NOTES ON DIFFERENTIAL IDEALS OF LASKERIAN RINGS

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Abstract. Let $R$ be a ring and $d$ a derivation of $R$. We consider the following three conditions: (a) every quasi-prime $d$-ideal of $R$ is prime, (b) any weak associated prime of every $d$-ideal of $R$ is a $d$-ideal and (c) every $d$-prime $d$-ideal of $R$ is prime. In this paper we show that if $R$ is a Laskerian ring, then the two conditions (a) and (b) are equivalent. Furthermore we show that if $R$ is a strongly Laskerian ring, then any $d$-prime $d$-ideal of $R$ is quasi-prime, and then the three conditions (a), (b) and (c) are equivalent.

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§1. Introduction

All rings in this paper are assumed to be commutative with a unit element. Let $R$ be a ring. A derivation $d$ of a ring $R$ is an additive endomorphism $d : R \to R$ such that $d(ab) = d(a)b + ad(b)$ for every $a, b \in R$. Let $d$ be a derivation of $R$. An ideal $I$ of $R$ is called a $d$-ideal if $d(I) \subseteq I$. A proper $d$-ideal $Q$ of $R$ is called a $d$-prime $d$-ideal if for $d$-ideals $I$ and $J$ of $R$ the relation $IJ \subseteq Q$ implies either $I \subseteq Q$ or $J \subseteq Q$. A proper $d$-ideal $Q$ of $R$ is called a quasi-prime $d$-ideal if there is a multiplicative subset $S$ of $R$ such that $Q$ is maximal among $d$-ideals disjoint from $S$. Some of the properties of the $d$-prime $d$-ideals and the quasi-prime $d$-ideals are given in [3], [8], [9], [11], [12], [14].

Let $I$ be an ideal of $R$. A prime ideal $P$ of $R$ is called a minimal prime divisor of $I$ if $P$ is minimal among the prime ideals containing $I$. A prime ideal $P$ of $R$ is called a weak associated prime of $I$ if there exists $x \in R$ such that $P$ is a minimal prime divisor of $I : (x)$; We denote by $\text{Ass}_f(R/I)$ the set of weak associated primes of $I$ (cf.[1, IV, §1, Exercise 17]). It is known that if $R$ is Noetherian, then the weak associated primes of $I$ coincides with the usual associated primes of $I$. If $I$ can be expressed as an intersection of a finite number of primary ideals, we say that $I$ has a primary decomposition. A ring $R$ is called Laskerian if every ideal of $R$ has a primary decomposition.
A Laskerian ring $R$ is called strongly Laskerian if each primary ideal of $R$ contains a power of its radical (cf.[1, IV, §2, Exercise 23, 28]).

Let $R$ be a ring and $d$ a derivation of $R$. A. Nowicki ([11], [12]) obtained the following results under the assumption that the ring $R$ is Noetherian:

(1) An ideal $Q$ of $R$ is a $d$-prime $d$-ideal if and only if $Q$ is a quasi-prime $d$-ideal.

(2) The following three conditions are equivalent:
   (a) every quasi-prime ideal of $R$ is a prime ideal,
   (b) any weak associated prime of every $d$-ideal of $R$ is a $d$-ideal,
   (c) every $d$-prime $d$-ideal of $R$ is a prime ideal.

The aim of this paper is to try to weaken the condition “Noetherian” of the ring $R$ in A. Nowicki’s results above. The results which we obtained are as follows:

If $R$ is a Laskerian ring, then the conditions (a) and (b) above are equivalent (see Theorem 4.1). Furthermore, if $R$ is a strongly Laskerian ring, the results (1) and (2) above hold (see Theorem 4.2). If the ring $R$ is not strongly Laskerian, then the result (1) is not necessarily true, and then the conditions (a) and (c) of (2) are not equivalent in general, even if $R$ is Laskerian (see Example 4.3).

§2. Preliminaries

Throughout this paper, let $R$ be a ring, $d$ a derivation of $R$ and $\mathbb{Z}$ the rational integers. In this section we record several lemmas for convenience, which are known.

