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Note on the Integral Curves of Pfaff's Equation.

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Note on the Integral Curves of Pfaff's Equation,

BY

K. OGURA in Sendai.

1. Lie and Appell proved that all the curves of a linear complex through any point have the same osculating plane at that point.(1) In this note I will treat in general the integral curves of Pfaff's equation

$$\xi\, dx + \eta\, dy + \zeta\, dz = 0 \tag{1}$$

passing the same point and having the same osculating plane at that point.

Let the coordinates of a curve be expressed in terms of the parameter s which is the arc measured along the curves from some fixed point, and let the first and second derivatives of x, y, z with respect to s be x', y', z' and x'', y'', z'' respectively.

Since all the integral curves of the equation

$$\xi\, x' + \eta\, y' + \zeta\, z' = 0 \tag{2}$$

through a point $P(x, y, z)$ have the same binormal at that point, the direction-cosines of the binormal λ, μ, ν satisfy

$$\lambda\, x' + \mu\, y' + \nu\, z' = 0, \tag{3}$$

and

$$\lambda\, x'' + \mu\, y'' + \nu\, z'' = 0. \tag{4}$$

Since equations (2) and (3) hold good for all values of $x' : y' : z'$ at $P(x, y, z)$, and therefore

$$\frac{\lambda}{\xi} = \frac{\mu}{\eta} = \frac{\nu}{\zeta}.$$

Hence equation (4) becomes

$$\xi\, x'' + \eta\, y'' + \zeta\, z'' = 0. \tag{5}$$

Now differentiating (2) with respect to the parameter s,

$$\left(\frac{\partial \xi}{\partial x} x' + \frac{\partial \eta}{\partial x} y' + \frac{\partial \zeta}{\partial x} z'\right) x' + \left(\frac{\partial \xi}{\partial y} x' + \frac{\partial \eta}{\partial y} y' + \frac{\partial \zeta}{\partial y} z'\right) y'$$

$$+ \left(\frac{\partial \xi}{\partial z} x' + \frac{\partial \eta}{\partial z} y' + \frac{\partial \zeta}{\partial z} z'\right) z' + \xi\, x'' + \eta\, y'' + \zeta\, z'' = 0,$$

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(1) P. Appell, Annales de l'École Normale Supérieure, 1876, p. 245; S. Lie—G. Scheffers, Geometrie der Berührungstransformationen, Bd. I, 1896, p. 230.

whence by (5)

$$\frac{\partial \xi}{\partial x} x'^2 + \frac{\partial \eta}{\partial y} y'^2 + \frac{\partial \zeta}{\partial z} z'^2 + \left(\frac{\partial \xi}{\partial y} + \frac{\partial \eta}{\partial x}\right) x' y'$$
$$+ \left(\frac{\partial \eta}{\partial z} + \frac{\partial \zeta}{\partial y}\right) y' z' + \left(\frac{\partial \zeta}{\partial x} + \frac{\partial \xi}{\partial z}\right) z' x' = 0. \qquad (6)$$

This equation must be true for all values of $x' : y' : z'$ satisfying (2). Eliminating z' between two equations (2) and (6),

$$x'^2 \left\{ \frac{\partial \xi}{\partial x} \zeta^2 - \left(\frac{\partial \xi}{\partial z} + \frac{\partial \zeta}{\partial x}\right) \xi \zeta + \frac{\partial \zeta}{\partial z} \xi^2 \right\}$$
$$+ x' y' \left\{ \left(\frac{\partial \xi}{\partial y} + \frac{\partial \eta}{\partial x}\right) \zeta^2 + 2 \frac{\partial \zeta}{\partial z} \xi \eta - \left(\frac{\partial \zeta}{\partial x} + \frac{\partial \xi}{\partial z}\right) \eta \zeta - \left(\frac{\partial \eta}{\partial z} + \frac{\partial \zeta}{\partial y}\right) \xi \zeta \right\}$$
$$+ y'^2 \left\{ \frac{\partial \eta}{\partial y} \zeta^2 - \left(\frac{\partial \eta}{\partial z} + \frac{\partial \zeta}{\partial y}\right) \eta \zeta + \frac{\partial \zeta}{\partial z} \eta^2 \right\} = 0, \qquad (7)$$

