HOMOCLINIC ORBITS FOR 3–DIMENSIONAL SYSTEMS

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Abstract. Suppose a dynamical system $dx/dt = F(x; \mu), x \in \mathbb{R}^s, \mu \in \mathbb{R}^m,$ has a hyperbolic saddle at $x = 0$ with a homoclinic loop, for $\mu = \mu^0$. When $\mu$ varies from $\mu^0$, the loop will be destroyed in general. For $s = 2$, Perko proved that, if $\mu$ varies on an $(m-1)$ dimensional hypersurface, then the system remains to admit homoclinic orbit. We consider here the same problem for $s = 3$. The result is: if $\mu$ varies on an $(m-2)$ hypersurface, then the system remains to admit homoclinic orbit.

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§1. Introduction

Consider a 3-dimensional dynamical system

$$
\begin{align*}
\frac{dx_1}{dt} &= F_1(x_1, x_2, x_3; \mu), \\
\frac{dx_2}{dt} &= F_2(x_1, x_2, x_3; \mu), \\
\frac{dx_3}{dt} &= F_3(x_1, x_2, x_3; \mu),
\end{align*}
$$

(1)

in which $\mu \in \mathbb{R}^m$, $m \geq 3$. $F_j$ are supposed to be of $C^2$-class with respect to both $x = \{x_1, x_2, x_3\}$ and $\mu = \{\mu_1, \ldots, \mu_m\}$.

Suppose that, for $\mu = \mu^0$, (1) has a hyperbolic saddle at $(0,0,0)$ with a homoclinic loop $\Gamma : x = \gamma(t)$. When $\mu$ varies from $\mu^0$, the loop will be destroyed in general. For standard exposition of these facts, see [2]. For 2-dimensional systems of $C^\infty$ or $C^\omega$ class, Perko [4] proved that, if $\mu$ varies on an $(m-1)$ dimensional hypersurface, then the system remains to admit homoclinic orbit. We consider here 3-dimensional case.
Now we suppose that, for $\mu = \mu^0$, $F = t(F_1, F_2, F_3)$ is expanded at $(0,0,0)$ as follows:

$$(2) \quad F(x, \mu^0) = \Lambda x + \Phi^0(x), \quad \Phi^0(x) = O(|x|^2),$$

in which

\[
\Lambda = \begin{pmatrix}
\lambda_1 & \epsilon & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{pmatrix}
\]

$\lambda_j$ are real and $\lambda_1 \leq \lambda_2 < 0 < \lambda_3$, $\epsilon = 0$ if $\lambda_1 \neq \lambda_2$.

Now we put

$$(3) \quad I_j = \int_{-\infty}^{\infty} \exp\left[- \int_0^t \left(\nabla F - tDF(\gamma(s))ds\right) \left(F \times \frac{\partial F}{\partial \mu_j}\right)(\gamma(t))dt\right]$$

$$= t(I_{j1}, I_{j2}, I_{j3}),$$

for $j = 1, \ldots, m$, in which we assume $\mu = \mu^0$.

We will prove the following theorem:

**Theorem.** Suppose that, for $\mu = \mu^0$, (1) has a hyperbolic saddle at $(0,0,0)$ with a homoclinic loop $\Gamma$, and $F = t(F_1, F_2, F_3)$ is expanded at $(0,0,0)$ as shown in (2), with the following condition ($\Lambda$):

$$\lambda_3 > \lambda_2 - \lambda_1.$$

Further, suppose that

$$(4) \quad \begin{vmatrix}
I_{11} & I_{21} \\
I_{12} & I_{22}
\end{vmatrix} \neq 0$$

Then there are $\delta > 0$ and two functions $h_1, h_2$ of $(\mu_3, \ldots, \mu_m)$ defined for $|\mu_3 - \mu_3^0| + \ldots + |\mu_m - \mu_m^0| < \delta$ such that, when $\mu = (\mu_1, \mu_2, \mu_3, \ldots, \mu_m)$ varies satisfying $\mu_j = h_j(\mu_3, \ldots, \mu_m)$, $j = 1, 2$, then (1) remains to admit homoclinic loop at $(0,0,0)$.

This is a 3-dimensional generalization of a theorem of Perko [4]. Generalizations to higher dimensional case will be further topics.
§2. Proof of the Theorem

For simplicity, we write \( F(x, \mu^0) \) as \( F_0(x) \), \( \partial F(x, \mu^0)/\partial \mu_j \) as \( \partial F_0(x)/\partial \mu_j \).