**Lemma 2.1 ([4, Proposition (1.4)])**. Let $I$ be a $d$-ideal of $R$ and $P$ a minimal prime divisor of $I$. Then the primary component $Q (= sat_P(I))$ of $I$ belonging to $P$ is also $d$-ideal.

**Lemma 2.2 ([9, Exercise 3, p.63])**. Any quasi-prime $d$-ideal of $R$ is primary. If $R$ contains the rational numbers, then every quasi-prime $d$-ideal of $R$ is prime.

For an ideal $I$ of $R$, we denote by $I_\#$ the biggest $d$-ideal contained in $I$. Note that $I_\# = \{ x \in I \mid d^n(x) \in I, \text{ for all } n \geq 1 \}$.

**Lemma 2.3 ([8, Proposition 2.2])**. For a $d$-ideal $Q$ of $R$, the following three conditions are equivalent:

1. $Q$ is quasi-prime.
2. $Q$ is primary and $Q = (\sqrt{Q})_\#$.
3. There is a prime ideal $P$ in $R$ such that $Q = P_\#$.

**Lemma 2.4 ([12, Proposition 2.1])**. Any quasi-prime $d$-ideal of $R$ is $d$-prime.
Lemma 2.5 ([8, Proposition 2.1]). The following four conditions are equivalent:

1. Every quasi-prime \( d \)-ideal of \( R \) is prime.
2. Any minimal prime divisor of every \( d \)-ideal of \( R \) is a \( d \)-ideal.
3. The radical of any \( d \)-ideal of \( R \) is a \( d \)-ideal.
4. For any prime ideal \( P \) of \( R \), the ideal \( P_\# \) is prime.

§3. \( d \)-prime \( d \)-ideals and quasi-prime \( d \)-ideals

In this section we study some conditions under which a \( d \)-prime \( d \)-ideal is primary. Furthermore we discuss a relation among quasi-prime ideals and prime ideals, and we give some examples.

For a ring \( R \) to be Laskerian, it is necessary and sufficient that it satisfies the following two conditions:

(LA\(_1\)) For every ideal \( I \) of \( R \) and every prime ideal \( P \) of \( R \), the saturation of \( I \) with respect to \( P \) in \( R \) is of the form \( I : (a) \) for some \( a \notin P \).

(LA\(_2\)) For every ideal \( I \) of \( R \), every decreasing sequence \( \text{sat}_{S_n}(I) \) (where \( (S_n) \) is any decreasing sequence of multiplicative subset of \( R \)) is stationary.

(cf.[1, IV, x2, Exercise 23]).

Proposition 3.1. Let \( R \) be a ring and \( d \) a derivation of \( R \). If \( R \) satisfies the conditions (LA\(_1\)) above or \( \text{char}(R) \neq 0 \), then every \( d \)-prime \( d \)-ideal of \( R \) is primary. In particular, if the ring \( R \) is Laskerian, then every \( d \)-prime \( d \)-ideal of \( R \) is primary.

Proof. First we assume that \( R \) satisfies the condition (LA\(_1\)). Let \( I \) be a \( d \)-prime \( d \)-ideal of \( R \) and \( P \) a minimal prime divisor of \( I \). Then the primary component \( Q \) of \( I \) belonging to \( P \) is \( d \)-ideal by Lemma 1.1. Since \( R \) satisfies the condition (LA\(_1\)), the saturation \( \text{sat}_P(I) = (Q) \) of \( I \) is of the form \( I : (a) \) for some \( a \notin P \). It follows that \( (x) \subset I : Q \) and so \( [x] \subset I : Q \), where \( [x] \) is the smallest \( d \)-ideal containing \( x \). Hence \( [x]Q \subset I \). Since \( [x] \notin I \), we have that \( Q \subset I \), and hence \( Q = I \). Therefore \( I \) is primary.

Next we assume that \( \text{char}(R) = n(\neq 0) \). Let \( I \) be a \( d \)-prime \( d \)-ideal. Suppose that \( xy \in I \) and \( x \notin \sqrt{I} \). Then we have \( x^n y \in I \) and hence \( y \in I : (x^n) \). Since \( I : (x^n) \) is a \( d \)-ideal, we have \( [y] \subset I : (x^n) \), where \( [y] \) is the smallest \( d \)-ideal containing \( y \). Therefore \( (x^n)[y] \subset I \). On the other hand, since \( (x^n) \notin I \), we have that \( [y] \subset I \), and hence \( y \in I \). Therefore \( I \) is primary.

