 - \frac{\chi}{2} \int_\Omega u_\lambda^2 v_\lambda.
\]
Let $\varepsilon \in (0, 1)$ be a constant satisfying

\[(16.3.18) \quad 1 - \varepsilon - \varepsilon \max \left\{ W_\lambda(0) + C_1, \frac{1}{\varepsilon} \right\} \geq \frac{1}{2}. \]

Now we note that $\varepsilon \in (0, 1)$ satisfying (16.3.18) can be taken from the intermediate value theorem, and that $\varepsilon \in (0, 1)$ satisfying (16.3.18) is independent of $\lambda \in (0, \lambda_0)$ from the fact $W_\lambda(0) = \int_\Omega (u_{\text{init}} \log u_{\text{init}} - \chi_{\text{init}} u_{\text{init}} + \frac{1}{2} (|\nabla u_{\text{init}}|^2 + v_{\text{init}}^2))$. Since the Gagliardo–Nirenberg inequality and its application (see [144, Lemma 3.5]) entail

\[\frac{\lambda}{2} \int_\Omega u_\lambda^3 \leq \varepsilon \|u_\lambda\|_{L^2(\Omega)}^2 \|u_\lambda \log u_\lambda\|_{L^1(\Omega)} + C_2(\|u_\lambda \log u_\lambda\|_{L^1(\Omega)}^3 + \|u_\lambda\|_{L^1(\Omega)}^2)\]

and

\[-\frac{\lambda}{2} \int_\Omega u_\lambda^2 (v_\lambda)_t \leq C_3 \lambda (\|(v_\lambda)_t\|_{L^2(\Omega)}^2 (\|\nabla u_\lambda\|_{L^2(\Omega)} \|u_\lambda\|_{L^2(\Omega)} + \|u_\lambda\|_{L^2(\Omega)}^2)) \leq \varepsilon \|\nabla u_\lambda\|_{L^2(\Omega)} + \left( C_4 \lambda^2 (\|(v_\lambda)_t\|_{L^2(\Omega)}^2 + \frac{1}{4}) \right) \|u_\lambda\|_{L^2(\Omega)}^2\]

with some $C_2, C_3, C_4 > 0$, the relation (16.3.17) with the nonnegativity of $v_\lambda$ tells us that

\[\frac{1}{2} \frac{d}{dt} \int_\Omega u_\lambda^2 + (1 - \varepsilon - \varepsilon \|u_\lambda \log u_\lambda\|_{L^1(\Omega)}) \int_\Omega |\nabla u_\lambda|^2 \leq \left( C_4 \lambda^2 (\|(v_\lambda)_t\|_{L^2(\Omega)}^2 + \frac{1}{4}) \right) \int_\Omega u_\lambda^2 + C_2(\|u_\lambda \log u_\lambda\|_{L^1(\Omega)}^3 + \|u_\lambda\|_{L^1(\Omega)}^2).\]

Noticing from the definition of $\varepsilon \in (0, 1)$ (see (16.3.18)) and (16.3.15) that

\[1 - \varepsilon - \varepsilon \|u_\lambda(\cdot, t) \log u_\lambda(\cdot, t)\|_{L^1(\Omega)} \geq \frac{1}{2}\]

holds for all $t \in (0, T_{\text{max, \lambda}})$ and $\lambda \in (0, \lambda_0)$, we infer from the application of the Gagliardo–Nirenberg inequality

\[\|u_\lambda\|_{L^2(\Omega)} \leq \|\nabla u_\lambda\|_{L^2(\Omega)}^2 + C_5 \|u_\lambda\|_{L^1(\Omega)}^2\]

with some $C_5 > 0$ that

\[(16.3.19) \quad \frac{1}{2} \frac{d}{dt} \int_\Omega u_\lambda^2 + \frac{1}{2} \left( \frac{1}{2} - C_4 \lambda^2 (\|(v_\lambda)_t\|_{L^2(\Omega)}^2) \right) \int_\Omega u_\lambda^2 \leq C_6(\|u_\lambda \log u_\lambda\|_{L^1(\Omega)}^3 + \|u_\lambda\|_{L^1(\Omega)}^2) \leq \frac{1}{2} L\]

with some $C_6 > 0$ and $L > 0$. Now we put

\[y(t) := \int_\Omega u_\lambda^2(\cdot, t)\]

212
and
\[
\phi(t) := \frac{1}{2} t - \frac{C_4 \lambda^2}{2} \int_0^t \|(v_\lambda)_t(\cdot, s)\|_{L^2(\Omega)}^2 \, ds.
\]
Then from the differential inequality (16.3.19), we establish that
\[
y(t) \leq y(0)e^{-\phi(t)} + Le^{-\phi(t)} \int_0^t e^{\phi(s)} \, ds
\]
for all \( t \in (0, T_{\text{max},\lambda}) \) and \( \lambda \in (0, \lambda_0) \).

Thus the boundedness of \( \phi(t) \)
\[
\frac{1}{2} t - \frac{C_4 \lambda_0}{2 \chi} (|W_\lambda(0)| + C_1) \leq \phi(t) \leq \frac{1}{2} t \quad (t \in (0, T_{\text{max},\lambda}), \ \lambda \in (0, \lambda_0))
\]
(from (16.3.16)) entails that there is \( C_7 = C_7(\lambda_0) > 0 \) such that
\[
y(t) \leq y(0)e^{-\phi(t)} + Le^{-\phi(t)} \int_0^t e^{\phi(s)} \, ds \leq C_7
\]
for all \( t > 0 \) and \( \lambda \in (0, \lambda_0) \), which means the end of the proof.

Thanks to Lemma 16.3.4, we attain global existence and uniform-in-\( \lambda \in (0, \lambda_0) \) boundedness of the solution \((u_\lambda, v_\lambda)\) to the 2-dimensional Keller–Segel system.

**Lemma 16.3.5.** Assume that \( u_{\text{init}} \) satisfies
\[
\|u_{\text{init}}\|_{L^1(\Omega)} < \frac{4\pi}{\chi},
\]
Then \( T_{\text{max},\lambda} = \infty \) holds, and for all \( \lambda_0 > 0 \) there exists \( C > 0 \) such that
\[
\|u_\lambda(\cdot, t)\|_{L^\infty(\Omega)} \leq C
\]
for all \( t \in (0, \infty) \) and all \( \lambda \in (0, \lambda_0) \).

**Proof.** A combination of Lemmas 16.2.2 and 16.3.4, along with the extensibility criterion leads to this lemma.

**16.4. Convergence**

In this section we will show that solutions of (16.1.1) converge to those of (16.1.2). Here we assume that there exists a unique global classical solution \((u_\lambda, v_\lambda)\) of (16.1.1) such that for all \( \lambda_0 > 0 \) there is \( C > 0 \) independent of \( \lambda \in (0, \lambda_0) \) such that,
\[
\|u_\lambda(\cdot, t)\|_{L^\infty(\Omega)} + \|v_\lambda(\cdot, t)\|_{W^{1,\theta}(\Omega)} \leq C
\]
for all \( t > 0 \) and all \( \lambda \in (0, \lambda_0) \), which is established by Lemmas 16.3.3 and 16.3.5.

Arguments in this section are based on those in the proof of [188, Theorem 1.1]; thus I shall only show brief proofs. We first confirm the following lemma which is a cornerstone of this work.

213
Lemma 16.4.1. For all sequences of numbers \( \{\lambda_n\}_{n \in \mathbb{N}} \subset (0, \lambda_0) \) satisfying \( \lambda_n \searrow 0 \) as \( n \to \infty \) there exist a subsequence \( \lambda_{n_j} \searrow 0 \) and functions
\[
\begin{align*}
u \in \mathcal{C}(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)) \quad \text{and} \quad \nu \in \mathcal{C}(\bar{\Omega} \times (0, \infty)) \cap L^\infty(0, \infty; W^{1,q}(\Omega))
\end{align*}
\]
such that
\[
\begin{align*}
u_{\lambda_{n_j}} & \to \nu \quad \text{in} \mathcal{C}_{\text{loc}}(\bar{\Omega} \times [0, \infty)), \\
v_{\lambda_{n_j}} & \to v \quad \text{in} \mathcal{C}_{\text{loc}}(\bar{\Omega} \times (0, \infty)) \cap L^2_{\text{loc}}((0, \infty); W^{1,2}(\Omega))
\end{align*}
\]
as \( j \to \infty \). Moreover, \( (u, v) \) solves (16.1.2) classically.

Remark 16.4.1. This lemma also gives that global existence and boundedness in (16.1.2) hold under the condition that there is a unique global bounded solution in (16.1.1) which is bounded uniformly in \( \lambda \in (0, \lambda_0) \).

Proof. Similar arguments in the proof of [188, Theorem 1.1] enable us to have this lemma; thus we only write a laugh sketch of the proof. From the assumption in this section and the standard parabolic regularity argument [156, Theorem 1.3] we see that \( \{u_\lambda\}_{\lambda \in (0, \lambda_0)} \) is bounded in \( C^{\alpha, \beta}_{\text{loc}}(\bar{\Omega} \times [0, \infty)) \) with some \( \alpha \in (0, 1) \). Thus the Arzelà–Ascoli theorem and the boundedness of \( \|\nabla u_\lambda\|_{L^\infty(0, \infty; W^{1,q}(\Omega))} \) yields that we can find a subsequence \( \lambda_{n_j} \searrow 0 \) and functions
\[
\begin{align*}u \in C^{\alpha, \beta}_{\text{loc}}(\bar{\Omega} \times [0, \infty)) \quad \text{and} \quad v \in L^\infty(0, \infty; W^{1,q}(\Omega))
\end{align*}
\]
satisfying
\[
\begin{align*}u_{\lambda_{n_j}} & \to u \quad \text{in} \mathcal{C}_{\text{loc}}(\bar{\Omega} \times [0, \infty)) \quad \text{and} \quad v_{\lambda_{n_j}} \overset{*}{\rightharpoonup} v \quad \text{in} \mathcal{C}_{\text{loc}}(\bar{\Omega} \times (0, \infty)) \cap L^2_{\text{loc}}((0, \infty); W^{1,2}(\Omega))
\end{align*}
\]
as \( j \to \infty \). Then these convergences with arguments in the proof of [188, Lemma 5.1] imply that \( v(\cdot, t) \in W^{1,2}(\Omega) \) and
\[
\begin{align*}\int_{\Omega} \nabla v(\cdot, t) \cdot \nabla \psi + \int_{\Omega} v(\cdot, t) \psi = \int_{\Omega} u(\cdot, t) \psi \quad \text{for all} \psi \in W^{1,2}(\Omega)
\end{align*}
\]
a.a. \( t \in (0, \infty) \). This identity with arguments in the proof of [188, Lemma 5.2] entails that for all \( T > 0 \) there is \( C_1(T) > 0 \) such that
\[
\|v(\cdot, t)\|_{W^{1,2}(\Omega)} \leq C_1(T) \quad \text{a.a.} \quad t \in (0, T),
\]
and moreover, on redefining \( v(\cdot, t) \) for \( t \) in a null set and \( t = 0, T \) if necessary, we have that \( v \in C^{\theta_1}_{\text{loc}}([0, \infty); W^{1,2}(\Omega)) \) with some \( \theta_1 \in (0, 1) \). Thanks to this regularity of \( v \) we can see that for all \( T > 0 \) there is \( C_2(T) > 0 \) such that
\[
\|v(\cdot, t)\|_{C^{2+\theta_2}_{\text{loc}}(\Omega)} \leq C_2(T) \quad \text{for all} \ t \in (0, T)
\]
with some \( \theta_2 \in (0, 1) \) and that \( v \) satisfies
\[
\begin{align*}
\begin{cases}
-\Delta v + v = u & \text{in} \ \Omega \times (0, \infty), \\
\nabla v \cdot \nu = 0 & \text{on} \ \partial \Omega \times (0, \infty)
\end{cases}
\end{align*}
\]

214
in the classical sense. Then through an estimate of \( \|v_t\|_{L^2_{\text{loc}}(\Omega \times (0,\infty))} \) (from arguments in
the proofs of [188, Lemmas 3.5,3.6 and 5.5]), we infer from arguments in the proof of
[188, Lemma 5.6] that

\[
(16.4.2) \quad v_{n_j} \to v \quad \text{in } L^\infty((0,\infty); L^2(\Omega)),\\
\nabla v_{n_j} \to \nabla v \quad \text{in } L^2_{\text{loc}}(\Omega \times (0,\infty))
\]
as \( \lambda = \lambda_{n_j} \downarrow 0 \). Therefore the convergences stated in (16.4.1) and (16.4.2) with similar
arguments in the proof of [188, Lemma 6.1] ensure that \( u \in C^{2+\theta_3,1+\frac{\theta_3}{2}}_{\text{loc}}(\Omega \times (0,\infty)) \) with
some \( \theta_3 \in (0,1) \) and that \( u \) satisfies

\[
\begin{cases}
    u_t = \Delta u - \chi \nabla \cdot (u \nabla v) & \text{in } \Omega \times (0,\infty), \\
    \nabla u \cdot \nu = 0 & \text{on } \partial \Omega \times (0,\infty), \\
    u(\cdot,0) = u_{\text{init}} & \text{in } \Omega,
\end{cases}
\]
which implies that this lemma holds.

We next verify the following lemma which implies that the pair of functions \((u,v)\) provided by Lemma 16.4.1 is independent of a choice of a sequence \( \lambda_n \downarrow 0 \).

**Lemma 16.4.2.** A solution \((\overline{u},\overline{v})\) of (16.1.2) satisfying

\[
\overline{u} \in C(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty))
\]
and

\[
\overline{v} \in C^{2,0}(\overline{\Omega} \times (0,\infty)) \cap L^\infty(0,\infty; W^{1,2}(\Omega))
\]
is unique.

**Proof.** Let \((\overline{u}_1,\overline{v}_1)\) and \((\overline{u}_2,\overline{v}_2)\) be solutions to (16.1.2) and put \( y(x,t) := \overline{u}_1(x,t) - \overline{u}_2(x,t) \)
for \((x,t) \in \Omega \times (0,\infty)\). Then aided by the Gronwall-type argument similar to that in the
proof of [164, Lemma 2.1], we infer that \( y(x,t) = 0 \), which concludes the proof.

Finally we shall establish convergence of the solution \((u_\lambda,v_\lambda)\) for (16.1.1) as \( \lambda \downarrow 0 \).

**Lemma 16.4.3.** The solution \((u_\lambda,v_\lambda)\) of (16.1.1) with \( \lambda \in (0,\lambda_0) \) satisfies that

\[
u_\lambda \to u \quad \text{in } C_{\text{loc}}(\overline{\Omega} \times [0,\infty)),
\]

\[
v_\lambda \to v \quad \text{in } C_{\text{loc}}(\overline{\Omega} \times (0,\infty)) \cap L^2_{\text{loc}}((0,\infty); W^{1,2}(\Omega))
\]
as \( \lambda \downarrow 0 \), where \((u,v)\) is the solution of (16.1.2) provided by Lemma 16.4.1.

**Proof.** Lemmas 16.4.1 and 16.4.2 yield that there exists the pair of the functions \((u,v)\)
such that for any sequences \( \{\lambda_n\}_{n \in \mathbb{N}} \subset (0,\lambda_0) \) satisfying \( \lambda_n \downarrow 0 \) as \( n \to \infty \) there is a
subsequence \( \lambda_{n_j} \downarrow 0 \) such that

\[
u_{\lambda_{n_j}} \to u \quad \text{in } C_{\text{loc}}(\overline{\Omega} \times [0,\infty)),
\]
and moreover that

\[
v_{\lambda_{n_j}} \to v \quad \text{in } C_{\text{loc}}(\overline{\Omega} \times (0,\infty)) \cap L^2_{\text{loc}}((0,\infty); W^{1,2}(\Omega))
\]
as \( j \to \infty \), which enables us to see this lemma.
Proof of Theorem 16.1.1. Lemmas 16.3.3 and 16.4.3 show Theorem 16.1.1.

Proof of Corollary 16.1.2. Put $\varepsilon_1 = \varepsilon_1(p, \chi, |\Omega|) := \sup_{q \in (n, \infty)} \varepsilon_0(p, q, \chi, |\Omega|)$ and let $u_{\text{init}}$ satisfy $\|u_{\text{init}}\|_{L^p(\Omega)} < \varepsilon_1$. Then we can pick $q > n$ such that
\[
\|u_{\text{init}}\|_{L^p(\Omega)} < \varepsilon_0(p, q, \chi, |\Omega|).
\]
Now we choose $v_{\text{init}} \in W^{1,q}(\Omega)$ satisfying $\|\nabla v_{\text{init}}\|_{L^q(\Omega)} < \varepsilon_0$. By virtue of Theorem 16.1.1, we can prove Corollary 16.1.2.

Proof of Theorem 16.1.3. A combination of Lemmas 16.3.5 and 16.4.3 enables us to obtain Theorem 16.1.3.

Proof of Corollary 16.1.4. Theorem 16.1.3 directly leads to Corollary 16.1.4.
Chapter 17

Convergence of solutions to chemotaxis system to solutions of Fisher–KPP equation

17.1. Motivation and results

The Fisher-KPP equation ([47, 92])

\( u_t = \Delta u + \mu u(1 - u) \)  \hspace{1cm} (17.1.1)

modelling the spread and growth of a biological population - or, in the original setting, of the prevalence of an advantageous gene within the population ([47]) - is well-studied and clearly of interest on its own, and there is a large corpus of literature bearing witness to this, ranging from articles on existence and speed ([47, 92]) or stability ([78]) of travelling waves, long-term behaviour of solutions and a 'hair-trigger effect' (i.e. instability of the rest state \( u \equiv 0 \)) ([3]) to treatments of system variants in a heterogeneous environment ([9]), with nonlinear ([106]) or fractional ([163]) diffusion or nonlocal interaction ([65]), spatio-temporal delays ([1]), or investigations of the spreading as free boundary problem ([39]), to name but a few. At the same time owing to the rather simple structure of the equation, it is no wonder that (17.1.1) makes an appearance as constituent of more complex models. In the present chapter, we shall view chemotaxis systems with logistic growth terms as perturbation of (17.1.1) and ask to what extent the behaviour of solutions to (17.1.1) is altered in the presence of weak chemotactic effects.

The Keller-Segel model ([89], see also the surveys [74, 5]) has arisen from the ambition to understand chemotaxis, i.e. the partially directed movement of cells (bacteria, slime mould, etc.; with density denoted by \( u \)) in the direction indicated by concentration gradient of a signal substance (concentration \( v \)) they themselves produce:

\[
\begin{align*}
  u_t &= \Delta u - \chi \nabla \cdot (u \nabla v) + f(u), \\
  \tau v_t &= \Delta v - v + u,
\end{align*}
\]  \hspace{1cm} (17.1.2)

Herein, the constant \( \chi \) stands for the chemotactic sensitivity and \( f \) is used to denote growth terms, one of the most natural forms (apart from \( f \equiv 0 \)) being \( f(u) = u - u^2 \).
The model plays an important role in the mathematical study of emergence of pattern and structure in many different biological contexts (see [71]), e.g. slime mould formation, bacterial patterning, embryonic development, progression of cancer, and has spawned an abundance of mathematical literature over the past decades (see [74, 5] and the references therein). In particular in the presence of logistic source terms like \( f(u) = \kappa u - \mu u^2 \) (cf. also [71, Sec. 2.8]) structure formation can be observed in (17.1.2), as witnessed by the numerical experiments in [154] or [41], attractor results in [152] and transient growth phenomena demonstrated in [200, 99, 205].

Let us recall some known results about this system: In the simplified parabolic-elliptic case (i.e. \( \tau = 0 \)), with \( f(u) \) generalizing \( f(u) = \kappa u - \mu u^2 \), Tello and Winkler proved the existence of global weak solutions and global classical solutions if \( \mu > \frac{u-2}{n} \). They also showed convergence of solutions to the constant steady state under a stricter condition on the source terms. Very weak solutions have been constructed for sources of the form \( f(u) = \kappa u - \mu u^a \) for \( a > 2 - \frac{1}{n} \) in [192].

As to the fully parabolic case of (17.1.2) \( (\tau = 1, \text{again with } f(u) = \kappa u - \mu u^2) \) it is known that globally bounded classical solutions exist in two-dimensional domains ([152]) or if \( \mu \) is sufficiently large ([195]); these solutions converge, provided a further largeness requirement is satisfied by \( \mu ([199]) \). For any positive \( \mu \), global weak solutions exist ([100]), which moreover in three-dimensional settings are known to become classical after some waiting time and enter an absorbing ball in \( C^{2+\alpha} \) if \( \kappa \) is sufficiently small ([100]).

Extensive studies regarding the interplay of exponents \( \alpha \) and \( \beta \) with respect to global existence of bounded solutions to the system obtained from (17.1.2) upon replacing the production term \( +u \) in the second equation by, roughly speaking, \( +u^2 \) and with \( f(u) = u - u^a \) have been conducted by Nakaguchi and Osaki ([146, 147]). The existence of very weak solutions to (17.1.2) with \( f(u) = u - u^a \), \( a > 2 - \frac{1}{n} \), has been established by Viglialoro ([181]) for bounded domains of arbitrary dimension. The convergence rate of solutions for both the parabolic-elliptic and the parabolic-parabolic variant of (17.1.2) has recently been studied by He and Zheng ([67]).

Letting different parameters in (17.1.2) tend to zero can help uncover dynamical properties in (17.1.2) and the relation to affiliated models. In [200, 99], considering \( \varepsilon \rightarrow 0 \) in

\[
\begin{align*}
  u_t &= \varepsilon \Delta u - \nabla \cdot (u \nabla v) + \kappa u - \mu u^2 \\
  0 &= \Delta v - v + u
\end{align*}
\]

(17.1.3)

was used to obtain insight into some transient growth phenomenon of solutions from a blow-up result in the hyperbolic-elliptic limit system (17.1.3) with \( \varepsilon = 0 \), both in the one-dimensional ([200]) and in the higher-dimensional radially symmetric case ([99]). For quite general choices of \( f \), system (17.1.2) had been suggested and investigated by Mimura and Tsujikawa ([125]). Inter alia, they considered the limit \( \varepsilon \rightarrow 0 \) of the time-rescaled system with Allee effect

\[
\begin{align*}
  u_t &= \varepsilon^2 \Delta u - \varepsilon \nabla \cdot (u \nabla v) + u(1 - u)(u - a), \\
  v_t &= \Delta v + u - v,
\end{align*}
\]

\( a \in (0, \frac{1}{2}) \), thus showing the existence of localized aggregating patterns.
In the present part, we want to investigate the disturbances to Fisher-KPP dynamics caused by weak chemotactic effects and hence consider the system

\begin{align}
(u_\varepsilon)_t &= \Delta u_\varepsilon - \varepsilon \nabla \cdot (u_\varepsilon \nabla v_\varepsilon) + \mu u_\varepsilon (1 - u_\varepsilon), \quad x \in \Omega, \ t > 0, \\
(v_\varepsilon)_t &= \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon, \quad x \in \Omega, \ t > 0, \\
\frac{\partial u_\varepsilon}{\partial \nu} &= 0, \quad x \in \partial \Omega, \ t > 0, \\
\left. u_\varepsilon \right|_{t=0} &= u_{\text{init}}(x), \quad \left. v_\varepsilon \right|_{t=0} = v_{\text{init}}(x), \quad x \in \Omega,
\end{align}

in a bounded convex domain $\Omega \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) with smooth boundary, where $\mu > 0$ and $\varepsilon > 0$ is to be small. We will compare its solutions to those of

\begin{align}
\left\{
\begin{array}{ll}
(u_t) = \Delta u + \mu (1 - u), & x \in \Omega, \ t > 0, \\
\frac{\partial u}{\partial \nu} = 0, & x \in \partial \Omega, \ t > 0, \\
\left. u \right|_{t=0} &= u_{\text{init}}(x), & x \in \Omega,
\end{array}
\right.
\end{align}

and where $\varepsilon > 0$ is to be small. We will compare its solutions to those of

Whereas large chemotaxis terms can cause significantly altered solution behaviour (cf. e.g. [177], Thm. 4.3), intuition leads to surmise that in presence of weak chemotactic effects, solutions to (17.1.4) should be close to solutions of (17.1.6). For example, as $\varepsilon \to 0$, one might expect convergence in some $L^p(\Omega)$ on each finite time interval. We will prove that the solutions converge \textit{uniformly} in $\Omega \times (0, \infty)$ and moreover show that this convergence is \textit{linear} in the chemotactic strength $\varepsilon$:

\textbf{Theorem 17.1.1.} Let $n \in \mathbb{N}$ and let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain with smooth boundary. Let $\mu > 0$ and suppose that $(u_{\text{init}}, v_{\text{init}})$ satisfies (17.1.5). Then there are $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in [0, \varepsilon_0]$, (17.1.4) has a global classical solution and that for all $\varepsilon \in (0, \varepsilon_0)$ the solution $u_\varepsilon$ to (17.1.4) and the solution $u$ to (17.1.6) satisfy

\begin{align}
\sup_{t > 0} \|u_\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^\infty(\Omega)} \leq C \varepsilon.
\end{align}

Having ensured global existence and some uniform bounds for $u_\varepsilon$ and $\nabla v_\varepsilon$ in Section 17.2, in Section 17.3 we will first take care of convergence of $u_\varepsilon$ to $u$ on finite time intervals $[0, T]$. The key to this proof lies in the derivation of a differential inequality for $\int_\Omega \omega_\varepsilon^{2k}$ for the difference $\omega_\varepsilon := u_\varepsilon - u$ and sufficiently large $k$ (Lemma 17.3.1). This inequality yields an $L^{2k}(\Omega)$-estimate for $\omega_\varepsilon$, which in Lemma 17.3.3 will be turned into a corresponding $L^\infty(\Omega)$-information. In preparation of a comparison from below (Lemma 17.4.5), we provide a uniform bound for $\|\Delta v_\varepsilon\|_{L^\infty(\Omega)}$ (Lemma 17.4.3) and a positive lower bound for $u_\varepsilon$ at some positive time (Lemma 17.4.4). The lower estimate obtained from Lemma 17.4.5 can then be inserted into the differential inequality from Lemma 17.3.1, dealing with the difference on the remaining interval $(T, \infty)$ and in Section 17.5 finally proving Theorem 17.1.1.
17.2. Global existence and uniform-in-ε boundedness

In this section we shall show global existence and uniform-in-ε boundedness of solutions to (17.1.4). Firstly we will recall the well-known result about local existence of solutions to (17.1.4) (see [195, Lemma 1.1], [199, Lemma 2.1]).

**Lemma 17.2.1.** Let $\varepsilon \in [0, \infty)$, $\mu > 0$, and suppose that $(u_{\text{init}}, v_{\text{init}})$ satisfies (17.1.5). Then there exist $T_{\text{max},\varepsilon} \in (0, \infty]$ and a classical solution $(u_\varepsilon, v_\varepsilon)$ of (17.1.4) in $\Omega \times (0, T_{\text{max},\varepsilon})$, which satisfy

\begin{equation}
\text{either } T_{\text{max},\varepsilon} = \infty \text{ or } \limsup_{t \uparrow T_{\text{max},\varepsilon}} \|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} = \infty.
\end{equation}

Moreover, this solution is uniquely determined in the class of function couples such that

\begin{align*}
&u_\varepsilon \in C(\overline{\Omega} \times [0, T_{\text{max},\varepsilon})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\text{max},\varepsilon})) \quad \text{and} \\
v_\varepsilon \in C^0(\overline{\Omega} \times [0, T_{\text{max},\varepsilon})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\text{max},\varepsilon})) \cap L^\infty((0, T_{\text{max},\varepsilon}); W^{1,\infty}(\Omega)).
\end{align*}

Throughout the sequel, we keep $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$, $\mu > 0$ and initial data $u_{\text{init}}$ and $v_{\text{init}}$ satisfying (17.1.5) fixed and, without loss of generality, assume $u_{\text{init}} \neq 0$. (If $u_{\text{init}} \equiv 0$, also $u \equiv 0$ and $u_\varepsilon \equiv 0$ for any $\varepsilon > 0$, and (17.1.7) trivially holds true.) Moreover, we let $T_{\text{max},\varepsilon}$ and $(u_\varepsilon, v_\varepsilon)$ be as given by Lemma 17.2.1. We also denote the solution of (17.1.6) by $u = u_0$.

To simplify notation we shall abbreviate the deviations from the nonzero homogeneous steady state by introducing

\begin{equation}
U_\varepsilon(x, t) := u_\varepsilon(x, t) - 1 \quad \text{and} \quad V_\varepsilon(x, t) := v_\varepsilon(x, t) - 1
\end{equation}

for $x \in \overline{\Omega}$ and $t > 0$. Then by straightforward computation it follows that $(U_\varepsilon, V_\varepsilon)$ solves

\begin{align}
(U_\varepsilon)_t &= \Delta U_\varepsilon - \varepsilon \nabla \cdot (u_\varepsilon \nabla V_\varepsilon) - \mu U_\varepsilon - \mu U_\varepsilon^2, \quad x \in \Omega, \ t > 0, \\
(V_\varepsilon)_t &= \Delta V_\varepsilon - V_\varepsilon + U_\varepsilon, \quad x \in \Omega, \ t > 0, \\
\frac{\partial u_\varepsilon}{\partial \nu} &= \frac{\partial v_\varepsilon}{\partial \nu} = 0, \quad x \in \partial \Omega, \ t > 0, \\
U_\varepsilon(x, 0) &= u_{\text{init}}(x) - 1, \ V_\varepsilon(x, 0) = v_{\text{init}}(x) - 1, \quad x \in \Omega.
\end{align}

We will prove global existence and boundedness of solutions to (17.1.4). For the pointwise comparison argument (cf. [199, Lemma 3.1]) used in this proof, convexity of the domain is essential.

**Lemma 17.2.2.** For any $\varepsilon \in [0, \frac{4\mu}{n})$, the solution of (17.1.4) exists globally. Moreover, there is $c_1 > 0$ such that

\begin{equation}
\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq c_1 e^{-t} + 1 + \frac{(\mu - 1)^2 + n\varepsilon}{4\mu - n\varepsilon}
\end{equation}

for all $t > 0$ and for all $\varepsilon \in [0, \frac{4\mu}{n})$. 

