POSITIVE SOLUTIONS OF IMPULSIVE INTEGRODIFFERENTIAL BOUNDARY VALUE PROBLEMS*

Xiyu LIU

(Received January 10, 1996)

AMS 1991 Mathematics Subject Classification. 34B.

Key words and phrases. impulsive integrodifferential equations, boundary value problems, fixed points, cones.

§1. INTRODUCTION

Impulsive differential equations arise naturally and often in engineering and physics, see [1]–[4] for example. Recently, various existence principles of such problems are obtained. Among these, Guo and Liu in [1] proved that at least two solutions exist for superlinear impulsive boundary value problems. In this paper, we consider the existence of positive solutions for impulsive integrodifferential boundary value problems, where the nonlinear term is sublinear at infinity and may have singular nature at the origin. Our results are new even in the non-impulsive case. Specifically, consider the following problem:

\begin{align}
(p(t)x'(t))' + p(t)f(t, x(t), (Hx)(t), (Sx)(t)) &= 0, \\
t \in (0, 1), t \neq t_k, k = 1, 2, ..., m, \\
\lim_{\varepsilon \to 0^+} [x(t_k + \varepsilon) - x(t_k - \varepsilon)] &= I_k(x(t_k)), \\
x(t) \text{ is left continuous at } t = t_k, k = 1, 2, ..., m, \\
\alpha x(0) - \beta \lim_{t \to 0} p(t)x'(t) &= \gamma x(1) + \delta \lim_{t \to 1} p(t)x'(t) = 0.
\end{align}

*This work is supported in part by NSF of China

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where \( f \in C([0, 1] \times R^+ \times R^1 \times R^1, R^+) \), \( R^+ = (0, \infty) \), \( p \in C[0, 1] \cap C^1(0, 1) \), \( p(t) > 0 \) for \( t \in (0, 1) \). We shall assume the following conditions throughout this paper:

\[
\int_0^1 \frac{1}{p(t)} dt < \infty.
\]

The operators \( H \) and \( S \) are given by

\[
(Hx)(t) = \int_0^t k(t,s)x(s)ds, \quad (Sx)(t) = \int_0^1 k_1(t,s)x(s)ds
\]

with \( k, k_1 \in C([0,1] \times [0,1], [0,\infty]) \), and \( \alpha, \beta, \gamma, \delta \geq 0, \beta \gamma + \alpha \delta + \alpha \gamma > 0, \beta \delta = 0, I_k \in C([0,\infty), [0,\infty]), k = 1, 2, ..., m, 0 < t_1 < t_2 < ... < t_m < 1 \). Note that the nonlinear term \( f(t,x,y,z) \) may be singular at \( t = 0,1 \) and \( x = 0 \), i.e., it may be unbounded when \( t \) tends to 0,1 or when \( x \) tends to 0. Let \( J = [0,1], PC(J) = \{ x : x \text{ is a function from } J \text{ to } R^1, \text{ continuous at } t \neq t_k, \text{ left continuous at } t = t_k, \text{ and right hand limit at } t = t_k \text{ exist for } k = 1, 2, ..., m \} \). Recall that \( PC(J) \) is a Banach space with norm \( \| x \| = \sup_{t \in J} |x(t)| \). Denote the normal cone of \( PC(J) \) by \( P = \{ x : x \in PC(J), x(t) \geq 0, t \in [0,1] \} \). A function \( x \) is called a positive solution of (1.1) if \( x(t) > 0, t \in (0,1), x \in PC(J) \) and satisfies (1.1). Throughout this paper, we use \( C \) to denote a generic constant, and \( C(\varepsilon) \) a constant dependent of \( \varepsilon \).

\section{The x-Nonsingular Case}

In this section, we assume that \( f(t,x,y,z) \) is nonsingular with respect to \( x \) at \( x = 0 \), and we shall prove the existence of positive solutions. Denote

\[
\tau_1(t) = \int_t^1 \frac{1}{p(s)} ds, \quad \tau_0(t) = \int_0^t \frac{1}{p(s)} ds
\]

then we have \( \tau_1, \tau_0 \in C[0,1] \). Let \( \rho^2 = \beta \gamma + \alpha \delta + \alpha \gamma \int_0^1 \frac{1}{p(s)} ds \), and write

\[
u(t) = \frac{1}{\rho} [\delta + \gamma \tau_1(t)], \quad v(t) = \frac{1}{\rho} [\beta + \alpha \tau_0(t)].
\]

Note that \( \gamma v + \alpha u \equiv \rho \). Define

\[
G(t,s) = \begin{cases} 
  u(t)v(s)p(s), & 0 \leq s \leq t \leq 1, \\
  v(t)u(s)p(s), & 0 \leq t \leq s \leq 1.
\end{cases}
\]