Proposition 3.2. Let \( R \) be a ring and \( d \) a derivation of \( R \). If a \( d \)-prime \( d \)-ideal \( I \) of \( R \) has a primary decomposition, then \( I \) is primary.

Proof. Let \( P \) be a minimal prime divisor of \( I \) and \( Q \) the primary component of \( I \) belonging to \( P \). Since \( I \) has a primary decomposition, it is clear that \( Q = I : (x) \) for some \( x \notin P \). Therefore by the same way as the proof of the first case of Proposition 3.1, we have that \( I \) is primary.
Proposition 3.3. Let $R$ be a ring of characteristic 0 and $d$ a derivation of $R$. Let $I$ be a $d$-prime $d$-ideal of $R$. If $I \cap \mathbb{Z} \neq (0)$, where $\mathbb{Z}$ is the rational integers, then $I$ is primary.

Proof. Put $I \cap \mathbb{Z} = (n)(n \neq 0)$. Then the residue ring $R/I$ is of characteristic $n$. Let $\overline{d}$ be the derivation of $R/I$ defined by $\overline{d}(x + I) = d(x) + I$ ($x \in R$). Since $I$ is a $d$-prime $d$-ideal of $R$, $(0)$ is a $\overline{d}$-prime $\overline{d}$-ideal of $R/I$. By Proposition 3.1, $(0)$ is a primary ideal of $R/I$, and thus $I$ is a primary ideal of $R$.

Remark. We do not know whether a $d$-prime $d$-ideal is a primary ideal in general.

Proposition 3.4. Let $R$ be a ring of characteristic 0 and $d$ a derivation of $R$. Let $Q$ be a quasi-prime $d$-ideal of $R$. If $Q \cap \mathbb{Z} = (0)$, then $Q$ is prime.

Proof. Let $R'$ be the quotient ring $S^{-1}R$ with respect to $S = \mathbb{Z} - \{0\}(\subset \mathbb{R})$ and $d'$ the derivation of $R'$ induced by $d$. Put $P = \sqrt{Q}$. Then $Q$ is $P$-primary and $Q = P_0 \subset P$. Put $Q' = QR'$ and $P' = PR'$. Then $Q'$ is a $d'$-ideal and $P'$-primary. Furthermore we have $(P')_0 = (P_0)R'$. Thus $(P')_0 = QR' = Q'$ and therefore $Q'$ is a quasi-prime $d'$-ideal of $R'$. Since $R'$ contains the rational integers, $Q'$ is a prime ideal by Lemma 2.2. It follows that $Q' = P'$ and so we have $Q = P$. Consequently $Q$ is prime.

In case of $\text{char}(R) \neq 0$, a quasi-prime $d$-ideal of $R$ is not necessarily prime as in the following example.

Example 3.5. Let $k$ be a field of characteristic $p > 0$ and $R = k[X]$ a polynomial ring over $k$. Let $d$ be a $k$-derivation of $R$ such that $d(X) = 1$. Put $P = (X)$ and $Q = (X^p)$. Then $Q$ is a $P$-primary ideal of $R$ and by a simple calculation we have $Q = P_0$. Thus $Q$ is a quasi-prime $d$-ideal by Lemma 2.3, but $Q$ is not a prime ideal.

In case of $\text{char}(R) = 0$, let $Q$ be a quasi-prime $d$-ideal of $R$ such that $Q \cap \mathbb{Z} \neq (0)$. Then $Q$ is not necessarily prime as shown in the following example.

Example 3.6. Let $R = \mathbb{Z}[X]$ be a polynomial ring over the rational integers $\mathbb{Z}$ and $d$ a derivation of $R$ such that $d(X) = 1$. Then $Q := (X^2, 2)$ is a $d$-ideal of $R$. Put $P = (X, 2)$. Then $Q$ is a $P$-primary ideal. It is clear that $Q = P_0$. Thus $Q$ is a quasi-prime $d$-ideal by Lemma 2.3, but $Q$ is not a prime ideal.