220
Proof. With $U_\varepsilon$ and $V_\varepsilon$ as defined in (17.2.2), we let

$$ z_\varepsilon(x,t) := U_\varepsilon(x,t) + \frac{\varepsilon}{2} |\nabla V_\varepsilon(x,t)|^2 \quad \text{for } x \in \overline{\Omega} \text{ and } t \in (0,T_{\text{max},\varepsilon}). $$

Then $z_\varepsilon$ satisfies

$$ (z_\varepsilon)_t - \Delta z_\varepsilon + z_\varepsilon = -\varepsilon |D^2 V_\varepsilon|^2 - \varepsilon u_\varepsilon \Delta V_\varepsilon - (\mu - 1) U_\varepsilon - \mu U_\varepsilon - \varepsilon |\nabla V_\varepsilon|^2 $$

$$ \leq \frac{n\varepsilon}{4} U_\varepsilon^2 + \frac{n\varepsilon}{2} U_\varepsilon + \frac{n\varepsilon}{4} - (\mu - 1) U_\varepsilon - \mu U_\varepsilon^2 $$

$$ \leq \frac{(\frac{n\varepsilon}{4} - \mu + 1)^2}{4 (\mu - \frac{n\varepsilon}{4})} + \frac{n\varepsilon}{4} $$

$$ = \frac{(\mu - 1)^2 + n\varepsilon}{4\mu - n\varepsilon} $$

for all $x \in \Omega$ and $t \in (0,T_{\text{max},\varepsilon})$, where we have used the condition $4\mu > n\varepsilon$ (for more detail, see [199, Lemma 3.1]). In order to derive an estimate for $z_\varepsilon$ itself from this, we note that since $\Omega$ is convex and $\frac{\partial v_\varepsilon}{\partial v} = 0$ on $\partial \Omega$, we have

$$ \frac{\partial |\nabla v_\varepsilon|^2}{\partial v} \leq 0 \quad \text{on } \partial \Omega $$

(see [116, Lemme 2.1]) and hence also $\frac{\partial v_\varepsilon}{\partial v} \leq 0$ on $\partial \Omega$. We define $y_{\varepsilon 0} := \|U_\varepsilon(\cdot,0)\|_{L^\infty(\Omega)} + \frac{n\varepsilon}{2}\|\nabla V_\varepsilon(\cdot,0)\|^2_{L^\infty(\Omega)} > 0$, and denote by $y_\varepsilon : [0,\infty) \to \mathbb{R}$ the function solving

$$ \begin{cases} 
 y'_\varepsilon(t) + y_\varepsilon(t) = \frac{(\mu-1)^2+n\varepsilon}{4\mu-n\varepsilon}, & t > 0, \\
 y_\varepsilon(0) = y_{\varepsilon 0}. 
 \end{cases} $$

By the comparison theorem we obtain that

$$ 0 \leq u_\varepsilon(\cdot,t) = U_\varepsilon(\cdot,t) + 1 \leq y(t) + 1 \leq c_1 e^{-t} + 1 + \frac{(\mu - 1)^2 + n\varepsilon}{4\mu - n\varepsilon}, $$

where $c_1 := \|U_\varepsilon(\cdot,0)\|_{L^\infty(\Omega)} + \frac{2\varepsilon}{n}\|\nabla V_\varepsilon(\cdot,0)\|^2_{L^\infty(\Omega)}$. In view of (17.2.1) we complete the proof, seeing that actually $T_{\text{max},\varepsilon} = \infty$. \hfill $\Box$

In the next lemma we aim at deriving a bound for $\nabla v_\varepsilon$. We restrict the admissible values for $\varepsilon$ to a smaller range than in Lemma 17.2.2 in order to establish the estimate independently of $\varepsilon$, in contrast to the right-hand side of (17.2.4).

Lemma 17.2.3. There exist $c_2 > 0$ and $\lambda_1 > 0$ such that

$$ \|\nabla v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq c_2 (1 + e^{-t} + e^{-(1+\lambda_1)t}) $$

for all $t \geq 0$ and all $\varepsilon \in [0,\frac{2\mu}{n})$.

Proof. We note that

$$ \frac{(\mu - 1)^2 + n\varepsilon}{4\mu - n\varepsilon} \leq \frac{(\mu - 1)^2 + 2\mu}{4\mu - 2\mu} = \frac{\mu^2 + 1}{2\mu} \quad \text{for all } \varepsilon \in \left[0, \frac{2\mu}{n}\right), $$

221
so that according to Lemma 17.2.2 the estimate
\[ \|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq c_1 e^{-t} + 1 + \frac{\mu^2 + 1}{2\mu} \leq c_1 + 1 + \frac{\mu^2 + 1}{2\mu} =: c_3 \]
holds for all \( t > 0 \) and all \( \varepsilon \in [0, \frac{2n}{m}) \). By means of a variation-of-constants representation for \( v \), we have
\[ \nabla v_\varepsilon(\cdot, t) = \nabla e^{t(\Delta - 1)} v_{\text{init}} + \int_0^t \nabla e^{(t-s)(\Delta - 1)} u_\varepsilon(\cdot, s) \, ds \quad \text{for all } t > 0. \]
Known smoothing estimates for the Neumann heat semigroup in \( \Omega \) (more precisely: the limit case \( p \to \infty \) in Lemma 2.2.2 (iii)) provide us with constants \( c_4 > 0 \) and \( c_5 > 0 \) such that
\[ \| \nabla e^{\tau(\Delta - 1)} \varphi \|_{L^\infty(\Omega)} \leq c_4 e^{-\lambda_1 \tau} \| \nabla \varphi \|_{L^\infty(\Omega)} \quad \text{for all } \tau > 0 \text{ and all } \varphi \in W^{1,\infty}(\Omega). \]
Accordingly,
\[ \| \nabla e^{(t-s)(\Delta - 1)} v_{\text{init}} \|_{L^\infty(\Omega)} \leq c_4 e^{-(1+\lambda_1)t} \| \nabla v_{\text{init}} \|_{L^\infty(\Omega)}. \]
Similarly employing Lemma 2.2.2, we gain \( c_5 > 0 \) such that
\[
\int_0^t \| \nabla e^{(t-s)(\Delta - 1)} u_\varepsilon(\cdot, s) \|_{L^\infty(\Omega)} \, ds \leq c_3 \int_0^t (t-s)^{-\frac{1}{2}} e^{-(1+\lambda_1)(t-s)} \| u_\varepsilon(\cdot, s) \|_{L^\infty(\Omega)} \, ds \\
\leq c_3 c_5 \int_0^t (t-s)^{-\frac{1}{2}} e^{-(1+\lambda_1)(t-s)} \, ds \\
\leq c_3 c_5 \int_0^\infty s^{-\frac{1}{2}} e^{-(1+\lambda_1)s} \, ds
\]
for all \( t > 0 \). Therefore we have (17.2.5) for all \( t > 0 \) and all \( \varepsilon \in [0, \frac{2n}{m}) \), with an obvious definition of \( c_2 > 0 \).

17.3. Local-in-time convergence to the Fisher–KPP equation

In this section we shall prove the convergence of solutions of (17.1.4) to those of the Fisher-KPP equation (17.1.6) on some interval \([0, T]\). We will begin with the key ingradient of both the proof on finite and on eventual time intervals: a differential inequality that will first lead to an estimate of \( L^{2k}\)-norms of the difference \( \omega_\varepsilon \):

**Lemma 17.3.1.** Let \( k \geq 1 \) be an integer. Then there is \( c_6(k) > 0 \) such that for all \( \varepsilon \in [0, \frac{2n}{m}) \)
\[ \omega_\varepsilon := u_\varepsilon - u \quad \text{in } \Omega \times (0, \infty) \]
satisfies
\[
(17.3.1) \quad \frac{d}{dt} \int_\Omega \omega_\varepsilon^{2k} \leq \varepsilon^{2k} c_6(k) + \mu k \int_\Omega \omega_\varepsilon^{2k} + 2k\mu \int_\Omega \omega_\varepsilon^{2k}(1 - u_\varepsilon - u) \]
on \((0, \infty)\).
Proof. We immediately see that \( \omega_\varepsilon \) satisfies
\[
(\omega_\varepsilon)_t = \Delta \omega_\varepsilon - \varepsilon \nabla \cdot (u_\varepsilon \nabla v_\varepsilon) + \mu \omega_\varepsilon - \mu (u_\varepsilon + u) \omega_\varepsilon \quad \text{in } \Omega \times (0, \infty).
\]

Multiplying the above equation by \( \omega_\varepsilon^{2k-1} \) and integrating over \( \Omega \), we can calculate
\[
\frac{d}{dt} \int_\Omega \omega_\varepsilon^{2k} = -2k(2k-1) \int_\Omega \omega_\varepsilon^{2k-2} |\nabla \omega_\varepsilon|^2 + \varepsilon 2k(2k-1) \int_\Omega \omega_\varepsilon^{2k-2} u_\varepsilon \nabla \omega_\varepsilon \cdot \nabla v_\varepsilon
\]
\[
+ 2k \mu \int_\Omega (1 - u_\varepsilon - u) \omega_\varepsilon^{2k}
\]
on \( (0, \infty) \). Two successive applications of Young’s inequality reveal that with some \( \varepsilon > 0 \) we have
\[
(17.3.2) \quad \varepsilon 2k(2k-1) \int_\Omega \omega_\varepsilon^{2k-2} u_\varepsilon \nabla \omega_\varepsilon \cdot \nabla v_\varepsilon \leq 2k(2k-1) \int_\Omega \omega_\varepsilon^{2k-2} |\nabla \omega_\varepsilon|^2
\]
\[
+ \frac{\varepsilon^2 2k(2k-1)}{2} \int_\Omega \omega_\varepsilon^{2k-2} u_\varepsilon^2 |\nabla v_\varepsilon|^2
\]
\[
\leq 2k(2k-1) \int_\Omega \omega_\varepsilon^{2k-2} |\nabla \omega_\varepsilon|^2 + k \mu \int_\Omega \omega_\varepsilon^{2k}
\]
\[
+ c_\gamma(k) \varepsilon 2k \int_\Omega u_\varepsilon^{2k} |\nabla v_\varepsilon|^{2k}
\]
for all \( \varepsilon \in [0, \frac{2n}{n}) \) and on \( (0, \infty) \), where thanks to Lemma 17.2.2 and Lemma 17.2.3 we may further estimate
\[
(17.3.3) \quad c_\gamma(k) \int_\Omega u_\varepsilon^{2k} |\nabla v_\varepsilon|^{2k} \leq c_6(k) \quad \text{on } (0, \infty)
\]
for some \( c_6(k) > 0 \) independent of \( \varepsilon \in [0, \frac{2n}{n}) \), so that a combination of (17.3.2), (17.3.3) and (17.3.4) finally yields (17.3.1). \( \Box \)

Corollary 17.3.2. Let \( k \geq 1 \) be an integer and let \( c_6(k) \) be as in Lemma 17.3.1. Then, for any \( \varepsilon \in [0, \frac{2n}{n}) \), the function \( u_\varepsilon \) satisfies
\[
\|u_\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^{2k}(\Omega)} \leq \sqrt{n} \varepsilon c_6(k) \varepsilon e^{\frac{3n}{2} t} \quad \text{for any } t > 0.
\]

Proof. Nonnegativity of \( u_\varepsilon \) and \( u \) together with Lemma 17.3.1 show that
\[
\frac{d}{dt} \int_\Omega \omega_\varepsilon^{2k} \leq 3k \mu \int_\Omega \omega_\varepsilon^{2k} + c_6(k) \varepsilon 2k \quad \text{on } (0, \infty),
\]
which upon an ODE comparison argument and radication readily results in the Corollary, due to the fact that \( \omega_\varepsilon(\cdot, 0) \equiv 0 \). \( \Box \)

We employ semigroup techniques to upgrade these estimates to uniform bounds.

223
Lemma 17.3.3. Let $q > \frac{n}{2}$ and $p > n$. There is $c_8 > 0$ such that for any $T > 0$ and any $z_0 \in C(\Omega)$, $f \in C((0,T); C^1(\partial\Omega; \mathbb{R}^N))$, $g \in C(\Omega \times (0,T))$, the solution $z$ of

$$z(\cdot,0) = z_0 \text{ in } \Omega, \quad \partial_\nu z|_{\partial\Omega} = 0, \quad z_t = \Delta z + \nabla \cdot f + g \text{ in } \Omega \times (0,T)$$

for all $t \in (0,T)$ satisfies

$$\|z(\cdot,t)\|_{L^\infty(\Omega)} \leq c_8 \left( \|f\|_{L^\infty((0,T); L^p(\Omega))} + \|g\|_{L^\infty((0,T); L^q(\Omega))} + \|z_0\|_{L^\infty(\Omega)} + \|\zeta\|_{L^\infty((0,T); L^q(\Omega))} \right).$$

Proof. Aided by $L^p$-$L^q$ estimates for the heat semigroup similar to those in Lemma 2.2.2 we let $c_9 > 0$ and $c_{10} > 0$ be such that

$$\|e^{\Delta t}w\|_{L^\infty(\Omega)} \leq c_9(1 + t^{-\frac{n}{2p}})\|w\|_{L^q(\Omega)} \quad \text{for all } w \in L^q(\Omega)$$

and

$$\|e^{\Lambda \nabla} \varphi\|_{L^\infty(\Omega)} \leq c_{10}(1 + t^{-\frac{n}{2p}})e^{-\lambda_t t}\|\varphi\|_{L^p(\Omega)} \quad \text{for all } \varphi \in L^p(\Omega; \mathbb{R}^n).$$

Then for $t \in (0,2] \cap (0,T)$ we obtain

$$\|z(\cdot,t)\|_{L^\infty(\Omega)} \leq \|z_0\|_{L^\infty(\Omega)} + c_9 \int_0^t (1 + (t-s)^{-\frac{n}{2p}})e^{-\lambda_1 (t-s)}\|f(\cdot,s)\|_{L^p(\Omega)} ds$$

$$+ c_{10} \int_0^t (1 + (t-s)^{-\frac{n}{2p}})\|g(\cdot,s)\|_{L^q(\Omega)} ds$$

$$\leq \left( 1 + c_{10} \int_0^t (1 + t^{-\frac{n}{2p}})e^{-\lambda_1 t} d\tau + c_9 \int_0^t (1 + t^{-\frac{n}{2p}}) d\tau \right) \cdot \left( \|z_0\|_{L^\infty(\Omega)} + \|f\|_{L^\infty((0,T); L^p(\Omega))} + \|g\|_{L^\infty((0,T); L^q(\Omega))} \right).$$

For $t \in (2, \infty) \cap (0,T)$, on the other hand, we have

$$z(\cdot,t) = e^{\Delta z(\cdot,t-1)} + \int_0^1 e^{(1-s)\Delta \nabla} \cdot f(\cdot, t-1+s) ds + \int_0^1 e^{(1-s)\Delta} g(\cdot, t-1+s) ds$$

and hence may estimate

$$\|z(\cdot,t)\|_{L^\infty(\Omega)} \leq 2c_9 \|z(\cdot, t-1)\|_{L^p(\Omega)}$$

$$+ \int_0^1 c_9(1 + (1-s)^{-\frac{n}{2p}})e^{-\lambda_1 (1-s)}\|f(\cdot, t-1+s)\|_{L^p(\Omega)} ds$$

$$+ \int_0^1 c_{10}(1 + (1-s)^{-\frac{n}{2p}})\|g(\cdot, t-1+s)\|_{L^q(\Omega)} ds$$

$$\leq \left( 2c_9 + \int_0^1 c_{10}(1 + t^{-\frac{n}{2p}})e^{-\lambda_1 t} d\tau + \int_0^1 c_9(1 + t^{-\frac{n}{2p}}) d\tau \right) \cdot \left( \|z\|_{L^\infty((0,T); L^q(\Omega))} + \|f\|_{L^\infty((0,T); L^p(\Omega))} + \|g\|_{L^\infty((0,T); L^q(\Omega))} \right).$$

A consequence for the model under consideration is the following.
Corollary 17.3.4. Let \( k > \frac{n}{2} \) be an integer. There is \( c_{11} > 0 \) such that for any \( \varepsilon \in [0, \frac{2n}{n}] \) and any \( t > 0 \)

\[
\| u_\varepsilon(\cdot, t) - u(\cdot, t) \|_{L^\infty(\Omega)} \leq c_{11} (\varepsilon + \| u_\varepsilon - u \|_{L^\infty((0,t);L^{2k}(\Omega))}).
\]

Proof. The function \( \omega_\varepsilon := u_\varepsilon - u \) solves \( (\omega_\varepsilon)_t = \Delta \omega_\varepsilon + \varepsilon \nabla \cdot (u_\varepsilon \nabla v_\varepsilon) + \mu \omega_\varepsilon - \mu \omega_\varepsilon (u_\varepsilon + u) \) and hence from Lemma 17.3.3 we can take \( c_{12} > 0 \) such that

\[
\| \omega_\varepsilon(\cdot, \tau) \|_{L^\infty(\Omega)} \leq c_{12} \varepsilon \| u_\varepsilon \nabla v_\varepsilon \|_{L^\infty(\Omega)} + c_{12} \| \mu \omega_\varepsilon - \mu \omega_\varepsilon (u_\varepsilon + u) \|_{L^\infty((0,t);L^{2k}(\Omega))}
\]

\[
\leq c_{12} \varepsilon \| u_\varepsilon \nabla v_\varepsilon \|_{L^\infty(\Omega \times (0,t))} + c_{12} (\mu (1 + \| u \|_{L^\infty(\Omega)} + \| u_\varepsilon \|_{L^\infty(\Omega)}) + 1) \| \omega_\varepsilon \|_{L^\infty((0,t);L^{2k}(\Omega))}
\]

holds for any \( \varepsilon \in [0, \frac{2n}{n}] \) and any \( \tau \in (0, t) \). Using the uniform bounds on \( u, u_\varepsilon, \nabla v_\varepsilon \) that have been provided by Lemma 17.2.2 and Lemma 17.2.3, we obtain the conclusion.

Corollary 17.3.5. For any \( T > 0 \)

\[
u_\varepsilon \to u \quad \text{uniformly in } \overline{\Omega} \times [0, T] \text{ as } \varepsilon \searrow 0.
\]

Proof. This results from straightforward combination of Corollary 17.3.2 and Corollary 17.3.4.

17.4. Large time behaviour in both systems

Corollary 17.3.5 takes care of convergence of \( u_\varepsilon \) to \( u \) on finite time intervals. Seeing that \( u_\varepsilon \) and \( u \) both converge to 1 as \( t \to \infty \), we still have hope that they will be close to each other on intervals of the form \((T, \infty)\). We will, nevertheless, need such information in a much more quantitative form – and this is what we prepare in the present section. After recalling a well-known estimate for the Laplacian in \( \Omega \) supplemented with homogeneous Neumann boundary conditions, we obtain bounds for \( U_\varepsilon \) in the domain of some fractional power of this operator and of \( \Delta v_\varepsilon \) in \( L^\infty(\Omega) \) that, together with the uniform lower bound of \( u_\varepsilon(\cdot, t) \) for some positive time \( t \) (Lemma 17.4.4), can consequently be turned into precisely those quantitative lower bounds for \( u_\varepsilon \) and \( u \) we will need in the proof of Theorem 17.1.1 in Section 17.5.

We fix any number \( \hat{\mu} \in (0, \mu) \cap (0, 1) \) and given \( p > 1 \) let \( A = A_p \) denote the realization of the operator \(-\Delta + \hat{\mu}\) under homogeneous Neumann boundary condition in \( L^p(\Omega) \).

Lemma 17.4.1. \( A \) is sectorial and thus possesses closed fractional powers \( A^\eta \) for arbitrary \( \eta > 0 \), and the corresponding domains \( D(A^\eta) \) are known to have the embedding property

\[(17.4.1) \quad D(A^\eta) \hookrightarrow W^{2,\infty}(\Omega) \quad \text{if } 2\eta - \frac{n}{p} > 2.
\]

Moreover, if \( (e^{-tA})_{t \geq 0} \) denotes the corresponding analytic semigroup, then for each \( \eta > 0 \) there exists \( c_{13}(p, \eta) > 0 \) such that

\[(17.4.2) \quad \| A^\eta e^{-tA} \varphi \|_{L^p(\Omega)} \leq c_{13}(p, \eta) t^{-\eta} \| \varphi \|_{L^p(\Omega)}
\]

for all \( t > 0 \) and each \( \varphi \in L^p(\Omega) \).
Proof. See [68, Theorem 1.4.3 and Theorem 1.6.1].

Lemma 17.4.2. For all $p > 1$ and any $\eta \in (0, \frac{1}{2})$ there exists $c_{14} > 0$ such that

\begin{equation}
||A^\eta U_{\varepsilon}(\cdot, t)||_{L^p(\Omega)} \leq c_{14}
\end{equation}

holds for all $t \geq 2$ and all $\varepsilon \in [0, \frac{2n}{m})$, where $U_{\varepsilon} \equiv u_{\varepsilon} - 1$.

Proof. According to standard estimates for the Neumann heat semigroup (see e.g. [52, Lemma 3.3]), we can find $c_{15} > 0$ such that

\begin{equation}
||e^{\tau \Delta} \nabla \cdot \varphi||_{L^p(\Omega)} \leq c_{15}(1 + \tau^{-\frac{1}{2}})||\varphi||_{L^p(\Omega)}
\end{equation}

all $\tau > 0$ and any $\varphi \in C^1(\overline{\Omega}; \mathbb{R}^n)$ such that $\varphi \cdot \nu = 0$ on $\partial \Omega$. We represent $U_{\varepsilon}$ according to

$$U_{\varepsilon}(\cdot, t) = e^{(t-1)(\Delta-\mu)}U_{\varepsilon}(\cdot, 1) - \varepsilon \int_1^t e^{(t-s)(\Delta-\mu)} \nabla \cdot (u_{\varepsilon} \nabla v_{\varepsilon}(\cdot, s))ds$$

$$- \mu \int_1^t e^{(t-s)(\Delta-\mu)} U_{\varepsilon}^2(\cdot, s)ds$$

$$= e^{-(\mu-\delta)(t-1)}e^{-(t-1)\Delta}U_{\varepsilon}(\cdot, 1) - \varepsilon \int_1^t e^{-(\mu-\delta)(t-s)} e^{\frac{\mu-\delta}{2} \Delta} \nabla \cdot (u_{\varepsilon} \nabla v_{\varepsilon}(\cdot, s))ds$$

$$- \mu \int_1^t e^{-(\mu-\delta)(t-s)} e^{-(t-s)\Delta} U_{\varepsilon}^2(\cdot, s)ds$$

for all $t > 1$.

Hence we can calculate that

$$||A^\eta U_{\varepsilon}(\cdot, t)||_{L^p(\Omega)} \leq e^{-(\mu-\delta)(t-1)}||A^\eta e^{-(t-1)\Delta}U_{\varepsilon}(\cdot, 1)||_{L^p(\Omega)}$$

$$+ \varepsilon \int_1^t e^{-(\mu-\delta)(t-s)}||A^\eta e^{\frac{\mu-\delta}{2} \Delta} \nabla \cdot (u_{\varepsilon} \nabla v_{\varepsilon}(\cdot, s))||_{L^p(\Omega)}ds$$

$$+ \mu \int_1^t e^{-(\mu-\delta)(t-s)}||A^\eta e^{-(t-s)\Delta} U_{\varepsilon}^2(\cdot, s)||_{L^p(\Omega)}ds$$

for all $t > 1$.

Herein, (17.4.2) and (17.4.4) allow us to estimate

$$||A^\eta e^{-(t-1)\Delta}U_{\varepsilon}(\cdot, 1)||_{L^p(\Omega)} \leq c_{13}(p, \eta)(t-1)^{-\eta}||U_{\varepsilon}(\cdot, 1)||_{L^p(\Omega)}$$

and

$$||A^\eta e^{-(t-s)\Delta} U_{\varepsilon}^2(\cdot, s)||_{L^p(\Omega)}$$

$$\leq c_{13}(p, \eta)c_{15}\left(\frac{t-s}{2}\right)^{-\eta}\left(1 + \left(\frac{t-s}{2}\right)^{-\frac{1}{2}}\right)||u_{\varepsilon} \nabla v_{\varepsilon}(\cdot, s)||_{L^p(\Omega)}$$

as well as

$$||A^\eta e^{-(t-s)\Delta} U_{\varepsilon}^2(\cdot, s)||_{L^p(\Omega)} \leq c_{13}(p, \eta)(t-s)^{-\eta}||U_{\varepsilon}^2(\cdot, s)||_{L^p(\Omega)}ds$$

for $1 < s < t$. Since $\int_0^\infty e^{-(\mu-\delta)\sigma} \sigma^{-\eta}(1 + \sigma^{-\frac{1}{2}})d\sigma < \infty$ and $\int_0^\infty e^{-(\mu-\delta)\sigma} \sigma^{-\eta}d\sigma < \infty$, Lemma 17.2.2 and Lemma 17.2.3 establish the existence of $c_{14} > 0$ such that (17.4.3) holds for all $t \geq 2$.\[\square\]
An important consequence of this estimate is that it provides some control over \( \Delta v_\varepsilon \):

**Lemma 17.4.3.** There exists \( c_{16} > 0 \) such that for all \( \varepsilon \in [0, \frac{2n}{n}) \)

\[
(17.4.5) \quad \|\Delta v_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq c_{16}
\]

for all \( t \geq 3 \).

**Proof.** We fix an arbitrary \( \gamma \in (1, \frac{3}{2}) \) and then can choose positive numbers \( \eta \) and \( p \) such that

\[
\gamma - 1 < \eta < \frac{1}{2}, \quad p > \frac{n}{2(\gamma - 1)}.
\]

Then

\[
2\gamma - \frac{n}{p} > 2\gamma - 2(\gamma - 1) = 2.
\]

According to a variation-of-constants formula associated with the second equation in (17.2.3), we can write

\[
V_\varepsilon(t) = e^{(t-2)(\Delta-1)}V_\varepsilon(\cdot, 2) + \int_2^t e^{(t-s)(\Delta-1)}U_\varepsilon(\cdot, s) \, ds
\]

\[
= e^{-(1-\bar{\rho})(t-2)}e^{-(t-2)A}V_\varepsilon(2) + \int_2^t e^{-(1-\bar{\rho})(t-s)}e^{-(t-s)A}U_\varepsilon(\cdot, s) \, ds,
\]

for all \( t \geq 2 \), and hence (17.4.1) implies that

\[
\|V_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq c_{17}\|A^\gamma V_\varepsilon(\cdot, t)\|_{L^p(\Omega)}
\]

\[
\leq c_{17}e^{-(1-\bar{\rho})(t-2)}\|A^\gamma e^{-(t-2)A}V_\varepsilon(\cdot, 2)\|_{L^p(\Omega)}
\]

\[
+ c_{17} \int_2^t e^{-(1-\bar{\rho})(t-s)}\|A^\gamma e^{-(t-s)A}U_\varepsilon(\cdot, s)\|_{L^p(\Omega)} \, ds
\]

\[
= c_{17}e^{-(1-\bar{\rho})(t-2)}\|A^\gamma e^{-(t-2)A}V_\varepsilon(\cdot, 2)\|_{L^p(\Omega)}
\]

\[
+ c_{17} \int_2^t e^{-(1-\bar{\rho})(t-s)}\|A^{\gamma-\eta}e^{-(t-s)A}A^\eta U_\varepsilon(\cdot, s)\|_{L^p(\Omega)} \, ds
\]

with some \( c_{17} > 0 \). Using (17.4.3) and (17.4.2) to estimate

\[
\|A^{\gamma-\eta}e^{-(t-s)A}A^\eta U_\varepsilon(\cdot, s)\|_{L^p(\Omega)} \leq c_{13}(p, \gamma - \eta)(t-s)^{-(\gamma-\eta)}c_{14}
\]

for any \( 2 < s < t \), and taking into account the boundedness of \( c_{17}e^{-(1-\bar{\rho})(t-2)}\|A^\gamma e^{-(t-2)A}V_\varepsilon(\cdot, 2)\|_{L^p(\Omega)} \) on \((3, \infty)\) due to

\[
\|A^\gamma e^{-(t-2)A}V_\varepsilon(\cdot, 2)\|_{L^p(\Omega)} \leq c_{13}(p, \gamma)(t-2)^{-\gamma}\|V_\varepsilon(\cdot, 2)\|_{L^p(\Omega)}
\]

\[
\leq c_{13}(p, \gamma) \left( \|e^{2(\Delta-1)(v_{\text{init}} - 1)}\|_{L^p(\Omega)} + \int_0^2 \|e^{(2-s)(\Delta-1)}U_\varepsilon(\cdot, s)\|_{L^p(\Omega)} ds \right), \quad t \in (3, \infty),
\]

and Lemma 17.2.2, we obtain \( c_{16} > 0 \) such that (17.4.5) holds. \( \square \)
We will now establish lower bounds for \( u_\varepsilon(x, 3) \), which are independent of \( \varepsilon \). In contrast to the final assertion of Theorem 17.1.1, this lemma relies on our assumption \( u_{\text{init}} \neq 0 \).

**Lemma 17.4.4.** There exist \( c_{18} > 0 \) and \( \varepsilon_1 \in (0, \frac{2\mu}{n}) \) such that
\[
u_\varepsilon(x, 3) \geq c_{18}
\]
for all \( x \in \Omega \) and for all \( \varepsilon \in (0, \varepsilon_1) \).

**Proof.** We will use a contradiction argument. If there exist \((\varepsilon_j)_{j \in \mathbb{N}}\) with \( \lim_{j \to \infty} \varepsilon_j = 0 \) and \((x_j)_{j} \subset \Omega \) such that
\[
\lim_{j \to \infty} u_{\varepsilon_j}(x_j, 3) = 0,
\]
then we can take a subsequence \((x_{jk})_{k} \subset (x_j)_{j}\) satisfying that there exists \( x_0 \in \overline{\Omega} \) such that
\[
\lim_{k \to \infty} x_{jk} = x_0.
\]
Thanks to Corollary 17.3.5 and (17.4.6) we deduce
\[
u(x_0, 3) = \lim_{k \to \infty} u_{\varepsilon_{jk}}(x_{jk}, 3) = 0.
\]
However, we obtain by the strong maximum principle that
\[
u(x_0, 3) > 0,
\]
which is contradiction. \( \square \)

We are now able to estimate \( u_\varepsilon \) and \( u \) from below. The following lemma can be viewed as a one-sided quantitative statement on the long-term behaviour of \( u_\varepsilon \) and \( u \).