Denote \( \theta(s) = \tau_1(s) \) for \( s \in (0,1) \) when \( \beta > 0, \delta = 0; \theta(s) = \tau_0(s) \) for \( s \in (0,1) \) when \( \beta = 0, \delta > 0; \theta(s) = \tau_0(s) \) for \( s \in [0, \frac{1}{2}] \), and \( \theta(s) = \tau_1(s) \) for \( s \in (\frac{1}{2}, 1] \) when \( \beta = 0, \delta = 0 \). Write
\[
\triangle(p x')\big|_{t_k} = \lim_{\varepsilon \to +0} [p(t_k + \varepsilon)x'(t_k + \varepsilon) - p(t_k - \varepsilon)x'(t_k - \varepsilon)],
\]
and introduce the following condition (see [1]):

\[\text{(2.4)} \quad \triangle(p x')\big|_{t_k} = -\frac{\gamma I_k(x(t_k))}{\delta + \gamma \tau_1(t_k)}, \quad k = 1, 2, \ldots, m.\]

Similar to Lemma 1 of [1], we have the following Lemma.

**Lemma 2.1.** If \(x \in P\) is a solution of the following integral equation

\[
x(t) = (Ax)(t) = \int_0^1 G(t, s)f(s, x(s), (Hx)(s), (Sx)(s))ds
+ (\delta + \gamma \tau_1(t)) \sum_{0 < t_k < t} I_k(x(t_k))
\]

(2.5)

then \(x\) is a positive solution of (1.1) satisfying condition (2.4).

**Lemma 2.2.** The following estimate holds

\[G(t, s) \leq \theta(s)p(s), \quad t, s \in [0, 1], s \neq 0, 1.\]

*Proof.* It is straightforward, see [5], or see Appendix. \(\square\)

In order to show the existence of positive solutions, we now make the following assumptions:

- **(H_1)** \(f(t, x, y, z) \leq \psi(t)\phi(x, y, z), \) where \(\psi \in C[[0, 1), R^+]\), \(\phi \in C[R^+ \times R^1 \times R^1, R^+]\) and \(\int_0^1 \theta(s)p(s)\psi(s)ds < \infty.\)
- **(H_2)** \(\theta(s)p(s)\) is bounded for \(s \in (0, 1).\)
- **(H_3)** \(\lim_{x \to \infty} \frac{I_k(x)}{x} = 0, k = 1, 2, \ldots, m.\)
- **(H_4)** \(\lim_{|x| + |y| + |z| \to \infty} \frac{\phi(x, y, z)}{|x| + |y| + |z|} \leq \lambda, \lambda > 0.\)
- **(H_5)** \(\lambda[1 + M + M_1] \int_0^1 \theta(s)p(s)ds < \frac{1}{m + 1}, \) where \(M = \max\{k(t, s) : t, s \in [0, 1]\}, M_1 = \max\{k_1(t, s) : t, s \in [0, 1]\}.\)
- **(H_6)** For any \(h > 0\), there exist \(y \in C[0, 1]\) with \(y(t) \geq 0\) for \(t \in [0, 1]\) and \(y(t) \neq 0\) such that \(f(t, x, y, z) \geq y(t)\) for \(t \in (0, 1), x, y, z \in (0, h].\)
Lemma 2.3. Assume \((H_1)\) holds, and \(\phi(x, y, z)\) is bounded on \((0, 1) \times [0, M] \times [0, M]\), where \(M > 0\) is arbitrary. Then the operator \(A\) maps \(P^*\) into \(P\) and is completely continuous, where \(P^* = \{x \in P : x(t) > 0, t \in (0, 1)\}\).

Proof. For any \(x \in Q\), where \(Q\) is a bounded subset of \(P\), we have that
\[
(2.6) \quad f(s, x(s), Hx, Sx) \leq C\psi(s),
\]
where \(C\) is a constant. Define
\[
y_1(t) = \int_0^1 G(t, s)f(s, x(s), (Hx)(s), (Sx)(s))ds = (A_1x)(t),
\]
\[
y_2(t) = (\delta + \gamma \tau_1(t)) \sum_{0 < t_k < t} \frac{I_k(x(t_k))}{\delta + \gamma \tau_1(t_k)} = (A_2x)(t).
\]

From [5] we know that \(y_1 \in C[0, 1]\) and \(A\) is continuous and maps bounded sets into bounded sets, where \(P^*\) and \(P\) have induced topology from \(PC(J)\), see Appendix for complete proof. When \(t \in (0, 1), t \neq t_k\), we can directly get
\[
(2.7) \quad -p(t)y_1'(t) = \frac{\gamma}{p} \int_0^t vpfds - \frac{\alpha}{p} \int_t^1 upfds.
\]

Because the proofs of other cases are similar, we now only consider the case of \(\beta = \delta = 0, \alpha = \gamma = 1\). Then from (2.6) we have
\[
|p(t)y_1'(t)| \leq C \int_0^t \tau_0 \psi ds + C \int_t^1 \tau_1 \psi ds.
\]