By taking suitable coordinates, we can assume that the local stable manifold \( S_0 \) and local unstable manifold \( U_0 \) are

\[
S_0 : x_3 = 0 \quad \text{and} \quad U_0 : x_1 = x_2 = 0,
\]

respectively, and that the condition (4) holds still. Then we have, in (2),

\[
\Phi^0_1(0, 0, x_3) = \Phi^0_2(0, 0, x_3) = \Phi^0_3(x_1, x_2, 0) = 0.
\]

For general \( \mu \), we write stable manifold and unstable manifold as \( M^S_\mu \) and \( M^U_\mu \), respectively. By the stable manifold theorem [3], these manifolds are \( C^2 \) continuous with respect to \( \mu \).

We assume that \( \gamma(0) = x_0 \in S_0 \). Let \( \Pi \) be a plane crossing with \( \Gamma \) at \( x_0 \). Take a point \( b \in U_0 \cap \Gamma \). Let \( a_\mu \in M^S_\mu \) and \( b_\mu \in M^U_\mu \) be points such that they depend on \( \mu \) as \( C^2 \)-class functions, and \( a_{\mu_0} = x_0, b_{\mu_0} = b \).

Now let \( \phi(t, \xi, \mu), \xi \in \mathbb{R}^3 \), denote the solution of (1) which satisfies the initial condition \( \phi(0, \xi, \mu) = \xi \). Let \( \tau^U_\mu \) be the time such that \( \phi(\tau^U_\mu, b_\mu, \mu) = x_0 \), and \( \tau^S_\mu, \tau^U_\mu \) be the times such that \( \phi(\tau^S_\mu, a_\mu, \mu) \in \Pi, \phi(\tau^U_\mu, b_\mu, \mu) \in \Pi \). The following lemma is proved easily, as in Perko [4].

**Lemma 1.** Under the hypothesis of Theorem, we can take \( \tau^S_\mu \) and \( \tau^U_\mu \) so that \( \tau^S_\mu \rightarrow 0 \) and \( \tau^U_\mu \rightarrow \tau^U \) as \( \mu \rightarrow \mu_0 \).

Write \( \phi(t + \tau^S_\mu, a_\mu, \mu) \) as \( x^S(t, \mu) \) and \( \phi(t + \tau^U_\mu, b_\mu, \mu) \) as \( x^U(t, \mu) \). Put

\[
x^S(0, \mu) = x^S_0(\mu) \quad \text{and} \quad x^U(0, \mu) = x^U_0(\mu),
\]

(5)

\[
d(\mu) = x^U_0(\mu) - x^S_0(\mu).
\]

If \( d(\mu) = 0 \), then \( x^S(t, \mu) = x^U(t, \mu) \) represents a homoclinic loop. Write \( x^S(t, \mu) \) or \( x^U(t, \mu) \) simply as \( x(t, \mu) \), and

\[
\xi_k(t, \mu) = \frac{\partial x(t, \mu)}{\partial \mu_k},
\]

\[
\rho_k(t, \mu) = \xi_k(t, \mu) \times F(x(t, \mu), \mu),
\]

then

\[
\frac{d\xi_k}{dt} = D F(x(t, \mu), \mu) \xi_k + \frac{\partial F(x(t, \mu), \mu)}{\partial \mu_j}
\]
\[
\frac{d\rho_k}{dt} = (\nabla F - {}^t D F)(x(t, \mu), \mu)\rho_k + \frac{\partial F}{\partial \mu_k} \times F(x(t, \mu), \mu).
\]

To see (6), writing \( \xi_k \) and \( \rho_k \) simply as \( \xi \) and \( \rho \), respectively, and differentiating \( \rho \) by \( t \),

\[
\frac{d\rho}{dt} = \frac{d\xi}{dt} \times F + \xi \times \frac{dF}{dt} = (\langle (DF)\xi + \frac{\partial F}{\partial \mu_k} \rangle \times F + \xi \times ((DF)F)).
\]

Let \( DF = (a_{ij}) \). Then the first component of \( \{(DF)\xi \times F + \xi \times ((DF)F)\} \) is, by an easy calculation,

\[
\begin{vmatrix}
\Sigma a_{2j} \xi_j & F_2 \\
\Sigma a_{3j} \xi_j & F_3
\end{vmatrix} + \begin{vmatrix}
\xi_2 & \Sigma a_{2j} F_j \\
\xi_3 & \Sigma a_{3j} F_j
\end{vmatrix} = (a_{22} + a_{33})(\xi \times F)_1 - a_{21}(\xi \times F)_2 - a_{31}(\xi \times F)_3.
\]