**Lemma 17.4.5.** Let \( \varepsilon_2 := \min \{\varepsilon_1, \frac{\mu}{2 c_{16}}\} \leq \frac{2\mu}{n} \) with \( \varepsilon_1 \) taken from Lemma 17.4.4 and \( c_{16} \) as defined in Lemma 17.4.3. Then there is \( c_{19} > 0 \) such that
\[
\inf_{x \in \Omega} \inf_{\varepsilon \in [0, \varepsilon_0]} u_\varepsilon(x, t) \geq \frac{1 - \varepsilon_0 c_{16}}{1 + c_{19} \varepsilon^{-\frac{1}{3}}} \quad \text{for all } t \geq 3.
\]

**Proof.** From the first equation in (17.1.4) and (17.4.5) we see that
\[
(u_\varepsilon)_t = \Delta u_\varepsilon - \varepsilon \nabla u_\varepsilon \cdot \nabla v_\varepsilon - \varepsilon u_\varepsilon \Delta v_\varepsilon + \mu u_\varepsilon - \mu u_\varepsilon^2 
\geq \Delta u_\varepsilon - \varepsilon \nabla u_\varepsilon \cdot \nabla v_\varepsilon + (\mu - c_{16} \varepsilon) u_\varepsilon - \mu u_\varepsilon^2 \quad \text{in } \Omega
\]
for \( t \geq 3 \). We choose \( y_0 \in (0, \frac{1}{2}) \) to be a positive number such that
\[
y_0 \leq \frac{\mu - \varepsilon_0 c_{16}}{\mu}
\]
228
and
\[ y_0 \leq \inf_{\varepsilon \in [0,\varepsilon_0]} \inf_{x \in \Omega} u_{\varepsilon}(x, 3) \]
(cf. Lemma 17.4.4) and put
\[ y_{\varepsilon}(t) := \frac{\mu - c_{16}\varepsilon}{\mu + (\mu - c_{16}\varepsilon - \mu)e^{-(\mu - c_{16}\varepsilon)(t-3)}}, \quad t \geq 3. \]
Then \( y_{\varepsilon} : [3, \infty) \to \mathbb{R} \) is the solution to the ODE initial value problem
\[
\begin{align*}
    y''_{\varepsilon}(t) &= (\mu - c_{16}\varepsilon)y_{\varepsilon}(t) - \mu y_{\varepsilon}(t)^2, \quad t > 3, \\
    y_{\varepsilon}(3) &= y_0.
\end{align*}
\]
Apparently, \( 0 < \frac{\mu - c_{16}\varepsilon}{y_0} - \mu \leq \frac{\mu}{y_0} \), and due to \( \varepsilon < \frac{\mu}{2c_{16}} \), we may employ the estimate
\[ e^{-(\mu - c_{16}\varepsilon)(t-3)} \leq e^{-\frac{\mu}{2}t + 2\varepsilon} \]
to see that
\[ y_{\varepsilon}(t) \geq \frac{\mu - c_{16}\varepsilon}{\mu + \frac{\mu}{y_0} e^{-\frac{\mu}{2}t} e^{-\frac{2}{2}t}} = \frac{1 - c_{16}\varepsilon}{1 + c_{19} e^{-\frac{\mu}{2}t}} \]
for all \( t \geq 3 \) if we let
\[ c_{19} := \frac{1}{y_0} e^{-\frac{3\varepsilon}{2}}. \]
In light of a comparison lemma we can deduce (17.4.7).

17.5. Global-in-time convergence: Proof of Theorem 17.1.1

With these explicit and quantitative uniform lower bounds for \( u_{\varepsilon} \) and \( u \), everything has been prepared to revisit the differential inequality of Lemma 17.3.1 and turn our attention to the proof of Theorem 17.1.1. In fact, we only have to show the following:

**Lemma 17.5.1.** There are \( c_{20} > 0 \) and \( \varepsilon_0 > 0 \) such that
\[ \|u_{\varepsilon}(\cdot, t) - u(\cdot, t)\|_{L^\infty(\Omega)} \leq c_{20}\varepsilon \quad \text{for any } t \in (0, \infty) \text{ and any } \varepsilon \in [0, \varepsilon_0). \]

**Proof.** We let \( k > \frac{n}{4} \) be an integer and with \( \varepsilon_2 \) as in Lemma 17.4.5 and \( c_{16} \) taken from Lemma 17.4.3 we set
\[ \varepsilon_0 := \min \left\{ \varepsilon_2, \frac{\mu}{8c_{16}} \right\}. \]
In accordance with Lemma 17.4.5, we then choose \( T > 0 \) such that on \( \Omega \times (T, \infty) \)
\[ u > \frac{7}{8} \quad \text{and} \quad u_{\varepsilon} > \frac{7}{8} \quad \text{for any } \varepsilon \in [0, \varepsilon_0). \]
By Corollary 17.3.2 and Corollary 17.3.4 there is \( c_{21} > 0 \) such that
\[ \|u_{\varepsilon} - u\|_{L^\infty(\Omega \times (0,T))} \leq c_{21}\varepsilon \quad \text{for all } \varepsilon \in [0, \varepsilon_0). \]
Moreover, Lemma 17.3.1 ensures that (with $c_6(k)$ as defined there)
\[
\frac{d}{dt} \int_\Omega \omega^2 k \leq \varepsilon^{2k} c_6(k) + \mu k \int_\Omega \omega^2 k + 2\mu k \int_\Omega (1 - u - u_e) \omega^2 k
\]
\[
\leq \varepsilon^{2k} c_6(k) - \frac{\mu k}{2} \int_\Omega \omega^2 k
\]
for all $\varepsilon \in [0, \varepsilon_0), t \in (T, \infty)$, where we have used that
\[
1 - u - u_e < 1 - \frac{7}{8} - \frac{7}{8} = -\frac{3}{4} \quad \text{on} \quad \Omega \times (T, \infty).
\]
Therefore,
\[
\int_\Omega \omega^2 k(\cdot, t) \leq \int_\Omega \omega^2 k(\cdot, T) e^{-\frac{\mu k}{2}(t-T)} + \frac{2c_6(k)}{\mu k} \varepsilon^{2k} \leq \left( c_{21}^2 + \frac{2c_6(k)}{\mu k} \right) \varepsilon^{2k}
\]
for all $t > T$ and all $\varepsilon \in [0, \varepsilon_0)$, and hence we have found $c_{22} > 0$ such that
\[
\|u_e(\cdot, t) - u(\cdot, t)\|_{L^2_\varepsilon(\Omega)} \leq c_{22} \varepsilon \quad \text{for all} \quad t > T, \varepsilon \in [0, \varepsilon_0).
\]
A further application of Corollary 17.3.4 shows that hence for some $c_{23} > 0$
\[
\|u_e(\cdot, t) - u(\cdot, t)\|_{L^\infty(\Omega)} \leq c_{23} \varepsilon \quad \text{for all} \quad t > T, \varepsilon \in [0, \varepsilon_0).
\]
Together with (17.5.1), this proves the lemma and hence also Theorem 17.1.1. \hfill \qed
Chapter 18

Global existence and boundedness in a chemotaxis-haptotaxis system with signal-dependent sensitivity

18.1. Motivation and results

There are many important properties which support our active and comfortable life. One of these important properties is *taxis*. Taxis is the property such that species react some stimuli and move towards, or away from, these stimuli. This property helps us such as, e.g., species move towards their foods, and move away from poison for them. Usually taxis is classified by “which stimuli species react on”. The case that species react on a chemical substance is called *chemotaxis*. Chemotaxis is related to e.g., the movement of sperm, the migration of neurons or lymphocytes: Thus chemotaxis is closely related with not only the beginning but also the maintenance of our life. The system which describes the aggregation of the species by chemotaxis

\[ u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \mu u(1 - u), \quad \tau v_t = \Delta v - v + u, \]

where \( \chi, \mu > 0 \) and \( \tau = 0, 1 \), is called a *chemotaxis system* and is studied intensively: in the two-dimensional setting for all \( \chi, \mu > 0 \) global existence and boundedness were established by Osaki et. al. [152], Tello and Winkler [177]; in the higher-dimensional setting with sufficiently large \( \mu > 0 \) Winkler [195, 199] obtained global classical bounded solutions; related works can be found in a survey [5, Section 3] by Bellomo et. al; models with nonlinear diffusion terms, for instance flux limited diffusion, were studied by Bellomo and Winkler [6, 7]. Recently, Chaplain and Lolas [32] researched a tumor invasion, and proposed the *chemotaxis-haptotaxis* system with constant sensitivity

\[ u_t = \Delta u - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + \mu u(1 - u - w), \]
\[ \tau v_t = \Delta v - v + u, \]
\[ w_t = -vw, \]

where \( \chi, \xi, \mu > 0 \) and \( \tau = 0, 1 \). This system mainly consists of the chemotaxis \( \nabla \cdot (u \nabla v) \) and the *haptotaxis* \( \nabla \cdot (u \nabla w) \). In this system there are some results about global existence
and boundedness: In the case that $\tau = 0$ global existence and boundedness were shown by Tao and Winkler \cite{173}; in the case that $\tau = 1$ and $\Omega \subset \mathbb{R}^2$ for all $\chi, \xi, \mu > 0$ global existence and boundedness were obtained by Tao \cite{167}; in the case that $\tau = 1$, $\Omega \subset \mathbb{R}^3$ and $\mu > 0$ is large, global existence of bounded solutions was shown by Cao \cite{26}.

In summary, in the case of constant sensitivity some results related to the above chemotaxis-haptotaxis system are known; however, the chemotaxis-haptotaxis system with signal-dependent sensitivity, i.e., $\nabla \cdot (u \nabla v)$ is replaced with $\nabla \cdot (u^{\chi(v)} \nabla v)$, which determines the chemotactic power depending on the stimuli, has not been studied. In this chapter we consider the chemotaxis-haptotaxis system with signal-dependent sensitivity

\[
\begin{aligned}
 u_t &= \Delta u - \nabla \cdot (\chi(v) u \nabla v) - \xi \nabla \cdot (u \nabla w) + \mu u(1 - u - w), \quad x \in \Omega, \ t > 0, \\
v_t &= \Delta v - v + u, \quad x \in \Omega, \ t > 0, \\
w_t &= -vw, \quad x \in \Omega, \ t > 0, \\
\frac{\partial u}{\partial v} - \chi(v) u \frac{\partial w}{\partial v} - \xi u \frac{\partial w}{\partial v} = 0, \\
 u(x, 0) = u_0(x), \ v(x, 0) = v_0(x), \ w(x, 0) = w_0(x), \ x \in \Omega,
\end{aligned}
\]  

(18.1.1)

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ ($n \geq 3$) with smooth boundary $\partial \Omega$, $\nu$ is the outward normal vector to $\partial \Omega$, $\xi, \mu > 0$ are constants and the initial data are regular enough and satisfy a standard compatibility condition in the sense that

\[
\begin{aligned}
u_0 \in C(\overline{\Omega}) \text{ with } u_0 \geq 0 \text{ and } u_0 \neq 0, \quad v_0 \in W^{1,\infty}(\Omega) \text{ with } v_0 \geq 0, \\
w_0 \in C^{2+\alpha}(\overline{\Omega}) \text{ (for } \alpha \in (0, 1)) \text{ with } w_0 > 0 \text{ and } \frac{\partial w_0}{\partial v} \big|_{\partial \Omega} = 0.
\end{aligned}
\]  

(18.1.2)

Here $u, v$ and $w$ denote the density of cells, the concentration of matrix-degrading enzyme (MDE) and the density of extracellular matrix (ECM), respectively. In the case that $w = 0$ in (18.1.1), which namely is the chemotaxis system with signal-dependent sensitivity, global existence and boundedness were established under the condition that

\[
\exists p > n; \quad \chi'(s) + \sqrt{p} |\chi(s)|^2 \leq 0
\]  

(18.1.3)

for all $s > 0$ (cf. Chapter 3) or under the condition that

\[
0 \leq \chi(s) \leq \frac{K}{(a + s)^k} \quad \text{for all } s > 0
\]  

(18.1.4)

with some $a \geq 0$, $k > 1$ and $K > 0$ satisfying

\[
K < k(a + \eta)^{k-1} \sqrt{\frac{2}{n}}
\]  

(18.1.5)

for some constant $\eta \geq 0$ (Chapters 13, 14). Here we note that the conditions (18.1.3) and (18.1.4)–(18.1.5) are independent of $\mu > 0$; therefore we expect global existence and boundedness in (18.1.1) under a condition similar to (18.1.3) or (18.1.4)–(18.1.5) instead of largeness conditions for $\mu > 0$. The purpose of this chapter is to establish global existence and boundedness in (18.1.1) under some conditions only for $\chi$.  

232
To attain the goal of this chapter we will suppose that \( \chi \) satisfies the following conditions:

\begin{align}
(18.1.6) \quad \chi & \in C^{1+\theta}([0, \infty)) \cap L^1(0, \infty) \quad (0 < \exists \theta < 1), \quad \chi > 0, \\
(18.1.7) \quad \exists C_1 > 0; \quad \chi(s) s \leq C_1 \quad \text{for all } s \geq 0, \\
(18.1.8) \quad \exists p_0 \in (n, n+1) \quad \chi'(s) + \alpha_{p_0} |\chi(s)|^2 \leq 0 \quad \text{for all } s > 0,
\end{align}

where \( \alpha_{p_0} \) is a positive constant defined as

\begin{equation}
(18.1.9) \quad \alpha_p := \frac{\eta(p, \varepsilon_1)}{p \varepsilon_1 (\varepsilon_1 + 1)}
\end{equation}

with some \( \varepsilon_1 \in (0, 1) \), where \( \eta(p, \varepsilon_1) \) is a constant given later (see (18.3.5)). Now the main result reads as follows.

**Theorem 18.1.1.** Let \( n \geq 3 \) and let \( \xi > 0, \mu > 0 \). Assume that \( \chi \) satisfies \((18.1.6)-(18.1.8)\). Then for any \((u_0, v_0, w_0)\) fulfilling \((18.1.2), (18.1.1)\) possesses a unique global classical solution

\begin{align*}
&u \in C(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)), \\
v \in C(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)) \cap L^\infty_{\text{loc}}([0, \infty); W^{1,q}(\Omega)), \\
w \in C(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty))
\end{align*}

for some \( q > n \). Moreover, the solution \((u, v, w)\) is bounded uniformly-in-time:

\[\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,q}(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)} \leq C\]

for all \( t > 0 \) with some \( C > 0 \).

**Remark 18.1.1.** This theorem gives global existence and boundedness in the case that \( n \geq 3 \). On the other hand, in the case that \( n = 1, 2 \), we can prove global existence of bounded solutions to \((18.1.1)\) under the condition that

\[\chi \in C^{1+\theta}([0, \infty)) \cap W^{1,\infty}(0, \infty)\]

by using the same strategy as in the proof of [167, Theorem 1.1] (see Remark 18.3.1).

The strategy for the proof of Theorem 18.1.1 is to build the \( L^p \)-estimate for \( u \) with some \( p > n \). In order to establish the desired estimate we use the energy estimate for \( \int_\Omega u^p f(v, w)^{-r} \) with some smooth function \( f \) and some constant \( r > 0 \). This strategy is based on ideas in Chapter 3 which overcome difficulties of the signal-dependent sensitivity. In the case that \( w = 0 \) in \((18.1.1)\), considering the estimate for \( \int_\Omega u^p \varphi(v)^{-r} \) with some smooth function \( \varphi \), we can see the \( L^p \)-estimate for \( u \) (Chapter 3). However, we cannot give the same energy estimate for \( \int_\Omega u^p \varphi(v)^{-r} \) because there is a new unknown function \( w \). Moreover, in the case that \( \chi(s) \equiv \chi \) we have already obtained a useful estimate for \( \Delta w \) ([26, Lemma 2.2]). Unfortunately \((18.1.1)\) includes \( \chi(v) \), and so we cannot use such estimate. Thus we put

\[f(v, w) := \varphi(v) \cdot e^{Cw}\]
with some $C > 0$ and obtain from an application of the estimate for $\Delta w$ (see Lemma 18.2.2) that there is $C > 0$ such that

$$\int_{\Omega} u^p (f(v, w)^{-r} + 1) \leq C,$$

which entails the $L^p$-estimate for $u$.

The plan of this chapter is as follows. In Section 18.2 we collect basic facts which will be used later. Section 18.3 is devoted to proving global existence and boundedness.

18.2. Local existence and basic facts

We first state a result on local existence of classical solutions.

**Lemma 18.2.1.** Let $n \in \mathbb{N}$, $\xi > 0$ and $\mu > 0$. Assume that $\chi$ satisfies (18.1.6)–(18.1.8). Then for any $(u_0, v_0, w_0)$ satisfying (18.1.2), there is $T_{\text{max}} \in (0, \infty]$ such that (18.1.1) admits a unique classical solution

$$u \in C(\bar{\Omega} \times [0, T_{\text{max}}]) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\text{max}})),
$$

$$v \in C(\bar{\Omega} \times [0, T_{\text{max}}]) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\text{max}})) \cap L^{\infty}_{\text{loc}}([0, T_{\text{max}}]; W^{1,q}(\Omega)) \quad (q > n),$$

$$w \in C(\bar{\Omega} \times [0, T_{\text{max}}]) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\text{max}}))$$

such that

$$u > 0, \quad v \geq 0 \quad \text{and} \quad 0 < w \leq \|w_0\|_{L^\infty(\Omega)} \quad \text{for all } t \in (0, T_{\text{max}}).$$

Moreover,

$$\text{if } T_{\text{max}} < \infty, \quad \text{then } \limsup_{t \uparrow T_{\text{max}}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \tag{18.2.1}$$

**Proof.** The proof of local existence of classical solutions to (18.1.1) is based on a standard contraction mapping argument which can be found in [172].

Let $s_0 \in (0, T_{\text{max}})$. According to the regularity of the solution (Lemma 18.2.1), we note that there exists $M > 0$ such that

$$\|u(\cdot, s_0)\|_{L^\infty(\Omega)} + \|v(\cdot, s_0)\|_{L^\infty(\Omega)} + \|w(\cdot, s_0)\|_{W^{2,\infty}(\Omega)} \leq M. \tag{18.2.2}$$

Then an application of the lower estimate for $\Delta w$ ([26, Lemmas 2.2 and 2.3]) leads to the following lemma.

**Lemma 18.2.2.** Let $\xi > 0$ and assume that (18.1.2) holds. Then for any $p > 1$, $s_0 \in (0, T_{\text{max}})$, some $\varepsilon_2 \in (0, p)$, $\varepsilon_3 \in (0, p(1 - \varepsilon_1))$, the solution of (18.1.1) satisfies

$$p(p - 1)\xi \int_{\Omega} u^{p-1} \nabla u \cdot \nabla w \leq b_1 \int_{\Omega} u^p + \mu \varepsilon_2 \int_{\Omega} u^{p+1} + (p - 1)\varepsilon_3 \int_{\Omega} u^{p-2} |\nabla u|^2 + b_2 \int_{\Omega} u^{p+1}$$

for all $t \in (s_0, T_{\text{max}})$ and some $b_1 = b_1(p, M, \varepsilon_3, \xi), b_2 = b_2(p, M, \xi, \mu, \varepsilon_2) > 0$. 

234
\textbf{Proof.} Recalling the inequality in [26, Lemmas 2.2 and 2.3], we have already known that
\[
p(p - 1)\xi \int_{\Omega} w^{p-1} \nabla u \cdot \nabla w \leq \left( 3M\xi + \frac{1}{e}M\xi \right) p \int_{\Omega} w^p + Mp\xi \int_{\Omega} w^p v + 2Mp(p - 1)\xi \int_{\Omega} w^{p-1}|\nabla u|,
\]
where in accordance with (18.2.2), M is an upper bound for $\|w(\cdot, s_0)\|_{W^{2,\infty}(\Omega)}$. Hence the Young inequality yields
\[
p(p - 1)\xi \int_{\Omega} w^{p-1} \nabla u \cdot \nabla w \leq \left( 3M\xi + \frac{1}{e}M\xi \right) p \int_{\Omega} w^p + \mu\varepsilon_2 \int_{\Omega} w^{p+1} + b_2 \int_{\Omega} v^{p+1}
+ (p - 1)\varepsilon_3 \int_{\Omega} w^{p-2}|\nabla u|^2 + b_0 \int_{\Omega} w^p
\]
for some $\varepsilon_2 \in (0, p)$, $\varepsilon_3 \in (0, p(1 - \varepsilon_1))$, $b_0 = b_0(p, M, \varepsilon_3, \xi)$, $b_2 = b_2(p, M, \xi, \mu, \varepsilon_2) > 0$ and all $t \in (s_0, T_{\max})$. Thus by putting
\[
b_1(p, M, \varepsilon_3, \xi) := \left( 3M\xi + \frac{1}{e}M\xi \right) p + b_0(p, M, \varepsilon_3, \xi),
\]
we have (18.2.3). \hfill \Box

The following lemma holds a key in the proof of Theorem 18.1.1.

\textbf{Lemma 18.2.3.} Let $r \geq 1$ and let $(u, v, w)$ be a solution to (18.1.1). Then for all $\varepsilon > 0$ there exists $C > 0$ such that
\[
(18.2.4) \quad \int_{s_0}^{t} \int_{\Omega} e^{t} |v(x, s)|^r \, dx \, ds \leq \varepsilon \int_{s_0}^{t} \int_{\Omega} e^{t} |u(x, s)|^r \, dx \, ds + Ce^{t} + \varepsilon
\]
for all $t \in (s_0, T_{\max})$.

\textbf{Proof.} Let $t \in (s_0, T_{\max})$ and $r \geq 1$. Noting the compact embedding $W^{2,r}(\Omega) \hookrightarrow L^r(\Omega)$ and the continuous embedding $L^r(\Omega) \hookrightarrow L^1(\Omega)$, we obtain from the Ehrling lemma that for all $\bar{e} > 0$ there exists $C_1 > 0$ such that
\[
(18.2.5) \quad \|e^{\bar{e}}v\|_{L^r(\Omega)} \leq \bar{e}\|e^{\bar{e}}v\|_{W^{2,r}(\Omega)} + C_1\|e^{\bar{e}}v\|_{L^1(\Omega)} \quad \text{for all } s \in (s_0, t).
\]
Now from the standard elliptic regularity argument we can find $C_2 > 0$ such that
\[
(18.2.6) \quad \|e^{\bar{e}}v\|_{W^{2,r}(\Omega)} \leq C_2\|e^{\bar{e}}\Delta v\|_{L^r(\Omega)} + C_2\|e^{\bar{e}}v\|_{L^r(\Omega)} \quad \text{for all } s \in (s_0, t).
\]
Then a combination of (18.2.5) and (18.2.6) means that for all $\bar{e} > 0$ there exists $C_3 > 0$ such that
\[
(18.2.7) \quad \|e^{\bar{e}}v\|_{L^r(\Omega)} \leq \bar{e}\|e^{\bar{e}}\Delta v\|_{L^r(\Omega)} + C_3e^{\bar{e}}.
\]
Recalling from an application of the maximal Sobolev regularity ([26, Lemma 2.5]) that there is $C_4 > 0$ such that
\[
\int_{s_0}^{t} \int_{\Omega} \Delta v(x, s)|^r \, dx \, ds \leq C_4 \int_{s_0}^{t} \int_{\Omega} u(x, s)|^r \, dx \, ds + C_4,
\]
235
we can derive from (18.2.7) that

\[
\int_{s_0}^{t} \int_{\Omega} e^{\frac{s}{2}} |v(x, s)|^r \, dx ds \leq \tilde{c} \int_{s_0}^{t} \int_{\Omega} e^{\frac{s}{2}} |\Delta v(x, s)|^r \, dx ds + C_3 \int_{s_0}^{t} e^{\frac{s}{2}} \, ds
\]

\[
\leq \tilde{c} C_4 \int_{s_0}^{t} \int_{\Omega} e^{\frac{s}{2}} |u(x, s)|^r \, dx ds + \tilde{c} C_4 + \frac{2C_3}{r} e^{\frac{t}{2}}
\]

holds, which implies this lemma.

We finally introduce the following lemma which gives us a strategy to show global existence and boundedness.

**Lemma 18.2.4.** Let \( p > n \). If there exists \( m_1 > 0 \) such that

\[
\|u(\cdot, t)\|_{L^p(\Omega)} \leq m_1 \quad \text{for all } t \in (s_0, T_{\max}),
\]

then there exists \( m_2 > 0 \) satisfying

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq m_2 \quad \text{for all } t \in (s_0, T_{\max}).
\]

**Proof.** Combination of (18.2.8) and \( L^p-L^q \) estimates for the Neumann heat semigroup on bounded domains (see [196, Lemma 2.4]) implies that there is \( c_1 > 0 \) such that

\[
\|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq c_1 \quad \text{for all } t \in (s_0, T_{\max}).
\]

Hence a combination of (18.2.10) and an argument similar to that in [26, Lemma 3.5] leads to (18.2.9).

---

**18.3. Global existence and boundedness**

In this section we show global existence and boundedness in (18.1.1) (Theorem 18.1.1). In the following let \((u, v, w)\) be the solution of (18.1.1) on \([0, T_{\max}]\) as in Lemma 18.2.1. Thanks to Lemma 18.2.4, it is sufficient to show the \( L^p \)-estimate for \( u \) with some \( p > n \). We first recall the following elementary estimates for \( u, v \).

**Lemma 18.3.1.** Let \( \xi > 0 \) and \( \mu > 0 \). Assume that \( \chi \) satisfies (18.1.6)–(18.1.8) and \((u_0, v_0, w_0)\) satisfies (18.1.2). Then there exists \( C > 0 \) such that

\[
\int_{\Omega} u(\cdot, t) \leq C \quad \text{for all } t \in (0, T_{\max})
\]

and

\[
\int_{\Omega} v(\cdot, t) \leq C \quad \text{for all } t \in (0, T_{\max}).
\]

**Proof.** By simply integrating the first and second equations in (18.1.1) on \( \Omega \) and using the Hölder inequality \((\int_{\Omega} u)^2 \leq |\Omega| \int_{\Omega} u^2\) we see the \( L^1 \)-boundedness of \( u \), and then of \( v \).
In order to obtain the desired estimate for $u$ we introduce the function $f = f(v, w)$ by

$$f(v, w) = \exp\left\{ \int_0^v \chi(s) ds + \xi w \right\}.$$  

Now we shall show the following lemma.

**Lemma 18.3.2.** Let $\xi, \mu, r > 0$ and assume that $\chi$ satisfies (18.1.6)–(18.1.7). Then for all $p > 1$ and $\varepsilon_1 > 0$,

$$\frac{d}{dt} \int_\Omega u^p + \frac{d}{dt} \int_\Omega u^p f^{-r} \leq I_1 + I_2 + I_3 + I_4 + (\mu p + C_1 r) \int_\Omega u^p f^{-r} - \mu p \int_\Omega u^{p+1} f^{-r}$$

$$+ r \xi \int_\Omega u^p f^{-r} vw - (p-1)(p-p\varepsilon_1 - \varepsilon_3) \int_\Omega u^{p-2} |\nabla u|^2$$

$$+ (b_1 + \mu p) \int_\Omega u^p - \mu (p-\varepsilon_2) \int_\Omega u^{p+1} + b_2 \int_\Omega v^{p+1}$$

for all $t \in (s_0, T_{\text{max}})$, where

$$I_1 := -p(p-1)\varepsilon_1 \int_\Omega u^{p-2} f^{-r} |\nabla u|^2;$$

$$I_2 := p(p-1) \int_\Omega u^{p-1} f^{-r} \chi(v) f' \nabla u \cdot \nabla v,$$

$$I_3 := p \int_\Omega u^{p-1} f^{-r} \nabla \cdot (\nabla u - \chi(v) u \nabla v - \xi u \nabla w),$$

$$I_4 := -r \int_\Omega u^{p-1} f^{-r} \chi(v) \Delta u.$$

**Proof.** Testing the first equation in (18.1.1) by $u^{p-1}$ ($p > 1$) and integrating it over $\Omega$ with positivity of $u$ imply that for any $\varepsilon_1 \in (0, 1)$,

$$\frac{d}{dt} \int_\Omega u^p = -p(p-1) \int_\Omega u^{p-2} |\nabla u|^2 + p(p-1) \int_\Omega u^{p-1} \chi(v) \nabla u \cdot \nabla v$$

$$+ (p(p-1)\xi \int_\Omega u^{p-1} \nabla u \cdot \nabla w + \mu p \int_\Omega u^p (1 - u - w)$$

$$= -p(p-1)\varepsilon_1 \int_\Omega u^{p-2} |\nabla u|^2 - p(p-1)(1 - \varepsilon_1) \int_\Omega u^{p-2} |\nabla u|^2$$

$$+ p(p-1) \int_\Omega u^{p-1} \chi(v) \nabla u \cdot \nabla v + p(p-1)\xi \int_\Omega u^{p-1} \nabla u \cdot \nabla w$$

$$+ \mu p \int_\Omega u^p (1 - u - w).$$

Noticing from the definition of $f$ that $f^{-r} \leq 1$ holds, we derive from the nonnegativity of
that
\[
\frac{d}{dt} \int_\Omega u^p \leq -p(p-1)\varepsilon_1 \int_\Omega u^{p-2} f^{-\tau} |\nabla u|^2 - p(p-1)(1-\varepsilon_1) \int_\Omega u^{p-2} |\nabla u|^2 \\
+ p(p-1) \int_\Omega u^{p-1} \chi(v) \nabla u \cdot \nabla v + p(p-1)\xi \int_\Omega u^{p-1} \nabla u \cdot \nabla w \\
+ \mu p \int_\Omega u^p(1 - u).
\]

Denoting by $I_1$ and $I_2$ the first and third terms on the right-hand side as
\[
I_1 := -p(p-1)\varepsilon_1 \int_\Omega u^{p-2} f^{-\tau} |\nabla u|^2
\]
and
\[
I_2 := p(p-1) \int_\Omega u^{p-1} f^{-\tau} \chi(v) f^r \nabla u \cdot \nabla v,
\]
we can write as
\[
(18.3.1) \quad \frac{d}{dt} \int_\Omega u^p \leq I_1 + I_2 - p(p-1)(1-\varepsilon_1) \int_\Omega u^{p-2} |\nabla u|^2 \\
+ p(p-1)\xi \int_\Omega u^{p-1} \nabla u \cdot \nabla w + \mu p \int_\Omega u^p - \mu p \int_\Omega u^{p+1}.
\]

Similarly, we have from straightforward calculations that
\[
\frac{d}{dt} \int_\Omega u^p f^{-r} \\
= p \int_\Omega u^{p-1} f^{-r} \nabla \cdot (\nabla u - \chi(v)u\nabla v - \xi u\nabla w) + \mu p \int_\Omega u^p f^{-r}(1 - u - w) \\
- r \int_\Omega u^p f^{-r}(\chi(v)(\Delta v - v) + \xi(-vw)).
\]

Then denoting by $I_3$ and $I_4$ as
\[
I_3 := p \int_\Omega u^{p-1} f^{-r} \nabla \cdot (\nabla u - \chi(v)u\nabla v - \xi u\nabla w)
\]
and
\[
I_4 := -r \int_\Omega u^p f^{-r} \chi(v) \Delta v,
\]
we obtain from the nonnegativity of the solution that
\[
(18.3.2) \quad \frac{d}{dt} \int_\Omega u^p f^{-r} \leq I_3 + I_4 + \mu p \int_\Omega u^p f^{-r}(1 - u) \\
- r \int_\Omega u^p f^{-r}(-v + u)\chi(v) + r\xi \int_\Omega u^p f^{-r}vw.
\]
In light of (18.1.7), (18.3.1), (18.3.2), we have from Lemma 18.2.2 that
\[
\frac{d}{dt} \int_{\Omega} u^p + \frac{d}{dt} \int_{\Omega} u^p f^{-r}
\]
\[
\leq I_1 + I_2 + I_3 + I_4 - p(p-1)(1-\varepsilon_1) \int_{\Omega} u^{p-2} |\nabla u|^2
\]
\[
+ b_1 \int_{\Omega} u^p + \mu \varepsilon_2 \int_{\Omega} u^{p+1} + (p-1)\varepsilon_3 \int_{\Omega} u^{p-2} |\nabla u|^2
\]
\[
+ b_2 \int_{\Omega} u^{p+1} + \mu p \int_{\Omega} u^p - \mu p \int_{\Omega} u^{p+1}
\]
\[
+ \mu p \int_{\Omega} u^p f^{-r} - \mu p \int_{\Omega} u^{p+1} f^{-r} + C_1 r \int_{\Omega} u^p f^{-r} + r \xi \int_{\Omega} u^p f^{-r} w
\]
\[
\leq I_1 + I_2 + I_3 + I_4 + (\mu p + C_1 r) \int_{\Omega} u^p f^{-r} - \mu p \int_{\Omega} u^{p+1} f^{-r}
\]
\[
+ r \xi \int_{\Omega} u^p f^{-r} w - (p-1)(p(1-\varepsilon_1) - \varepsilon_3) \int_{\Omega} u^{p-2} |\nabla u|^2
\]
\[
+ (b_1 + \mu p) \int_{\Omega} u^p - \mu (p-\varepsilon_2) \int_{\Omega} u^{p+1} + b_2 \int_{\Omega} u^{p+1},
\]
which implies the end of the proof.