Notice the fact that
\[
\int_0^1 \frac{1}{p(t)}(\int_0^t \tau_0 \psi ds)dt = \int_0^1 \tau_0 \tau_1 \psi p dt < \infty,
\]
\[
\int_0^1 \frac{1}{p(t)}(\int_t^1 \tau_1 \psi ds)dt = \int_0^1 \tau_0 \tau_1 \psi p dt < \infty.
\]

Hence \(\{y_1(t)\}\) is pre-compact for \(x \in Q\). Similarly we can prove that \(\{y_2(t)\}\) is pre-compact for \(x \in Q\). As a result, \(A\) is completely continuous. The proof is complete.

\[\square\]

Theorem 2.4. Suppose \((H_1)\)–\((H_5)\) hold and \(\phi(x, y, z)\) is bounded on \((0, 1) \times [0, M] \times [0, M]\) for arbitrary \(M > 0\). Then problem (1.1) has a positive solution \(x \in PC(J)\).
Proof. From the definition of $A_2$ and condition (H₃), it is easy to show that
\begin{equation}
\|A_2x\| \leq \varepsilon C\|x\| + C(\varepsilon)
\end{equation}
Let $M$ and $M_1$ be as in condition (H₅). From our assumptions we know that
\[
\phi(x, y, z) \leq (\lambda + \varepsilon)(|x| + |y| + |z|) + C(\varepsilon), \quad x, y, z > 0.
\]
Hence
\[
\|A_1x\| \leq \int_0^1 \theta \psi p\phi(x, Hx, Sx)ds
\leq \int_0^1 \theta \psi ps((\lambda + \varepsilon)(|x| + M\|x\| + M_1\|x\|) + C(\varepsilon)).
\]
Choose $\varepsilon$ such that $\int_0^1 \theta \psi ps((\lambda + \varepsilon)(1 + M + M_1) < 1$ and we obtain
\[
\lim_{\|x\| \to \infty} \frac{\|A_1x\|}{\|x\|} < 1.
\]
Then the fixed point theorem of cone compression (see [6]) yields the required solution. The proof is complete.

§3. THE $x$-SINGULAR CASE

In this section we will give an existence principle when the function $f(t, x, y, z)$ is unbounded. First, consider the following approximate problem:

\begin{equation}
\begin{aligned}
(p(t)x'(t))' + p(t)f_n(t, x(t), (Hx)(t), (Sx)(t)) = 0, \\
t \in (0, 1), \quad t \neq t_k, k = 1, 2, \ldots, m, \\
\lim_{\varepsilon \to 0} [x(t_k + \varepsilon) - x(t_k - \varepsilon)] = I_k(x(t_k)), \\
x(t) \text{ is left continuous at } t = t_k, \quad k = 1, 2, \ldots, m, \\
ax(0) - \beta \lim_{t \to 0} p(t)x'(t) = \gamma x(1) + \delta \lim_{t \to 1} p(t)x'(t) = 0
\end{aligned}
\end{equation}

where $f_n(t, x, y, z) = f(t, \max\{\frac{1}{n}, x\}, y, z)$. Suppose (H₁)–(H₅) hold. Then from Theorem 2.4 we know that problem (3.1) is solvable for any integer $n \leq 1$. Moreover, the solutions of (3.1) satisfy (2.4).

Lemma 3.1. Suppose (H₁)–(H₅) hold. Then there exists a constant $R > 0$ independent of $n$ such that $0 \leq x(t) \leq R$, $t \in [0, 1]$ for any positive solution $x$ of (3.1).
Proof. Let $x$ be a positive solution of (3.1).