§4. Main results

We are now ready to prove the main results.

Theorem 4.1. Let $R$ be a Laskerian ring and $d$ a derivation of $R$. The following two conditions are equivalent:

(a) Every quasi-prime $d$-ideal of $R$ is prime.
(b) Any weak associated prime of every d-ideal of R is a d-ideal.

Proof. (a) $\implies$ (b). Let I be a d-ideal of R. First, we consider the case $\text{char}(R) \neq 0$ or $\text{char}(R) = 0$ and $I \cap \mathbb{Z} \neq (0)$. Then I can be written as an irredundant intersection of a finite number of primary d-ideals $Q_i (i = 1, \ldots, n)$ by [5, Theorem 2 and Proposition 6]. Furthermore we have that irredundant intersection of a finite number of primary non-zero integer ideals. Thus every weak associated prime of $I$ is a d-ideal.

Next, suppose that $\text{char}(R) = 0$ and $I \cap \mathbb{Z} = (0)$. Let $I = Q_1 \cap \cdots \cap Q_n$ be an irredundant primary decomposition such that $P_i \cap \mathbb{Z} = (0) (i = 1, \ldots, t)$ and $P_i \cap \mathbb{Z} \neq (0) (i = t + 1, \ldots, n)$, where $P_i = \sqrt{Q_i} (i = 1, \ldots, n)$. Note that $\text{Ass}_f(R/I) = \{ P_1, \ldots, P_n \}$. By [5, Theorem 1], $P_i (i = 1, \ldots, t)$ are d-ideals. Put $I_1 = Q_1 \cap \cdots \cap Q_t$ and $I_2 = Q_{t+1} \cap \cdots \cap Q_n$. Then $I_2 \cap \mathbb{Z} = (q)$ for some non-zero integer $q$. Put $I_2' = qR + I$. Then $I_2'$ is a d-ideal and $I \subset I_2' \subset I_2$. Thus we have $I = I_1 \cap I_2'$ and $I_2$ can be written as an intersection $Q_1' \cap \cdots \cap Q_m'$ of primary d-ideals $Q_i' (i = 1, \ldots, m)$ by [5, Proposition 6]. Therefore we have that $I = Q_1 \cap \cdots \cap Q_n \cap Q_1' \cap \cdots \cap Q_m'$. By the same reason as the first step, each $\sqrt{Q_i}$ is a d-ideal. For any $i (t + 1 \leq i \leq n)$, $P_i = \sqrt{Q_j'}$ for some $j (1 \leq j \leq m)$. Thus $P_i (1 \leq i \leq n)$ are d-ideals.

(b) $\implies$ (a). Let I be a d-ideal of R and P a minimal prime divisor of I. Then clearly P is a weak associated prime of I. Thus P is a d-ideal. Therefore, the assertion follows from Lemma 1.5.

Remark. By the same way as the proof of Theorem 4.1, we get the following result:

Let $R$ be a Laskerian ring of characteristic $0$, d a derivation of $R$ and I a proper d-ideal of $R$. Let $\mathbb{Z}$ be the rational integers. Then I can be represented as an irredundant intersection $Q_1 \cap \cdots \cap Q_t \cap \cdots \cap Q_n (0 \leq t \leq n)$ of primary ideals $Q_i$ of $R$ such that: (1) $P_i \cap \mathbb{Z} = (0) (i = 1, \ldots, t)$, $P_j \cap \mathbb{Z} \neq (0) (j = t + 1, \ldots, n)$ (where $P_i = \sqrt{Q_i}$). (2) $P_i (i = 1, \ldots, t)$, $Q_j (j = t + 1, \ldots, n)$ are d-ideals. Obviously, (i) if the ring $R$ contains the rational numbers, then the number $t$ equal to $n$, and (ii) if $I \cap \mathbb{Z} \neq (0)$, then the number $t$ equal to 0.

When $R$ is a Noetherian ring, the following Theorem 4.2 was proved by A. Nowicki in [11] and [12].

Theorem 4.2. Let $R$ be a strongly Laskerian ring and $d$ a derivation of $R$, then the following statements hold.