We next show the following lemma to obtain a differential inequality which derives the estimate for \( \int_{\Omega} u^p + \int_{\Omega} u^p f^{-r} \).

**Lemma 18.3.3.** Let \( \xi, \mu > 0 \) and assume that \( \chi \) satisfies
\[
\chi'(s) + \alpha_p |\chi(s)|^2 \leq 0 \quad \text{for all} \quad s > 0
\]
with some \( p \in (1, n+1) \), where \( \alpha_p \) is a positive constant defined as (18.1.9). Then there exists \( r = r(p) > 0 \) such that
\[
I_1 + I_2 + I_3 + I_4 \leq 0. \tag{18.3.3}
\]

**Proof.** Due to straightforward calculations, we obtain
\[
I_3 = -p \int_{\Omega} \nabla (u^{p-1} f^{-r}) \cdot (\nabla u - \chi(v) u \nabla v - \xi u \nabla w)
\]
\[
= -p(p-1) \int_{\Omega} u^{p-2} f^{-r} \nabla u \cdot (\nabla u - \chi(v) u \nabla v - \xi u \nabla w)
\]
\[
+ pr \int_{\Omega} u^{p-1} f^{-r} (\chi(v) \nabla v + \xi \nabla w) \cdot (\nabla u - \chi(v) u \nabla v - \xi u \nabla w)
\]
and
\[
I_4 = r \int_{\Omega} \nabla (u^p f^{-r} \chi(v)) \cdot \nabla v
\]
\[
= pr \int_{\Omega} u^{p-1} f^{-r} \chi(v) \nabla u \cdot \nabla v
\]
\[
- r^2 \int_{\Omega} u^p f^{-r} \chi(v) (\chi(v) \nabla v + \xi \nabla w) \cdot \nabla v + r \int_{\Omega} u^p f^{-r} \chi'(v) |\nabla v|^2.
\]
Therefore it follows that

\[ I_1 + I_2 + I_3 + I_4 = -p(p - 1)\varepsilon_1 \int_\Omega u^{p-2} f^{-r} |\nabla u|^2 + p(p - 1) \int_\Omega u^{p-1} f^{-r} \chi(v) f' \nabla u \cdot \nabla v - p(p - 1) \int_\Omega u^{p-2} f^{-r} |\nabla u|^2 + p(p - 1) \int_\Omega u^{p-1} f^{-r} \chi(v) \nabla u \cdot \nabla v + p(p - 1) \xi \int_\Omega u^{p-1} f^{-r} \nabla u \cdot \nabla w + pr \int_\Omega u^{p-1} f^{-r} \chi(v) \nabla u \cdot \nabla v + pr \int_\Omega u^{p-1} f^{-r} \nabla u \cdot \nabla w + pr \int_\Omega u^{p-1} f^{-r} \chi(v) \nabla u \cdot \nabla w - pr\xi \int_\Omega u^{p-1} f^{-r} \nabla u \cdot \nabla w - pr\xi \int_\Omega u^{p-1} f^{-r} \chi(v) |\nabla v|^2 - pr\xi \int_\Omega u^{p-1} f^{-r} \chi(v) |\nabla v|^2 - pr\xi \int_\Omega u^{p-1} f^{-r} \nabla u \cdot \nabla v - r^2 \int_\Omega u^{p-1} f^{-r} \nabla u \cdot \nabla v - r^2 \xi \int_\Omega u^{p-1} f^{-r} \chi(v) \nabla u \cdot \nabla v + r \int_\Omega u^{p-1} f^{-r} \chi'(v) |\nabla v|^2. \]

Noting by assumption that

\[ \chi'(v) \leq -\alpha_p |\chi(v)|^2, \]

we obtain

\[
I_1 + I_2 + I_3 + I_4 \leq \int_\Omega a_1 u^{p-2} f^{-r} |\nabla u|^2 + \int_\Omega a_2 u^{p-1} f^{-r} |\nabla u| |\nabla v| + \int_\Omega a_3 u^{p-1} f^{-r} |\nabla u| |\nabla w| + \int_\Omega a_4 u^{p-1} f^{-r} |\nabla v|^2 + \int_\Omega a_5 u^{p-1} f^{-r} |\nabla v| |\nabla w| + \int_\Omega a_6 u^{p-1} f^{-r} |\nabla w|^2 = \int_\Omega u^{p-1} f^{-r} (a_1 x^2 + a_2 x y + a_3 x z + a_4 y^2 + a_5 y z + a_6 z^2),
\]

where \( x, y, z \) and \( a_i \) \((i = 1, 2, 3, 4, 5, 6)\) are given as

\[
\begin{align*}
    x &= u^{-1} |\nabla u|, & y &= |\nabla v|, & z &= |\nabla w|, \\
    a_1 &= -p(p - 1)(1 + \varepsilon_1), & a_2 &= p((p - 1) + 2r + (p - 1) f'') \chi(v), & a_3 &= p((p - 1) + r) \xi, \\
    a_4 &= -r(p + r + \alpha_p) |\chi(v)|^2, & a_5 &= -r(2p + r) \xi \chi(v), & a_6 &= -pr \xi^2.
\end{align*}
\]

Thus our goal is to find some \( r > 0 \) such that

\[ a_1 x^2 + a_2 x y + a_3 x z + a_4 y^2 + a_5 y z + a_6 z^2 \leq 0. \]
We shall show (18.3.4) by using the Sylvester criterion. We first confirm that \(a_1a_6 - \frac{a_3^2}{4} > 0\) holds with some \(r > 0\). Indeed, we can see that

\[
a_1a_6 - \frac{a_3^2}{4} = p(p - 1)(1 + \varepsilon_1)(pr\xi^2) - p^2 \left(\frac{(p - 1) + r}{4}\right)^2 \xi^2
\]

\[
= p^2\xi^2\left(-\frac{r^2}{4} + (p - 1)(1 + \varepsilon_1)r - \frac{(p - 1)}{2}r - \frac{(p - 1)^2}{4}\right)
\]

\[
= p^2\xi^2\left(-\frac{r^2}{4} + (p - 1)\left(\frac{1}{2} + \varepsilon_1\right)r - \frac{(p - 1)^2}{4}\right)
\]

\[
= p^2\xi^2\left(-\frac{1}{4}(r - 2(p - 1)\left(\frac{1}{2} + \varepsilon_1\right))^2 + (p - 1)^2\varepsilon_1(\varepsilon_1 + 1)\right).
\]

Thus by choosing \(r = 2(p - 1)(\frac{1}{2} + \varepsilon_1)\) we can verify that \(a_1a_6 - \frac{a_3^2}{4} > 0\). We next confirm that

\[
A_3 := \begin{vmatrix}
a_1 & a_2 & a_3 \\
a_2 & a_6 & a_3 \\
a_3 & a_5 & a_4
\end{vmatrix} \leq 0.
\]

Straightforward calculations and the fact that \(a_1a_6 - \frac{a_3^2}{4} = p^2\xi^2(p - 1)^2\varepsilon_1(\varepsilon_1 + 1)\) when \(r = 2(p - 1)(\frac{1}{2} + \varepsilon_1)\) lead to

\[
A_3 = \left(a_1a_6 - \frac{a_3^2}{4}\right)a_4 + \frac{a_2a_3a_5}{4} - \frac{a_2^2a_6}{4} - \frac{a_1a_5}{4}
\]

\[
= -p^2\xi^2(p - 1)^3\varepsilon_1(\varepsilon_1 + 1)(1 + 2\varepsilon_1)(p + (p - 1)(1 + 2\varepsilon_1) + \alpha_p)|\chi(v)|^2
\]

\[
- \frac{p^2}{4}r((p - 1) + 2r + (p - 1)f')((p - 1) + r)(2p + r)\xi^2|\chi(v)|^2
\]

\[
+ \frac{p^2}{4}(p - 1) + 2r + (p - 1)f')^2(pr\xi^2)|\chi(v)|^2
\]

\[
+ \frac{p(p - 1)(1 + \varepsilon_1)}{4}r^2(2p + r)\xi^2|\chi(v)|^2.
\]

Now letting \(d := 1 + 2\varepsilon_1\) and noting that \(r = (p - 1)(1 + 2\varepsilon_1) = (p - 1)d\), we deduce that

\[
A_3 = -p^2(p - 1)^3\varepsilon_1(\varepsilon_1 + 1)d(p + (p - 1)d + \alpha_p)\xi^2|\chi(v)|^2
\]

\[
- \frac{p^2}{4}(p - 1)^3d(1 + 2d + f^{(p-1)d})(1 + d)(2p + (p - 1)d)\xi^2|\chi(v)|^2
\]

\[
+ \frac{p^2}{4}(p - 1)^3d(1 + 2d + f^{(p-1)d})\xi^2|\chi(v)|^2
\]

\[
+ \frac{p}{4}(p - 1)^3d^2(1 + \varepsilon_1)(2p + (p - 1)d)\xi^2|\chi(v)|^2
\]

\[
= p(p - 1)^3d\xi^2|\chi(v)|^2(-p\varepsilon_1(\varepsilon_1 + 1)\alpha_p + \psi(p, \varepsilon_1)),
\]

241
where \( \psi(p, \varepsilon) \) is given by
\[
\psi(p, \varepsilon) := -p\varepsilon_1(\varepsilon_1 + 1)(p + (p - 1)d)
- \frac{p}{4}(1 + 2d + f^{(p-1)d})(1 + d)(2p + (p - 1)d)
+ \frac{p^2}{4}(1 + 2d + f^{(p-1)d})^2
+ \frac{d}{4}(1 + \varepsilon_1)(2p + (p - 1)d)^2.
\]

Then since the boundedness of \( f(v, w) \)
\[
1 \leq f(v(x, t), w(x, t)) \leq k \quad \text{for all } x \in \Omega \text{ and all } t > 0
\]
with
\[
k = \exp \left\{ \int_0^\infty \chi(s)ds + \xi\|w_0\|_{L^\infty} \right\} > 1
\]
and the fact
\[
(p - 1)d \leq nd
\]
implication
(18.3.5) \[
\psi(p, \varepsilon_1) \leq -p\varepsilon_1(\varepsilon_1 + 1)(p + (p - 1)d)
- \frac{p}{4}(1 + 2d + 1)(1 + d)(2p + (p - 1)d)
+ \frac{p^2}{4}(1 + 2d + k^{nd})^2
+ \frac{d}{4}(1 + \varepsilon_1)(2p + (p - 1)d)^2
= -\varepsilon_1(\varepsilon_1 + 1)((1 + d)p - d)p
- \frac{1}{4}(1 + 2d + 1)(1 + d)((2 + d)p - d)p
+ \frac{1}{4}(1 + 2d + k^{nd})^2p^2 + \frac{d}{4}(1 + \varepsilon_1)((2 + d)p - d)^2
= -\varepsilon_1(\varepsilon_1 + 1)(1 + d)p^2 + \varepsilon_1(\varepsilon_1 + 1)dp - \frac{d + 1}{2}(1 + d)(2 + d)p^2
+ \frac{d + 1}{2}(1 + d)dp + \frac{1}{4}(1 + 2d + k^{nd})^2p^2
+ \frac{d}{4}(1 + \varepsilon_1)(2 + d)^2p^2 - \frac{d}{2}(1 + \varepsilon_1)(2 + d)p^2
+ \frac{d}{4}(1 + \varepsilon_1)d^2
=: \eta(p, \varepsilon_1),
\]
we can see from the definition of \( \alpha_p \) (see (18.1.9)) that
\[
A_3 \leq p(p - 1)^3|\chi(v)|^2\xi^2(1 + 2\varepsilon_1)(-p\varepsilon_1(\varepsilon_1 + 1)\alpha_p + \eta(p, \varepsilon_1)) = 0.
\]
Thus noticing that \( a_1 = -p(p - 1)(1 + \varepsilon_1) < 0 \), from the Sylvester criterion we have
(18.3.4), which yields (18.3.3).
Now we are in a position to show the desired $L^p$-estimate for $u$ with some $p > n$.

**Lemma 18.3.4.** Let $n \geq 3$ and let $\xi, \mu > 0$. Assume that $\chi$ satisfies (18.1.6)–(18.1.7) and

$$\chi'(s) + \alpha_p|\chi(s)|^2 \leq 0 \quad \text{for all } s > 0$$

with some $p \in (1, n + 1)$, where $\alpha_p$ is a positive constant defined as (18.1.9). Then there exists $C > 0$ such that

$$\|u(\cdot, t)\|_{L^p(\Omega)} \leq C \quad \text{for all } t \in (s_0, T_{\max}).$$

**Proof.** Lemmas 18.3.2 and 18.3.3 yield

\begin{equation}
\frac{d}{dt} \int_{\Omega} u^p + \frac{d}{dt} \int_{\Omega} u^p f^{-r} \\
\leq (\mu p + C_1 r) \int_{\Omega} u^p f^{-r} - \mu p \int_{\Omega} u^{p+1} f^{-r} + (b_1 + \mu p) \int_{\Omega} u^p \\
- (\mu p - \mu \varepsilon_2) \int_{\Omega} u^{p+1} - (p - 1)(p(1 - \varepsilon_1) - \varepsilon_3) \int_{\Omega} u^{p-2} |\nabla u|^2 \\
+ b_2 \int_{\Omega} u^{p+1} + r \xi \int_{\Omega} u^p f^{-r} v w
\end{equation}

with

$$r = 2(p - 1)\left(\frac{1}{2} + \varepsilon_1\right)$$

and some $C_1 > 0$. Noting that

$$w(x, t) \leq \|w_0\|_{L^\infty} \quad \text{for all } x \in \Omega \text{ and all } t > 0$$

and

$$f^{-r}(v(x, t), w(x, t)) \leq 1 \quad \text{for all } x \in \Omega \text{ and all } t > 0,$$

we derive from the Young inequality that

$$r \xi \int_{\Omega} u^p f^{-r} v w = (p - 1)(1 + 2\varepsilon_1) \xi \int_{\Omega} u^p f^{-r} v w \\
\leq \varepsilon_4 \int_{\Omega} u^{p+1} f^{-r} + \ell \int_{\Omega} u^{p+1}$$

with some $\varepsilon_4 \in (0, \mu p)$ and $\ell = \ell(\xi, \|w_0\|_{L^\infty}, p, \varepsilon_1, \varepsilon_4) > 0$. Thus a combination of (18.3.6) and the definitions of $\varepsilon_2, \varepsilon_3, \varepsilon_4$ enables us to see that

\begin{equation}
\frac{d}{dt} \int_{\Omega} u^p + \frac{d}{dt} \int_{\Omega} u^p f^{-r} \\
\leq c_1 \int_{\Omega} u^p f^{-r} - c_2 \int_{\Omega} u^{p+1} f^{-r} + c_3 \int_{\Omega} u^p - c_4 \int_{\Omega} u^{p+1} + c_5 \int_{\Omega} u^{p+1} \\
\leq c_6 \int_{\Omega} u^p(f^{-r} + 1) - c_7 \int_{\Omega} u^{p+1}(f^{-r} + 1) + c_5 \int_{\Omega} u^{p+1}
\end{equation}
with some $c_1, c_2, c_3, c_4, c_5, c_6, c_7 > 0$. Here we put
\[ y_q(t) := \int_\Omega |u(\cdot, t)|^q (f^{-r}(v(\cdot, t), w(\cdot, t)) + 1) \]
for $q \geq 1$ and $t \in (s_0, T_{\text{max}})$. Noting from the Hölder inequality, the Young inequality and
\[ f^{-r}(v, w) \leq 1 \text{ on } \Omega \times (s_0, T_{\text{max}}) \]
that there is $c_8 > 0$ satisfying
\[ \left( c_6 + \frac{p+1}{2} \right) y_p \leq \frac{c_7}{2} y_{p+1} + c_8 \text{ on } (s_0, T_{\text{max}}), \]
we infer that
\[
\begin{align*}
(18.3.8) \quad \frac{d}{dt} y_p(t) &\leq c_6 y_p(t) - c_7 y_{p+1}(t) + c_5 \int_\Omega |v(\cdot, t)|^{p+1} \\
&= -\frac{p+1}{2} y_p(t) + \left( c_6 + \frac{p+1}{2} \right) y_p(t) - c_7 y_{p+1}(t) + c_5 \int_\Omega |v(\cdot, t)|^{p+1} \\
&\leq -\frac{p+1}{2} y_p(t) - \frac{c_7}{2} y_{p+1}(t) + c_8 + c_5 \int_\Omega |v(\cdot, t)|^{p+1}
\end{align*}
\]
for all $t \in (s_0, T_{\text{max}})$ with some $c_8 > 0$. Then thanks to Lemma 18.2.3 with $\varepsilon \in (0, \frac{c_7}{2c_5})$ and the relation
\[ \int_\Omega u^{p+1} \leq y_{p+1}, \]
for all $t \in (s_0, T_{\text{max}})$ integrating (18.3.8) over $(s_0, t)$ implies that
\[
\begin{align*}
e^{\frac{p+1}{2} t} y_p(t) &\leq e^{\frac{p+1}{2} s_0} y_p(s_0) - \frac{c_7}{2} \int_{s_0}^t e^{\frac{p+1}{2} s} y_{p+1}(s) \, ds + c_8 \int_{s_0}^t e^{\frac{p+1}{2} s} \, ds \\
&\quad + c_5 \left( \varepsilon \int_{s_0}^t \int_\Omega e^{\frac{p+1}{2} |u(x, s)|^{p+1}} \, dx \, ds + c_9 e^{\frac{p+1}{2} t} + \varepsilon \right) \\
&\leq e^{\frac{p+1}{2} s_0} y_p(s_0) - \left( \frac{c_7}{2} - \varepsilon c_5 \right) \int_{s_0}^t e^{\frac{p+1}{2} s} y_{p+1}(s) \, ds + \frac{2c_8}{p+1} e^{\frac{p+1}{2} t} + c_9 e^{\frac{p+1}{2} t} \\
&\quad + \varepsilon c_5
\end{align*}
\]
with some $c_9 > 0$. Thus we can verify that
\[
y_p(t) \leq e^{-\frac{p+1}{2} (t-s_0)} y_p(s_0) + \frac{2c_8}{p+1} + c_9 e^{-\frac{p+1}{2} t} + c_5 e^{-\frac{p+1}{2} t}
\]
\[
\leq c_{10}
\]
for all $t \in (s_0, T_{\text{max}})$ with some $c_{10} > 0$, which concludes the proof. \qed
Remark 18.3.1. In the case that $n \geq 3$, in order to obtain the $L^p$-estimate for $u$, we use the energy estimate for $\int_\Omega u^p \varphi(v)$ with some function $\varphi$. On the other hand, in the case that $n = 2$, by considering the energy estimate for $\int_\Omega u \log u + \int_\Omega j \nabla v_j^2$ we can attain the $L^p$-estimate for $u$. Here we give a short proof. Since a combination of the Gagliardo–Nirenberg inequality and the $L^2$-estimate for $\nabla v$ (from [167, (3.11)] with the boundedness of $\int_0^{T_{\text{max}}} \int_\Omega u^2$ for all $t \in (0, T_{\text{max}} - \tau)$ with $\tau := \min\{1, \frac{1}{2}T_{\text{max}}\}$) yields

$$\int_\Omega |v|^4 \leq c_1 \left( \int_\Omega |\nabla v|^2 + 1 \right)$$

with some $c_1 > 0$, integration by parts and the Young inequality imply that

$$\langle 18.3.9 \rangle \quad \int_\Omega \chi(v) \nabla u \cdot \nabla v = -\int_\Omega \chi'(v) u |\nabla v|^2 - \int_\Omega \chi(v) u \Delta v$$

$$\leq c_1 \|\chi'\|_{L^\infty(\Omega)}^2 \int_\Omega u^2 + \frac{1}{4c_1} \int_\Omega |\nabla v|^4 + \|\chi\|_{L^\infty(\Omega)} \int_\Omega u^2 + \frac{1}{4} \int_\Omega |\Delta v|^2$$

$$\leq (c_1 \|\chi'\|_{L^\infty(\Omega)}^2 + \|\chi\|_{L^\infty(\Omega)}^2) \int_\Omega u^2 + \frac{1}{2} \int_\Omega |\Delta v|^2 + \frac{1}{4}$$

on $(0, T_{\text{max}})$. Therefore noting

$$\frac{d}{dt} \int_\Omega u \log u + \int_\Omega \frac{|\nabla u|^2}{u} = \int_\Omega \chi(v) \nabla u \cdot \nabla v + \xi \int_\Omega \nabla u \cdot \nabla w$$

$$+ \mu \int_\Omega u (1 + \log u) (1 - u - w),$$

and applying (18.3.9) to the first term on the right-hand-side of the above identity, from arguments similar to those in the proof of [167, Lemma 3.3] we can find a constant $c_2 > 0$ such that

$$\int_\Omega u \log u + \int_\Omega |\nabla v|^2 \leq c_2 \quad \text{in} \quad (0, T_{\text{max}}).$$

Thus due to a generalized Gagliardo–Nirenberg inequality, the boundedness of the energy function leads to existence of $c_3 > 0$ such that

$$\|u\|_{L^\infty(0, T_{\text{max}}; L^2(\Omega))} \leq c_3$$

and

$$\|\nabla v\|_{L^\infty(0, T_{\text{max}}; L^4(\Omega))} \leq c_3.$$
Thanks to this boundedness, from a standard testing argument (see [167, Lemma 3.10]) we can attain that
\[ \|u\|_{L^\infty(0,T_{\max};L^p(\Omega))} \leq c_5 \]
with some \(c_5 > 0\) and \(p > 2\). Here we note that these arguments do not work in the case that \(n \geq 3\); because of lacking the generalized Gagliardo–Nirenberg inequality, we could not obtain the estimate for \(\|u\|_{L^\infty(0,T_{\max};L^n(\Omega))}\) from the boundedness of the energy function. As we mentioned before, in the case that \(n = 2\) the estimate for the energy function yields the \(L^p\)-estimate for \(u\). On the other hand, in the case that \(n = 1\), we do not have to use an energy function; since we have already established the boundedness of \(\|u\|_{L^\infty(0,T_{\max};L^1(\Omega))}\), arguments similar to those in the case that \(n = 2\) lead to \(\|u\|_{L^\infty(0,T_{\max};L^p(\Omega))}\) for some \(p > 1\).

**Proof of Theorem 18.1.1.** Under the condition that \(\chi\) satisfies (18.1.6)–(18.1.8), a combination of Lemmas 18.2.4 and 18.3.4, along with (18.2.1) directly leads to Theorem 18.1.1. \(\Box\)
Chapter 19

Singular sensitivity in a Keller–Segel–fluid system

19.1. Introduction

19.1.1. Chemotaxis–fluid models

One of the assumptions underlying many mathematical models dealing with chemotaxis (i.e. the partially directed movement of cells in the presence of a chemical signal substance) is that – apart from response to or production of the signal – interaction with the environment can be neglected. For transport effects in liquid surroundings this may be justified in the case of single bacteria, but should no longer be assumed in presence of a larger number of cells, as experimentally shown in [38]. Accordingly, inspired by the model suggested in [179] and starting from the construction of weak solutions in [120, 119], over the past few years, the analysis of chemotaxis–fluid models has begun to flourish, in which cells and signal substance are assumed to be transported by a fluid, whose motion is driven by gravitational forces induced by density differences between bacteria and the fluid, and whose velocity $u$ and pressure $P$ hence obey

$$u_t + \kappa (u \cdot \nabla)u = \Delta u + \nabla P + n \nabla \phi, \quad \nabla \cdot u = 0,$$

where $\phi \in C^2(\Omega)$ is a given gravitational potential and the parameter $\kappa \in \{0, 1\}$ distinguishes between Stokes- and Navier–Stokes–governed fluid motion. The equations describing the time evolution of the population density $n$ and signal concentration $c$ differ from their more classical counterparts by a transport term only and are, in a general form, given by

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n \chi(n, c) \nabla c), \\ c_t + u \cdot \nabla c = \Delta c + g(n, c). \end{cases}$$

Bacteria of the species Bacillus subtilis that was used in the experiments in [38, 179] chemotactically respond to oxygen which they consume. A prototypical and popular choice of the functions in the above system is accordingly given by $g(n, c) = -nc$ and the classical $\chi(n, c) = \chi = \text{const}$. This choice was, for convex domains, first covered by the existence results in [197], also extension to nonconvex domains ([84]), results on
convergence of solutions ([201]) and its rate ([212]) are available. The most advanced developments dealing with the full chemotaxis-Navier-Stokes system in three-dimensional domains are constituted by the construction of global weak solutions in [204] and the proof of their eventual regularization and convergence in [206]. Also model variants involving logistic source terms ([22, 101]), several species (Chapters 9, 11), rotational sensitivity functions ([203, 30, 118]) and/or nonlinear diffusion ([214, 80]) have been studied and the interested reader can find additional information and references there or in [5, Section 4.1].

Chemotaxis-fluid systems describing a signal being produced by the cells themselves, as with the Keller–Segel type choice $g(n, c) = +n - c$, up to now have been studied in much fewer works. Global solutions were found to exist in a whole-space setting in the sense of mild solutions [93], in systems with sensitivity functions that obey an estimate of the form $\chi(n, c) \leq (1 + n)^{-\alpha}$ for some $\alpha > 0$ ([189, 191]), in the presence of additional logistic source terms ([176]), nonlinear diffusion ([110]) or for sublinear signal production, that is, $g$ generalizing $g(n, c) = n^\alpha - c$ for some $\alpha \in (0, 1)$ ([15]).

19.1.2. Singular sensitivity

Another important class of chemotaxis models is formed by those with a singular sensitivity function, like $\chi(n, c) = \frac{1}{c}$ ($\chi$ being a positive constant). This form is suggested by the Weber–Fechner law of stimulus perception (see [90]) and supported by experimental ([86]) and theoretical ([209]) evidence. Its characteristics are shaped by the chemotactic effects becoming very strong at fast-varying small signal concentrations – and, indeed, for sufficiently large values of the coefficient $\chi$ (namely, $\chi > \frac{2N}{N-2}$), radial solutions undergoing blow-up within finite time have been found in the corresponding parabolic-elliptic fluid-free setting ([142]). On the other hand, for small values of $\chi$ and in the absence of fluid, classical solutions are known to exist globally in bounded domains of dimension two ([11, 144]) or arbitrary dimension ([196]) and to be bounded ([50]; for a generalization involving different diffusion coefficients see also [216]), where the precise condition $\chi < \sqrt{\frac{2}{N}}$ imposed on the chemotactic coefficient in these works is known to be not strict: For two-dimensional domains, global existence and boundedness of classical solutions were shown for any slightly larger $\chi$ in [102]. There is still a range of values for $\chi$ where it is unknown whether blow-up can occur. Attempts at gaining insight here include the consideration of system variants where either component is assumed to diffuse slowly (though not infinitely slowly) if compared to the other ([54] and [55]) and the constructions of solutions within a weaker framework ([196, 165, 105]) that at least cannot undergo blow-up in form of a persistent delta-type singularity.

19.1.3. The combination of fluid and singular sensitivity. Results of this work

The study of chemotaxis systems accounting for both singular sensitivity functions and fluid has only just begun. In a signal-consumptive setting, the existence of weak solutions ([186]) and their eventual smoothness ([16]) have been shown as well as the existence of
global classical solutions under appropriate smallness conditions on the initial data ([16]).

To our knowledge, however, the corresponding system with production of the signal substance has not been treated yet and we wish to initiate these investigations with the present work and consider

\[
\begin{aligned}
    n_t + u \cdot \nabla n &= \Delta n - \chi \nabla \cdot (\frac{n}{\varepsilon} \nabla c), \\
    c_t + u \cdot \nabla c &= \Delta c - c + n, \\
    u_t + \kappa (u \cdot \nabla)u &= \Delta u + \nabla P + n \nabla \phi, \quad \nabla \cdot u = 0 \quad \text{in } \Omega \times (0, T)
\end{aligned}
\]

in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \in \{2, 3\}$, with smooth boundary, and on the time interval $(0, T)$, $T \in (0, \infty]$, supplemented with the usual boundary conditions

\[
\begin{aligned}
    \partial_n n &= \partial_n c = 0, \quad u = 0 \quad \text{in } \partial \Omega \times (0, T)
\end{aligned}
\]

and initial data

\[
\begin{aligned}
    n(\cdot, 0) &= n_0, \quad c(\cdot, 0) = c_0, \quad u(\cdot, 0) = u_0 \quad \text{in } \Omega,
\end{aligned}
\]

that satisfy

\[
\begin{aligned}
    0 &\leq n_0 \in C^0(\overline{\Omega}), \\
    c_0 &\in W^{1, \vartheta}(\Omega), \quad \inf_{x \in \Omega} c_0(x) > 0, \\
    u_0 &\in D(A^\alpha)
\end{aligned}
\]

for some $\vartheta > N$, $\alpha \in (\frac{N}{4}, 1)$, with $A := -\mathcal{P}\Delta$ denoting the Stokes operator in $L^2_2(\Omega)$ under homogeneous Dirichlet boundary conditions. Moreover, we will assume

\[
\begin{aligned}
    \phi &\in C^2(\overline{\Omega}).
\end{aligned}
\]

We shall ask ourselves the question to what extent results and methods of the fluid-free case can be transferred to the present, more complex situation and will, indeed, recover the result on global existence for the same range of parameters as known from [196] for the fluid-free case:

**Theorem 19.1.1.** For $N \in \{2, 3\}$ let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Suppose that $n_0, c_0, u_0, \phi$ fulfil (19.1.4)–(19.1.7) and $\chi > 0$ satisfies

\[
\chi < \sqrt{\frac{2}{N}}.
\]

Moreover assume that $\kappa \in \{0, 1\}$ if $N = 2$ and that $\kappa = 0$ if $N = 3$. Then there exist functions

\[
\begin{aligned}
    n &\in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)), \\
    c &\in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)) \cap L^\infty([0, \infty); W^{1, \vartheta}(\Omega)), \\
    u &\in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)), \\
    P &\in C^{1,0}(\overline{\Omega} \times (0, \infty))
\end{aligned}
\]

which solve (19.1.1)–(19.1.3) classically in $\Omega \times [0, \infty)$. Moreover, the solution $(n, c, u, P)$ of (19.1.1)–(19.1.3) is unique, up to addition of spatially constant functions to $P$. 249
19.1.4. Technical challenges and plan of this chapter

While the main idea of deriving (local-in-time-)boundedness from the functional

\[
\int_{\Omega} n^p c^{-r}
\]

for suitable values of \(p\) and \(r\) (employed in [196] as well as [50]) remains applicable (see Lemma 19.2.3), the presence of the transport terms poses obstacles in several respects: Firstly, the starting point for the iteration procedure underlying existence and boundedness proofs in [196] and [50], respectively, was to consider the equation for \(c\) as inhomogeneous heat equation and to use the apparent bound on \(\|n(\cdot, t)\|_{L^1(\Omega)}\) together with semigroup estimates. Now, however, the source term is not \(n\) anymore, but \(n - u \cdot \nabla c\), and a priori little is known about bounds for \(u\) or even \(\nabla c\), so that the reasoning of [50, Lemma 2.4] or [196, Lemma 2.4] cannot be used here. We will hence resort to a differential inequality for \(\int_{\Omega} c^q\) for \(q \in [1, \infty)\) (Lemma 19.2.4) and estimate the production term arising therein by means of

\[
\int_{\Omega} n c^{q-1} \leq \left( \int_{\Omega} n^p c^{-r} \right)^{\frac{1}{p}} \left( \int_{\Omega} c^{\frac{q(r-1)}{p}} \right)^{\frac{p-1}{r}},
\]

where the first factor can be controlled according to the previously obtained bound of \(\int_{\Omega} n^p c^{-r}\) and the second is susceptible to an application of the Gagliardo–Nirenberg inequality and subsequent absorption by the diffusion term (see proof of Lemma 19.2.5). This will enable us to transform the information on \(\int_{\Omega} n^p c^{-r}\) into a boundedness assertion on \(\|n\|_{L^p(\Omega)}\) for some suitably large \(p\) (Lemma 19.2.6). Having derived (time-local) \(L^\infty(\Omega)\)-bounds for the fluid velocity field in the Stokes- and Navier–Stokes settings in Sections 19.3.1 and 19.3.2, respectively, in Section 19.4 we can, mainly leaning on semigroup estimates, conclude global existence of solutions.

