On Some Models of Universal Expansion in General Relativity Using Otsuki Connections

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Abstract. An Otsuki connection $\Gamma$ is a cross section of the bundle $T(M) \otimes \mathcal{D}^2(M)$ and they can be understood as generalized objects of the affine connections. Briefly, the difference between these two theories is that an Otsuki connection $\Gamma$ is an affine connection if and only if the principal part $\lambda(\Gamma)$ of $\Gamma$, which is a homomorphism of the tangent bundle $T(M)$, is the identity map. We consider some special class $\Gamma(\Psi, G)$ of $\Gamma$. Using $\Gamma(\Psi, G)$, this paper presents universal expansion-like models which are exact solutions of some partial differential equations.

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§1. Basic Concepts and Preliminaries

A cross section $\Gamma$ on the vector bundle $T(M) \otimes \mathcal{D}^2(M)$ is called an Otsuki connection, where $T(M)$ and $\mathcal{D}^2(M)$ are the tangent bundle and the cotangent bundle of order 2 on a smooth manifold $M$ respectively. Using local coordinates $(u^\lambda)$, $\Gamma$ is written as follows:

$$\Gamma = \frac{\partial}{\partial u^\lambda} \otimes \left( P^\lambda_{\mu} d^{2}u^\mu + \Gamma^\lambda_{\mu\nu} d^{\mu}u^\mu \otimes d\nu^\lambda \right).$$

It is easy to see that the $P = (P^\lambda_{\mu})$ transforms as a tensor field of type $(1, 1)$ under coordinates changes. The tensor field $P = (P^\lambda_{\mu})$, which is denoted by $\lambda(\Gamma)$, is called the principal part of $\Gamma$. According to Otsuki [6, 7], the covariant derivative $\Gamma_X Y$ is defined by

$$\Gamma_X Y = (X(Y^\lambda) P^\mu_{\lambda} + \Gamma^\mu_{\lambda\nu} X^\lambda Y^\nu) \frac{\partial}{\partial u^\mu},$$

where $X, Y$ and $\Gamma_X Y$ are tangent vector fields on $M$. The operator $\Gamma_X$ has the following properties:
1. \( \Gamma_{fX+gY}Z = f\Gamma_XZ + g\Gamma_YZ, \)
2. \( \Gamma_X(Y + Z) = \Gamma_XY + \Gamma_XZ, \)
3. \( \Gamma_X fY = X(f)P(Y) + f\Gamma_XY, \) where \( f,g \) are functions on \( M. \)

It is routine work to extend the covariant derivative to arbitrary tensor fields. For example, if \( F \) is a tensor field of type \( (0,2) \), then \( \Gamma_X F \) is defined by

\[
\Gamma_X F(Y, Z) = X(F(PY, PZ)) - F(\Gamma_XY, PZ) - F(PY, \Gamma_XZ).
\]

We put

\[
T(X, Y) = \Gamma_X Y - \Gamma_Y X - P[X, Y].
\]

This element becomes a tensor field of type \( (0,2) \) and is called the torsion tensor field of \( \Gamma \). Any geodesic \( \gamma \) in \( M \) with an Otsuki connection \( \Gamma \) is given by a solution of the system of the ordinary differential equation of order 2 on \( M: \)

\[
P_\mu \frac{d^2 u^\mu}{ds^2} + \Gamma^\lambda_{\mu\nu} \frac{du^\mu}{ds} \frac{du^\nu}{ds} = 0,
\]

where \( s \) is an affine parameter of the connection.

Let \( P = (P_\mu^\lambda) \) and \( G = (g_{\mu\nu}) \) be a regular tensor field of type \( (1,1) \) and a non-singular tensor field of type \( (0,2) \) on \( M \). We put

\[
\bar{G}(X, Y) = G(PX, PY).
\]

Using the terminology of Otsuki connections, the Levi-Civita connection \( \tilde{\nabla} \) with respect to \( \bar{G} \) is written by

\[
\tilde{\nabla} = \frac{\partial}{\partial u^\lambda} \otimes \left( \delta^\lambda_{\mu\nu} d^2 u^\mu + \tilde{\Gamma}^\lambda_{\mu\nu} du^\mu \otimes du^\nu \right),
\]

where \( \tilde{\Gamma}^\lambda_{\mu\nu} \) are the Christoffel symbols of \( \bar{G} = (g_{\mu\nu}). \) We define an Otsuki connection

\[
P\tilde{\nabla} = \frac{\partial}{\partial u^\lambda} \otimes \left( P_\mu^\lambda d^2 u^\mu + P_\delta^\lambda \tilde{\Gamma}^\delta_{\mu\nu} du^\mu \otimes du^\nu \right).
\]

It is easy to see the following fundamental properties of \( P\tilde{\nabla}. \)