(1) Suppose $\|x\| = x(0)$. From the boundary condition we know 
\[ \beta \lim_{t \to 0^+} p(t)x'(t) = \alpha x(0) \geq 0. \]
Obviously $\lim_{t \to 0^+} p(t)x'(t) \leq 0$, thus $\beta \lim_{t \to 0^+} p(t)x'(t) = 0$. As a result we can deduce $\alpha = 0$, and furthermore, $\beta > 0$ with 
\[ \lim_{t \to 0^+} p(t)x'(t) = 0. \]
As stated above, in this case $\delta = 0$, henceforth $x(1) = 0$. Then from (3.1) and (2.4) we get $x'(t) < 0$ for $t \in (0, 1), t \neq t_k$. Choose $T > 1$ such that
\[
\phi(x, y, z) \leq (\lambda + \varepsilon)(|x| + |y| + |z|) \text{ for } |x| + |y| + |z| \geq T
\]
Without loss of generality we can assume that there exists $t^*_k \in (t_{k-1}, t_k]$ with $x(t^*_k) = T$, and $\|x\| > T$. In the following we assume that $t_0 = 0, t_{m+1} = 1$ for convenience. Now we begin with the first interval $[0, t_1]$. In the case of $x(t_1) \leq T$, we then choose $t^*_1 \in (0, t_1]$ such that $x(t^*_1) = T$. By integration we get for $t < t^*_1$ that
\[
-p(t)x'(t) \leq \int_0^t p(s)f_n(s, x(s), Hx, Sx)ds 
\leq (\lambda + \varepsilon)\int_0^t p(s)\psi(s)(x + Hx + Sx)ds
\leq (\lambda + \varepsilon)\|x\|(1 + M + M_1)\int_0^t p(s)\psi(s)ds.
\]
Thus we have
\[
x(0) - T \leq (\lambda + \varepsilon)\|x\|(1 + M + M_1)\int_0^{t^*_1} p(s)\psi(s)\tau_1(s)ds
\leq (\lambda + \varepsilon)\|x\|(1 + M + M_1)\int_0^{t^*_1} p(s)\psi(s)\theta(s)ds.
\]
From condition (H5), by letting $\varepsilon$ be small enough we can get the required constant $R > 0$ such that $\|x\| \leq R$. If on the other hand $x(t_1) > T$, then we integrate (3.3) on $[0, t_1]$. Thus we get
\[
x(0) - x(t_1) \leq (\lambda + \varepsilon)\|x\|(1 + M + M_1)\int_0^{t_1} \psi(s)\theta(s)p(s)ds.
\]
Because $x$ satisfies (2.4) we know $px'|_{t_1+0} \leq 0$. Then integration on $[t_1, t]$ with $t < t^*_2$ yields
\[
-p(t)x'(t) \leq -px'|_{t_1+0} + (\lambda + \varepsilon)\|x\|(1 + M + M_1)\int_{t_1}^{t} p\psi ds
\]
where $t^*_2$ belongs to $(t_1, t_2)$ or $t^*_2 = t_2$. From (2.4) we get
\[
-px'|_{t_1+0} = -px'|_{t_1-0} + \frac{\gamma I_1(x(t_1))}{\delta + \gamma \tau_1(t_1)}.
\]
From (H\textsubscript{3}) we have $C(\varepsilon) > 0$ dependent of $\varepsilon$ such that

$$I_k(x) < \varepsilon x + C(\varepsilon), \quad x > 0, k = 1, 2, ..., m.$$  \hfill (3.8)

Moreover (3.3) yields

$$-px'\big|_{t_1-0} \leq (\lambda + \varepsilon)\|x\| (1 + M + M_1) \int_0^{t_1} \psi ds.$$  \hfill (3.9)

Together with (3.7) we can deduce that

$$-px'\big|_{t_1+0} \leq (\lambda + \varepsilon)\|x\| (1 + M + M_1) \int_0^{t_1} \psi ds + \varepsilon C\|x\| + C(\varepsilon).$$  \hfill (3.10)

Combining (3.6) and (3.10) yields (Note that $\theta = \tau_1$ in this case):

$$x(t_1 + 0) - x(t_2^*)$$

$$\leq (\lambda + \varepsilon)\|x\| (1 + M + M_1) \int_{t_1}^{t_2} \frac{1}{p(s)} \int_0^{t_1} \psi ds + \varepsilon C\|x\| + C(\varepsilon)$$

$$+ (\lambda + \varepsilon)\|x\| (1 + M + M_1) \int_{t_1}^{t_2} \left[ \frac{1}{p(t)} \int_{t_1}^{t} \psi \psi ds \right] dt$$

$$\leq (\lambda + \varepsilon)\|x\| (1 + M + M_1) \int_0^{t_1} \psi \psi ds + \varepsilon C\|x\| + C(\varepsilon)$$

$$+ (\lambda + \varepsilon)\|x\| (1 + M + M_1) \int_{t_1}^{t_2} \psi \psi ds.$$  \hfill (3.11)

Using $\Delta x\big|_{t_1} > 0$, (2.4) and (3.5) we then obtain

$$x(0) \leq x(t_2^*) + (\lambda + \varepsilon)\|x\| (1 + M + M_1) \int_0^{t_1} \psi \psi ds + \varepsilon C\|x\|$$

$$+ C(\varepsilon) + (\lambda + \varepsilon)\|x\| (1 + M + M_1) \int_0^{t_1} \psi \psi ds.$$  \hfill (3.12)

If $x(t_2^*) = T$, then the proof is complete by letting $\varepsilon$ be small enough. If $x(t_2) > T$ and $t_2^* = t_2$ we can get (3.11) for $t_2^* = t_2$. Then inequalities similar to (3.5)–(3.11) hold. Thus by induction we finally have

$$x(0) \leq x(t_k^*) + (\lambda + \varepsilon)\|x\| (1 + M + M_1) \int_0^{t_k} \psi \psi ds + \varepsilon C\|x\|$$

$$+ C(\varepsilon) + (k - 1)(\lambda + \varepsilon)\|x\| (1 + M + M_1) \int_0^{t_k} \psi \psi ds.$$  \hfill (3.12)

By (H\textsubscript{5}) we can choose $\varepsilon$ small, and the required $R$ exists.