The second regard in which the fluid poses an obstacle concerns the crucial uniform-in-time lower bound for \(c\) constituting the core of the boundedness proof in [50]. Again viewing the equation for \(c\) as inhomogeneous heat equation with source term \(n\), in [50, Lemma 2.2] estimates of the heat kernel provide this uniform positive lower bound on \(c\). These estimates rely on the nonnegativity of \(n\), whereas no information about the sign of \(u \cdot \nabla c\) (and hence of \(n - u \cdot \nabla c\)) seems available. That we are hence lacking the corresponding time-global positive lower bound of \(c\) is the main reason why we have to leave open the question of boundedness of the solutions obtained in Theorem 19.1.1.

19.2. Basic properties and estimates

We first recall a local existence result. We also give some lower estimate for \(c\), which plays an important role to in avoiding the difficulty of the singular sensitivity function.

**Lemma 19.2.1.** Let \(N \in \{2, 3\}\), \(\chi > 0\), \(\kappa \in (0, 1)\), \(\vartheta > N\), \(\alpha \in \left(\frac{N}{\vartheta}, 1\right)\) and let \(\Omega \subset \mathbb{R}^N\) be a bounded domain with smooth boundary. Assume that \(n_0, c_0, u_0, \phi\) satisfy (19.1.4)–(19.1.7). Then there exist \(T_{\text{max}} \in (0, \infty]\) and a classical solution \((n, c, u, P)\) of (19.1.1)–
(19.1.3) in $\Omega \times (0, T_{\text{max}})$ such that

\[ n \in C^0(\overline{\Omega} \times [0, T_{\text{max}}]) \cap C^{2,1}(\Omega \times (0, T_{\text{max}})), \]

\[ c \in C^0(\overline{\Omega} \times [0, T_{\text{max}}]) \cap C^{2,1}(\Omega \times (0, T_{\text{max}})) \cap L^\infty_{\text{loc}}([0, T_{\text{max}}]; W^{1,\delta}(\Omega)), \]

\[ u \in C^0(\overline{\Omega} \times [0, T_{\text{max}}]) \cap C^{2,1}(\Omega \times (0, T_{\text{max}})), \]

\[ P \in C^{1,0}(\overline{\Omega} \times (0, T_{\text{max}})) \]

and

\[ T_{\text{max}} = \infty \quad \text{or} \quad \lim_{t \to T_{\text{max}}} \left( \| n(\cdot, t) \|_{L^\infty(\Omega)} + \| c(\cdot, t) \|_{W^{1,\delta}(\Omega)} + \| A^\alpha u(\cdot, t) \|_{L^2(\Omega)} \right) = \infty. \]

Also, the solution is unique, up to addition of spatially constant function to $P$ and, moreover, has the properties

\[ n(x, t) \geq 0 \quad \text{and} \quad c(x, t) \geq \left( \min_{x \in \overline{\Omega}} c_0(x) \right) e^{-t} \quad \text{for all} \ t \in (0, T_{\text{max}}). \]  

(19.2.1)

\[ n(x, t) \geq 0 \quad \text{and} \quad c(x, t) \geq \left( \min_{x \in \Omega} c_0(x) \right) e^{-t} \quad \text{for all} \ t \in (0, T_{\text{max}}). \]

Proof. With adaptations akin to those used in [102, Theorem 2.3 i]) to deal with the singular sensitivity, the usual reasoning (see [5, Theorem 3.1] and [197, Lemma 2.1]) based on the Banach fixed point theorem applied in a closed bounded set in $L^\infty((0, T); C^0(\Omega) \times W^{1,\delta}(\Omega) \times D(A^\alpha))$ for suitably small $T > 0$, followed by regularity arguments, proves this local existence and uniqueness result. The estimates in (19.2.1) immediately follow from the comparison principle.

In the following, we will always assume $N, \Omega, \kappa, \chi, n_0, c_0, u_0, \phi, \theta, \alpha$ to satisfy the conditions of Lemma 19.2.1 and to be fixed. By $(n, c, u, P)$ we will denote the corresponding solution to (19.1.1)–(19.1.3) given by Lemma 19.2.1 and by $T_{\text{max}}$ its maximal existence time.

Our study of these solutions begins with the following simple $L^1(\Omega)$-information:

**Lemma 19.2.2.** For all $t \in (0, T_{\text{max}})$ the mass identity

\[ \int_{\Omega} n(\cdot, t) = \int_{\Omega} n_0 \]

is satisfied. Moreover, there exists $C > 0$ such that

\[ \| c(\cdot, t) \|_{L^1(\Omega)} \leq C \quad \text{for all} \ t \in (0, T_{\text{max}}). \]

Proof. This results from integration of the first and second equation of (19.1.1) due to $\nabla \cdot u = 0$ in $\Omega \times (0, T_{\text{max}})$ and (19.1.2). \( \square \)

As in related situations (see [196, 50], but also [105]), the key for establishing estimates significantly going beyond those of Lemma 19.2.2 lies in the following:
Lemma 19.2.3. If $\chi < 1$, $p \in (1, \frac{1}{\chi})$ and $r \in I_p$, where

$$I_p := \left( \frac{p-1}{2} \left( 1 - \sqrt{1 - p\chi^2} \right), \frac{p-1}{2} \left( 1 + \sqrt{1 - p\chi^2} \right) \right),$$

then there are $C_1 > 0$, $C_2 > 0$ such that

$$\int_{\Omega} n(\cdot, t)^p c(\cdot, t)^{-r} \leq C_1 e^{C_2 t} \quad \text{for all } t \in (0, T_{\text{max}})$$

(19.2.3)

and, moreover, for any finite $T \in (0, T_{\text{max}}]$ there exists $C(T) > 0$ such that

$$\int_0^T \int_{\Omega} |\nabla (n^r e^{-\frac{\chi}{2}})|^2 \leq C(T).$$

(19.2.4)

Proof. Using (19.1.1) and integration by parts, we have that

$$\frac{d}{dt} \int_{\Omega} n^p c^{-r} = -p(p-1) \int_{\Omega} n^{p-2} c^{-r} |\nabla n|^2$$

$$+ (2pr + \chi p(p-1)) \int_{\Omega} n^{p-1} c^{-r-1} \nabla n \cdot \nabla c$$

$$- (\chi pr + r(r+1)) \int_{\Omega} n^{p} c^{-r-2} |\nabla c|^2 + r \int_{\Omega} n^{p} c^{-r}$$

$$- r \int_{\Omega} n^{p+1} c^{-r-1} - \int_{\Omega} u \cdot \nabla (n^p c^{-r}) \quad \text{on } (0, T_{\text{max}}).$$

(19.2.5)

The condition $r \in I_p$ entails that

$$0 > 4p \left( r^2 - (p-1)r + \frac{p(p-1)^2}{4} \chi^2 \right) = (2pr + \chi p(p-1))^2 - 4p(p-1)(\chi pr + r(r+1))$$

and hence

$$(2pr + \chi p(p-1))^2 < 4p(p-1)(\chi pr + r(r+1)).$$

Thus, we apply the Young inequality to the summand $(2pr + \chi p(p-1)) \int_{\Omega} n^{p-1} c^{-r} \nabla n \cdot \nabla c$ in (19.2.5) and with some small $\varepsilon > 0$ obtain

$$\frac{d}{dt} \int_{\Omega} n^p c^{-r} + \varepsilon \int_{\Omega} n^{p-2} c^{-r} |\nabla n|^2 + \varepsilon \int_{\Omega} n^{p} c^{-r-2} |\nabla c|^2 \leq r \int_{\Omega} n^{p} c^{-r} \quad \text{in } (0, T_{\text{max}}),$$

(19.2.6)

where we already have taken into account that $\int_{\Omega} u \cdot \nabla (n^p c^{-r}) = 0$ and $-r \int_{\Omega} n^{p+1} c^{-r-1} \leq 0$ in $(0, T_{\text{max}})$. Integration of (19.2.6) shows (19.2.3) and, since

$$\frac{p^2}{2} n^{p-2} c^{-r} |\nabla n|^2 + \frac{r^2}{2} n^{p} c^{-r-2} |\nabla c|^2 \geq |\nabla (n^r e^{-\frac{\chi}{2}})|^2 \quad \text{in } \Omega \times (0, T_{\text{max}}),$$

also results in (19.2.4).
In order to extract helpful boundedness information concerning \( n \) from (19.2.3), we first require estimates for higher norms of \( c \), whose source is the following lemma:

**Lemma 19.2.4.** For all \( q \geq 1 \),

\[
\int_0^{T_{\max}} \frac{1}{q} \frac{d}{dt} \int_\Omega c^q = -(q - 1) \int_\Omega c^{q-2} |\nabla c|^2 - \int_\Omega c^q + \int_\Omega nc^{q-1}
\]

holds on \((0, T_{\max})\).

**Proof.** From the second equation in (19.1.1) we obtain that

\[
\frac{1}{q} \frac{d}{dt} \int_\Omega c^q = \int_\Omega c^{q-1} \Delta c - \int_\Omega c^q + \int_\Omega nc^{q-1} - \int_\Omega c^{q-1} u \cdot \nabla c
\]

on \((0, T_{\max})\). Here we note from \( \nabla \cdot u = 0 \) in \( \Omega \times (0, T_{\max}) \) that

\[
\int_\Omega c^{q-1} u \cdot \nabla c = \frac{1}{q} \int_\Omega u \cdot \nabla (c^q) = 0
\]

on \((0, T_{\max})\) and therefore the last term in (19.2.8) vanishes, so that (19.2.7) holds.

We now turn our attention to the derivation of \( L^q(\Omega) \)-estimates for \( c \). In light of the differential inequality from Lemma 19.2.4, our main objective will be the estimate of \( \int_\Omega nc^{q-1} \).

**Lemma 19.2.5.** If \( \chi = \sqrt{\frac{2}{N}}, \) for all \( q \in (1, \infty) \) and any finite \( T \in (0, T_{\max}] \) there exists a constant \( C(q,T) > 0 \) such that

\[
\|c(\cdot, t)\|_{L^q(\Omega)} \leq C(q,T) \quad \text{for all} \; t \in (0,T).
\]

**Proof.** Without loss of generality we may assume \( q \geq 2 \). We put \( p \in (\frac{N}{2}, \min\{\frac{1}{\chi^2}, 3\}) \) and \( r = \frac{p}{2} \), so that the inequality \( q > p - r \) holds and, clearly, \( r \in I_p \). Therefore Lemma 19.2.3 is applicable and asserts the existence of \( C_1 > 0 \) and \( C_2 > 0 \) such that, by the Hölder inequality,

\[
\int_\Omega nc^{q-1} \leq \left( \int_\Omega n^p c^{-r} \right)^{\frac{1}{p}} \left( \int_\Omega c^{\frac{p(q-2)}{p-1}} \right)^{\frac{p-1}{p}} \leq C_1 c_{c^2}^T \|c(\cdot, t)\|_{L^\frac{2(pq-p+r)}{pq-p+r}(\Omega)}^a
\]

in \((0,T)\), where \( a := \frac{2pq}{pq-p+r} \). Due to the Gagliardo–Nirenberg inequality (see Lemma 2.1.1) there is \( C_{GN} > 0 \) such that

\[
\|c(\cdot, t)\|_{L^\frac{2(pq-p+r)}{pq-p+r}(\Omega)} \leq C_{GN} \|\nabla c(\cdot, t)\|^b_{L^2(\Omega)} \|c(\cdot, t)\|_{L^2(\Omega)}^{a(1-b)} + C_{GN} \|c(\cdot, t)\|_{L^2(\Omega)}^c
\]

for all \( t \in (0, T) \), where \( b := \frac{N(q-p+r)}{2(pq-p+r)} \). Noting from \(-N(p-r) < (2p-N)q\) that

\[
ab = \frac{2(pq-p+r)}{pq} \cdot \frac{N(q-p+r)}{2(pq-p+r)} = \frac{N(q-p+r)}{pq} < 2,
\]

253
we infer from the Young inequality that with some \( C_3(q, T) > 0 \)
\[
(19.2.11) \quad \| \nabla c(\cdot, t) \|_{L^2(\Omega)}^{a(1-b)} \| c(\cdot, t) \|_{L^2(\Omega)}^{b} \leq \frac{q-1}{2CGNC_1e^{CT^2}} \| \nabla c(\cdot, t) \|_{L^2(\Omega)}^{a(1-b)} + C_3(q, T) \| c(\cdot, t) \|_{L^2(\Omega)}^{b} + \frac{2a(1-b)pq}{2pq - N(q-p+r)}
\]
for all \( t \in (0, T) \). Here since \( N \in \{2, 3\} \) implies \((N-1)r < p\), we can confirm that
\[
\frac{2a(1-b)pq}{2pq - N(q-p+r)} = \frac{2(2p + (N-2)p - Nq + (N-2)r)}{2pq - N(q-p+r)} < 2.
\]
Thus, relying once more on the Young inequality we can derive from (19.2.11) that there exists \( C_4(q, T) > 0 \) such that
\[
(19.2.12) \quad \| \nabla c(\cdot, t) \|_{L^2(\Omega)}^{a(1-b)} \| c(\cdot, t) \|_{L^2(\Omega)}^{b} \leq \frac{q-1}{2^{a+1}CGNC_1e^{CT^2}} \| \nabla c(\cdot, t) \|_{L^2(\Omega)}^{a(1-b)} + \frac{1}{2^{a+1}CGNC_1e^{CT^2}} \| c(\cdot, t) \|_{L^2(\Omega)}^{b} + C_4(q, T)
\]
is valid for all \( t \in (0, T) \). Combination of (19.2.7), (19.2.9), (19.2.10) and (19.2.12) with Lemma 19.2.2 implies that with some \( C_5(q, T) > 0 \)
\[
\frac{1}{q} \frac{d}{dt} \int_\Omega c^a \leq -\frac{q-1}{2} \int_\Omega c^{a-2} |\nabla c|^2 - \frac{1}{2} \int_\Omega c^a + C_5(q, T)
\]
on \( (0, T) \), which means that there exists \( C_6(q, T) > 0 \) such that
\[
\| c(\cdot, t) \|_{L^q(\Omega)} \leq C_6(q, T) \quad \text{for all } t \in (0, T).
\]

Now we have already prepared all tools to obtain an \( L^p(\Omega) \)-estimate for \( n \), for some \( p > \frac{N}{2} \), which is an important stopover on the route to the \( L^\infty(\Omega) \)-estimate for \( n \) and will be of particular importance in the proofs of Lemma 19.4.1 and Lemma 19.3.2.

**Lemma 19.2.6.** We assume \( \chi < \sqrt{\frac{2}{N}} \). For any \( p \in (1, \frac{1}{\chi}) \) and any finite \( T \in (0, T_{\text{max}}] \) there is \( C(p, T) > 0 \) such that
\[
(19.2.13) \quad \| n(\cdot, t) \|_{L^p(\Omega)} \leq C(p, T) \quad \text{for all } t \in (0, T).
\]

**Proof.** Without loss of generality we assume \( p \in \left( \frac{N}{2}, \frac{1}{\chi} \right) \), let \( p_0 \in (p, \frac{1}{\chi}) \) and set \( r_0 := \frac{p}{p_0-1} \in I_p \). Then we can see from the Hölder inequality that
\[
\int_\Omega n^p = \int_\Omega \left( n^{p_0} e^{-r_0} \right)^{\frac{p}{p_0}} \leq \left( \int_\Omega n^{p_0} e^{-r_0} \right)^{\frac{p}{p_0}} \left( \int_\Omega \frac{r_0}{p_0} \right)^{\frac{p_0-p}{p_0}}
\]
on \( (0, T) \), which by Lemmas 19.2.3 and 19.2.5 implies (19.2.13). \( \square \)
19.3. Boundedness for $u$

Having obtained $L^p(\Omega)$-bounds for $n$ and hence for the driving force in the fluid equation, we devote this section to the derivation of estimates for the fluid velocity. We begin with the following $L^2(\Omega)$-information on $u$ and $L^2(\Omega \times (0, T))$-estimate for $\nabla u$, before we separately consider the cases of Stokes- and Navier-Stokes-fluids in Subsections 19.3.1 and 19.3.2.

Lemma 19.3.1. If $\chi < \sqrt{\frac{2}{N}}$, for any finite $T \in (0, T_{\text{max}}]$ there exists $C(T) > 0$ such that

\begin{equation}
\int_{\Omega} |u(t, \cdot)|^2 \leq C(T) \quad \text{for all } t \in (0, T) \tag{19.3.1}
\end{equation}

and

\begin{equation}
\int_{0}^{T} \int_{\Omega} |\nabla u(t, \cdot)|^2 \leq C(T). \tag{19.3.2}
\end{equation}

Proof. Testing the third equation in (19.1.1) by $u$ and integrating by parts, we see that

\begin{equation}
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u|^2 + \int_{\Omega} |\nabla u|^2 = \int_{\Omega} nu \cdot \nabla \phi \quad \text{on } (0, T), \tag{19.3.3}
\end{equation}

because $\kappa \int_{\Omega} u \cdot (u \cdot \nabla)u = -\kappa \int_{\Omega} \nabla \cdot u |u|^2 = 0$. We let $p \in \left(\frac{N+2}{2N}, \frac{1}{\chi^2}\right)$ so that Lemma 19.2.6 provides us with $C_1 > 0$ such that $\|u(t, \cdot)\|_{L^p(\Omega)} \leq C_1$ for all $t \in (0, T)$. Then setting $p' := \frac{p}{p-1}$ we have $p' \in \left[1, \frac{2N}{N+2}\right)$ and may rely on the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^{p'}(\Omega)$ to obtain $C_2 > 0$ satisfying $\|w\|_{L^{p'}(\Omega)} \leq C_2 \|\nabla w\|_{L^2(\Omega)}$ for all $w \in W^{1,2}_0(\Omega)$. From the Hölder inequality, this embedding, and the Young inequality, we can conclude that

\begin{equation}
\int_{\Omega} nu \cdot \nabla \phi \leq C_2 \|\nabla \phi\|_{L^\infty(\Omega)} \|n\|_{L^p(\Omega)} \|\nabla u\|_{L^2(\Omega)} \leq \frac{1}{2} \int_{\Omega} |\nabla u|^2 + C_2^2 C_2^2 \|\nabla \phi\|_{L^\infty(\Omega)}^2 \tag{19.3.4}
\end{equation}

holds on $(0, T)$, which by (19.3.3) implies that there exists $C_3 > 0$ such that

\begin{equation}
\frac{d}{dt} \int_{\Omega} |u|^2 + \int_{\Omega} |\nabla u|^2 \leq C_3 \quad \text{on } (0, T). \tag{19.3.4}
\end{equation}

Thus thanks to the Poincaré inequality we can find a constant such that (19.3.1) holds. Combination of (19.3.4) and (19.3.1) then also entails (19.3.2) with some $C(T)$. \hfill \Box

19.3.1. The case $\kappa = 0$

In the case $\kappa = 0$ the regularity properties for $n$ and $u$ already established in Lemma 19.2.6 and Lemma 19.3.1, respectively, will be sufficient to prove the boundedness of $u$ even in the case of $N = 3$. It is well known that the regularity of solutions to the Stokes subsystem $u_t + Au = P(n \nabla \phi)$ appearing in (19.1.1) is only contingent on the regularity of the forcing term $P(n \nabla \phi)$. Arguments appearing in the proof of the lemma below have been previously used in e.g. [202] and [190] and rely on semigroup estimates for the Stokes semigroup.
Lemma 19.3.2. If $\chi < \sqrt{\frac{2}{N}}$, for any finite $T \in (0, T_{\max}]$ and any $\alpha_0 \in (\frac{N}{4}, \alpha]$ satisfying
\[ \alpha_0 < 1 - \frac{N}{2} \chi^2 + \frac{N}{4}, \]
there is $C(T) > 0$ such that
\[ (19.3.5) \quad \|A^{\alpha_0} \left( e^{-tA}u_0 + \int_0^t e^{-(t-s)A}P(n(\cdot, s)\nabla\phi)ds \right) \|_{L^2(\Omega)} \leq C(T) \quad \text{for all } t \in (0, T). \]

Proof. We pick $p \in (\frac{N}{2}, \min\{\frac{1}{2}, 2\})$ and $\delta \in (0, 1)$ sufficiently small such that
\[ \alpha_0 + \delta < 1 - \frac{N}{2} \left( \frac{1}{p} - \frac{1}{2} \right) \]
holds. We then fix $p_0 > p$ satisfying
\[ 2\delta - \frac{N}{p} > -\frac{N}{p_0} \]
and note that
\[ (19.3.6) \quad \alpha_0 + \delta + \frac{N}{2} \left( \frac{1}{p_0} - \frac{1}{2} \right) < 1. \]
Since $u_0 \in D(A^{\alpha_0})$ by $\alpha_0 \leq \alpha$, we can use that $A^{\alpha_0}$ and $e^{-tA}$ commute ([162, p. 206, (1.5.16)]) and thereby find $C_1 > 0$ such that
\[ (19.3.7) \quad \|A^{\alpha_0}e^{-tA}u_0\|_{L^2(\Omega)} = \|e^{-tA}A^{\alpha_0}u_0\|_{L^2(\Omega)} \leq \|A^{\alpha_0}u_0\|_{L^2(\Omega)} \leq C_1 \]
for all $t \in (0, T_{\max})$. To treat the integrand in (19.3.5), we recall that by standard regularity estimates for the Stokes semigroup (e.g. [202, Lemma 3.1], [30, Lemma 2.3]) there exist $C_2 > 0$ and $\lambda_1 > 0$ such that
\[ (19.3.8) \quad \|A^{\alpha_0+\delta}e^{-(t-s)A}A^{-\delta}P((n(\cdot, s)\nabla\phi))\|_{L^2(\Omega)} \leq C_2(t-s)^{-\alpha_0-\delta-\frac{N}{p_0}-\frac{1}{2}}e^{-\lambda_1(t-s)} \|A^{-\delta}P(n(\cdot, s)\nabla\phi)\|_{L^{p_0}(\Omega)} \]
for all $s \in (0, t)$. Additionally, our choice of $2\delta - \frac{N}{p} > -\frac{N}{p_0}$ implies that for any $s \in (0, t)$ we have
\[ (19.3.9) \quad \|A^{-\delta}P(n(\cdot, s)\nabla\phi)\|_{L^{p_0}(\Omega)} \leq C_3\|n(\cdot, s)\nabla\phi\|_{L^p(\Omega)} \leq C_3\|\nabla\phi\|_{L^\infty(\Omega)}\|n(\cdot, s)\|_{L^p(\Omega)}, \]
with some $C_3 > 0$ (see also [202, Lemma 3.3] and [190, Lemma 2.3]). Due to the fact that $p \in (\frac{N}{2}, \frac{1}{\chi})$, Lemma 19.2.6 provides $C_4 > 0$ such that $\|n(\cdot, t)\|_{L^p(\Omega)} \leq C_4$ for all $t \in (0, T)$ and therefore a combination of (19.3.7)–(19.3.9) shows that
\[ (19.3.10) \quad \|A^{\alpha_0} \left( e^{-tA}u_0 + \int_0^t e^{-(t-s)A}P(n(\cdot, s)\nabla\phi)ds \right) \|_{L^2(\Omega)} \leq C_1 + C_2C_3C_4\|\nabla\phi\|_{L^\infty(\Omega)}\int_0^t (t-s)^{-\alpha_0-\delta-\frac{N}{p_0}-\frac{1}{2}}e^{-\lambda_1(t-s)}ds \]
holds for all $t \in (0, T)$, which in view of (19.3.6) and the fact that $\phi \in C^2(\overline{\Omega})$ implies (19.3.5). \[\Box\]
**Lemma 19.3.3.** Assume \( \chi < \sqrt{\frac{2}{N}} \) and let \( \alpha_0 \in (\frac{\chi}{2}, \alpha] \) satisfy \( \alpha_0 \leq 1 - \frac{N}{2} \chi^2 + \frac{N}{2} \chi \). If \( \kappa = 0 \), corresponding to any finite \( T \in (0, T_{\text{max}}] \) there exists \( C(T) > 0 \) such that

\[
\|A^{\alpha_0}u(\cdot, t)\|_{L^2(\Omega)} \leq C(T) \quad \text{for all } t \in (0, T)
\]

and

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C(T) \quad \text{for all } t \in (0, T).
\]

**Proof.** Since for \( \kappa = 0 \) the solution \( u \) is given by

\[
u(\cdot, t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}P(n(\cdot, s)\nabla \phi) \, ds,
\]
due to the compatibility of the choice of \( \alpha_0 \) with the requirements of Lemma 19.3.2, this lemma immediately yields (19.3.10), whereupon (19.3.11) is a consequence of the embedding \( D(A^{\alpha_0}) \hookrightarrow C^{\gamma}(\Omega) \) for arbitrary \( \gamma \in (0, 2\alpha_0 - \frac{N}{2}) \).

---

**19.3.2. The case \( \kappa = 1 \) and \( N = 2 \)**

Here we will focus on the case that \( \kappa = 1 \) and \( N = 2 \). In this setting we can make use of arguments previously employed in [197].

The proof of Lemma 19.3.5 will require some spatio-temporal integrability of \( n \) of higher order than directly guaranteed by Lemma 19.2.6. We therefore prepare the following:

**Lemma 19.3.4.** Assume that \( \kappa = 1 \) and \( N = 2 \). If \( \chi < \sqrt{\frac{2}{N}} \), for any finite \( T \in (0, T_{\text{max}}] \) there is \( C(T) > 0 \) such that

\[
\int_0^T \int_\Omega n^2 \leq C(T).
\]

**Proof.** Let \( p \in (1, \frac{1}{\chi^2}) \) and \( r := \frac{p-1}{2} \in I_p \). First we recall from (19.2.4) that there are \( C_1 > 0 \) and \( C_2 > 0 \) such that

\[
\int_0^T \int_\Omega |\nabla(n^\frac{p}{2}c^{-\frac{r}{2}})|^2 \leq C_1 \quad \text{and} \quad \sup_{t \in (0, T)} \int_\Omega n^pc^{-r}(\cdot, t) \leq C_2
\]

hold. The Gagliardo–Nirenberg inequality (see Lemma 2.1.1) hence provides \( C_3 > 0 \) such that

\[
\int_0^T \|n(\cdot, t)c(\cdot, t)^{-\frac{r}{2}}\|_{L^{2p}(\Omega)}^{2p} \, dt = \int_0^T \|n(\cdot, t)^{\frac{r}{2}}c(\cdot, t)^{-\frac{r}{2}}\|_{L^4(\Omega)}^4 \, dt \\
\leq C_3 \int_0^T \|\nabla(n(\cdot, t)^{\frac{r}{2}}c(\cdot, t))^{-\frac{r}{2}}\|_{L^2(\Omega)}^2 \|n(\cdot, t)^{\frac{r}{2}}c(\cdot, t)^{-\frac{r}{2}}\|_{L^2(\Omega)}^2 \, dt \\
+ C_3 \int_0^T \|n(\cdot, t)^{\frac{r}{2}}c(\cdot, t)^{-\frac{r}{2}}\|_{L^2(\Omega)} \, dt \leq C_3C_2C_1 + C_3C_2C_2T,
\]

257
which means that
\[ \int_0^T \int_\Omega n^{2p} e^{-(p-1)} \leq C \] (19.3.12)
holds. Thanks to the Young inequality, we can estimate
\[ \int_0^T \int_\Omega n^2 \leq \int_0^T \int_\Omega n^{2p} e^{-(p-1)} + \int_0^T \int_\Omega c, \]
so that (19.3.12) and the $L^1(\Omega)$-boundedness of $c$ as asserted by Lemma 19.2.2 finish the proof.