which must be satisfied for any values of $x' : y'$. Hence

$$\left.\begin{aligned}
&\frac{\partial \xi}{\partial x} \zeta^2 - \left(\frac{\partial \xi}{\partial z} + \frac{\partial \zeta}{\partial x}\right) \xi \zeta + \frac{\partial \zeta}{\partial z} \xi^2 = 0, \\
&\left(\frac{\partial \xi}{\partial y} + \frac{\partial \eta}{\partial x}\right) \zeta^2 + 2 \frac{\partial \zeta}{\partial z} \xi \eta - \left(\frac{\partial \zeta}{\partial x} + \frac{\partial \xi}{\partial z}\right) \eta \zeta \\
&\qquad - \left(\frac{\partial \eta}{\partial z} + \frac{\partial \zeta}{\partial y}\right) \xi \zeta = 0, \\
&\frac{\partial \eta}{\partial y} \zeta^2 - \left(\frac{\partial \eta}{\partial z} + \frac{\partial \zeta}{\partial y}\right) \eta \zeta + \frac{\partial \zeta}{\partial z} \eta^2 = 0.
\end{aligned}\right\} \qquad (8)$$

Multiplying the second equation by $\xi \eta$ and comparing the result with the first and third equations, we obtain

$$\frac{\partial \xi}{\partial x} \eta^2 - \left(\frac{\partial \xi}{\partial y} + \frac{\partial \eta}{\partial x}\right) \xi \eta + \frac{\partial \eta}{\partial y} \xi^2 = 0. \qquad (9)$$

Conversely, if ξ, η, ζ satisfy equation (9) and the first and the third of equations (8), then equation (7) and then (6) hold good for all values of $x' : y' : z'$. Moreover, differentiating (2) with respect to s and comparing the result with (6), we have

$$\xi x'' + \eta y'' + \zeta z'' = 0.$$

Hence ξ, η, ζ must be proportional to the direction-cosines of the binormal of all the integral curves of (2) at $P(x, y, z)$, and these curves have the same osculating plane at this point.

Therefore it is necessary and sufficient, that ξ, η, ζ should satisfy the three equations

$$\left.\begin{aligned}
&\frac{1}{\xi^2} \frac{\partial \xi}{\partial x} - \frac{1}{\xi \eta}\left(\frac{\partial \xi}{\partial y} + \frac{\partial \eta}{\partial x}\right) + \frac{1}{\eta^2} \frac{\partial \eta}{\partial y} = 0, \\
&\frac{1}{\eta^2} \frac{\partial \eta}{\partial y} - \frac{1}{\eta \zeta}\left(\frac{\partial \eta}{\partial z} + \frac{\partial \zeta}{\partial y}\right) + \frac{1}{\zeta^2} \frac{\partial \zeta}{\partial z} = 0, \\
&\frac{1}{\zeta^2} \frac{\partial \zeta}{\partial z} - \frac{1}{\zeta \xi}\left(\frac{\partial \zeta}{\partial x} + \frac{\partial \xi}{\partial z}\right) + \frac{1}{\xi^2} \frac{\partial \xi}{\partial x} = 0,
\end{aligned}\right\}$$

in order to secure that all the integral curves of Pfaff's equation

$$\xi\, dx + \eta\, dy + \zeta\, dz = 0$$

passing the same point have the same osculating plane at that point.

2. It should be noticed that in the particular case where equation (6) is an identity, the curves must be the curves of a linear complex. For, in this case we have

$$\frac{\partial \xi}{\partial x} = 0, \quad \frac{\partial \eta}{\partial y} = 0, \quad \frac{\partial \zeta}{\partial z} = 0;$$

$$\frac{\partial \xi}{\partial y} + \frac{\partial \eta}{\partial x} = 0, \quad \frac{\partial \eta}{\partial z} + \frac{\partial \zeta}{\partial y} = 0, \quad \frac{\partial \zeta}{\partial x} + \frac{\partial \xi}{\partial z} = 0;$$

which may be written as follows:

$$\begin{aligned}
&\frac{\partial \xi}{\partial x} = 0, && \frac{\partial \xi}{\partial y} = -C, && \frac{\partial \xi}{\partial z} = B, \\
&\frac{\partial \eta}{\partial x} = C, && \frac{\partial \eta}{\partial y} = 0, && \frac{\partial \eta}{\partial z} = -A, \\
&\frac{\partial \zeta}{\partial x} = -B, && \frac{\partial \zeta}{\partial y} = A, && \frac{\partial \zeta}{\partial z} = 0.
\end{aligned}$$