1. \( P = \lambda(\Gamma). \)
2. \( (P\tilde{\nabla})_X G = 0. \)
3. \( P\tilde{\nabla} \) is torsion free.
4. \( \gamma \) is a geodesic of \( \tilde{\nabla} \) if and only if it is a geodesic of \( P\tilde{\nabla}. \)
Conversely $\Gamma = P \hat{\nabla}$ is uniquely determined by the above 1~3, which we write

$$\Gamma = P \hat{\nabla} = \Gamma(P, G).$$

The Otsuki connection $\Gamma$, which we will consider in this paper, is a case $P = \Psi I$ and denoted by $\Gamma(\Psi, G)$, where $\Psi$ is a function on $M$ and $I$ is the fundamental unit tensor field of type $(1, 1)$. $\Gamma(\Psi, G)$ seems to have a meaning only where $\Psi$ does not vanish, but any function $\Psi$ on $M$ is available for $\Gamma(\Psi, G)$ because it can be written locally as follows:

$$\Gamma(\Psi, G) = \frac{\partial}{\partial u^\lambda} \otimes \left( \Psi du^\lambda + \Gamma^\lambda_{\mu \nu} du^\mu \otimes du^\nu \right),$$

$$\Gamma^\lambda_{\mu \nu} = \Psi \{\lambda\}_\mu + \left( \frac{\partial \Psi}{\partial u^\mu} \delta^\lambda_{\nu} + \frac{\partial \Psi}{\partial u^\nu} \delta^\lambda_{\mu} + \frac{\partial \Psi}{\partial u^\sigma} g^{\lambda \mu} g_{\mu \nu} \right),$$

where we use the apparatuses on a Riemannian manifold $(M, G)$. Using an affine parameter $s$ of $\Gamma(\Psi, G)$, equations of a geodesic become

$$(1) \quad \Psi \frac{d^2 u^\lambda}{ds^2} + \Gamma^\lambda_{\mu \nu} \frac{du^\mu}{ds} \frac{du^\nu}{ds} = 0.$$

We define a set $Sing(\Gamma) \subset M$ and a metric $\hat{G}$ by

$$Sing(\Gamma) = \{ x \in M \mid \Psi(x) = 0 \}, \quad \hat{G} = \Psi^2 G.$$

The next lemma is a special case of the above property 4, which says that a space $M \setminus Sing(\Gamma)$ with an Otsuki connection $\Gamma = \Gamma(\Psi, G)$ and a Riemannian manifold $(M \setminus Sing(\Gamma), \hat{G})$ are geodesically equivalent in the following sense.

**Lemma 1.** A curve $\gamma(s)$ in $M \setminus Sing(\Gamma)$ is a geodesic in the sense of Otsuki geometry of $\Gamma(\Psi, G)$ if and only if it is a geodesic in the sense of Riemannian geometry of $(M \setminus Sing(\Gamma), \hat{G})$.

Lemma 1 shows that $\hat{G} = \Psi^2 G$ has an important meaning in $\Gamma(\Psi, G)$ geometry, which we call the essential metric of an Otsuki connection $\Gamma(\Psi, G)$.

In the paper [4] we define a function $S_{\Gamma(\Psi, G)}$ and the condition (A) as follows:

$$(A) \quad \delta \int_M S_{\Gamma(\Psi, G)} dv_G = 0.$$

Using local coordinates $(u^\lambda)$ and the apparatuses on the Riemannian manifold $(M, G)$, the condition (A) becomes as follows:

$$\Psi(R^{\mu \nu} - \frac{1}{2} g^{\mu \nu} S - \frac{1}{2} \nabla_\lambda \nabla_\kappa (\Psi^2) (g^{\mu \lambda} g^{\nu \kappa} - g^{\mu \nu} g^{\lambda \kappa})$$

$$+ 12(\nabla_\lambda \Psi)(\nabla_\kappa \Psi) (g^{\mu \lambda} g^{\nu \kappa} - \frac{1}{2} g^{\mu \nu} g^{\lambda \kappa}) \Psi = 0,$$

$$\Psi(\Delta - \frac{1}{8} S) + \frac{1}{2} g^{\mu \nu}(\nabla_\mu \Psi)(\nabla_\nu \Psi) = 0,$$
where $S$, $\triangle$ are the scalar curvature, the Laplace-Beltrami operator of $(M, G)$. It appears very difficult to find non-trivial solutions, which means solutions with $\Psi$ not being constant, of the above equations. However, using a function $h$ and a metric $\overline{G}$, which are defined by
\[ h = \frac{\sqrt{\triangle}}{2} \log \Psi, \quad \overline{G} = \Psi^3 G, \]
the equations become the following simpler forms [5]:
\[ \mathcal{R}_{\mu \nu} - \frac{1}{2} g_{\mu \nu} S + (g^{\mu \rho} g^{\alpha \beta} - g^{\alpha \beta} g^{\mu \rho})(\nabla_\alpha h)(\nabla_\beta h) = 0, \]
\[ \overline{\Delta}(h) = 0, \]
where we use the apparatus of Riemannian manifold $(M, G)$. Rewriting these equations to the covariant forms, we have
\begin{align}
R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} S &= (2 \delta^\alpha_\mu \delta^\beta_\nu - g_{\mu \nu} g^{\alpha \beta})(\nabla_\alpha h)(\nabla_\beta h), \\
\triangle(h) &= 0,
\end{align}
where we use the apparatus on $(M \setminus Sing(\Gamma), \overline{G})$ but abbreviate the bar for convenience, which we do not think causes any confusion and we will use these notation from now on. (2) and (3) are the Euler-Lagrange equations of a Lagrange density $L(h, \overline{G})$ which is defined as follows [5]:
\[ L(h, \overline{G}) = g^{\mu \nu} \left[ \{^\alpha_\mu \} \{^\beta_\nu \} - \{^\alpha_\nu \} \{^\beta_\mu \} \right] + 2(\nabla_\mu h)(\nabla_\nu h) \sqrt{-\overline{G}}, \]
where $(M \setminus Sing(\Gamma), \overline{G})$ is a 4-dimensional Lorentz manifold and $|\overline{G}| = det(g_{\lambda \mu})$.
In the same paper [5] we look for solutions $\Gamma(\Psi, G)$ of the above equations under the condition that $\Gamma(\Psi, G)$ has the spherical symmetry and find two interesting families of Otsuki connections, one of which is the Schwartzschild spacetime and the other is peculiar to the theory of Otsuki connections $\Gamma(\Psi, G)$.
In Section 2 we will find exact solutions $\Gamma(\Psi, G)$ of the equations, whose essential metric has the following form:
\[ \overline{G} = -dw^2 + R^2(w) D(r)(dx^2 + dy^2 + dz^2). \]
Taking the results in advance, $D(r)$ becomes:
\[ D(r) = \left(1 + \frac{\varepsilon r^2}{4}\right)^{-2}, \]
where $\varepsilon = -1, 0, 1$. Some elementary properties of the functions $R(w)$ will be discussed in Section 3.
The following ranges of indices are used throughout this paper:
\[ 1 \leq i, j, k, \cdots \leq 3, \quad 0 \leq \alpha, \beta, \gamma, \cdots \leq 3. \]
§2. Universal Expansion Models by $\Gamma(\Psi, G)$