Lemma 19.3.5. If $\chi < \sqrt{\frac{2}{N}}$, $\kappa = 1$, $N = 2$, then for any finite $T \in (0, T_{\text{max}}]$ there exists $C(T) > 0$ such that
\[ \int_\Omega |\nabla u(\cdot, t)|^2 \leq C(T) \quad \text{for all } t \in (0, T) \]
and
\[ \int_0^T \int_\Omega |Au|^2 \leq C(T). \] (19.3.13)

Proof. Testing the third equation in (19.1.1) by $Au$ and using the Young inequality, we obtain that
\[ \frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla u|^2 + \int_\Omega |Au|^2 = \int_\Omega (n\nabla \phi)Au - \int_\Omega (u \cdot \nabla)uAu \leq \frac{1}{2} \int_\Omega |Au|^2 + C_1 \int_\Omega n^2 + \int_\Omega |u|^2|\nabla u|^2 \quad \text{on } (0, T) \] (19.3.14)
with some $C_1 > 0$. Concerning the last term herein, we follow [197, Proof (of Theorem 1.1)] and employ the Gagliardo–Nirenberg inequality, Lemma 19.3.1 and the Young inequality in obtaining $C_2 > 0$, $C_3 > 0$ and $C_4 > 0$ such that
\[ \int_\Omega |u|^2|\nabla u|^2 \leq \|u\|_{L^\infty(\Omega)}^2 \|\nabla u\|_{L^2(\Omega)}^2 \]
\[ \leq C_2 \|u\|_{W^{1,2}(\Omega)} \|u\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)}^2 \]
\[ \leq C_2 C_3 \|u\|_{W^{1,2}(\Omega)} \|\nabla u\|_{L^2(\Omega)}^2 \]
\[ \leq \frac{1}{4} \||Au||_{L^2(\Omega)}^2 + C_4 \|\nabla u\|_{L^2(\Omega)}^4 \quad \text{on } (0, T). \]
From (19.3.14) we can hence see that with $C_5 := \max\{C_1, C_4\} > 0$
\[ \frac{d}{dt} \int_\Omega |\nabla u|^2 + \frac{1}{4} \int_\Omega |Au|^2 \leq C_5 \left( \left( \int_\Omega |\nabla u|^2 \right)^2 + \int_\Omega n^2 \right) \quad \text{on } (0, T). \]
If we put \( y(t) := \int_\Omega |\nabla u(\cdot, t)|^2, \ t \in (0, T) \), then \( y \) satisfies

\[
y'(t) \leq C_5 \left( \int_\Omega |\nabla u(\cdot, t)|^2 y(t) + \int_\Omega n^2(\cdot, t) \right) \quad \text{for all } t \in (0, T),
\]

which implies

\[
y(t) \leq y(0) e^{\int_0^t \int_\Omega |\nabla u(\cdot, s)|^2 ds} + C_5 \int_0^t e^{\int_0^s \int_\Omega |\nabla u(\cdot, \sigma)|^2 d\sigma} \left( \int_\Omega n^2(x, s) dx \right) ds, \quad t \in (0, T).
\]

Here noting from Lemmas 19.3.1 and 19.3.4 that

\[
\int_0^T \int_\Omega |\nabla u(\cdot, \sigma)|^2 d\sigma < \infty
\]

and

\[
\int_0^T \int_\Omega n^2(\cdot, s) ds < \infty,
\]

we find \( C_6(T) > 0 \) such that

\[
\int_\Omega |\nabla u|^2 \leq C_6(T) \quad \text{on } (0, T).
\]

Thanks to this boundedness, we can establish (19.3.13).

Then the same argument as in [197, Proof (of Theorem 1.1)] leads to the \( L^\infty(\Omega) \)-estimate for \( u \):

**Lemma 19.3.6.** If \( \chi < \sqrt{\frac{2}{\pi}} \), \( \kappa = 1 \), \( N = 2 \), then for any finite \( T \in (0, T_{\text{max}}] \) and any \( \alpha_0 \in \left( \frac{N}{4}, \alpha \right] \) there exists \( C_1(T) > 0 \) such that

\[
\| A^{\alpha_0} u(\cdot, t) \|_{L^2(\Omega)} \leq C_1(T) \quad \text{for all } t \in (0, T).
\]

Moreover, there exists \( C_2(T) > 0 \) satisfying

\[
\| u(\cdot, t) \|_{L^\infty(\Omega)} \leq C_2(T) \quad \text{for all } t \in (0, T).
\]

**Proof.** Since \( \alpha_0 \) satisfies the condition of Lemma 19.3.2, we see that

\[
\| A^{\alpha_0} u(\cdot, t) \|_{L^2(\Omega)} \leq \| A^{\alpha_0} \left( e^{-tA} u_0 + \int_0^t e^{-(t-s)A} \mathcal{P} \left( u(\cdot, s) \nabla \phi \right) ds \right) \|_{L^2(\Omega)}
\]

\[
+ \int_0^t \| A^{\alpha_0} e^{-(t-s)A} \mathcal{P} \left( (u \cdot \nabla) u \right)(\cdot, s) \|_{L^2(\Omega)} ds
\]

\[
\leq C_3 + \int_0^t \| A^{\alpha_0} e^{-(t-s)A} \mathcal{P} \left( (u \cdot \nabla) u \right)(\cdot, s) \|_{L^2(\Omega)} ds \quad \text{for all } t \in (0, T),
\]

with \( C_3 > 0 \) given by said lemma. Again following [197, Proof (of Theorem 1.1)], we choose \( p > 2 \) so large that \( p' := \frac{p}{p-1} \) satisfies \( p' \alpha_0 < 1 \). Then by a well-known estimate for
the norm of $A^{\alpha_0}e^{-tA}$ (see e.g. [30, Lemma 2.3i]) and the H"older inequality, we find that with some $C_A > 0$

$$
\int_0^t \| A^{\alpha_0}e^{-(t-s)A}(u \cdot \nabla)u \|_{L^2(\Omega)} ds 
\leq C_A \int_0^t (t-s)^{-\alpha_0} \| (u(\cdot, s) \cdot \nabla)u(\cdot, s) \|_{L^2(\Omega)} ds 
\leq C_A \left( \int_0^t (t-s)^{-p\alpha_0} ds \right)^{\frac{1}{p}} \left( \int_0^t \| (u(\cdot, s) \cdot \nabla)u(\cdot, s) \|_{L^2(\Omega)}^p ds \right)^{\frac{1}{p}}
$$

for all $t \in (0, T)$. Since $W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ due to $N = 2$, from this embedding and the Gagliardo–Nirenberg inequality we obtain $C_5 > 0$ and $C_6 > 0$, respectively, such that from the H"older inequality we can infer

$$
\int_0^t \| (u(\cdot, s) \cdot \nabla)u(\cdot, s) \|_{L^2(\Omega)} ds \leq \int_0^T \| u(\cdot, s) \|_{L^p(\Omega)}^p \| \nabla u(\cdot, s) \|_{L^{\frac{2p}{2-p}}(\Omega)}^p ds
\leq C_5 \int_0^T \| \nabla u(\cdot, s) \|_{L^2(\Omega)}^p \| \nabla u(\cdot, s) \|_{L^{\frac{2p}{2-p}}(\Omega)}^p ds
\leq C_5 C_6 \int_0^T \| \nabla u(\cdot, s) \|_{L^2(\Omega)}^p \| \Delta u(\cdot, s) \|_{L^{\frac{2p}{p-2}}(\Omega)}^2 ds
\leq C_5 C_6 \left( \sup_{t \in (0,T)} \| \nabla u(\cdot, t) \|_{L^2(\Omega)}^{2p-2} \right) \int_0^T \| \Delta u(\cdot, s) \|_{L^2(\Omega)}^2 ds.
$$

Here an application of Lemma 19.3.5 finishes the proof of (19.3.15), which by the embedding of $D(A^{\alpha_0})$ into $L^\infty(\Omega)$ entails (19.3.16), too.

\[ \square \]

19.4. Boundedness for $n$

The goal of this section will be to establish an $L^\infty(\Omega)$-estimate for $n$ by combination of previously obtained estimates and to finally prove Theorem 19.1.1. Control of the cross-diffusion term in the equation for $n$ will be supplied by the following boundedness statement concerning $\nabla c$.

Lemma 19.4.1. If $\kappa = 1$, assume that $N = 2$. Let $1 \leq p \leq q < \infty$ satisfy $q < \vartheta$ and $\frac{1}{2} + \frac{N}{2} (\frac{1}{p} - \frac{1}{q}) < 1$. Then for any finite $T \in (0, T_{\text{max}}]$ there exists a constant $C(T) > 0$ such that

\begin{equation}
\| \nabla c(\cdot, t) \|_{L^q(\Omega)} \leq C(T) \left( 1 + \sup_{s \in (0,T)} \| n(\cdot, s) \|_{L^p(\Omega)} \right) \quad \text{for all } t \in (0, T).
\end{equation}

In particular, if $\chi < \sqrt{\frac{2}{N}}$, then for any $q \in [1, \frac{1}{\chi^2 - \frac{N}{N}}] \cap [1, \vartheta)$ and any finite $T \in (0, T_{\text{max}}]$ there is $C(q, T) > 0$ such that

\begin{equation}
\| \nabla c(\cdot, t) \|_{L^q(\Omega)} \leq C(q, T) \quad \text{for any } t \in (0, T).
\end{equation}
Proof. Applying the variation of constants formula for \( c \), we have
\[
(19.4.3) \quad c(\cdot, t) = e^{(\Delta - 1) t} c_0 + \int_0^t e^{(t-s)(\Delta - 1)} (n(\cdot, s) + u(\cdot, s) \cdot \nabla c(\cdot, s)) \, ds, \quad t \in (0, T_{\max}).
\]
In light of standard semigroup estimates for the Neumann heat semigroup (see Lemma 2.2.2) we find \( C_1 > 0 \) such that
\[
(19.4.4) \quad \| \nabla e^{(\Delta - 1) t} c_0 \|_{L^p(\Omega)} \leq C_1 \| \nabla c_0 \|_{L^q(\Omega)} \quad \text{for all } t \in (0, T_{\max}).
\]
Similarly, the semigroup estimates of [194, Lemma 1.3 (ii)] provide us with \( C_2 > 0 \), for any \( t \in (0, T) \) fulfilling
\[
(19.4.5) \quad \int_0^t \| e^{(t-s)(\Delta - 1)} n(\cdot, s) \|_{L^p(\Omega)} \, ds \\
\leq C_2 \int_0^t \left( 1 + (t-s)^{-\frac{1}{2}} \frac{N}{2} \frac{1}{2} \right) e^{-\sigma (t-s)} \| n(\cdot, s) \|_{L^p(\Omega)} \, ds \\
\leq C_2 \sup_{s \in (0, T)} \| n(\cdot, s) \|_{L^p(\Omega)} \int_0^t \left( 1 + \sigma^{-\frac{1}{2}} \frac{N}{2} \frac{1}{2} \right) e^{-\sigma} \, d\sigma,
\]
wherein the last integral is finite since \( \frac{1}{2} + \frac{N}{2} \frac{1}{2} < 1 \). Now we put \( r \geq q, \eta \leq \frac{1}{2} \)
satisfying \( \eta + \frac{N}{2} (\frac{1}{2} - \frac{1}{q}) < \eta \) and \( \delta \in (0, \frac{1}{2} - \eta) \). Since \( 2\eta - \frac{N}{r} > 1 - \frac{N}{q} \), the domain \( D((-\Delta + 1)^\eta) \) of the fractional power of the operator \(-\Delta + 1\) is continuously embedded into \( W^{1, q}(\Omega) \) (see Lemma 2.2.1) and we can hence find \( C_3 > 0 \) such that
\[
\| w \|_{W^{1, q}(\Omega)} \leq C_3 \| (-\Delta + 1)^\eta w \|_{L^r(\Omega)} \quad \text{for any } w \in D((-\Delta + 1)^\eta).
\]
Aided by Lemma 2.2.1 we moreover fix \( C_4 > 0 \) such that
\[
\| (-\Delta + 1)^\eta e^{-\tau (-\Delta + 1)^\eta} \nabla \cdot w \|_{L^r(\Omega)} \leq C_4 \tau^{-\frac{1}{2} \delta} e^{-\lambda \tau} \| w \|_{L^r(\Omega)} \quad \text{for all } \tau > 0 \text{ and } w \in L^r(\Omega).
\]
Additionally relying on \( \nabla \cdot u = 0 \) in \( \Omega \times (0, T_{\max}) \) we then find
\[
(19.4.6) \quad \int_0^t \| \nabla e^{(t-s)(\Delta - 1)} u(\cdot, s) \cdot \nabla c(\cdot, s) \|_{L^q(\Omega)} \, ds \\
\leq \int_0^t \| e^{(t-s)(\Delta - 1)} \nabla \cdot (c(\cdot, s) u(\cdot, s)) \|_{W^{1, q}(\Omega)} \, ds \\
\leq C_3 \int_0^t \| (-\Delta + 1)^\eta e^{(t-s)(\Delta - 1)} \nabla \cdot (c(\cdot, s) u(\cdot, s)) \|_{L^r(\Omega)} \, ds \\
\leq C_3 C_4 \int_0^t (t-s)^{-\frac{1}{2} \delta} \| c(\cdot, s) u(\cdot, s) \|_{L^r(\Omega)} \, ds \quad \text{for all } t \in (0, T_{\max}).
\]
Since by Lemma 19.2.5, and Lemma 19.3.3 or Lemma 19.3.6 there is \( C_5 > 0 \) such that
\[
\| c(\cdot, s) u(\cdot, s) \|_{L^r(\Omega)} \leq C_5 \quad \text{for all } s \in (0, T),
\]
261
we can conclude from $\eta + \frac{1}{2} + \delta < 1$ and (19.4.6) that with $C_6 := C_3 C_3 C_5 \int_0^T \sigma^{-\eta - \frac{1}{2} - \delta} d\sigma \in (0, \infty)$ we have

$$
(19.4.7) \quad \int_0^t \|\nabla e^{(t-s)(\Delta-1)} u(\cdot, s) \cdot \nabla c(\cdot, s)\|_{L^q(\Omega)} ds \leq C_6 \quad \text{for all } t \in (0, T).
$$

Combination of (19.4.3), (19.4.4), (19.4.5) and (19.4.7) establishes the asserted inequality (19.4.1). If $q < \frac{1}{\chi^2 - \frac{1}{N}}$ and hence

$$
1 > \frac{1}{2} + \frac{N}{2} \left( \frac{\chi^2 - \frac{1}{N}}{q} \right),
$$

it is possible to choose $p \in \left( \frac{N}{2}, \frac{1}{\chi^2} \right) \cap \left( \frac{N}{2}, q \right]$ such that still

$$
1 > \frac{1}{2} + \frac{N}{2} \left( \frac{1}{p} - \frac{1}{q} \right)
$$

and (19.4.2) results from (19.4.1) and Lemma 19.2.6.

Now we shall establish a temporally local $L^\infty(\Omega)$-estimate for $n$.

**Lemma 19.4.2.** Assume that $\chi < \sqrt{\frac{2}{N}}$. If $\kappa = 1$, additionally suppose that $N = 2$. Then for any finite $T \in (0, T_{\text{max}})$ there exists a constant $C(T) > 0$ satisfying

$$
\|n(\cdot, t)\|_{L^\infty(\Omega)} \leq C(T) \quad \text{for all } t \in (0, T).
$$

**Proof.** From (19.2.1) and (19.2.2) we obtain $C_1 > 0$ and $C_2 > 0$ such that

$$
(19.4.8) \quad \inf_{x \in \Omega} c(x, t) \geq \frac{1}{C_1} \quad \text{and} \quad \int_{\Omega} n(\cdot, t) = C_2 \quad \text{for all } t \in (0, T).
$$

We pick $q, r \in (1, \vartheta)$ such that

$$
\frac{1}{\chi^2 - \frac{1}{N}} > q > r > N
$$

and let $C_3 > 0$ fulfil

$$
(19.4.9) \quad \|\nabla c(\cdot, t)\|_{L^q(\Omega)} \leq C_3 \quad \text{for all } t \in (0, T)
$$

by Lemma 19.4.1. Now for all $T' \in (0, T)$ we note that

$$
M(T') := \sup_{t \in (0, T')} \|n(\cdot, t)\|_{L^\infty(\Omega)}
$$

is finite and that, furthermore, for any $p \in [1, \infty)$,

$$
(19.4.10) \quad \|n(\cdot, t)\|_{L^p(\Omega)} \leq C_2^\vartheta \left( M(T') \right)^{\frac{1}{p} - \frac{1}{\vartheta}} \quad \text{for all } t \in (0, T').
$$

262
In order to obtain an estimate for $M(T')$, for $t \in (0, T')$, we set $t_0 := (t-1)_+$ and represent $n$ according to
\[ n(\cdot, t) = e^{(t-t_0)\Delta} n(\cdot, t_0) - \int_{t_0}^{t} e^{(t-s)\Delta} \left( \frac{n(\cdot, s)}{c(\cdot, s)} \nabla c(\cdot, s) + n(\cdot, s) u(\cdot, s) \right) ds. \]

If $t_0 = 0$ (that is, $t \leq 1$), then
\[ \|e^{(t-t_0)\Delta} n(\cdot, t_0)\|_{L^\infty(\Omega)} = \|e^{t\Delta} n_0\|_{L^\infty(\Omega)} \leq \|n_0\|_{L^\infty(\Omega)}, \]
whereas, if $t_0 > 0$ (i.e. $t > 1$), then with the constant $C_4 > 0$ yielded by the semigroup estimate (see Lemma 2.2.2) we have
\[ \|e^{(t-t_0)\Delta} n(\cdot, t_0)\|_{L^\infty(\Omega)} \leq C_4 \left(1 + (t - t_0)^{-\frac{4}{r} - \frac{4}{q}}\right) \|n(\cdot, t_0)\|_{L^1(\Omega)} = 2C_2C_4, \]
because $t - t_0 = 1$. With $C_5 := \max\{\|n_0\|_{L^\infty(\Omega)}, 2C_2C_4\}$ and some $C_6 > 0$ obtained from the semigroup estimate in [30, Lemma 2.1 iv], we have
(19.4.11) \[ \|n(\cdot, t)\|_{L^\infty(\Omega)} \leq C_5 + C_6 \int_0^1 \left(1 + (t - s)^{-\frac{4}{r} - \frac{4}{q}}\right) \|\left(\frac{n}{c} \nabla c\right)(\cdot, s) - (nu)(\cdot, s)\|_{L^r(\Omega)} ds \]
for all $t \in (0, T_{\max})$. The definitions of $C_1$ and $C_3$ in (19.4.8) and (19.4.9) together with the H"older inequality and (19.4.10) imply
\[ \|\left(\frac{n}{c} \nabla c\right)(\cdot, t)\|_{L^r(\Omega)} \leq C_1 \|n(\cdot, t)\|_{L^\infty(\Omega)} \|\nabla c(\cdot, t)\|_{L^r(\Omega)} \leq C_1 C_2^{\frac{1}{r} - \frac{1}{q}} C_3 (M(T'))^{1 + \frac{4}{r} - \frac{4}{q}} \]
for all $t \in (0, T')$ and with $C_6$ obtained from either Lemma 19.3.3 or Lemma 19.3.6
\[ \|n(\cdot, t)u(\cdot, t)\|_{L^r(\Omega)} \leq C_2^{\frac{1}{r} C_6 (M(T'))^{1 - \frac{1}{r}}, \quad \text{for } t \in (0, T'). \]
Estimate (19.4.11) is hence transformed into
\[ \|n(\cdot, t)\|_{L^\infty(\Omega)} \leq C_4 + C_7 (M(T'))^{1 + \frac{1}{r} - \frac{1}{q} + \frac{1}{r}} + C_7 (M(T'))^{1 - \frac{1}{r}} \quad \text{for all } t \in (0, T'), \]
where
\[ C_7 = C_5 \int_0^1 \sigma^{-\frac{4}{r} - \frac{4}{q}} d\sigma \max\{C_1 C_2^{\frac{1}{r} - \frac{1}{q}} C_3, C_2^{\frac{1}{r} C_6}\} \]
is finite due to $r > N$. Accordingly, for any $T' \in (0, T)$ we have that
\[ M(T') \leq \sup \left\{ \xi \in \mathbb{R} \mid \xi \leq C_4 + C_7 \xi^{1 + \frac{1}{r} - \frac{1}{q} + \frac{1}{r}} + C_7 \xi^{1 - \frac{1}{r}} \right\}, \]
which is a finite number due to $1 + \frac{1}{r} - \frac{1}{q} < 1$ and $1 - \frac{1}{r} < 1$. This concludes the proof. \(\Box\)

263
Proof of Theorem 19.1.1. With $\alpha_0 \in (\frac{N}{2}, \alpha]$ satisfying $\alpha_0 < 1 - \frac{N}{T} \chi^2 + \frac{N}{T}$, we may view $u_0$ as an element of $D(A^{\alpha_0})$ and apply Lemma 19.2.1 so as to obtain a solution that either exists globally or satisfies

$$\lim_{t \to T_{\text{max}}} \left( \|n(\cdot, t)\|_{L^\infty(\Omega)} + \|c(\cdot, t)\|_{W^{1,q}(\Omega)} + \|A^{\alpha_0}u(\cdot, t)\|_{L^2(\Omega)} \right) = \infty. \tag{19.4.12}$$

If $T_{\text{max}}$ were finite, we could apply Lemma 19.3.3 or Lemma 19.3.6 and Lemma 19.4.2 with $T = T_{\text{max}}$ so as to see that there is $C_1 > 0$ such that

$$\|n(\cdot, t)\|_{L^\infty(\Omega)} + \|A^{\alpha_0}u(\cdot, t)\|_{L^2(\Omega)} \leq C_1$$

for all $t \in (0, T_{\text{max}})$. Combining the boundedness of $n$ with Lemma 19.4.1 and invoking Lemma 19.2.5, again for $T = T_{\text{max}}$, then would give rise to a contradiction to (19.4.12). Therefore, $T_{\text{max}} = \infty$. \qed
Chapter 20

Existence of global weak solutions to a 3-dimensional degenerate and singular chemotaxis-Navier–Stokes system with logistic source

20.1. Background and results

In the study of partial differential equations mathematical models describing the natural phenomena, e.g., the heat equation, the Fisher–KPP equation and so on, are one of the important topics in the mathematical analysis, and studied by many mathematicians. Recently, there have been many variations of systems of partial differential equations which describe complicated phenomena. One of systems which describe important biological phenomena related to animals life is the Keller–Segel system

\[ n_t = \Delta n^m - \chi \nabla \cdot (n^{q-1} \nabla c), \]
\[ c_t = \Delta c - c + n, \quad x \in \Omega, \quad t > 0, \]

where \( \Omega \subset \mathbb{R}^N \) (\( N \in \mathbb{N} \)), \( m > 0 \), \( \chi \geq 0 \), \( q \geq 2 \), which describes migration of species by chemotaxis which is the property such that species move towards higher concentration of the chemical substance. Here \( \Delta n^m \) with \( m = 1 \), that is, \( \Delta n \) is called a linear diffusion and \( \Delta n^m \) with \( m \neq 1 \) is called a nonlinear diffusion. In other words, the case that \( m > 1 \) is said to be a degenerate diffusion or a porous medium diffusion, and the case that \( 0 < m < 1 \) is said to be a singular diffusion or a fast diffusion. This system with \( m = 1 \) and \( q = 2 \) was first proposed by Keller–Segel [89] and then the system with \( m, q \in \mathbb{R} \) was suggested by Hillen–Painter [71]. In the case that \( m = 1 \) and \( q = 2 \) it is known that the size of initial data determines behaviour of solutions to the 2-dimensional Keller–Segel system. More precisely, there exists some constant \( C > 0 \) such that if an initial data \( n_0 \) satisfies \( \|n_0\|_{L^1(\Omega)} < C \) then global bounded classical solutions exist ([144]), moreover, for any \( M > C \) there exist initial data such that \( \|n_0\|_{L^1(\Omega)} = M \) and the corresponding solution blows up in finite time ([76, 127]), where \( C > 0 \) can be given as \( C = \frac{8\chi}{\lambda} \) in the radial setting and \( C = \frac{4\chi}{\lambda} \) in the nonradial setting.
In the other dimensional case we can see that there are many global/blow-up solutions. In the case that $m = 1$, $q = 2$ and $N = 1$ Osaki–Yagi [153] showed that global existence and boundedness hold for all smooth initial data: This means that there is not a blow-up solution in the 1-dimensional setting. On the other hand, it is known that the 3-dimensional case has many blow-up solutions: Winkler [198] established that for all $m > 0$ there are initial data $n_0$ such that $\|n_0\|_{L^1(\Omega)} = m$ and the corresponding solution blows up in finite time. To obtain global existence of classical solutions we need some additional conditions for initial data: Winkler [194] and Cao [25] proved existence of global boundedness classical solutions under conditions that initial data $(n_0, c_0)$ is small enough with respect to a suitable Lebesgue norm.

On the other hand, in the case that $m \geq 1$ and $q \geq 2$, it is known that relations between $m$ and $q$ determine whether solutions of the Keller–Segel system exist globally or not; in the case that $m > q - \frac{2}{N}$ Ishida–Seki–Yokota [81] obtained global existence of solutions; on the other hand, in the case that $m < q - \frac{2}{N}$ Ishida–Yokota [82] proved that there exist initial data such that the corresponding solution blows up in finite or infinite time, and recently Hashira–Ishida–Yokota [66] found initial data such that the solution blows up in finite time.

As a generalized problem, the nonlinear Keller–Segel system with logistic source in which the first equation in the above system is replaced with

$$n_t = \Delta n^m - \chi \nabla \cdot (n^{q-1} \nabla c) + \kappa n - \mu n^2,$$

where $m, \mu > 0$, $\chi, \kappa \geq 0$, $q \geq 2$, was also studied, and it is shown that the logistic source $\kappa n - \mu n^2$ suppresses blow-up phenomenon. When $m = 1$ and $q = 2$, in the 2-dimensional setting Osaki et al. [152] obtained global existence and boundedness for all smooth initial data, and in the higher dimensional setting Winkler [195] proved existence of global classical solutions under largeness conditions for $\mu > 0$; in the Keller–Segel system with logistic source global existence of solutions holds even though the $L^1$-norm of the initial data is large enough. Moreover, Lankeit [100] established global existence of weak solutions without largeness condition for $\mu > 0$. On the other hand, in the case that $q = 2$ a resent result established by Zheng–Wang [219] asserted that the condition that $m > 1 + \frac{(N-2)\lambda}{N+2}$ derives global existence and boundedness.

As we confirmed before, in the Keller–Segel system the relation between $m$ and $q$ strongly affects behaviour of solutions, and the logistic source often relaxes conditions for global existence. Thus “what is the condition for $m$ to derive existence of global/blow-up solutions?” makes one of the main topics. This will be also one of the important topics in the study of the chemotaxis-(Navier–Stokes) system with logistic source

$$n_t + u \cdot \nabla n = \Delta n^m - \chi \nabla \cdot (n^{q-1} \nabla c) + \kappa n - \mu n^2,$$
$$c_t + u \cdot \nabla c = \Delta c - nc,$$
$$u_t + \lambda (u \cdot \nabla) u = \Delta u + \nabla P + n \nabla \Phi, \ \nabla \cdot u = 0, \quad x \in \Omega, \ t > 0,$$

where $m, \mu > 0$, $\chi, \kappa \geq 0$, $q \geq 2$, $\lambda = 0$ (the chemotaxis-Stokes system) or $\lambda = 1$ (the chemotaxis-Navier–Stokes system). In the case that $m = 1$, $q = 2$ and moreover $\kappa = \mu = 0$
global existence of classical solutions in the 2-dimensional setting was established by Winkler [197] and existence of global weak solutions in the 3-dimensional setting was shown by Winkler [204] and He–Zhang [67]. In the case that \( q = 2 \) and \( \kappa = \mu = 0 \) Tao–Winkler [171] and Chung–Kang [34] showed global existence under the condition that \( m > 1 \).

On the other hand, in the chemotaxis-(Navier-)Stokes system with logistic source results similar to those in the Keller–Segel system with logistic source hold. In the case that \( q = 2 \) and \( \kappa = \mu = 0 \) Tao–Winkler [171] and Chung–Kang [34] showed global existence under the condition that \( m > 1 \).

In order to attain this purpose we consider the following degenerate and singular chemotaxis-Navier–Stokes system with logistic term:

\[
\begin{align*}
n_t + u \cdot \nabla n &= \Delta n^m - \chi \nabla \cdot (n \nabla c) + \kappa n - \mu n^2, & x \in \Omega, & t > 0, \\
c_t + u \cdot \nabla c &= \Delta c - nc, & x \in \Omega, & t > 0, \\
u_t + (u \cdot \nabla)u &= \Delta u + \nabla P + n \nabla \Phi, & \nabla \cdot u = 0, & x \in \Omega, & t > 0, \\
\partial_n n^m &= \partial_n c = 0, & u = 0, & x \in \partial \Omega, & t > 0, \\
n(x, 0) &= n_0(x), & c(x, 0) &= c_0(x), & u(x, 0) &= u_0(x), & x \in \Omega,
\end{align*}
\]
where $\Omega$ is a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial \Omega$ and $\partial_\nu$ denotes differentiation with respect to the outward normal of $\partial \Omega$; $\chi, \kappa \geq 0$ and $\mu, m > 0$ are constants; $n_0, c_0, u_0, \Phi$ are known functions satisfying

\begin{align}
0 < n_0 &\in C(\overline{\Omega}), \quad 0 < c_0 \in W^{1,q}(\Omega), \quad u_0 \in D(A^\theta), \quad (20.1.2) \\
\Phi &\in C^{1+\beta}(\overline{\Omega}) \quad (20.1.3)
\end{align}

for some $q > 3$, $\theta \in (\frac{3}{4}, 1)$, $\beta > 0$ and $A$ denotes the realization of the Stokes operator under homogeneous Dirichlet boundary conditions in the solenoidal subspace $L^2_\sigma(\Omega)$ of $L^2(\Omega)$.

Before stating the main theorem, we define weak solutions of (20.1.1).

**Definition 20.1.1.** A triplet $(n, c, u)$ is called a (global) weak solution of (20.1.1) if

\begin{align}
n &\in L^2_{\text{loc}}([0, \infty); L^2(\Omega)), \\
n^m &\in L^2_{\text{loc}}([0, \infty); W^{1,\frac{q}{2}}(\Omega)), \\
c &\in L^2_{\text{loc}}([0, \infty); W^{1,2}(\Omega)), \\
u &\in L^2_{\text{loc}}([0, \infty); W^{1,2}_\sigma(\Omega))
\end{align}

and the identities

\begin{align}
-\int_0^\infty \int_\Omega n\varphi_t - \int_\Omega n_0\varphi(\cdot, 0) - \int_0^\infty \int_\Omega n\cdot \nabla \varphi
&=- \int_0^\infty \int_\Omega \nabla n^m \cdot \nabla \varphi + \chi \int_0^\infty \int_\Omega n\nabla c \cdot \nabla \varphi + \int_0^\infty \int_\Omega (\kappa n - \mu n^2) \varphi, \\
-\int_0^\infty \int_\Omega c\varphi_t - \int_\Omega c_0\varphi(\cdot, 0) - \int_0^\infty \int_\Omega cu \cdot \nabla \varphi
&= - \int_0^\infty \int_\Omega c \cdot \nabla \varphi - \int_0^\infty \int_\Omega nc \varphi,
\end{align}

hold for all $\varphi \in C^\infty_0(\overline{\Omega} \times [0, \infty))$ and all $\psi \in C^\infty_0(\Omega \times [0, \infty))$, respectively.

The main result reads as follows. The following theorem gives existence of global weak solutions to (20.1.1).

**Theorem 20.1.1.** Let $\Omega \subset \mathbb{R}^3$ be a bounded smooth domain and let $\chi, \kappa \geq 0$ and $\mu, m > 0$. Assume that $n_0, c_0, u_0$ and $\Phi$ satisfy (20.1.2)–(20.1.3) with some $q > 3$, $\theta \in (\frac{3}{4}, 1)$ and $\beta \in (0, 1)$. Then there exists a weak solution of (20.1.1), which can be approximated by a sequence of solutions $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ of an approximate problem (see Section 20.2) in a pointwise manner.

**Remark 20.1.1.** This result means existence of weak solutions to (20.1.1) for all $m > 0$; which implies that we could construct weak solutions of (20.1.1) in not only the case that $m > 1$ (the case of a porous medium diffusion) but also $0 < m < 1$ (the case of a fast diffusion).
The proof of Theorem 20.1.1 can be applied to a nondegenerate chemotaxis-Navier-Stokes system which namely is the case that \( \Delta n^m \) is replaced with \( \Delta (n+1)^m \), and enables us to see the following result.