1. For a d-ideal $Q$, $Q$ is d-prime if and only if $Q$ is quasi-prime.
2. The following three conditions are equivalent:
   a. Every quasi-prime d-ideal of $R$ is prime.
   b. Any weak associated prime of every d-ideal of $R$ is a d-ideal.
   c. Every d-prime d-ideal of $R$ is prime.

Proof. (1) In virtue of Lemma 1.4, it suffices to show that if a d-ideal $Q$ is d-prime, then $Q$ is quasi-prime. Put $\sqrt{Q} = P$. Then $P$ is prime by Proposition
3.1. Furthermore we have that $Q \subseteq P^\# \subseteq P$. Since $Q$ is $P$-primary, $P^n \subseteq Q$ for some $n \geq 1$, and hence $(P^\#)^n \subseteq Q$. Since $Q$ is $d$-prime, we have that $P^\# \subseteq Q$ and therefore $Q = P^\#$. Thus $Q$ is quasi-prime by Lemma 1.3.

(2) The equivalence of (a) and (b) follows from Theorem 4.1 and the equivalence of (a) and (c) follows from (1).

Remarks. (1) Let $R$ be a ring and $d$ a derivation of $R$. If every quasi-prime $d$-ideal of $R$ is prime, then $R$ is called a $d$-MP ring (cf. [11]), or a special differential ring (cf. [8]).

(2) In Example 4.3 below we show that there is a Laskerian ring $R$ which is not a strongly Laskerian and there is a derivation $d$ of $R$ such that $R$ has a $d$-prime $d$-ideal which is neither prime nor quasi-prime. Therefore Example 4.3 shows that if $R$ is not strongly Laskerian, Theorem 4.2 is not necessarily true even if $R$ is Laskerian.

Example 4.3 (cf. [2, Example 2.1]). Let $T = k[X_1, X_2, \ldots]$ be a polynomial ring over the field $k(= \mathbb{Z}/(p))$ of prime characteristic $p$. For the ideal $A = (X_1^p, X_2^p, \ldots)$, put $R = T/A = k[x_1, x_2, \ldots]$, where $x_n = X_n + A$. Then $R$ is a local ring with the maximal ideal $M = (x_1, x_2, \ldots)$. Let $d$ be a derivation of $R$ such that $d(x_n) = x_{n+1}$ for every $n \geq 1$.

In this situation, the following properties hold.

1. $(0)$ is a $d$-prime $d$-ideal of $R$, but it is not prime.
2. $M$ is the only one quasi-prime $d$-ideal of $R$.
3. Every quasi-prime $d$-ideal of $R$ is prime.
4. $R$ is a Laskerian ring.
5. $R$ is not a strongly Laskerian ring.

Proof. (1) Assume that $I$ and $J$ are $d$-ideals of $R$ such that $IJ = (0)$. If $I \neq (0)$ and $J \neq (0)$, then $I \ni x_1^{p-1} \cdots x_n^{p-1}$ and $J \ni x_{n+1}^{p-1} \cdots x_{n+m}^{p-1}$ for some $n \geq 1$ and $m \geq 1$ (see the proof of Lemma 2.3 (p. 291) of [2]). Hence we have $IJ \ni x_1^{p-1} \cdots x_{n+m}^{p-1} \neq 0$, which is a contradiction. Consequently, $(0)$ is a $d$-prime $d$-ideal.

(2) Since Spec$(R) = \{ M \}$ and $M$ is a $d$-ideal, $M$ is the only one quasi-prime $d$-ideal of $R$.

(3) This is an immediate consequence of (2).

(4) Let $I$ be any ideal of $R$. Then $\sqrt{I} = M$, and so $I$ is primary. Hence $R$ is a Laskerian ring.

(5) Note that $(0)$ is a $M$-primary ideal of $R$ and $\{ x_1, x_2, \ldots \}$ is a $p$-basis of $R$ over $R^p$. For every $n \geq 1$, $M^n$ contains $x_1 x_2 \cdots x_n \neq 0$, and hence we have $M^n \neq (0)$. Therefore $R$ is not a strongly Laskerian ring.

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