We consider a metric $\overrightarrow{G}$ and a function $h$ of the following forms:

$$\overrightarrow{G} = g_{\lambda\mu}dx^\lambda dx^\mu$$
$$= -B(t,r)dt^2 + A(t,r)(dx^2 + dy^2 + dz^2),$$
$$h = h(t),$$

where $r^2 = x^2 + y^2 + z^2$ and we often use $t,x,y,z$ instead of $x^0,x^1,x^2,x^3$. Using these forms, the Christoffel symbols $\{\gamma_{\mu\nu}^\lambda\} = \{\gamma_{\nu\mu}^\lambda\}$ of $\overrightarrow{G}$ become as follows:

$$\begin{align*}
\{t_{t0}\} &= \{0_{00}\} = \frac{1}{2} B_t \frac{x^i}{r}, \\
\{0_{ij}\} &= \{0_{ji}\} = \frac{1}{2} A_r \delta_{ij}, \\
\{0_{ij}\} &= \{0_{ji}\} = \frac{1}{2} A_r \delta_{ij}, \\
\{i_{00}\} &= \{0_{00}\} = \frac{1}{2} B_r \frac{x^i}{r}, \\
\{i_{j0}\} &= \{i_{0j}\} = \frac{1}{2} A_r \frac{B_r}{A} \left( \frac{x^k}{r} \delta_{ij} + \frac{x^j}{r} \delta_{ik} - \frac{x^i}{r} \delta_{jk} \right),
\end{align*}$$

where $A_t = \frac{\partial A}{\partial t}$, $A_r = \frac{\partial A}{\partial r}$, etc. Components of Ricci tensor and the scalar curvature $S$ of $\overrightarrow{G}$ are given as follows:

$$R_{00} (\equiv R_{tt}) = -\frac{3}{2} \left( \frac{A_t}{A} \right)_t - \frac{3}{2} \left( \frac{A_t}{A} \right)^2 + \frac{3}{4} \left( \frac{A_r}{A} \right) \left( \frac{B_t}{B} \right)$$
$$+ \frac{1}{2} \frac{B_t}{A} \left\{ \left( \frac{B_1}{B} \right)_1 + \left( \frac{B_2}{B} \right)_2 + \left( \frac{B_3}{B} \right)_3 \right\}$$
$$+ \frac{1}{4} \frac{B_t}{A} \left\{ \left( \frac{A_1}{A} \right) \left( \frac{B_1}{B} \right) + \left( \frac{A_2}{A} \right) \left( \frac{B_2}{B} \right) + \left( \frac{A_3}{A} \right) \left( \frac{B_3}{B} \right) \right\}$$
$$+ \frac{1}{4} \frac{B_t}{A} \left\{ \left( \frac{B_1}{B} \right)^2 + \left( \frac{B_2}{B} \right)^2 + \left( \frac{B_3}{B} \right)^2 \right\},$$

$$R_{0k} (\equiv R_{k0}) = \left( \frac{A_t}{A} \right)_k + \frac{1}{2} \left( \frac{A_t}{A} \right) \left( \frac{B_k}{B} \right),$$

$$R_{kk} = \frac{1}{2} \frac{A}{B} \left( \frac{A_t}{A} \right)_k + \frac{3}{4} \frac{A}{B} \left( \frac{A_t}{A} \right)^2 - \frac{3}{4} \frac{A}{B} \left( \frac{A_r}{A} \right) \left( \frac{B_t}{B} \right)$$
$$- \frac{1}{2} \left\{ \left( \frac{A_1}{A} \right)_1 + \left( \frac{A_2}{A} \right)_2 + \left( \frac{A_3}{A} \right)_3 \right\}$$
$$- \frac{1}{4} \left\{ \left( \frac{A_1}{A} \right)^2 + \left( \frac{A_2}{A} \right)^2 + \left( \frac{A_3}{A} \right)^2 \right\}$$
$$- \frac{1}{2} \left\{ \left( \frac{A_k}{A} \right)_k + \left( \frac{B_k}{B} \right)_k \right\} + \frac{1}{4} \left( \frac{A_k}{A} \right)^2 - \frac{1}{4} \left( \frac{B_k}{B} \right)^2,$$
\[ R_{km} \quad (= R_{mk}) \quad = \quad -\frac{1}{2} \left( \frac{A_m}{A} \right)_k - \frac{1}{2} \left( \frac{B_m}{B} \right)_k + \frac{1}{4} \left( \frac{A_k}{A} \right) \left( \frac{A_m}{A} \right)
\quad + \frac{1}{4} \left( \frac{A_k}{A} \right) \left( \frac{B_m}{B} \right) + \frac{1}{4} \left( \frac{A_m}{A} \right) \left( \frac{B_k}{B} \right) - \frac{1}{4} \left( \frac{B_k}{B} \right) \left( \frac{B_m}{B} \right) , \]