**Corollary 20.1.2.** Let \( \Omega \subset \mathbb{R}^3 \) be a bounded smooth domain and let \( \chi, \kappa \geq 0 \) and \( \mu, m > 0 \). Assume that \( n_0, c_0, u_0 \) and \( \Phi \) satisfy (20.1.2)–(20.1.3) with some \( q > 3 \), \( \theta \in (\frac{2}{3}, 1) \) and \( \beta \in (0, 1) \). Then there exists a weak solution of the nondegenerate chemotaxis-Navier-Stokes system.

The strategy for the proof of Theorem 20.1.1 is described as follows. We start with the construction of local approximate solutions of (20.1.1). We next derive estimates for the Stokes system. This together with the identity

\[
\int_0^T \int_{\Omega} (n_\varepsilon + \varepsilon)^{m-1} |\nabla n_\varepsilon|^2 \frac{\varepsilon}{n_\varepsilon} = \left( \frac{2}{m+1} \right)^2 \int_0^T \int_{\Omega} |\nabla (n_\varepsilon + \varepsilon)^{\frac{m+1}{4}}|^2 \leq C_1(T)
\]

for all \( \varepsilon \in (0, 1) \) with some constant \( C_1(T) > 0 \). Aided by this estimate we can have that \( (n_\varepsilon + \varepsilon)^{\frac{m+1}{4}} \) is bounded in \( L^\frac{4}{3}([0, T]; W^{1, \frac{4}{3}}(\Omega)) \); however, it seems to be difficult to obtain the estimate for \( \partial_t (n_\varepsilon + \varepsilon)^{\frac{m+1}{4}} \) for all \( m > 0 \). Thus we need additional estimates for approximate solutions. Here the inequality (20.1.4) ensures that

\[
\int_0^T \int_{\Omega} (n_\varepsilon + \varepsilon)^{m-2} |\nabla n_\varepsilon|^2 \leq \int_0^T \int_{\Omega} \frac{(n_\varepsilon + \varepsilon)^{m-1}}{n_\varepsilon} |\nabla n_\varepsilon|^2 \leq C_1(T)
\]

for all \( \varepsilon \in (0, 1) \). This together with the identity

\[
\int_0^T \int_{\Omega} |\nabla (n_\varepsilon + \varepsilon)^\frac{m}{4}|^2 = \frac{m^2}{4} \int_0^T \int_{\Omega} (n_\varepsilon + \varepsilon)^{m-2} |\nabla n_\varepsilon|^2
\]

means that \( (n_\varepsilon + \varepsilon)^\frac{m}{4} \) is bounded in \( L^2_{\text{loc}}([0, \infty); W^{1, 2}(\Omega)) \). We moreover see that

\[
\|\partial_t (n_\varepsilon + \varepsilon)^\frac{m}{4}\|_{L^1(0,T; (W^{2,4}_0(\Omega))^*)} \leq C_2(T)
\]

for all \( \varepsilon \in (0, 1) \) with some \( C_2(T) > 0 \), which derives that \( (\partial_t (n_\varepsilon + \varepsilon)^\frac{m}{4})_{\varepsilon \in (0,1)} \) is bounded in \( L^1(0,T; (W^{2,4}_0(\Omega))^*) \). Then, aided by the Lions–Aubin theorem, we can show convergences of solutions of the approximation of (20.1.1) and we can prove Theorem 20.1.1.
20.2. Global existence in an approximate problem

We start by considering the following approximate problem with parameter \( \varepsilon > 0 \):

\[
\begin{align*}
    &n_{\varepsilon} + u_{\varepsilon} \cdot \nabla n_{\varepsilon} = \Delta n_{\varepsilon} + \varepsilon^m - \chi \nabla \cdot \left( \frac{n_{\varepsilon}}{1 + n_{\varepsilon}} \nabla c_{\varepsilon} \right) + \kappa n_{\varepsilon} - \mu n_{\varepsilon}^3, \\
    &c_{\varepsilon} + u_{\varepsilon} \cdot \nabla c_{\varepsilon} = \Delta c_{\varepsilon} - c_{\varepsilon} \frac{1}{\varepsilon} \log (1 + \varepsilon n_{\varepsilon}), \\
    &u_{\varepsilon} + (Y_{\varepsilon} u_{\varepsilon} \cdot \nabla) u_{\varepsilon} = \Delta u_{\varepsilon} + \nabla P_{\varepsilon} + n_{\varepsilon} \nabla \Phi, \quad \nabla \cdot u_{\varepsilon} = 0, \\
    &\partial_{\nu} n_{\varepsilon} |_{\partial \Omega} = \partial_{\nu} c_{\varepsilon} |_{\partial \Omega} = 0, \quad u_{\varepsilon} |_{\partial \Omega} = 0, \\
    &n_{\varepsilon}(\cdot, 0) = n_0, \quad c_{\varepsilon}(\cdot, 0) = c_0, \quad u_{\varepsilon}(\cdot, 0) = u_0,
\end{align*}
\]

(20.2.1)

where \( Y_{\varepsilon} = (1 + \varepsilon A)^{-1} \). In this section we shall show global existence of solutions to the approximate problem (20.2.1). We first give the following result which states local existence in (20.2.1).

**Lemma 20.2.1.** Let \( \chi, \kappa \geq 0, \mu > 0, m > 0 \) and let \( \Phi \in C^{1+\beta}(\overline{\Omega}) \) for some \( \beta \in (0, 1) \). Assume that \( n_0, c_0, u_0 \) satisfy (20.1.2) with some \( q > 3, \theta (\frac{3}{4}, 1) \). Then for each \( \varepsilon > 0 \) there exist \( T_{\text{max}, \varepsilon} \) and uniquely determined functions:

\[
\begin{align*}
    &n_{\varepsilon} \in C^0(\overline{\Omega} \times [0, T_{\text{max}, \varepsilon}]) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\text{max}, \varepsilon})), \\
    &c_{\varepsilon} \in C^0(\overline{\Omega} \times [0, T_{\text{max}, \varepsilon}]) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\text{max}, \varepsilon})) \cap L^\infty([0, T_{\text{max}, \varepsilon}); W^{1,q}(\Omega)), \\
    &u_{\varepsilon} \in C^0(\overline{\Omega} \times [0, T_{\text{max}, \varepsilon}]) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\text{max}, \varepsilon})),
\end{align*}
\]

which together with some \( P_{\varepsilon} \in C^{1,0}(\overline{\Omega} \times (0, T_{\text{max}, \varepsilon})) \) solve (20.2.1) classically. Moreover, \( n_{\varepsilon} \) and \( c_{\varepsilon} \) are positive and the following alternative holds: \( T_{\text{max}, \varepsilon} = \infty \) or

\[
\| n_{\varepsilon}(\cdot, t) \|_{L^\infty(\Omega)} + \| c_{\varepsilon}(\cdot, t) \|_{W^{1,q}(\Omega)} + \| A^\theta u_{\varepsilon}(\cdot, t) \|_{L^2(\Omega)} \rightarrow \infty
\]

as \( t \nearrow T_{\text{max}, \varepsilon} \).

**Proof.** This lemma can be shown by a standard fixed point theorem with parabolic regularity arguments. More precisely, a combination of the proofs of [168, Lemma 2.1] and [197, Lemma 2.1] enables us to obtain local existence in (20.2.1).

In the following for all \( \varepsilon \in (0,1) \) we denote by \( (n_{\varepsilon}, c_{\varepsilon}, u_{\varepsilon}) \) the corresponding solution of (20.2.1) given by Lemma 20.2.1 and by \( T_{\text{max}} \) its maximal existence time. Then we shall see global existence of solutions to the approximate problem (20.2.1) and their useful estimates. We first recall basic inequalities which are often used in studies of the chemotaxis-Navier–Stokes system.

**Lemma 20.2.2.** There exists a constant \( C_1 > 0 \) such that

\[
\int_{\Omega} n_{\varepsilon}(\cdot, t) \leq C_1 \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon}) \text{ and all } \varepsilon > 0.
\]

Moreover, there exists \( C_2 > 0 \) satisfying

\[
\int_{t}^{t+\tau} \int_{\Omega} n_{\varepsilon}^2 \leq C_2
\]

holds for all \( t \in (0, T_{\text{max}, \varepsilon} - \tau) \) and all \( \varepsilon > 0 \), where \( \tau \in (0, T_{\text{max}, \varepsilon}) \).
Proof. Integrating the first equation in (20.2.1) shows this lemma. □

Lemma 20.2.3. The function \( t \mapsto \|c(\cdot, t)\|_{L^\infty(\Omega)} \) is nonincreasing. In particular,

\[
\|c(\cdot, t)\|_{L^\infty(\Omega)} \leq \|c_0\|_{L^\infty(\Omega)}
\]

holds for all \( t \in (0, T_{\text{max}, \varepsilon}) \) and all \( \varepsilon > 0 \). Moreover, we have

\[
\int_0^{T_{\text{max}, \varepsilon}} \int_\Omega |\nabla c_\varepsilon|^2 \leq \frac{1}{2} \int_\Omega |c_0|^2 \quad \text{for all } \varepsilon > 0.
\]

Proof. Applying the maximum principle to the second equation in (20.2.1) (see e.g., [199, Lemma 2.1]), we can establish the \( L^1 \)-estimate for \( c_\varepsilon \). Moreover, multiplying the third equation in (20.2.1) by \( c_\varepsilon \) and integrating it over \( \Omega \times (0, T_{\text{max}, \varepsilon}) \) imply this lemma. □

We next consider an estimate for the energy function \( \mathcal{F}_\varepsilon : (0, T_{\text{max}}) \to \mathbb{R} \) defined as

\[
\mathcal{F}_\varepsilon(t) := \int_\Omega n_\varepsilon(\cdot, t) \log n_\varepsilon(\cdot, t) + \frac{\chi}{2} \int_\Omega \frac{|\nabla c_\varepsilon(\cdot, t)|^2}{c_\varepsilon(\cdot, t)} + K\chi \int_\Omega |u(\cdot, t)|^2
\]

with some \( K > 0 \), which plays an important role not only in the case that \( m = 1 ([101]) \) but also in the case that \( m > 0 \). In order to see an estimate for \( \mathcal{F}_\varepsilon \) we provide the following 3 lemmas.

Lemma 20.2.4. There exists a constant \( C > 0 \) such that for all \( \varepsilon > 0 \),

\[
\frac{d}{dt} \int_\Omega n_\varepsilon \log n_\varepsilon + \frac{\mu}{2} \int_\Omega n_\varepsilon^2 \log n_\varepsilon + \frac{4m}{(m+1)^2} \int_\Omega \frac{|\nabla (n_\varepsilon + \varepsilon)|^{m+1}}{n_\varepsilon} \leq \chi \int_\Omega \nabla n_\varepsilon \cdot \nabla c_\varepsilon + C
\]

on \( (0, T_{\text{max}, \varepsilon}) \).

Proof. We first obtain from \( \nabla \cdot u_\varepsilon = 0 \) in \( \Omega \times (0, T_{\text{max}}) \) and straightforward calculations that

\[(20.2.2) \quad \frac{d}{dt} \int_\Omega n_\varepsilon \log n_\varepsilon = \int_\Omega \log n_\varepsilon \Delta (n_\varepsilon + \varepsilon)^m - \chi \int_\Omega \log n_\varepsilon \nabla \cdot \left( \frac{n_\varepsilon}{1 + \varepsilon n_\varepsilon} \nabla c_\varepsilon \right) + \kappa \int_\Omega n_\varepsilon \log n_\varepsilon - \mu \int_\Omega n_\varepsilon^2 \log n_\varepsilon + \kappa \int_\Omega n_\varepsilon = \mu \int_\Omega n_\varepsilon^2.
\]

Then, noting from the boundedness of the functions \( s \mapsto (\kappa s - \frac{\mu}{2} s_2^2) \) log \( s + \kappa s - \mu s_2^2 \) on \((0, \infty)\) that

\[
\kappa \int_\Omega n_\varepsilon \log n_\varepsilon - \frac{\mu}{2} \int_\Omega n_\varepsilon^2 \log n_\varepsilon + \kappa \int_\Omega n_\varepsilon - \mu \int_\Omega n_\varepsilon^2 \leq C_1
\]

with some \( C_1 > 0 \), we can see from (20.2.2) with the relation

\[
\int_\Omega \log n_\varepsilon \Delta (n_\varepsilon + \varepsilon)^m = -m \int_\Omega (n_\varepsilon + \varepsilon)^{m-1} \frac{\nabla n_\varepsilon}{n_\varepsilon} = -\frac{4m}{(m+1)^2} \int_\Omega \frac{|\nabla (n_\varepsilon + \varepsilon)|^{m+1}}{n_\varepsilon}^2
\]

that this lemma holds. □
The following 2 lemmas have already been proved in the proofs of \([101, \text{Lemmas} 2.8 \text{and } 2.9]\). Thus we only recall statements of lemmas.

**Lemma 20.2.5.** There exist $K, C, k > 0$ such that for all $\varepsilon > 0$,
\[
\frac{d}{dt} \int_\Omega \frac{\nabla c_\varepsilon}{c_\varepsilon} |^2 + k \int_\Omega c_\varepsilon |D^2 \log c_\varepsilon|^2 + k \int_\Omega \frac{\nabla c_\varepsilon}{c_\varepsilon} |^4 \leq C + K \int_\Omega |\nabla u_\varepsilon|^2 - 2 \int_\Omega \frac{\nabla c_\varepsilon \cdot \nabla n_\varepsilon}{1 + \varepsilon n_\varepsilon} \quad \text{on } (0, T_{\max, \varepsilon}).
\]

**Lemma 20.2.6.** For all $\eta > 0$ there exists $C_\eta > 0$ such that for all $\varepsilon > 0$,
\[
\frac{d}{dt} \int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 \leq \eta \int_\Omega n_\varepsilon^2 \log n_\varepsilon + C_\eta \quad \text{on } (0, T_{\max, \varepsilon}).
\]

Thanks to these lemmas, we can establish the estimate for $\frac{dF_\varepsilon}{dt}$ which enables us to derive the desired estimate for $F_\varepsilon$.

**Lemma 20.2.7.** Let $K$ be the constant obtained in Lemma 20.2.5. Then there exist $C, k_0 > 0$ satisfying
\[
(20.2.3) \quad \quad \frac{d}{dt} \left[ \int_\Omega n_\varepsilon \log n_\varepsilon + \frac{\chi}{2} \int_\Omega \frac{\nabla c_\varepsilon}{c_\varepsilon} |^2 + K \chi \int_\Omega |u_\varepsilon|^2 \right] + \frac{\mu}{4} \int_\Omega n_\varepsilon^2 \log n_\varepsilon + \frac{4m}{(m + 1)^2} \int_\Omega \frac{\nabla (n_\varepsilon + \varepsilon)^{m+1}}{n_\varepsilon}^2 + k_0 \int_\Omega c_\varepsilon |D^2 \log c_\varepsilon|^2 + k_0 \int_\Omega \frac{\nabla c_\varepsilon}{c_\varepsilon} |^4 + k_0 \int_\Omega |\nabla u_\varepsilon|^2 \leq C \quad \text{on } (0, T_{\max, \varepsilon}) \quad \text{for all } \varepsilon > 0.
\]

**Proof.** This lemma can be derived by a combination of Lemmas 20.2.4, 20.2.5 and 20.2.6 with $\eta := \frac{\mu}{4K\chi}$.

Now we are in a position to see the estimate for $F_\varepsilon$ uniformly-in-$\varepsilon$.

**Lemma 20.2.8.** Let $K$ be the constant obtained in Lemma 20.2.5. Then there exists $C > 0$ such that
\[
F_\varepsilon(t) = \int_\Omega n_\varepsilon(\cdot, t) \log n_\varepsilon(\cdot, t) + \frac{\chi}{2} \int_\Omega \frac{\nabla c_\varepsilon(\cdot, t)}{c_\varepsilon(\cdot, t)} |^2 + K \chi \int_\Omega |u_\varepsilon(\cdot, t)|^2 \leq C
\]
for all $t \in (0, T_{\max})$ and all $\varepsilon > 0$ and
\[
\int_t^{t+\tau} \int_\Omega n_\varepsilon^2 \log n_\varepsilon + \int_t^{t+\tau} \int_\Omega \frac{\nabla (n_\varepsilon + \varepsilon)^{m+1}}{n_\varepsilon}^2 + \int_t^{t+\tau} \int_\Omega c_\varepsilon |D^2 \log c_\varepsilon|^2 \leq C,
\]
\[
\int_t^{t+\tau} \int_\Omega \frac{\nabla c_\varepsilon}{c_\varepsilon} |^4 + \int_t^{t+\tau} \int_\Omega |\nabla u_\varepsilon|^2 \leq C,
\]
\[
\int_t^{t+\tau} \int_\Omega \nabla (n_\varepsilon + \varepsilon)^{m+1} |^4 + \int_\Omega |\nabla c_\varepsilon|^2 + \int_t^{t+\tau} \int_\Omega |\nabla c_\varepsilon|^4 + \int_t^{t+\tau} \int_\Omega n_\varepsilon^2 \leq C
\]
for all $t \in [0, T_{\max, \varepsilon} - \tau)$ and all $\varepsilon > 0$, where $\tau := \min \{1, \frac{1}{2} T_{\max, \varepsilon}\}$.
Proof. The proof is based on that of [101, Lemma 2.11]. Noticing from the inequalities $s \log s \leq \frac{1}{2x} + s^2 \log s$, $\int_{\Omega} \frac{|\nabla c|^2}{c^4} \leq \|c_0\|_{L^\infty(\Omega)} \int_{\Omega} \frac{|\nabla c|^4}{c^4} + \|\Omega\|$ (from Lemma 20.2.3) and $\int_{\Omega} |u_\varepsilon|^2 \leq C_1 \int_{\Omega} |\nabla u_\varepsilon|^2$ with some $C_1 > 0$ (from the Poincaré inequality) that Lemma 20.2.7 implies

$$\frac{dF_\varepsilon}{dt} + k_1 F_\varepsilon \leq k_2$$

with some $k_1, k_2 > 0$, we establish the boundedness of $F_\varepsilon$ on $(0, T_{\max})$. Then for $\tau = \min\{1, \frac{1}{2} T_{\max}\}$ integrating the inequality (20.2.3) over $(t, t + \tau)$ with Lemmas 20.2.2 and 20.2.3 implies this lemma.

Then we shall establish global existence in approximate problem (20.2.1) by using a Moser–Alikakos-type procedure.

**Lemma 20.2.9.** For all $\varepsilon \in (0, 1)$, $T_{\max, \varepsilon} = \infty$ holds.

Proof. The proof is based on that of [204, Lemma 3.9]. Assume that $T_{\max, \varepsilon} < \infty$ and put $p := \min\{3 + m, 4\}$. We shall first verify the $L^p$-estimate for $n_\varepsilon$. We see from the first equation and the fact $\nabla \cdot u_\varepsilon = 0$ on $\Omega \times (0, \infty)$ that

$$\frac{1}{p} \frac{d}{dt} \int_{\Omega} n_\varepsilon^p = \int_{\Omega} n_\varepsilon^{p-1} \nabla \cdot \left( m(n_\varepsilon + \varepsilon)^{m-1} \nabla n_\varepsilon - \frac{n_\varepsilon}{1 + \varepsilon n_\varepsilon} \nabla c_\varepsilon \right)$$

$$+ \int_{\Omega} n_\varepsilon^{p-1} (C_\varepsilon - \mu n_\varepsilon^4) - \frac{1}{p} \int_{\Omega} u_\varepsilon \cdot \nabla n_\varepsilon^p$$

$$= -m(p - 1) \int_{\Omega} n_\varepsilon^{p-2}(n_\varepsilon + \varepsilon)^{m-1} |\nabla n_\varepsilon|^2$$

$$+ \chi(p - 1) \int_{\Omega} \frac{n_\varepsilon^{p-1}}{1 + \varepsilon n_\varepsilon} \nabla n_\varepsilon \cdot \nabla c_\varepsilon + \kappa \int_{\Omega} n_\varepsilon^p - \mu \int_{\Omega} n_\varepsilon^{p+1}.$$

Here, since $2p - 4 + 2(1 - m)_+ < p + 1$, the Young inequality yields that

$$\chi(p - 1) \int_{\Omega} \frac{n_\varepsilon^{p-1}}{1 + \varepsilon n_\varepsilon} \nabla n_\varepsilon \cdot \nabla c_\varepsilon$$

$$\leq \frac{\chi(p - 1)}{\varepsilon} \int_{\Omega} n_\varepsilon^{p-2}(n_\varepsilon + \varepsilon)^{m-1} |\nabla n_\varepsilon| n_\varepsilon^{p-1}(n_\varepsilon + \varepsilon)^{-m-1} |\nabla c_\varepsilon|$$

$$\leq \frac{m(p - 1)}{2} \int_{\Omega} n_\varepsilon^{p-2}(n_\varepsilon + \varepsilon)^{m-1} |\nabla n_\varepsilon|^2 + \int_{\Omega} n_\varepsilon^{2p-4}(n_\varepsilon + \varepsilon)^{2(1-m)} + C_1 \int_{\Omega} |\nabla c_\varepsilon|^4$$

$$\leq \frac{m(p - 1)}{2} \int_{\Omega} n_\varepsilon^{p-2}(n_\varepsilon + \varepsilon)^{m-1} |\nabla n_\varepsilon|^2 + \frac{\mu}{2} \int_{\Omega} n_\varepsilon^{p+1} + C_2 + C_1 \int_{\Omega} |\nabla c_\varepsilon|^4$$

on $(0, T_{\max, \varepsilon})$ with some $C_1 = C_1(\varepsilon) > 0$ and $C_2 = C_2(\varepsilon) > 0$, where we used the inequalities $(a + b)^r \leq 2^r(a^r + b^r)$ $(a, b \geq 0, r > 0)$ and $(a + b)^r \leq b^r (a, b \geq 0, r \leq 0)$ to obtain the last inequality. Therefore we obtain from the positivity of $n_\varepsilon$ that

$$\frac{1}{p} \frac{d}{dt} \int_{\Omega} n_\varepsilon^p \leq C_1 \int_{\Omega} |\nabla c_\varepsilon|^4 + \kappa \int_{\Omega} n_\varepsilon^p + C_2.$$
Thus it follows from Lemma 20.2.8 that
\[
\int_{\Omega} n_{\varepsilon}^p \leq C_3 \quad \text{on } (0, T_{\max, \varepsilon}),
\]
where \( C_3 = C_3(\varepsilon) > 0 \). Then, aided by the \( L^2 \)-estimate for \( \nabla u_{\varepsilon} \) (from a testing argument), we can obtain that
\[
\| A^6 u_{\varepsilon}(\cdot, t) \|_{L^2(\Omega)} \leq C_4
\]
for all \( t \in (0, T_{\max}) \) with some \( C_4 = C_4(\varepsilon) > 0 \). Then the continuous embedding \( D(A^6) \hookrightarrow L^\infty(\Omega) \) implies the \( L^\infty \)-estimate for \( u_{\varepsilon} \). Using these estimates a standard \( L^p-L^q \) estimate for the Neumann heat semigroup on bounded domains and the inequality
\[
\| u(\cdot, t)\nabla c(\cdot, t) \|_{L^2(\Omega)} \leq \| u(\cdot, t) \|_{L^\infty(\Omega)} \| \nabla c(\cdot, t) \|_{L^2(\Omega)} \| \nabla c(\cdot, t) \|_{L^2(\Omega)}
\]
and the \( L^2 \)-estimate for \( \nabla c_{\varepsilon} \) from Lemma 20.2.8 imply the \( L^6 \)-estimate for \( \nabla c_{\varepsilon} \) (cf. an argument in the proof of Lemma 11.3.10). Finally we shall verify the \( L^\infty \)-estimate for \( n_{\varepsilon} \).

We can see from \( \nabla \cdot u_{\varepsilon} = 0 \) in \( \Omega \times (0, T_{\max}) \) that for all \( r \geq 1 \)
\[
\frac{d}{dt} \int_{\Omega} (n_{\varepsilon} + \varepsilon)^r + mr(r - 1) \int_{\Omega} (n_{\varepsilon} + \varepsilon)^{r+m-3} |\nabla (n_{\varepsilon} + \varepsilon)|^2 \leq r(r - 1) \chi \int_{\Omega} (n_{\varepsilon} + \varepsilon)^{r-2} \frac{n_{\varepsilon}}{1 + \varepsilon n_{\varepsilon}} \nabla n_{\varepsilon} \cdot \nabla c_{\varepsilon} + \kappa \int_{\Omega} (n_{\varepsilon} + \varepsilon)^{r-1} n_{\varepsilon}
\]
on \( (0, T_{\max}) \). Thus, noting that
\[
\frac{n_{\varepsilon}}{1 + \varepsilon n_{\varepsilon}} \nabla c_{\varepsilon} \in L^\infty(0, T_{\max, \varepsilon}; L^6(\Omega)), \quad n_{\varepsilon} \in L^\infty(0, T_{\max, \varepsilon}; L^{\min\{3+m,4\}}(\Omega)),
\]
from a Moser–Alikakos-type procedure (see the proof of [169, Lemma A.1]), we can attain that
\[
\| n_{\varepsilon}(\cdot, t) \|_{L^\infty(\Omega)} \leq C_5
\]
for all \( t \in (0, T_{\max}) \) with some \( C_5 = C_5(\varepsilon) > 0 \), which with extensibility criterion shows \( T_{\max} = \infty \) for each \( \varepsilon \in (0, 1) \).

\( \square \)

### 20.3. Uniform-in-\( \varepsilon \) estimates

In this section we collect lemmas which are needed to show convergence of solutions of (20.2.1) as \( \varepsilon \searrow 0 \). Here the case that \( m = 1 \) has already been dealt with in [101]. Thus we shall consider the case that \( m > 0 \) with \( m \neq 1 \). From Lemma 20.2.8 we only know not some estimate for \( \nabla n_{\varepsilon} \) but \( L^\frac{4}{3}_{\text{loc}}([0, \infty); W^{1,\frac{4}{3}}(\Omega)) \)-boundedness of \( (n_{\varepsilon} + \varepsilon)^\frac{m+1}{m} )_{\varepsilon \in (0, 1)} \). However, it seems to be difficult to derive an enough regularity of \( \partial_t (n_{\varepsilon} + \varepsilon)^\frac{m+1}{m} \) for each \( m > 0 \). Therefore we need to establish an estimate for \( \nabla (n_{\varepsilon} + \varepsilon)^\gamma \) with some \( \gamma < \frac{m+1}{2} \).

The following lemma is a cornerstone in the proof of Theorem 20.1.1.

**Lemma 20.3.1.** For all \( T > 0 \) there exists a constant \( C = C(T) > 0 \) such that
\[
\int_0^T \int_{\Omega} |\nabla (n_{\varepsilon} + \varepsilon)^\frac{\gamma}{2}|^2 \leq C \quad \text{on } (0, T).
\]
Proof. In light of Lemma 20.2.8 we see that there exists $C_1 = C_1(T) > 0$ such that
\[
\int_0^T \int_{\Omega} |\nabla (n_\varepsilon + \varepsilon)\frac{m}{m+1}|^2 = \left(\frac{m}{m+1}\right)^2 \int_0^T \int_{\Omega} |\nabla (n_\varepsilon + \varepsilon)\frac{m+1}{2}|^2
\leq \int_0^T \int_{\Omega} |\nabla (n_\varepsilon + \varepsilon)\frac{m+1}{2}|^2 \leq C_1
\]
with $C_1 = C_1(T) > 0$.

We next confirm the following lemma which will play an important role in deriving some time regularity of $(n_\varepsilon + \varepsilon)\frac{m}{2}$.

**Lemma 20.3.2.** Let $m > 0$ be such that $m \neq 1$. Then for all $T > 0$ there exists a constant $C = C(T) > 0$ such that for all $\varepsilon \in (0,1)$,
\[
\int_0^T \int_{\Omega} (n_\varepsilon + \varepsilon)^m \leq C
\]
on $(0,T)$ and
\[
\int_0^T \int_{\Omega} (n_\varepsilon + \varepsilon)^{2m-3}|\nabla n_\varepsilon|^2 \leq C
\]
hold.

Proof. From the first equation in (20.1.1) with $\nabla \cdot u_\varepsilon = 0$ in $\Omega \times (0,\infty)$ we have
\[
\frac{1}{m} \frac{d}{dt} \int_{\Omega} (n_\varepsilon + \varepsilon)^m = -m(m-1) \int_{\Omega} (n_\varepsilon + \varepsilon)^{2m-3}|\nabla n_\varepsilon|^2 + (m-1)\chi \int_{\Omega} (n_\varepsilon + \varepsilon)^{m-2} \frac{n_\varepsilon}{1 + \varepsilon n_\varepsilon} \nabla n_\varepsilon \cdot \nabla c_\varepsilon + \int_{\Omega} (n_\varepsilon + \varepsilon)^{m-1}(\kappa n_\varepsilon - \mu n_\varepsilon^2).
\]
Here we note from the Young inequality that
\[
(20.3.1) \quad \left|\chi(m-1) \int_{\Omega} (n_\varepsilon + \varepsilon)^{m-2} \frac{n_\varepsilon}{1 + \varepsilon n_\varepsilon} \nabla n_\varepsilon \cdot \nabla c_\varepsilon\right|
\leq \chi |m-1| \int_{\Omega} (n_\varepsilon + \varepsilon)^{m-1} |\nabla n_\varepsilon||\nabla c_\varepsilon|
\leq \frac{m|m-1|}{2} \int_{\Omega} (n_\varepsilon + \varepsilon)^{2m-3} |\nabla n_\varepsilon|^2 + \int_{\Omega} (n_\varepsilon + \varepsilon)^2 + C_1 \int_{\Omega} |\nabla c_\varepsilon|^4,
\]
where $C_1 > 0$. In the case that $m > 1$, since $m - 1 > 0$, we obtain that
\[
\frac{1}{m} \frac{d}{dt} \int_{\Omega} (n_\varepsilon + \varepsilon)^m + \frac{m(m-1)}{2} \int_{\Omega} (n_\varepsilon + \varepsilon)^{2m-3} |\nabla n_\varepsilon|^2
\leq \int_{\Omega} (n_\varepsilon + \varepsilon)^2 + C_1 \int_{\Omega} |\nabla c_\varepsilon|^4 + \kappa \int_{\Omega} (n_\varepsilon + \varepsilon)^m,
\]
275
which together with Lemma 20.2.8 shows that there is $C_2 = C_2(T) > 0$ such that

$$\int_{\Omega} (n_\varepsilon + \varepsilon)^m \leq C_2$$

on $(0, T)$ for all $\varepsilon \in (0, 1)$ and

$$\int_0^T \int_{\Omega} (n_\varepsilon + \varepsilon)^{2m-3} |\nabla n_\varepsilon|^2 \leq C_2$$

for all $\varepsilon \in (0, 1)$. On the other hand, in the case that $0 < m < 1$, we have from (20.3.1) that

$$\frac{1}{m} \frac{d}{dt} \int_{\Omega} (n_\varepsilon + \varepsilon)^m \geq \frac{m(1-m)}{2} \int_{\Omega} (n_\varepsilon + \varepsilon)^{2m-3} |\nabla n_\varepsilon|^2$$

$$- \int_{\Omega} (n_\varepsilon + \varepsilon)^2 - C_1 \int_{\Omega} |\nabla c_\varepsilon|^4 - \mu \int_{\Omega} (n_\varepsilon + \varepsilon)^{m+1}.$$ 

Thus, integrating it over $(0, T)$, we derive from applications of the Young inequality

$$\int_{\Omega} (n_\varepsilon + \varepsilon)^m \leq m \int_{\Omega} (n_\varepsilon + 1) + (1 - m)|\Omega|$$

and

$$\int_0^T \int_{\Omega} (n_\varepsilon + \varepsilon)^{m+1} \leq \frac{m+1}{2} \int_0^T \int_{\Omega} (n_\varepsilon + 1)^2 + \frac{1-m}{2} |\Omega|T$$

with Lemma 20.2.2 that

$$\frac{m(1-m)}{2} \int_0^T \int_{\Omega} (n_\varepsilon + \varepsilon)^{2m-3} |\nabla n_\varepsilon|^2 \leq C_3$$

with some $C_3 = C_3(T) > 0$. □

In order to see some time regularity of $(n_\varepsilon + \varepsilon)^{\frac{m}{2}}$ we will give the following lemma.