Now from the equations

$$\frac{\partial C}{\partial x} = -\frac{\partial^2 \xi}{\partial y \partial x} = 0 \quad \text{and} \quad \frac{\partial C}{\partial y} = \frac{\partial^2 \eta}{\partial x \partial y} = 0,$$

it follows that C may depend upon z only. Similarly A and B may depend upon only x and y respectively.

But from

$$-\frac{dC}{dz} = \frac{\partial^2 \xi}{\partial y \partial z} = \frac{dB}{dy} = -\frac{\partial^2 \zeta}{\partial x \partial y}$$

and

$$\frac{dC}{dz} = \frac{\partial^2 \eta}{\partial x \partial z} = -\frac{dA}{dx} = -\frac{\partial^2 \zeta}{\partial y \partial x},$$

we have
$$\frac{dA}{dx}=\frac{dB}{dy}=\frac{dC}{dz}=0.$$

Hence A, B and C must be constants; and we must have
$$\xi=Bz-Cy+D,\quad \eta=Cx-Az+E,\quad \zeta=Ay-Bx+F,$$
where D, E, F are arbitrary constants.

Therefore equation (1) must take the form
$$(Bz-Cy+D)\,dx+(Cx-Az+E)\,dy+(Ay-Bx+F)\,dz=0$$
representing the curves of a linear complex.

3. Let
$$\xi x'+\eta y'+\zeta z'=0 \qquad \overset{10}{(7)}$$
be the orthogonal trajectory of the congruence of curves Γ
$$\frac{dx}{\xi}=\frac{dy}{\eta}=\frac{dz}{\zeta},\qquad \text{where}\quad \xi^2+\eta^2+\zeta^2=1,$$
and let $\dfrac{1}{\rho_s}$, $\dfrac{1}{R_s}$ and $\dfrac{1}{h_s}$ be respectively the curvature, geodesic curvature, and normal curvature, at a point $P(x, y, z)$, of a trajectory drawn in any given direction at that point. Then we have the relation
$$\frac{1}{\rho_s^2}=\frac{1}{R_s^2}+\frac{1}{h_s^2}.\ (^1)$$

If the normal curvature of the trajectories at any point be always equal to zero, it follows that
$$\frac{1}{h_s}=\xi x''+\eta y''+\zeta z''=0. \qquad \overset{11}{(7)}$$

Differentiating ($\overset{10}{6}$) with respect to s, and comparing the result with ($\overset{11}{7}$), we have for all values of $x':y':z'$
$$\frac{\partial\xi}{\partial x}x'^2+\frac{\partial\eta}{\partial y}y'^2+\frac{\partial\zeta}{\partial z}z'^2+\left(\frac{\partial\xi}{\partial y}+\frac{\partial\eta}{\partial x}\right)x'y'$$
$$+\left(\frac{\partial\eta}{\partial z}+\frac{\partial\zeta}{\partial y}\right)y'z'+\left(\frac{\partial\zeta}{\partial x}+\frac{\partial\xi}{\partial z}\right)z'x'=0.$$

Hence, by Art. 1, we have the following result:

If the geodesic curvature of any orthogonal trajectory K of a congruence of curves Γ at any point be equal to the curvature of the curve K at that point, all the trajectories have the same osculating plane at that point. Or, if any orthogonal trajectory K of a congruence

(1) Lilienthal, Math. Ann., 52 (1899), p. 415.

of curves Γ at any point be an asymptotic curve (1) of the congruence, all the trajectories at that point have the same osculating plane.

As a particular case, it will be seen that if any geodesic line of a congruence of curves Γ at any point be a straight line, all the orthogonal trajectories of the congruence at that point have the same osculating plane.