where \( k \neq m, \)

\[ S \quad = \quad \frac{3}{B} \left( \frac{A_t}{A} \right)_t + \frac{3}{B} \left( \frac{A_t}{A} \right)^2 - \frac{3}{2} \left( \frac{A_t}{A} \right) \left( \frac{B_t}{B} \right)
\quad - \frac{2}{A} \left\{ \left( \frac{A_1}{A} \right)_1 + \left( \frac{A_2}{A} \right)_2 + \left( \frac{A_3}{A} \right)_3 \right\}
\quad - \frac{1}{A} \left\{ \left( \frac{B_1}{B} \right)_1 + \left( \frac{B_2}{B} \right)_2 + \left( \frac{B_3}{B} \right)_3 \right\}
\quad - \frac{11}{2} \frac{A}{A} \left\{ \left( \frac{A_1}{A} \right)^2 + \left( \frac{A_2}{A} \right)^2 + \left( \frac{A_3}{A} \right)^2 \right\}
\quad - \frac{1}{4} \left\{ \left( \frac{A_1}{A} \right) \left( \frac{B_1}{B} \right) + \left( \frac{A_2}{A} \right) \left( \frac{B_2}{B} \right) + \left( \frac{A_3}{A} \right) \left( \frac{B_3}{B} \right) \right\}
\quad - \frac{11}{2} \frac{B}{B} \left\{ \left( \frac{B_1}{B} \right)^2 + \left( \frac{B_2}{B} \right)^2 + \left( \frac{B_3}{B} \right)^2 \right\} , \]

where we use the following formulas and notations:

\[ R_{\mu \nu} \quad = \quad R_{\nu \mu} \quad = \quad \frac{\partial}{\partial x^\lambda} \{^\lambda \}_{\mu \nu} \quad - \quad \frac{\partial}{\partial x^\sigma} \{^\sigma \}_{\mu \lambda} \quad - \quad \{^\lambda \}_{\kappa \lambda} \{^\kappa \}_{\mu \nu} \quad - \quad \{^\lambda \}_{\mu \sigma} \{^\sigma \}_{\kappa \lambda} , \]

\[ S \quad = \quad g^{\mu \nu} R_{\mu \nu} , \]

\[ A_1 \quad = \quad \frac{\partial A}{\partial x^1} , \quad A_2 \quad = \quad \frac{\partial A}{\partial x^2} , \quad A_3 \quad = \quad \frac{\partial A}{\partial x^3} \quad etc. \]

Using these equalities, the left side of (2), which are denoted by \( G_{\mu \nu} \) become as follows:

\[ G_{tt} \quad = \quad R_{tt} - \frac{1}{2} g_{tt} S \quad = \quad R_{tt} + \frac{1}{2} BS \]
\[ = \quad \frac{3}{4} \left( \frac{A_t}{A} \right)^2 - \frac{A}{B} \left\{ \left( \frac{A_1}{A} \right)_1 + \left( \frac{A_2}{A} \right)_2 + \left( \frac{A_3}{A} \right)_3 \right\}
\quad - \frac{1}{4} \frac{B}{A} \left\{ \left( \frac{A_1}{A} \right)^2 + \left( \frac{A_2}{A} \right)^2 + \left( \frac{A_3}{A} \right)^2 \right\}
\quad + \frac{1}{8} \frac{B}{A} \left\{ \left( \frac{A_1}{A} \right) \left( \frac{B_1}{B} \right) + \left( \frac{A_2}{A} \right) \left( \frac{B_2}{B} \right) + \left( \frac{A_3}{A} \right) \left( \frac{B_3}{B} \right) \right\} , \]