The second and third components are obtained similarly, and we have

\[
((DF)\xi \times F + \xi \times ((DF)F)) = (\nabla F - {}^t D F)(\xi \times F),
\]

which shows (6). Write

\[
\nabla F - {}^t D F = H, \quad H(\mu = \mu_0) = H_0.
\]

Then (6) can be written as

\[
(6') \quad \frac{d\rho_k}{dt} = H\rho_k + \frac{\partial F}{\partial \mu_k} \times F.
\]

For \( \rho_k = \rho_k^S \) with \( \mu = \mu_0 \) we have, solving the first order linear differential equation (6'),

\[
\left[ \exp\left[-\int_0^t H_0(\gamma(s))ds\rho_k^S(t, \mu_0)\right]\right]_{t_0}^{t_1} = \int_{t_0}^{t_1} \exp\left[-\int_0^t H_0(\gamma(s))ds\right] \left\{ \frac{\partial F}{\partial \mu_k} \times F \right\} (\gamma(t))dt.
\]

Letting \( t_0 = 0, \ t_1 \to \infty \), we get

\[
\lim_{t \to \infty} \left\{ \exp\left[-\int_0^t H_0(\gamma(s))ds\rho_k^S(t, \mu_0)\right] - \rho_k^S(0, \mu_0) \right\}
\]
\[ \int_0^\infty \exp\left[ -\int_0^\infty H_0(\gamma(s))ds \right] \left\{ \frac{\partial F}{\partial \mu_k} \times F \right\} (\gamma(t))dt. \]

Similarly we have
\[ \lim_{t \to -\infty} \left\{ \exp\left[ -\int_0^t H_0(\gamma(s))ds \right] \rho_k^S(t, \mu_0) \right\} - \rho_k^U(0, \mu^0) = \int_{-\infty}^0 \exp\left[ -\int_0^t H_0(\gamma(s))ds \right] \left\{ \frac{\partial F}{\partial \mu_k} \times F \right\} (\gamma(t))dt. \]

By the condition (Λ), we obtain that
\( (7) \) the first and second components of
\[ \exp[-\int_0^t H_0(\gamma(s))ds] \rho_k^S(t, \mu_0) \] tend to 0 as \( t \to \infty \), and that
\( (7') \) \( \lim_{t \to -\infty} \exp[-\int_0^t H_0(\gamma(s))ds] \rho_k^U(t, \mu_0) = 0 \), respectively, which will be shown later. Then we get
\( (8) \)
\[ \rho_k^U(0, \mu^0) - \rho_k^S(0, \mu^0) = \frac{\partial \mathbf{d}(\mu_0)}{\partial \mu_k} \times \mathbf{F}_0(\mathbf{x}_0) = \int_0^\infty \exp\left[ -\int_0^t H_0(\gamma(s))ds \right] \left\{ \frac{\partial \mathbf{F}}{\partial \mu_k} \times \mathbf{F} \right\} (\gamma(t))dt + \begin{pmatrix} 0 \\ 0 \\ c_k \end{pmatrix}, \]

Since there holds, for vectors \( \mathbf{A}, \mathbf{B}, \mathbf{F} \),
\[ (\mathbf{A} \times \mathbf{F}) \times (\mathbf{B} \times \mathbf{F}) = ((\mathbf{A} \times \mathbf{B}) \cdot \mathbf{F}) \mathbf{F}, \]
the third components of
\[ \mathbf{I}_1 \times \mathbf{I}_2 \text{ and } \left\{ \frac{\partial \mathbf{d}(\mu_0)}{\partial \mu_1} \times \frac{\partial \mathbf{d}(\mu_0)}{\partial \mu_2} \right\} \cdot \mathbf{F}_0(\mathbf{x}_0) \mathbf{F}_0(\mathbf{x}_0) \]
coincide. If (4) holds, then \( [\partial \mathbf{d}(\mu^0)/\partial \mu_1] \times [\partial \mathbf{d}(\mu^0)/\partial \mu_2] \neq 0 \). Therefore, we may take, for example, that

\[
\begin{vmatrix}
\partial d_1(\mu^0)/\partial \mu_1 & \partial d_1(\mu^0)/\partial \mu_2 \\
\partial d_2(\mu^0)/\partial \mu_1 & \partial d_2(\mu^0)/\partial \mu_2 \\
\end{vmatrix} \neq 0.
\]