Conversely, if all the orthogonal trajectories of a congruence of curves Γ, at any point, have the same osculating plane, then the geodesic curvature of any trajectory K of the congruence Γ at any point is equal to the curvature of the curve K at that point.

4. It is necessary and sufficient that all the orthogonal trajectories at any point have the same osculating plane, in order to secure that the two consecutive curves Γ_1 and Γ_2 in a congruence have the shortest distance P_1P_2, Γ_1 being the curve through any point P_1, and Γ_2 the curve through any point P_2 consecutive to P_1.

Let (ξ, η, ζ) be the direction-cosines of the tangent P_1T_1 of the curve Γ_1 at $P_1(x, y, z)$ and let $(\xi+\partial\xi, \eta+\partial\eta, \zeta+\partial\zeta)$ be the direction-cosines of the tangent P_2T_2 of the curve Γ_2, consecutive to Γ_1, at $P_2(x+\partial x, y+\partial y, z+\partial z)$. Now a necessary and sufficient condition that P_1P_2 is the shortest distance between Γ_1 and Γ_2 is that P_1P_2 cuts orthogonally both P_1T_1 and P_2T_2, that is,
$$\Sigma\,\xi\,\partial x=0\quad\text{and}\quad\Sigma\,(\xi+\partial\xi)\,\partial x=0;$$
which may be written as follows:
$$\Sigma\,\xi\,\partial x=0\quad\text{and}\quad\Sigma\,\partial\xi\,\partial x=0.$$
But since the latter equation is nothing but equation (6), P_1 and P_2 lie on the orthogonal trajectory and all the orthogonal trajectories at P_1 have the same osculating plane.

5. Let us constitute a surface F from any ∞^1 curves in the congruence Γ whose orthogonal trajectories at any point have the same osculating plane. Then there exist ∞^1 curves K on this surface, which belong to the orthogonal trajectories of Γ.

Now the principal normal at any point P of any curve K is normal to the curves K and Γ at this point P; but the normal at P to the surface F is also normal to the curves K and Γ at P. Hence the geodesic curvature at all points of the curve K on the surface F must be zero; which shows us the curve K is a geodesic on the surface F.

(1) Lilienthal, Grundlagen einer Krümmungslehre d. Curvenscharen, 1896, p. 50.

In this way we can constitute ∞^1 surfaces F having any curve Γ_0 in common; and then the ∞^1 geodesics, one upon one surface F respectively, passing through any point P on the curve Γ_0, have the same osculating plane at that point.

For example, (1) in the congruence of helices

$$x=v\cos u, \quad y=v\sin u, \quad z=ku+w,$$

where k is a constant and v, w are two parameters, we have

$$\frac{dx}{-y}=\frac{dy}{x}=\frac{dz}{k}.$$

If we take

$$w=f(v),$$

we obtain a helicoid F whose linear element is given by

$$ds^2=(v^2+k^2)\,du^2+2kf'(v)\,du\,dv+\{1+f'^2(v)\}dv^2;$$

when putting

$$U=u+k\int\frac{f'(v)}{v^2+k^2}\,dv, \quad V=\int\sqrt{1+\frac{v^2f'^2(v)}{v^2+k^2}}\,dv,$$

the linear element becomes

$$ds^2=dV^2+(v^2+k^2)\,dU^2.$$

Hence the orthogonal trajectories $U=$const. of the ∞^1 helices $V=$const., i.e. $v=$const., are the geodesics on the helicoid F; and along the geodesics

$$dU=du+k\frac{f'(v)}{v^2+k^2}\,dv=0,$$

we have

$$-y\,dx+x\,dy+k\,dz=0,$$

which shows us that the geodesics $U=$const. are the curves of a linear complex.

Now if we take, for example,

$$w=m(v-1),$$

m being a parameter, we get ∞^1 helicoids having the helix Γ_0

$$x=\cos u, \quad y=\sin u, \quad z=ku$$

in common; and the ∞^1 geodesics, one upon one helicoid F respectively, passing through any point $P(x_0, y_0, z_0)$ on the helix Γ_0, have the same osculating plane

$$-y_0(x-x_0)+x_0(y-y_0)+k(z-z_0)=0.$$

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(1) Compare Lie-Scheffers, ibid, p. 685.

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