\[ G_{kk} \quad = \quad R_{kk} - \frac{1}{2} g_{kk} S \quad = \quad R_{kk} - \frac{1}{2} AS \]
\[ = \quad -\frac{A}{B} \left( \frac{A_k}{A} \right)_t - \frac{3 A}{4 B} \left( \frac{A_t}{A} \right)^2 + \frac{1}{2} \frac{A}{B} \left( \frac{A_k}{A} \right) \left( \frac{B_t}{B} \right) \]
+ \frac{1}{2} \left\{ \left( \frac{A_1}{A} \right)_1 + \left( \frac{A_2}{A} \right)_2 + \left( \frac{A_3}{A} \right)_3 \right\} \\
+ \frac{1}{2} \left\{ \left( \frac{B_1}{B} \right)_1 + \left( \frac{B_2}{B} \right)_2 + \left( \frac{B_3}{B} \right)_3 \right\} \\
+ \frac{1}{8} \left\{ \left( \frac{A_1}{A} \right) \left( \frac{B_1}{B} \right) + \left( \frac{A_2}{A} \right) \left( \frac{B_2}{B} \right) + \left( \frac{A_3}{A} \right) \left( \frac{B_3}{B} \right) \right\} \\
+ \frac{1}{4} \left\{ \left( \frac{B_1}{B} \right)^2 + \left( \frac{B_2}{B} \right)^2 + \left( \frac{B_3}{B} \right)^2 \right\} \\
- \frac{1}{2} \left( \frac{A_k}{A} \right)_k - \frac{1}{2} \left( \frac{B_k}{B} \right)_k + \frac{1}{4} \left( \frac{A_k}{A} \right)^2 - \frac{1}{4} \left( \frac{B_k}{B} \right)^2 \\
G_{km} = R_{km} - \frac{1}{2} g_{km} S = R_{km} \\
= - \frac{1}{2} \left( \frac{A_m}{A} \right)_k - \frac{1}{2} \left( \frac{B_m}{B} \right)_k + \frac{1}{4} \left( \frac{A_k}{A} \right) \left( \frac{A_m}{A} \right) \\
+ \frac{1}{4} \left( \frac{A_k}{A} \right) \left( \frac{B_m}{B} \right) + \frac{1}{4} \left( \frac{A_m}{A} \right) \left( \frac{B_k}{B} \right) - \frac{1}{4} \left( \frac{B_k}{B} \right) \left( \frac{B_m}{B} \right),
\]
where \( k \neq m \). On the other hand, the right sides of (2), which are denoted by \( H_{\mu\nu} \), become as follows:
\[
H_{tt} = (2 \delta_t^\alpha \delta_t^\beta - g_{tt} g^{\alpha\beta}) \nabla_\alpha (h) \nabla_\beta (h) = (h_t)^2, \\
H_{kk} = (2 \delta_k^\alpha \delta_k^\beta - g_{kk} g^{\alpha\beta}) \nabla_\alpha (h) \nabla_\beta (h) = \frac{A}{B} (h_t)^2, \\
the \ others \ = \ 0,
\]
where \( h_t = \frac{\partial h}{\partial t} \). (2) becomes as follows:
\[
(4) \quad G_{00} = (h_t)^2, \\
(5) \quad G_{kk} = \frac{A}{B} (h_t)^2, \\
(6) \quad G_{0k} = G_{k0} = 0, \\
(7) \quad G_{km} = G_{mk} = 0,
\]
where \( k \neq m \). Now we assume that \( A(t, r) \) has a form of \( A(t, r) = C(t) D(r) \), then (6) becomes \( B_r C_t = 0 \). We assume \( B_r = 0 \) i.e. \( B = B(t) \) and rewrite \( \overline{G} \) and \( h \) using the following new variable \( \overline{t} \) such that \( \overline{t} = \int^t \sqrt{B(s)} ds \) for \( t \). Now \( \overline{G} \) and \( h \) become as follows:
\[
\overline{G} = - d\overline{t}^2 + \tilde{A}(\overline{t}, r) (dx^2 + dy^2 + dz^2), \\
h(\overline{t}) = \tilde{h}(\overline{t}),
\]
where \( \tilde{A}(\overline{t}, r) = C(t) D(r) = \tilde{C}(\overline{t}) D(r) \). Without loss of generality, we can assume as follows:
\[
\overline{G} = - d\overline{t}^2 + A(t, r) (dx^2 + dy^2 + dz^2),
\]
\[ h = h(t), \]

where \( A(t,r) = C(t)D(r) \). Using these forms of \( C \) and \( h \), (4), (5) and (7) become as follows:

\begin{align*}
(8) \quad & \frac{3}{4} \left( \frac{C_t}{C} \right)^2 + \frac{1}{CD^2} \left\{ D_{rr} + \frac{2}{r} D_r - \frac{3}{4D} (D_r)^2 \right\} = (h_t)^2, \\
(9) \quad & - \left\{ C_t \frac{1}{4C} (C_t)^2 \right\} + \frac{1}{2D} \left\{ D_{rr} + \frac{1}{r} D_r - \frac{1}{D} (D_r)^2 \right\} = CD(h_t)^2, \\
(10) \quad & -\frac{1}{2D} \left\{ D_{rr} - \frac{1}{r} D_r - \frac{3}{2D} (D_r)^2 \right\} = 0.
\end{align*}

Equality (6) becomes as follows:

\begin{align*}
(11) \quad & h_{tt} + \frac{3}{2C^2C_t} (C_t h_t) = \frac{1}{2h_t C^3} \left\{ (h_t)^2 C^3 \right\}_t = 0.
\end{align*}

(8)~(11) are the fundamental equations of this paper.