(2) Suppose $\|x\| = x(1)$, then $\gamma = 1, \delta = 0$, and $\lim_{t \to 1} p(t)x'(t) = 0$. The rest of the proof is similar to step (1).
(3) Suppose \( \|x\| = x(t_0) \), where \( t_0 \in (t_{k-1}, t_k) \). First assume \( \alpha = 0, \beta > 0 \), then \( x'(t_0) = 0 \), and \( \theta(s) = \tau_1(s) \). So the proof is similar to case (1). If \( \beta = \delta = 0, \alpha = \gamma = 1 \), then we can distinguish between two cases: \( t_0 \leq \frac{1}{2} \) and \( t_0 \geq \frac{1}{2} \). Integration on \([t_0, 1]\) and \([0, t_0]\) respectively will yield the required estimate.

(4) Suppose \( \|x\| = x(t_k + 0), 1 \leq k \leq m \). Then \( x'(t_k + 0) \leq 0 \), and \( x'(t_0) < 0 \) for \( t \in (t_k, 1) \). If \( \lim_{t \to 0} p(t)x'(t) = 0 \) we can prove as in case (1) that \( x(0) \leq R \).

Thus condition \( (H_3) \) implies that \( \|x\| \) is bounded.

Now assume that \( x' \) has one zero to in \([0, t_k]\) (including right limit zeros). From equation (3.1) and condition (2.4), we know that \( x'(t) \leq 0 \) for \( t \in (t_0, 1) \setminus \{t_1, t_2, ..., t_m\} \). Then from the impulse conditions \( \lim_{\varepsilon \to +0} [x(t_j + \varepsilon) - x(t_j - \varepsilon)] = I_j(x(t_j)) \) for \( j = 1, 2, ..., m \) and (3.8), we get

\[
\|x\| = x(t_k + 0) \leq m \varepsilon \|x\| + C(\varepsilon) + x(t_{**} + 0), \quad t_{**} \in [t_0, t_k).
\]

By essentially the same way as in the proof of (3.12) of step (1), we get

\[
x(t_0 + 0) \leq x(t_k^*) + (\lambda + \varepsilon) \|x\| \int_0^{t_k^*} \psi p \theta ds + \varepsilon C \|x\| + C(\varepsilon)
\]

\[
+ (k-1) (\lambda + \varepsilon) \|x\| \int_0^{t_k} \psi p \theta ds
\]

where \( t_k^* \) satisfies \( t_k^* = t_k \), or \( x(t_k^*) = T \) and \( t_k^* \in (t_0, t_k) \). If \( x(t_k^*) = T \) then (3.14) becomes

\[
x(t_0 + 0) \leq T + (\lambda + \varepsilon) \|x\| \int_0^{t_k^*} \psi p \theta ds + \varepsilon C \|x\| + C(\varepsilon)
\]

\[
+ (k-1) (\lambda + \varepsilon) \|x\| \int_0^{t_k} \psi p \theta ds.
\]

Thus the inequalities (3.13) and (3.15) yield

\[
\|x\| \leq T + (\lambda + \varepsilon) \|x\| \int_0^{t_k} \psi p \theta ds + \varepsilon C \|x\| + C(\varepsilon)
\]

\[
+ (k-1) (\lambda + \varepsilon) \|x\| \int_0^{t_k} \psi p \theta ds.
\]

If on the other hand \( t_k^* = t_k \), then let \( t_{**} = t_k \) in (3.13), and together with (3.14) we get that (3.16) holds. By letting \( \varepsilon \) be small enough we know that the required estimate holds.

Finally let \( x' \) have no zeros in \([0, t_k]\). If \( x'(t_k - 0) \geq 0 \), then from (2.4) and (3.8) we have

\[
0 \leq -px'_t |_{t_k + 0} \leq -\Delta(px') |_{t_k} \leq C \varepsilon \|x\| + C(\varepsilon),
\]
(3.18) \[ 0 \leq -px' \mid_{t_k} \leq -\Delta(px') \mid_{t_k} \leq C\varepsilon \|x\| + C(\varepsilon). \]

Since the proofs are similar we only consider the case of \( \beta > 0, \delta = 0. \) Thus \( \gamma = 0 \) and \( x(1) = 0. \) By integration, instead of \( \lim_{t \to 0} p(t)x'(t) = 0 \) (or \( x'(t_0) = 0 \)), we can use inequality (3.17). Then similar to (3.3)–(3.12) we can easily obtain

\[
\|x\| \leq T + (\lambda + \varepsilon)\|x\|(1 + M + M_1) \int_{t_k}^1 \psi_p\theta ds + \varepsilon C\|x\| + C(\varepsilon)
\]

\[
+ M\varepsilon\|x\| + (m - k)(\lambda + \varepsilon)\|x\|(1 + M + M_1) \int_{t_k}^1 \psi_p\theta ds.
\]