**Lemma 20.3.3.** Let $m > 0$ be such that $m \neq 1$. Then for all $T > 0$ there exists a constant $C = C(T) > 0$ such that for all $\varepsilon \in (0, 1)$,

$$\int_{\Omega} (n_\varepsilon + \varepsilon)^{m-1} \leq C$$

on $(0, T)$ and

$$\int_0^T \int_{\Omega} (n_\varepsilon + \varepsilon)^{2m-4} |\nabla n_\varepsilon|^2 \leq C$$

hold.
Proof. We derive from the first equation in (20.2.1) and integration by parts that

\begin{equation}
(20.3.2) \quad \frac{d}{dt} \int_\Omega (n_\varepsilon + \varepsilon)^{m-1} = -m(m-1)(m-2) \int_\Omega (n_\varepsilon + \varepsilon)^{2m-4} |\nabla n_\varepsilon|^2 \\
+ \chi(m-1)(m-2) \int_\Omega (n_\varepsilon + \varepsilon)^{m-3} \frac{n_\varepsilon}{1 + \varepsilon n_\varepsilon} \nabla n_\varepsilon \cdot \nabla c_\varepsilon \\
+ \kappa(m-1) \int_\Omega (n_\varepsilon + \varepsilon)^{m-2} n_\varepsilon - \mu(m-1) \int_\Omega (n_\varepsilon + \varepsilon)^{m-2} n_\varepsilon^2.
\end{equation}

Now we note from the Young inequality and Lemma 20.2.8 that

\begin{equation}
(20.3.3) \quad \left| \chi(m-1)(m-2) \int_\Omega (n_\varepsilon + \varepsilon)^{m-3} \frac{n_\varepsilon}{1 + \varepsilon n_\varepsilon} \nabla n_\varepsilon \cdot \nabla c_\varepsilon \right| \\
\leq \frac{m|(m-1)(m-2)|}{2} \int_\Omega (n_\varepsilon + \varepsilon)^{2m-4} |\nabla n_\varepsilon|^2 + C_1 \int_\Omega |\nabla c_\varepsilon|^2 \\
\leq \frac{m|(m-1)(m-2)|}{2} \int_\Omega (n_\varepsilon + \varepsilon)^{2m-4} |\nabla n_\varepsilon|^2 + C_2
\end{equation}

with $C_1, C_2 > 0$. We first consider the cases that $m > 2$ and $0 < m < 1$. Since it follows that $(m-1)(m-2) > 0$ and that if $m > 2$ then

\[ \kappa(m-1) \int_\Omega (n_\varepsilon + \varepsilon)^{m-2} n_\varepsilon - \mu(m-1) \int_\Omega (n_\varepsilon + \varepsilon)^{m-2} n_\varepsilon^2 \leq \kappa(m-1) \int_\Omega (n_\varepsilon + \varepsilon)^{m-1} \]

and if $0 < m < 1$ then

\[ \kappa(m-1) \int_\Omega (n_\varepsilon + \varepsilon)^{m-2} n_\varepsilon - \mu(m-1) \int_\Omega (n_\varepsilon + \varepsilon)^{m-2} n_\varepsilon^2 \leq \mu(1-m) \int_\Omega (n_\varepsilon + \varepsilon)^m \leq C_3 \]

with some $C_3 = C_3(T) > 0$ (from Lemma 20.3.2), we infer from (20.3.2) that

\[ \frac{d}{dt} \int_\Omega (n_\varepsilon + \varepsilon)^{m-1} + \frac{m(m-1)(m-2)}{2} \int_\Omega (n_\varepsilon + \varepsilon)^{2m-4} |\nabla n_\varepsilon|^2 \]

\[ \leq \kappa m - 1 \int_\Omega (n_\varepsilon + \varepsilon)^{m-1} + C_4 \]

with some $C_4 = C_4(T) > 0$, and hence there exists a constant $C_5 = C_5(T) > 0$ such that for all $\varepsilon \in (0, 1)$, $\int_\Omega (n_\varepsilon + \varepsilon)^{m-1} \leq C_5$ on $(0, T)$ and $\int_\Omega (n_\varepsilon + \varepsilon)^{2m-4} |\nabla n_\varepsilon|^2 \leq C_5$. On the other hand, in the case that $1 < m < 2$, we see from (20.3.2) and (20.3.3) that

\[ \frac{d}{dt} \int_\Omega (n_\varepsilon + \varepsilon)^{m-1} \geq \frac{m(m-1)(2-m)}{2} \int_\Omega (n_\varepsilon + \varepsilon)^{2m-4} |\nabla n_\varepsilon|^2 - C_6 \]

\[ - \mu(m-1) \int_\Omega (n_\varepsilon + \varepsilon)^m \]

with some $C_6 > 0$. Hence, noticing that $\int_\Omega (n_\varepsilon + \varepsilon)^{m-1} \leq (m-1) \int_\Omega (n_\varepsilon + 1) + (2 - m)|\Omega|$ and $(m-1)(2-m) > 0$, we derive from Lemmas 20.2.2 and 20.3.2 that

\[ \frac{m(m-1)(2-m)}{2} \int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^{2m-4} |\nabla n_\varepsilon|^2 \leq C_7 \]

with some $C_7 = C_7(T) > 0$. Finally, in the case that $m = 2$, Lemma 20.3.1 implies this lemma. \qed
The following time regularity of \((n_\varepsilon + \varepsilon)^\frac{m}{2}\) will be useful in applying a Lions–Aubin type lemma later.

**Lemma 20.3.4.** Let \(m > 0\) be such that \(m \neq 1\). Then for all \(T > 0\) there exists a constant \(C = C(T) > 0\) satisfying

\[ \|\partial_t (n_\varepsilon + \varepsilon)^\frac{m}{2}\|_{L^1(0,T;W_0^{2,4}(\Omega))} \leq C \quad \text{for all } \varepsilon \in (0,1). \]

**Proof.** Let \(T > 0\) and let \(\psi \in L^\infty(0,T;W_0^{2,4}(\Omega))\). The first equation in (20.2.1) and integration by parts yield that

\[
\int_0^T \int_\Omega (\partial_t (n_\varepsilon + \varepsilon)^\frac{m}{2}) \psi = - \int_0^T \int_\Omega \psi u_\varepsilon \cdot \nabla (n_\varepsilon + \varepsilon)^\frac{m}{2} \]
\[
- \frac{m^2}{m+1} (\frac{m}{2} - 1) \int_0^T \int_\Omega \psi \frac{\nabla (n_\varepsilon + \varepsilon)^\frac{m+1}{2}}{(n_\varepsilon + \varepsilon)^\frac{1}{2}} \cdot (n_\varepsilon + \varepsilon)^{-2} \nabla n_\varepsilon \]
\[
- \frac{m^2}{2} \int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^{m-\frac{3}{2}} \nabla n_\varepsilon \cdot (n_\varepsilon + \varepsilon)^\frac{m-3}{2} \nabla \psi \]
\[
+ \frac{m\chi}{m+1} (\frac{m}{2} - 1) \int_0^T \int_\Omega \frac{n_\varepsilon}{(1 + \varepsilon n_\varepsilon)(n_\varepsilon + \varepsilon)} \frac{\nabla (n_\varepsilon + \varepsilon)^\frac{m+1}{2}}{(n_\varepsilon + \varepsilon)^\frac{1}{2}} \cdot \nabla c_\varepsilon \]
\[
+ \frac{m\kappa}{2} \int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^\frac{m}{2} - 1 n_\varepsilon \psi - \frac{m\mu}{2} \int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^\frac{m}{2} - 1 n_\varepsilon^2 \psi.
\]

Then, noting from the Young inequality that

\[
\left| \int_0^T \int_\Omega \psi u_\varepsilon \cdot \nabla (n_\varepsilon + \varepsilon)^\frac{m}{2} \right| \leq \frac{\|\psi\|_{L^\infty(\Omega \times (0,T))}}{2} \left( \int_0^T \int_\Omega |u_\varepsilon|^2 + \int_0^T \int_\Omega |\nabla (n_\varepsilon + \varepsilon)^\frac{m}{2}|^2 \right),
\]
\[
\left| \int_0^T \int_\Omega \psi \frac{\nabla (n_\varepsilon + \varepsilon)^\frac{m+1}{2}}{(n_\varepsilon + \varepsilon)^\frac{1}{2}} \cdot (n_\varepsilon + \varepsilon)^{-2} \nabla n_\varepsilon \right|
\]
\[
\leq \frac{\|\psi\|_{L^\infty(\Omega \times (0,T))}}{2} \int_0^T \int_\Omega \left( \frac{|\nabla (n_\varepsilon + \varepsilon)^\frac{m+1}{2}|^2}{n_\varepsilon} + \int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^{2m-4} |\nabla n_\varepsilon|^2 \right)
\]

and

\[
\left| \int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^{m-\frac{3}{2}} \nabla n_\varepsilon \cdot (n_\varepsilon + \varepsilon)^\frac{m-3}{2} \nabla \psi \right|
\]
\[
\leq \frac{\|\nabla \psi\|_{L^\infty(\Omega \times (0,T))}}{2} \left( \int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^{2m-3} |\nabla n_\varepsilon|^2 + \int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^{m-1} \right),
\]
\[
\left| \int_0^T \int_\Omega \frac{n_\varepsilon}{(1 + \varepsilon n_\varepsilon)(n_\varepsilon + \varepsilon)} \frac{\nabla (n_\varepsilon + \varepsilon)^\frac{m+1}{2}}{(n_\varepsilon + \varepsilon)^\frac{1}{2}} \cdot \nabla c_\varepsilon \right|
\]
\[
\leq \frac{\|\psi\|_{L^\infty(\Omega \times (0,T))}}{2} \left( \int_0^T \int_\Omega \frac{|\nabla (n_\varepsilon + \varepsilon)^\frac{m+1}{2}|^2}{n_\varepsilon} + \int_0^T \int_\Omega |\nabla c_\varepsilon|^2 \right).
\]

278
as well as
\[
\left| \int_0^T \int_\Omega \left( n_\varepsilon + \varepsilon \right)^{m/2} \left( n_\varepsilon + \varepsilon \right)^{n/2} \nabla c_\varepsilon \cdot \nabla \psi \right| \leq \| \nabla \psi \|_{L^\infty(\Omega \times (0,T))} \left( \int_0^T \int_\Omega \left( n_\varepsilon + \varepsilon \right)^m + \int_0^T \int_\Omega |\nabla c_\varepsilon|^2 \right)
\]

with
\[
\left| \frac{mK}{2} \int_0^T \int_\Omega \left( n_\varepsilon + \varepsilon \right)^{m-1} n_\varepsilon \psi - \frac{m\mu}{2} \int_0^T \int_\Omega \left( n_\varepsilon + \varepsilon \right)^{m-1} n_\varepsilon^2 \psi \right| \leq \| \psi \|_{L^\infty(\Omega \times (0,T))} \left( \int_0^T \int_\Omega \left( n_\varepsilon + \varepsilon \right)^{\max\{m,2\}} + C_1 T \right)
\]

for some $C_1 > 0$ (from the fact that $\frac{m}{2} + 1 \leq \max\{m, 2\}$), we obtain from Lemmas 20.2.8, 20.3.1, 20.3.2 and 20.3.3 together with the continuous embedding $W_0^{2,4}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ that
\[
\int_0^T \int_\Omega \left( \partial_t (n_\varepsilon + \varepsilon)^m \right) \psi \leq C_2 \| \psi \|_{L^\infty(0,T;W^{2,4}(\Omega))}
\]

with some $C_2 = C_2(T) > 0$. Therefore a standard duality argument enables us to see this lemma.

We also give the following lemma concerned with time regularities of $c_\varepsilon$ and $u_\varepsilon$.

**Lemma 20.3.5.** For all $T > 0$ there exists $C = C(T) > 0$ satisfying
\[
\| (c_\varepsilon)_t \|_{L^2(0,T;W_0^{1,2}(\Omega)^*)} \leq C \quad \text{and} \quad \| (u_\varepsilon)_t \|_{L^2(0,T;W^{1,3}(\Omega)^*)} \leq C \quad \text{for all } \varepsilon \in (0,1).
\]

**Proof.** This lemma can be proved from the same arguments as those in the proofs of [101, Lemmas 2.14 and 2.15].

Finally we give an estimate for $\nabla (n_\varepsilon + \varepsilon)^m$ to see convergence of $\int_0^T \int_\Omega \nabla (n_\varepsilon + \varepsilon)^m \cdot \nabla \varphi$ for all $\varphi \in C_0^\infty(\Omega \times [0,\infty))$.

**Lemma 20.3.6.** Let $m > 0$ be such that $m \neq 1$. Then for all $T > 0$ and all $r \in (1, \frac{4}{3}]$ there exists a constant $C = C(T) > 0$ such that
\[
\| \nabla (n_\varepsilon + \varepsilon)^m \|_{L^r(0,T;L^r(\Omega))} \leq C \quad \text{for all } \varepsilon \in (0,1).
\]

**Proof.** Let $r = \frac{4}{3}$. Since $\frac{r}{2-r} = 2$, the Young inequality yields
\[
\int_0^T \int_\Omega |\nabla (n_\varepsilon + \varepsilon)^m|^r = m^r \int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^{(m-1)r} |\nabla n_\varepsilon|^r \leq \int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^{\frac{2m^{r}}{2-r}} + C_1 \int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^{2m-3} |\nabla n_\varepsilon|^2 \leq \int_0^T \int_\Omega (n_\varepsilon + 1)^2 + C_1 \int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^{2m-3} |\nabla n_\varepsilon|^2
\]

with some $C_1 > 0$. Therefore Lemmas 20.2.8 and 20.3.2 lead to this lemma. \[\square\]
20.4. Convergence: Proof of Theorem 20.1.1

In this section we consider convergence of solutions of approximate problem (20.2.1) and then prove Theorem 20.1.1. We first state the following result which can be obtained from the previous estimates in Section 20.3.

**Lemma 20.4.1.** There exist a sequence \((\varepsilon_j)_{j \in \mathbb{N}}\) such that \(\varepsilon_j \downarrow 0\) as \(j \to \infty\) and functions \(n, c, u\) such that

\[
\begin{align*}
n &\in L^2_{\text{loc}}([0, \infty); L^2(\Omega)), \\
c &\in L^2_{\text{loc}}([0, \infty); W^{1,2}(\Omega)), \\
u &\in L^2_{\text{loc}}([0, \infty); W^{1,2}_{0,\sigma}(\Omega))
\end{align*}
\]

and that for all \(p \in [1, 6]\),

\[
\begin{align*}
(20.4.1) \quad & (n_\varepsilon + \varepsilon)^\frac{m}{2} \to n \quad \text{in } L^2_{\text{loc}}([0, \infty); L^p(\Omega)) \text{ and a.e. in } \Omega \times (0, \infty), \\
(20.4.2) \quad & n_\varepsilon \to n \quad \text{in } L^2_{\text{loc}}([0, \infty); L^2(\Omega)), \\
(20.4.3) \quad & c_\varepsilon \to c \quad \text{in } C^0_{\text{loc}}([0, \infty); L^p(\Omega)), \\
(20.4.4) \quad & u_\varepsilon \to u \quad \text{in } L^2_{\text{loc}}([0, \infty); L^p(\Omega)), \\
(20.4.5) \quad & \nabla c_\varepsilon \to \nabla c \quad \text{weakly in } L^4_{\text{loc}}([0, \infty); L^4(\Omega)), \\
(20.4.6) \quad & \nabla u_\varepsilon \to \nabla u \quad \text{weakly in } L^2_{\text{loc}}([0, \infty); L^2(\Omega)), \\
(20.4.7) \quad & Y_\varepsilon u_\varepsilon \to u \quad \text{in } L^p_{\text{loc}}([0, \infty); L^2(\Omega))
\end{align*}
\]

as \(\varepsilon = \varepsilon_j \downarrow 0\).

**Proof.** Let \(T > 0\). Thanks to Lemmas 20.3.1, 20.3.2 and 20.3.4, we have that

\[
\left( (n_\varepsilon + \varepsilon)^\frac{m}{2} \right)_{\varepsilon \in (0, 1)} \text{ is bounded in } L^2(0, T; W^{1,2}(\Omega))
\]

and

\[
\left( \partial_t (n_\varepsilon + \varepsilon)^\frac{m}{2} \right)_{\varepsilon \in (0, 1)} \text{ is bounded in } L^1(0, T; (W^{2,4}_{0,\sigma}(\Omega))^*)
\]

Therefore, aided by the compact embedding \(W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)\) for all \(p \in [1, 6]\) and the continuous embedding \(L^p(\Omega) \hookrightarrow (W^{2,4}_{0,\sigma}(\Omega))^*\), we can see from a Lions–Aubin type lemma (see [161, Corollary 4]) that \((n_\varepsilon + \varepsilon)^\frac{m}{2}\) is relatively compact in \(L^2(0, T; L^p(\Omega))\), which means that there are a sequence \((\varepsilon_j)_{j \in \mathbb{N}} \downarrow 0\) and a function \(v \in L^2(0, T; L^p(\Omega))\) such that \((n_\varepsilon + \varepsilon)^\frac{m}{2} \to v\) in \(L^2(0, T; L^p(\Omega))\) as \(\varepsilon = \varepsilon_j \downarrow 0\). Then by putting \(n := v^\frac{2}{m}\) we have (20.4.1), which yields that \(n_\varepsilon \to n\) a.e. in \(\Omega \times (0, \infty)\) as \(\varepsilon = \varepsilon_j \downarrow 0\). The rest of the proof is mainly based on arguments in the proof of [101, Proposition 2.1]; thus we will give a short proof. Since a uniform bound on \(\int_0^T \int_\Omega \Phi(n_\varepsilon^2)\), where \(\Phi(s) := \frac{s}{2} \log(s)\) for \(s > 0\), derives from the Dunford–Pettis theorem (cf. [40, Lemma IV.8.9]) that \((n_\varepsilon^2)_{\varepsilon \in (0, 1)}\) is weakly relatively precompact in \(L^1(\Omega \times (0, T))\), we obtain that there is a subsequence of \((\varepsilon_j)_{j \in \mathbb{N}}\) such that \(\int_0^T \int_\Omega n_\varepsilon^2 \to \int_0^T \int_\Omega n^2\) as \(\varepsilon = \varepsilon_j \downarrow 0\). This together with the convergence \(n_\varepsilon \to n\) weakly in \(L^2(0, T; L^2(\Omega))\) as \(\varepsilon = \varepsilon_j \downarrow 0\) (from Lemma 20.2.8) yields that (20.4.2) along a further subsequence. On the other hand, by virtue of Lemmas 20.2.3, 20.2.8 and 20.3.5, we can establish that \((c_\varepsilon)_{\varepsilon \in (0, 1)}\) and \((u_\varepsilon)_{\varepsilon \in (0, 1)}\) are bounded in \(L^\infty(0, T; W^{1,2}(\Omega))\) and

280
in \(L^2(0, T; (W^{1,2}_0(\Omega))^*)\), respectively, as well as \((u_{\varepsilon})_{\varepsilon \in (0, 1)}\) and \((u(t))_{t \in (0, 1)}\) are bounded in \(L^2(0, T; W^{1,2}_0(\Omega))\) and in \(L^2(0, T; (W^{1,3}(\Omega))^*)\), respectively. Thus [161, Corollary 4] again implies (20.4.3) and (20.4.4), and then Lemma 20.2.8 leads to the convergences (20.4.5) and (20.4.6) along a further subsequence. Finally, noticing that \(\|Y_{\varepsilon}u_{\varepsilon}(t) - u(t)\|_{L^2(\Omega)} \to 0\) as \(\varepsilon = \varepsilon_j \searrow 0\) a.e. \(t > 0\) and \(\|Y_{\varepsilon}u_{\varepsilon}(t) - u(t)\|_{L^2(\Omega)}^2 \leq C\) for all \(t > 0\) and all \(\varepsilon > 0\) in view of Lemma 20.2.8, we can establish from the dominated convergence theorem that (20.4.7) along a further subsequence.

We then provide convergence of \(\nabla(n_{\varepsilon} + \varepsilon)^m\) from Lemma 20.3.6.

**Lemma 20.4.2.** Let \(m > 0\) be such that \(m \neq 1\). Then the function \(n\) obtained in Lemma 20.4.1 satisfies that \(n^m \in L^{\frac{3}{4}}_{\text{loc}}([0, \infty); W^{1,\frac{4}{3}}(\Omega))\) and

\[
\nabla(n_{\varepsilon} + \varepsilon)^m \to \nabla n^m \quad \text{weakly in } L^{\frac{3}{4}}_{\text{loc}}([0, \infty); L^{\frac{4}{3}}(\Omega))
\]
as \(\varepsilon = \varepsilon_j \searrow 0\).

**Proof.** Let \(T > 0\). Since the Poincaré–Wirtinger inequality yields that

\[
\int_0^T \|(n_{\varepsilon} + \varepsilon)^m\|_{L^{\frac{3}{4}}(\Omega)}^\frac{4}{3} \leq C_1 \int_0^T \|\nabla(n_{\varepsilon} + \varepsilon)^m\|_{L^{\frac{4}{3}}(\Omega)}^\frac{4}{3} + C_1 |\Omega|^{-\frac{1}{3}} \int_0^T \|(n_{\varepsilon} + \varepsilon)^m\|_{L^1(\Omega)}^\frac{4}{3}
\]
with some \(C_1 > 0\), we first note from the Fatou lemma and Lemmas 20.3.2, 20.3.6 that

\[
\int_0^T \|n^m\|_{L^{\frac{3}{4}}(\Omega)}^\frac{4}{3} \leq \liminf_{\varepsilon \searrow 0} \int_0^T \|(n_{\varepsilon} + \varepsilon)^m\|_{L^{\frac{3}{4}}(\Omega)}^\frac{4}{3} \leq C_2
\]
with some \(C_2 = C_2(T) > 0\), which implies that \(n^m \in L^{\frac{3}{4}}(0, T; L^{\frac{4}{3}}(\Omega))\). We next have from Lemma 20.3.6 that there exist a subsequence of \((\varepsilon_j)_{j \in \mathbb{N}}\) obtained in Lemma 20.4.1 (again denoted by \((\varepsilon_j)_{j \in \mathbb{N}}\)) and a function \(w \in L^{\frac{3}{4}}(0, T; L^{\frac{4}{3}}(\Omega))\) such that

\[
\nabla(n_{\varepsilon} + \varepsilon)^m \to w \quad \text{weakly in } L^{\frac{3}{4}}(0, T; L^{\frac{4}{3}}(\Omega))
\]
as \(\varepsilon = \varepsilon_j \searrow 0\). In order to verify that \(w = \nabla n^m\), it is enough to confirm that \((n_{\varepsilon} + \varepsilon)^m \to n^m\) in \(L^1(0, T; L^1(\Omega))\) as \(\varepsilon = \varepsilon_j \searrow 0\). Now, since we have already known that \((n_{\varepsilon} + \varepsilon)^m\) is uniform integrable (from Lemma 20.3.2) and \((n_{\varepsilon} + \varepsilon)^m \to n^m\) a.e. in \(\Omega \times (0, \infty)\) as \(\varepsilon = \varepsilon_j \searrow 0\), the Vitali convergence theorem entails that \((n_{\varepsilon} + \varepsilon)^m \to n^m\) in \(L^1(0, T; L^1(\Omega))\) as \(\varepsilon = \varepsilon_j \searrow 0\). Thanks to this strong convergence, we can verify that \(w = \nabla n^m\), which together with \(w \in L^{\frac{3}{4}}(0, T; L^{\frac{4}{3}}(\Omega))\) shows that \(n^m \in L^{\frac{3}{4}}(0, T; W^{1,\frac{4}{3}}(\Omega))\).

We will establish global existence of weak solutions to (20.1.1) from convergences obtained in Lemmas 20.4.1 and 20.4.2.

**Proof of Theorem 20.1.1.** Let \(\varphi \in C_0^\infty(\overline{\Omega} \times [0, \infty))\) and \(\psi \in C_0^\infty(\Omega \times [0, \infty))\). Testing each equations in (20.2.1) by these functions and using integration by parts, we can see
that

\begin{align*}
(20.4.8) & \quad -\int_0^\infty \int_\Omega n_\varepsilon \varphi_t - \int_\Omega n_0 \varphi(\cdot,0) - \int_0^\infty \int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla \varphi \\
& \quad = -\int_0^\infty \int_\Omega \nabla (n_\varepsilon + \varepsilon)^m \cdot \nabla \varphi + \chi \int_0^\infty \int_\Omega \frac{n_\varepsilon}{1 + \varepsilon n_\varepsilon} \nabla c_\varepsilon \cdot \nabla \varphi \\
& \quad \quad + \int_0^\infty \int_\Omega (\kappa n_\varepsilon - \mu n_\varepsilon^2) \varphi,
\end{align*}

\begin{align*}
(20.4.9) & \quad -\int_0^\infty \int_\Omega c_\varepsilon \varphi_t - \int_\Omega c_0 \varphi(\cdot,0) - \int_0^\infty \int_\Omega c_\varepsilon u_\varepsilon \cdot \nabla \varphi \\
& \quad = -\int_0^\infty \int_\Omega \nabla c_\varepsilon \cdot \nabla \varphi - \int_0^\infty \int_\Omega \frac{1}{\varepsilon} (\log(1 + \varepsilon n_\varepsilon)) c_\varepsilon \varphi,
\end{align*}

\begin{align*}
(20.4.10) & \quad -\int_0^\infty \int_\Omega u_\varepsilon \cdot \psi_t - \int_\Omega u_0 \cdot \psi(\cdot,0) - \int_0^\infty \int_\Omega Y \psi u_\varepsilon \otimes u_\varepsilon \cdot \nabla \psi \\
& \quad = -\int_0^\infty \int_\Omega \nabla u_\varepsilon \cdot \nabla \psi + \int_0^\infty \int_\Omega n_\varepsilon \nabla \varphi \cdot \psi
\end{align*}

hold. Now, since the dominated convergence theorem enables us to have \(\frac{1}{1 + \varepsilon n_\varepsilon} \to 1\) in \(L^1_{\text{loc}}([0,\infty); L^4(\Omega))\) as \(\varepsilon = \varepsilon_j \searrow 0\), the convergences of \(n_\varepsilon\) in \(L^2_{\text{loc}}([0,\infty); L^2(\Omega))\) and \(\nabla c_\varepsilon\) weakly in \(L^4_{\text{loc}}([0,\infty); L^4(\Omega))\) (see Lemma 20.4.1) derive

\begin{equation}
(20.4.11) \quad \frac{n_\varepsilon}{1 + \varepsilon n_\varepsilon} \nabla c_\varepsilon = n_\varepsilon \cdot \frac{1}{1 + \varepsilon n_\varepsilon} \nabla c_\varepsilon \to n \nabla c \quad \text{weakly in } L^1_{\text{loc}}([0,\infty); L^1(\Omega))
\end{equation}

as \(\varepsilon = \varepsilon_j \searrow 0\). On the other hand, to confirm convergence of \(\frac{1}{\varepsilon} (\log(1 + \varepsilon n_\varepsilon)) c_\varepsilon\) in \(L^1_{\text{loc}}([0,\infty); L^1(\Omega))\) we shall show that \(f_\varepsilon(n_\varepsilon) \to n\) in \(L^2_{\text{loc}}([0,\infty); L^2(\Omega))\) as \(\varepsilon = \varepsilon_j \searrow 0\), where \(f_\varepsilon(s) := \frac{1}{\varepsilon} \log(1 + \varepsilon s)\) for \(s \geq 0\). Noticing from (20.4.2) that \(|f_\varepsilon(n) - n|^2 \to 0\) a.e. in \(\Omega \times (0,T)\) as \(\varepsilon = \varepsilon_j \searrow 0\) and from the inequality \(f_\varepsilon(s) \leq s (s \geq 0)\) that \(|f_\varepsilon(n) - n|^2 \leq 2n^2\), we deduce from the dominated convergence theorem that for all \(T > 0\),

\[\|f_\varepsilon(n) - n\|_{L^2(0,T; L^2(\Omega))} = \int_0^T \int_\Omega |f_\varepsilon(n) - n|^2 \to 0\]

as \(\varepsilon = \varepsilon_j \searrow 0\). Therefore we can see from the inequality \(0 < f_\varepsilon'(s) = \frac{1}{1 + \varepsilon s} \leq 1\) and (20.4.2) that for all \(T > 0\),

\[\|f_\varepsilon(n) - n\|_{L^2(0,T; L^2(\Omega))} \leq \|f_\varepsilon(n) - f_\varepsilon(n)\|_{L^2(0,T; L^2(\Omega))} + \|f_\varepsilon(n) - n\|_{L^2(0,T; L^2(\Omega))} + \|f_\varepsilon(n) - n\|_{L^2(0,T; L^2(\Omega))} \to 0\]

as \(\varepsilon = \varepsilon_j \searrow 0\). This together with (20.4.3) enables us to obtain that

\begin{equation}
(20.4.12) \quad \frac{1}{\varepsilon} (\log(1 + \varepsilon n_\varepsilon)) c_\varepsilon = f_\varepsilon(n_\varepsilon)c_\varepsilon \to nc \quad \text{in } L^1_{\text{loc}}([0,\infty); L^1(\Omega))
\end{equation}

as \(\varepsilon = \varepsilon_j \searrow 0\). Then all convergences in Lemmas 20.4.1, 20.4.2 as well as (20.4.11), (20.4.12) make sure to pass to the limit in all integrals in (20.4.8)–(20.4.10), which means that the triplet \((n,c,u)\) is a global weak solution of (20.1.1). \(\square\)
References


287


295


List of original papers