Using a new function: \( f = f^2 \), (10) becomes as follows:

\[ -\frac{2}{f^3} \left( f_{rr} - \frac{1}{r} f_r \right) = 0, \]

where \( f_r = \frac{\partial f}{\partial r} \), \( f_{rr} = \frac{\partial^2 f}{\partial r^2} \). Integrating this equality, we have

\[ f(r) = 1 + \frac{\epsilon r^2}{4}, \quad \epsilon = -1,0,1, \]

and \( D(r) \) becomes as follows:

\[ D(r) = \left( 1 + \frac{\epsilon r^2}{4} \right)^{-2}, \quad \epsilon = -1,0,1, \]

where we use a boundary condition such that

\[ \lim_{r \to +\infty} D(r) = 1. \]

Putting these \( D(r) \) into (8),(9) and (11), we have the following:

\begin{align*}
(12) \quad & \frac{3}{4} \left( \frac{C_t}{C} \right)^2 + \frac{3\epsilon}{C} = (h_t)^2, \\
(13) \quad & \frac{C_{tt}}{C} - \frac{1}{4} \left( \frac{C_t}{C} \right)^2 + \frac{\epsilon}{C} = -(h_t)^2, \\
(14) \quad & \left\{ (h_t)^2 C^3 \right\}_t = 0.
\end{align*}
Differentiating (12) by \( t \), we have

\[
\frac{3}{2} \left( \frac{C_t}{C} \right) \left( \frac{C_{tt}}{C} \right) - \frac{3}{2} \left( \frac{C_t}{C} \right)^3 - \frac{3\varepsilon}{C} \left( \frac{C_t}{C} \right) = 2(h_t)(h_{tt}).
\]

Using (14) to the right side of the above equality, we have as follows:

\[
\begin{align*}
\frac{3}{2} \left( \frac{C_t}{C} \right) & \left\{ \frac{3}{4} \left( \frac{C_t}{C} \right)^2 + \frac{3\varepsilon}{C} - (h_t)^2 \right\} \\
+ \frac{3}{2} \left( \frac{C_t}{C} \right) & \left\{ \frac{C_{tt}}{C} - \frac{1}{4} \left( \frac{C_t}{C} \right)^2 + \frac{\varepsilon}{C} + (h_t)^2 \right\} = 0.
\end{align*}
\]

This equality shows that (12) and (14) imply (13). Integrating (14), we have

\[
(h_t)^2 = \frac{3\eta^2}{C^3}, \quad \eta \geq 0.
\]

Substituting this equality into (12), we have

\[
\frac{1}{4} \left( \frac{C_t}{C} \right)^2 + \frac{\varepsilon}{C} - \frac{\eta^2}{C^3} = 0.
\]

Summerizing the preceding results of this section, we have

**Lemma 2.** Let \( \Gamma(\Psi, G) \) be an Otsuki connection with the condition (A) and \( \overline{\mathcal{G}} \) and \( h \) have forms such that

\[
\begin{align*}
\overline{\mathcal{G}} &= -B(t, r)dt^2 + A(t, r)(dx^2 + dy^2 + dz^2), \\
h &= h(t),
\end{align*}
\]

where \( B_t \neq 0 \) and \( A(t, r) = C(t)D(r) \). Then \( \overline{\mathcal{G}} \) becomes

\[
\overline{\mathcal{G}} = -dt^2 + C(t)D(r)(dx^2 + dy^2 + dz^2)
\]

and \( h(t), C(t) \) and \( D(r) \) satisfy the following equalities:

\[
\begin{align*}
(15) \quad \left( h_t \right)^2 &= \frac{3\eta^2}{C^3}, \quad \eta \geq 0, \\
(16) \quad D(r) &= \left( 1 + \frac{\varepsilon r^2}{4} \right)^{-2}, \quad \varepsilon = -1, 0, 1, \\
(17) \quad \frac{1}{4} \left( \frac{C_t}{C} \right)^2 + \frac{\varepsilon}{C} - \frac{\eta^2}{C^3} &= 0.
\end{align*}
\]
A case $\eta = 0$ in Lemma 2 is easy to treat. Then, (15) implies $h \equiv 0$, which means that $\Gamma(\Psi, G)$ is the Levi-Civita connection of Riemannian manifold $(M, G)$, where $M \subseteq \mathbb{R}^4$. (17) implies that $C(t) = \dot{t}^2$ for $\varepsilon = -1$, $C(t) \equiv 1$ for $\varepsilon = 0$ and no solution for $\varepsilon = 1$. Now we have

**Lemma 3.** For a case $\eta = 0$ in Lemma 2, $\Gamma(\Psi, G)$ becomes the Levi-Civita connection of Riemannian manifold $(M, G)$, $M \subseteq \mathbb{R}^4$, such that $G$ is either of the following two:

\[
G = -dt^2 + \dot{t}^2 \left(1 - \frac{\eta^2}{4}\right)^{-2} (dx^2 + dy^2 + dz^2),
\]

\[
G = -dt^2 + dx^2 + dy^2 + dz^2.
\]

From now on we suppose $\eta > 0$ and we define a function $\zeta(t)$ by

\[
C(t) = \eta^2 \zeta^2(t).
\]

It is tedious to display every detail of calculations for all the cases $\varepsilon = -1, 0, 1$, but we give details of calculations for one case $\varepsilon = -1$ and only the results for other two. Using $\zeta(t)$, (17) is written as follows:

\[
(\zeta_t)^2 = \frac{1}{\eta^2} \left(1 + \frac{\zeta^4}{\zeta^2}\right).
\]

Defining a non-negative function $\vartheta(t)$ by

\[
\zeta^2(t) = \sinh(\vartheta(t))
\]

and inserting it into the above equality, we have

\[
\cosh^2(\vartheta)(\vartheta_t)^2 = \frac{4}{\eta^2} \left(1 + \frac{\zeta^4}{\zeta^2}\right).
\]

This equality is rewritten as follows:

\[
(18) \quad dt = \frac{1}{2} \eta \sqrt{\sinh(\vartheta)} d\vartheta.
\]

(18) shows that $\vartheta(t)$ is the inverse function of

\[
t(\vartheta) = \frac{1}{2} \int_0^\vartheta \sqrt{\sinh(s)} ds.
\]