Consequently our lemma is true. Next from the boundary condition, we have \( ax(0) = \beta \lim_{t \to 0} p(t)x'(t) \). If \( \alpha = 0 \), then \( \beta > 0 \) and \( \lim_{t \to 0} p(t)x'(t) = 0. \) Thus from equation (3.1) and condition (2.4) we know that \( \|x\| = x(0), \) and this is exactly case (1). If \( \alpha > 0, x(0) = 0, \) then \( \lim_{t \to 0} p(t)x'(t) \geq 0. \) When \( \lim_{t \to 0} p(t)x'(t) = 0, \) this is again case (1). Now we need to consider the case of \( \lim_{t \to 0} p(t)x'(t) > 0. \) If on the other hand \( \alpha > 0, x(0) > 0, \) then \( \beta > 0, \) and \( \lim_{t \to 0} p(t)x'(t) > 0. \) In both cases we finally know that \( x'(t) > 0 \) when \( t \) belongs to some neighbourhood of zero. Therefore we can choose \( t_j \) with \( 1 \leq j \leq k - 1 \) such that \( x'(t_j + 0) \leq 0 \) and \( x'(t_j - 0) \geq 0 \) (Notice that we have already assumed that \( x'(t) \) has no zeros). Moreover \( x'(t) \leq 0 \) for \( t \in (t_j, 1) \setminus \{t_1, t_2, ..., t_m\}. \) Note that we can use inequalities (3.17) and (3.18) instead of the condition \( x'(t_0) = 0. \) Let \( t_0 = t_j \) in inequality (3.13). Then by essentially the same way as in the proof of inequalities (3.13)–(3.16), we can prove that (3.13)–(3.16) still hold. The only difference is using \( t_0 \) instead of \( t_j. \) The proof is complete. \( \square \)

**Lemma 3.2.** Let \((H_1)-(H_6)\) hold. Then there exists \( x^* \) such that \( x^*(t) > 0 \) for \( t \in (0, 1) \) and \( x(t) \geq x^*(t), t \in (0, 1), \) where \( x \) is any solution of the problem (3.1).

**Proof.** From (2.5), Lemma 3.1 and \((H_6),\) let \( h = 1 + R + M + M_1 \) and

\[
x^*(t) = \int_0^1 G(t, s)y(s)ds.
\]

Then it is easy to show that \( x^* \) is the required function. \( \square \)

Now we assume that \((H_1)-(H_6)\) hold, and \( \phi(x, y, z) \geq 1 \) without loss of generality. Let \( a(x, y, z) = \frac{\phi(x, y, z)}{x}, \) and

\[
b(u) = \sup\{a(x, y, z) : x \in (u, R + 1], y, z \in [0, (R + 1)(1 + M + M_1)]\}.
\]

\[
T(u) = \int_0^u \frac{1}{b(v)}dv.
\]
Lemma 3.3. Let $x$ be a solution of (3.1). Denote by $t^0_2$ the zeros of $x'(t)$. Then there exists $\eta$ independent of $n$ such that

1. $t^0_2 \leq 1 - \eta$, when $\beta > 0, \delta = 0$.
2. $t^0_2 \geq \eta$, when $\delta > 0, \beta = 0$.
3. $\eta \leq t^0_2 \leq 1 - \eta$, when $\beta = \delta = 0$.

where $n$ is any integer.

Proof. The proof of this lemma is essentially the same as that of Lemma 4.4 of [5]. Thus it is omitted. □

Lemma 3.4. Assume $(H_1)$–$(H_6)$ hold. Then there exists $w(t) \in L^1(0,1)$ such that

$$\left| \frac{d}{dt} T(x(t)) \right| \leq w(t), \quad t \neq t_k, k = 1, 2, ..., m$$

where $x$ is any solution of (3.1).

Proof. From the proof of Lemma 3.1 we know that there are only three cases of $x(t)$ to be considered.

(1) $\lim_{t \to 0} p(t)x'(t) = 0$ or $\lim_{t \to 1} p(t)x'(t) = 0$. Since the proofs of two cases are similar, we only consider the case of $\lim_{t \to 0} p(t)x'(t) = 0$, hence $\alpha = 0, \beta > 0$, and $x$ decreases on $(t_{k-1}, t_k), k = 1, 2, ..., m$. Integration on $(0, t_1)$ yields

$$-px' \leq C \int_0^t p\psi b(x) ds \leq C b(x) \int_0^t \psi ds.$$

Let $z(t) = T(x(t))$. Then from $(H_1)$ we know

$$|z'(t)| \leq \frac{C}{p(t)} \int_0^t \psi ds \in L^1(0, t_1).$$

From (2.4) we have $-px'|_{t_{1+0}} \leq C$, hence for $t \in (t_1, t_2)$ we have

$$|px'| \leq C + \int_{t_1}^{t_2} \psi p ds \leq C.$$

Similarly $|x'| \leq C$ for $t \in [t_1, t_m]$ and $|z'| \leq C$. For $t \in (t_m, 1)$, we have (Notice that $b \geq \frac{1}{1+R}$)

$$|px'| \leq C + C \int_{t_m}^t \psi p ds \leq C + b(x) \int_{t_m}^t \psi ds,$$

$$|z'| \leq \frac{C}{p(t)} + \frac{C}{p(t)} \int_{t_m}^t \psi ds \in L^1(0, 1).$$
(2) $x'$ has one zero $t^0$ in $(0, 1)$, including limit zeros. According to the
boundary conditions, we have the following three cases to be considered:

(i) $\beta > 0, \delta = 0$,

(ii) $\delta > 0, \beta = 0$,

(iii) $\beta = \delta = 0$.