Then, by the implicit function theorem, there are two functions \( h_1, h_2 \) of \( (\mu_3, ..., \mu_m) \), defined for \( |\mu_3 - \mu_3^0| + ... + |\mu_m - \mu_m^0| < \delta \) with sufficiently small \( \delta > 0 \), such that, when \( \mu = (\mu_1, \mu_2, \mu_3, ..., \mu_m) \) varies satisfying \( \mu_j = h_j(\mu_3, ..., \mu_m), \ j = 1, 2 \), then \( d_1(\mu) = d_2(\mu) = 0 \). Since \( \mathbf{d}(\mu) \) moves on the plane \( \Pi \), we obtain that \( \mathbf{d}(\mu) = 0 \), which proves the existence of homoclinic loop.

It remains to prove (7) and (7').

On the local stable manifold for \( \mu = \mu^0 \), we have \( x_3 = 0 \) and

\[
\phi_3^0(x_1, x_2, 0) = 0, \quad \frac{\partial \phi_3^0(x_1, x_2, 0)}{\partial x_1} = \frac{\partial \phi_3^0(x_1, x_2, 0)}{\partial x_2} = 0.
\]

Then (1,3) and (2,3) elements \( h_{13}^0 \) and \( h_{23}^0 \) of \( \mathbf{H}_0 \) are zero. As \( x_1^S(t, \mu^0) = \exp[\lambda_1 t](a + o(1)), x_2^S(t, \mu^0) = \exp[\lambda_2 t](b + o(1)) \), we get, when \( t \to \infty \),

\[
\mathbf{H}_0 = \begin{pmatrix}
\lambda_2 + \lambda_3 & \lambda_3 + \lambda_1 \\
\lambda_3 + \lambda_1 & \lambda_1 + \lambda_2 \\
\end{pmatrix} + \begin{pmatrix}
O(\exp[\lambda_2 t]) & 0 \\
O(\exp[\lambda_2 t]) & 0 \\
O(\exp[\lambda_2 t]) & 0 \\
\end{pmatrix},
\]

\[
- \int_0^t \mathbf{H}_0 ds = \begin{pmatrix}
-(\lambda_2 + \lambda_3) t & -(\lambda_3 + \lambda_1) t \\
-(\lambda_3 + \lambda_1) t & -(\lambda_1 + \lambda_2) t \\
\end{pmatrix} + \begin{pmatrix}
O(1) & 0 \\
O(1) & 0 \\
\end{pmatrix},
\]

hence

\[
\exp[- \int_0^t \mathbf{H}_0 ds] = \begin{pmatrix}
O(\exp[-(\lambda_3 + \lambda_1) t]) & 0 \\
O(\exp[-(\lambda_3 + \lambda_1) t]) & 0 \\
O(\exp[-(\lambda_1 + \lambda_2) t]) & 0 \\
\end{pmatrix}.
\]

As \( \rho_k^S(t) = O(\exp[\lambda_2 t]) \), we have

\[
\exp[- \int_0^t \mathbf{H}_0 ds] \rho_k^S(t) = \begin{pmatrix}
O(\exp[-(\lambda_1 + \lambda_2 - \lambda_3) t]) & 0 \\
O(\exp[-(\lambda_1 + \lambda_2 - \lambda_3) t]) & 0 \\
O(\exp[-(\lambda_1 t)]) & 0 \\
\end{pmatrix}.
\]

Since \( -\lambda_1 + \lambda_2 - \lambda_3 < 0 \) by (A), the first and second elements of the right side tend to 0 as \( t \to \infty \), which proves (7).
Next, as $t \to -\infty$, we have

$$\exp[- \int_0^t H_0 ds] = O(\exp[-(\lambda_2 + \lambda_3)t]) + O(1),$$

$$\rho_k^U(t) = O(\exp[\lambda_3 t]).$$

$$\exp[- \int_0^t H_0 ds] \rho_k^U(t) = O(\exp[-\lambda_2 t]) + O(\exp[\lambda_3 t]).$$

Since $\lambda_2 < 0 < \lambda_3$, the right side tends to 0 as $t \to -\infty$, which proves (7').

**Remark.** When $F$ is expanded at $(0,0,0)$ as follows:

$$F(x, \mu^0) = \Lambda x + O(|x|^2),$$

$$\Lambda = \begin{pmatrix} \lambda_1 & -\nu & 0 \\ \nu & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

$\lambda_j$, $\nu$ are real, and $\lambda_1 < 0 < \lambda_3$,

then we can obtain also a similar result as above.

**References**


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