Using

\[
C(t) = \eta^2 \zeta^2(t) = \eta^2 \sinh(\vartheta(t))
\]
and (18) in (15), we have

\[
(h_t)^2 = \frac{3}{\eta^2 \sinh^3(\theta)},
\]

\[
h_{\vartheta} = h_t \frac{dt}{d\vartheta} = \pm \frac{\sqrt{3}}{2} \left( \frac{1}{\sinh(\vartheta)} \right),
\]

where \( h_{\vartheta} = \frac{\partial h}{\partial \vartheta} \). Integrating the second equality of the above, we have

\[
h(t) = \pm \frac{\sqrt{3}}{2} \int_{\vartheta(t)}^{\infty} \frac{ds}{\sinh(s)}.
\]

Now we find \( \mathcal{G} \) and \( h \), i.e. Otsuki connections \( \Gamma(\Psi, G) \), as follows:

\[
\mathcal{G} = -dt^2 + \eta^2 \sinh(\vartheta(t)) \left( 1 - \frac{r^2}{4} \right)^{-2} (dx^2 + dy^2 + dz^2),
\]

\[
h(t) = \pm \frac{\sqrt{3}}{2} \int_{\vartheta(t)}^{\infty} \frac{ds}{\sinh(s)}.
\]

The other cases are similar to the above calculations.

**Theorem 1.** Under the same conditions as Lemma 2, \( \mathcal{G} \) and \( h(t) \) are given as follows:

**Riemannian Type**

1. \( \varepsilon = 0 \)

\[
\mathcal{G} = -dt^2 + dx^2 + dy^2 + dz^2.
\]

2. \( \varepsilon = -1 \)

\[
\mathcal{G} = -dt^2 + t^2 \left( 1 - \frac{r^2}{4} \right)^{-2} (dx^2 + dy^2 + dz^2).
\]

**Otsuki Type**

1. \( \varepsilon = -1 \)

\[
\mathcal{G} = -dt^2 + \eta^2 \sinh(\vartheta(t)) \left( 1 - \frac{r^2}{4} \right)^{-2} (dx^2 + dy^2 + dz^2),
\]

\[
h(t) = \pm \frac{\sqrt{3}}{2} \int_{\vartheta(t)}^{\infty} \frac{ds}{\sinh(s)}.
\]
2. $\varepsilon = 0 \ (t > 0)$

$$\mathcal{G} = -dt^2 + \mu^2 t^\frac{1}{3} \left( 1 - \frac{r^2}{4} \right)^{-2} (dx^2 + dy^2 + dz^2), \quad \mu^3 = 3\eta^2,$$

$$h(t) = \pm \frac{1}{\sqrt{3}} \log(t).$$

3. $\varepsilon = 1 \ (0 \leq \vartheta \leq \pi)$

$$\mathcal{G} = -dt^2 + \eta^2 \sin(\vartheta(t)) \left( 1 + \frac{r^2}{4} \right)^{-2} (dx^2 + dy^2 + dz^2),$$

$$h(t) = \pm \frac{\sqrt{3}}{2} \int_{\vartheta(t)}^{\frac{\pi}{2}} \frac{ds}{\sin(s)},$$

where $\vartheta(t)$ is the inverse function of

$$t(\vartheta) = \frac{1}{2} \int_{0}^{\vartheta} \sqrt{\sinh(s)} ds.$$

§3. The Essential Metrics, the Variable $w$ and the Function $R(w)$

As discussed in Section 1, neither a metric $G$ nor $\mathcal{G}$ but $\tilde{G} \equiv \Psi^2 G = \Psi^{-1}\mathcal{G}$, which is called the essential metric of $\Gamma(\Psi, G)$, has an important meaning on a manifold $M$ with an Otsuki connection $\Gamma(\Psi, G)$. Lemma 1 says that any geodesic in $M\setminus \text{Sing}(\Gamma)$ with Otsuki connection $\Gamma(\Psi, G)$ is a geodesic in a Riemannian manifold $(M\setminus \text{Sing}(\Gamma), \tilde{G})$ and vice versa, where $\text{Sing}(\Gamma) \subseteq M$ is defined as follows:

$$\text{Sing}(\Gamma) = \{x \in M \mid \Psi(x) = 0\}.$$

Using a new variable $w$, which is defined by

$$w(t) = \int_{0}^{t} \Psi^{-\frac{1}{2}}(s) ds,$$

the essential metric $\tilde{G}$ i.e.

$$\tilde{G} \equiv \Psi^{-1}G = -\Psi^{-1}(t)dt^2 + \Psi^{-1}(t)C(t) \left( 1 + \frac{r^2}{4} \right)^{-2} (dx^2 + dy^2 + dz^2),$$
becomes as follows:

$$\bar{G} = -dw^2 + R^2(w) \left( 1 + \frac{\varepsilon r^2}{4} \right)^{-2} (dx^2 + dy^2 + dz^2),$$

where $R^2(w) = \Psi^{-1}(t)C(t)$. The induced metric of $\bar{G}$ on a hyperplane $H_w \subset M \subset \mathbb{R}^4$ such that

$$H_w = \{(w, x, y, z) \in M \mid w = \text{const.}\} \hookrightarrow M$$

is given as follows:

$$i^*(\bar{G}) = R^2 \left( 1 + \frac{\varepsilon r^2}{4} \right)^{-2} (dx^2 + dy^2 + dz^2) = \left( 1 + \frac{\varepsilon r^2}{4R^2} \right)^{-2} (dx^2 + dy^2 + dz^2),$$

where $R = R(w), \bar{x} = Rx, \bar{y} = Ry, \bar{z} = Rz, \bar{r}^2 = \bar{x}^2 + \bar{y}^2 + \bar{z}^2$. Since this metric is just that of a sphere with a radius $R = R(w)$ for $\varepsilon = 1$, $R = R(w)$, which can be understood as the radius of the model at $w$. Rewriting Theorem 1 using the variable $w$, we have