However, proofs of the lemma in these three cases are similar. So for brevity
we only consider the third case, i.e., the case of $\beta = \delta = 0$ and $\alpha = \gamma = 1$.
Then Lemma 3.3 yields $\eta \leq t^0 \leq 1 - \eta$. Now we can assume $\eta \leq t_1, 1 - \eta \geq t_m$
for convenience. By integration and condition (3.4) we get $C > 0$ independent
of $n$ such that

\[
|x'(t)| \leq C, \quad t \in [\eta, 1 - \eta].
\]

Since $b$ decreases and $x$ increases, integration of (3.1) on $(t, t_1]$ yields:

\[
p(t)x'(t) - p(t_1)x'(t_1) \leq \int_t^{t_1} p(s)\psi(s) \max\{\frac{1}{n}, x(s)\} a(\max\{\frac{1}{n}, x(s)\}, (Hx)(s), (Sx)(s)) ds
\]

\[
\leq (R + 1) \int_t^{t_1} p(s)\psi(s)b(x(s)) ds.
\]

Thus from (3.19) we have

\[
|px'| \leq C + C \int_t^{t_1} \psi b(x) ds \leq C + Cb(x) \int_t^{t_1} \psi ds.
\]

Let $z(t) = T(x(t))$, then

\[
|z'| \leq C + \frac{1}{p} \int_t^{t_1} \psi ds \in L^1(0, t_1).
\]

Similarly we can get

\[
|z'| \leq C + \frac{1}{p} \int_{t_m}^{t} \psi ds \in L^1(t_m, 1).
\]

(3) There exists $t_k, 1 \leq k \leq m$, such that $x'(t_k + 0) < 0, x'(t_k - 0) > 0$. As
in the proof of Lemma 3.1, we know in this case that

\[
|x'(t_k) + 0| \leq C.
\]

where $C$ is independent of $n$. Thus we can also prove that (3.19) is true. Following the above steps (3.20)–(3.23), it is easy to get the required estimate. The proof is complete.

Now we come to our main theorem of this section.
**Theorem 3.5.** Suppose $(H_1)-(H_6)$ hold. Then problem (1.1) has a positive solution satisfying (2.4).

**Proof.** Let $x_n$ be solutions of (3.1), and $z_n = T(x_n)$. From the above Lemma 3.1, Lemma 3.4 and the Arzela-Ascoli theorem we may assume $z_n$ converges. Thus we know $x_n \to x$ in $C[0,1]$. Because $p(t)$ is positive on $(0,1)$ we can assume without loss of generality that $x_n \to x$ in $PC^1[\varepsilon,1-\varepsilon]$, where $\varepsilon \in (0,1)$ is arbitrary, and where $PC^1[\varepsilon,1-\varepsilon] = \{ x : x \text{ is continuously differentiable at } t \neq t_k, \text{ and left continuous at } t = t_k, \text{ and } x(t_k+0), x(t_k-0), x'(t_k+0), x'(t_k-0) \text{ exist for } k = 1,2,\ldots,m \}$, equipped with the norm

$$||x||_1 = \max\{||x||, ||x'||\}.$$ 

Hence $x$ satisfies (2.4) and the impulsive conditions. If $\beta = \delta = 0, \alpha = \gamma = 1$, then $x$ satisfies the boundary conditions and by integration we have

$$p(\eta)x_n'(\eta) - p(t)x_n'(t) = \int_{\eta}^{t} p(s)f_n(s,x_n,Hx_n,Sx_n)ds.$$ 

From the continuity of $f$ we obtain

$$p(\eta)x'(\eta) - p(t)x'(t) = \int_{\eta}^{t} p(s)f(s,x,Hx,Sx)ds.$$ 

Thus $x$ is a solution of (1.1). If $\beta > 0, \delta = 0$, then similar to [5] we can prove $x$ is a solution of (1.1).

**Remark 3.6.** Our conditions are weaker than those of [5], [7] even in the non-impulsive cases.

**APPENDIX**

In this appendix, we will give detailed proof of Lemma 2.2, and will prove that the operator $A$ in page 70 is continuous and maps bounded sets into bounded sets.

**Proof of Lemma 2.2.** First from the definiton of $u(t), v(t)$, and the condition

$$\int_{0}^{1} \frac{1}{p(t)}dt < \infty,$$ 

we know that $u(t)$ is decreasing and $v(t)$ is increasing, and $u, v \in C[0,1]$. Thus from the definition of $G(t,s)$ we have that

$$(A_1) \quad G(t,s) \leq u(s)v(s)p(s), \quad \text{for } t, s \in [0,1].$$
Case (1). Suppose that $\beta > 0$, $\delta = 0$. Then

$$\rho^2 = \beta \gamma + \alpha \gamma \int_0^1 \frac{1}{p(t)} dt,$$

$$u(s) = \frac{\gamma}{\rho} \tau_1(s), \quad v(s) = \frac{1}{\rho} [\beta + \alpha \tau_0(s)],$$

and $\theta(s) = \tau_1(s)$. Thus from (A$_1$) we have that

$$G(t, s) \leq \frac{1}{\rho^2} [\beta + \alpha \tau_0(s)] \gamma \tau_1(s) p(s) \leq p(s) \theta(s).$$

Case (2). Suppose that $\beta = 0$, $\delta > 0$. Proof of this case is similar to that of the case (1).

Case (3). Suppose that $\beta = \delta = 0$. Then

$$\rho^2 = \alpha \gamma \int_0^1 \frac{1}{p(t)} dt,$$

$$u(s) = \frac{\gamma}{\rho} \tau_1(s), \quad v(s) = \frac{\alpha}{\rho} \tau_0(s),$$

and $\theta(s) = \tau_0(s)$, for $s \in (0, \frac{1}{2})$; $\theta(s) = \tau_1(s)$, for $s \in (\frac{1}{2}, 1)$. Thus from (A$_1$) we have that

$$G(t, s) \leq \frac{1}{\rho^2} \alpha \gamma \tau_0(s) \tau_1(s) p(s) \leq p(s) \theta(s).$$

The proof is complete.

Proof of the properties of the operator $A$ in page 70: Now we shall prove that the operator $A$ in page 70 is continuous and maps bounded sets into bounded sets, where the domain of $A$ is $P^*$, and the range of $A$ is $P$; both have induced topology from $PC(J)$. Let the operators $A_1, A_2$ be the same as those in the proof of Lemma 2.3. We will give the whole proof in the following three steps:

1. $A_1$ maps $P^*$ into $P$.

In fact, let $x \in P^*$, $y_1(t) = (A_1 x)(t)$. Because $\phi(x, y, z)$ is bounded on $(0, 1) \times [0, M] \times [0, M]$, where $M$ is arbitrary, we can choose constant $C$ such that

$$G(t, s) f(s, x(s), (Hx)(s), (Sx)(s)) \leq C \theta(s) p(s) \psi(s) \in L^1(0, 1).$$

It is clear that $G(t, s)$ is continuous on $[0, 1] \times [0, 1]$. Thus from Lebesgue's convergence theorem of dominance we know that $y_1(t)$ is continuous, and thus belongs to $P$.

2. $A_1$ is continuous and maps bounded sets into bounded sets.

In fact, let $E$ be a bounded set in $P^*$. Then we can choose a constant $C$ as in step (1) such that the estimate (A$_2$) holds, which immediately yields that $A_1(E)$ is a bounded set in $P$. Next let $x_0 \in P^*$ be fixed and $x \in P^*$ such that
\[ |(A_1x)(t) - (A_1x_0)(t)| \]
\[ \leq C \int_0^\varepsilon p(s)\theta(s)\psi(s)ds + C \int_{1-\varepsilon}^1 p(s)\theta(s)\psi(s)ds \]
\[ + \int_{\varepsilon}^{1-\varepsilon} p(s)\theta(s)\psi(s) \left| f(s, x, Hx, Sx) - f(s, x_0, Hx_0, Sx_0) \right| ds. \]

Since \( f(s, x, y, z) \) is continuous on \((0, 1) \times (0, \infty) \times (0, \infty) \times [0, \infty)\), it is easy to show that \( A_1 \) is continuous.

(3) Clearly \( A_2 \) maps \( P^* \) into \( P \), and from the continuity of \( I_k, k = 1, 2, ..., m \) we know that there are constants such that for \( x, x_0 \in P^* \) the following estimates hold:

\[ |(A_2x)(t) - (A_2x_0)(t)| \leq C \sum_{k=1}^m |I_k(x(t_k)) - I_k(x_0(t_k))| \]
\[ |(A_2x)(t)| \leq C \sum_{k=1}^m |I_k(x(t_k))| \]

Hence it is easy to show that \( A_2 \) is continuous and maps bounded sets into bounded sets. The proof is complete.

**ACKNOWLEDGMENTS**

The author thanks the referee for his useful comments and suggestions for the first and second revisions.

**References**


Xiyu LIU
Department of Mathematics, Shandong Normal University
Ji-Nan, Shandong 250014, People’s Republic of China