**Theorem 2.** Let $\Gamma(\Psi, G)$ be an Otsuki connection with the condition (A) which has the forms as follows:

$$\bar{G} = -dt^2 + C(t)D(r)(dx^2 + dy^2 + dz^2),$$

$$h = h(t)$$

and $\Gamma(\Psi, G)$ be an Otsuki type i.e. $h \neq \text{const.}$, then the corresponding essential metric $\bar{G}$ and the function $R(w)$ become as follows:

$$\bar{G} = -dt^2 + \eta^2 \sinh(\vartheta(t)) \left( 1 - \frac{\eta^2}{4} \right)^{-2} (dx^2 + dy^2 + dz^2),$$

$$h(t) = \frac{\sqrt{3}}{2} \int_{\vartheta(t)}^{\infty} \frac{ds}{\sinh(s)},$$

$$\bar{G} = -dw^2 + R^2(w) \left( 1 - \frac{\eta^2}{4} \right)^{-2} (dx^2 + dy^2 + dz^2),$$

$$R^2(w) = \eta^2 \exp \left( \int_{\vartheta(t)}^{\infty} \frac{ds}{\sinh(s)} \right) \sinh(\vartheta(t)), $$

where $w(t) = \int_{0}^{t} \Psi^{-1}(s)ds = \int_{0}^{t} d\exp \left( -\int_{\vartheta(s)}^{\infty} \frac{du}{\sinh(u)} \right)$ and $\vartheta(t)$ is the inverse function of $t(\vartheta) = \frac{1}{2} \eta \int_{0}^{\vartheta} \sinh^{\frac{1}{2}}(s)ds$. 


\[ II \]

\[ \mathcal{G} = -dt^2 + \eta^2 \sinh(\theta(t)) \left( 1 - \frac{r^2}{4} \right)^{-2} (dx^2 + dy^2 + dz^2), \]

\[ h(t) = -\frac{\sqrt{3}}{2} \int_{\theta(t)}^{\infty} \frac{ds}{\sinh(s)}, \]

\[ \tilde{G} = -dw^2 + R^2(\omega) \left( 1 - \frac{r^2}{4} \right)^{-2} (dx^2 + dy^2 + dz^2), \]

\[ R^2(\omega) = \eta_0^2 \exp \left( -\int_{\theta(t)}^{\infty} \frac{ds}{\sinh(s)} \right) \sinh(\theta(t)), \]

where \( w(t) = \int_0^t \psi^{-\frac{1}{2}}(s) ds = \int_0^t ds \exp \left( + \int_{\theta(s)}^{\infty} \frac{du}{\sinh(u)} \right) \)
and \( \theta(t) \) is the inverse function of \( t(\theta) = \frac{1}{2\eta} \int_0^\theta \sinh^2(s) ds \).

\[ III \]

\[ \mathcal{G} = -dt^2 + \sqrt{3} \eta \sqrt{t}(dx^2 + dy^2 + dz^2), \]

\[ h(t) = \frac{1}{\sqrt{3}} \log(t), \]

\[ \tilde{G} = -dw^2 + (3\eta^2)^{\frac{2}{3}}(dx^2 + dy^2 + dz^2), \]

\[ R^2(\omega) = (3\eta^2)^{\frac{2}{3}} = \text{const.}, \]

where \( w(t) = \int_0^t \psi^{-\frac{1}{2}}(s) ds = \frac{3}{2} t^{\frac{2}{3}}. \)

\[ IV \]

\[ \mathcal{G} = -dt^2 + (3\eta^2)^{\frac{2}{3}} t^{\frac{2}{3}}(dx^2 + dy^2 + dz^2), \]

\[ h(t) = -\frac{1}{\sqrt{3}} \log(t), \]

\[ \tilde{G} = -dw^2 + \frac{4}{3} (3\eta^2)^{\frac{2}{3}} w(dx^2 + dy^2 + dz^2), \]

\[ R^2(\omega) = \frac{4}{3} (3\eta^2)^{\frac{2}{3}} w, \]

where \( w(t) = \int_0^t \psi^{-\frac{1}{2}}(s) ds = \frac{3}{4} t^{\frac{1}{3}}. \)

\[ V \]

\[ \mathcal{G} = -dt^2 + \eta^2 \sin(\theta(t)) \left( 1 + \frac{r^2}{4} \right)^{-2} (dx^2 + dy^2 + dz^2), \]

\[ h(t) = \frac{\sqrt{3}}{2} \int_{\theta(t)}^{\infty} \frac{ds}{\sinh(s)}. \]
\[
\dot{G} = -dw^2 + R^2(w) \left( 1 + \frac{\eta^2}{4} \right)^{-2} (dx^2 + dy^2 + dz^2),
\]
\[
R^2(w) = \eta^2 \exp \left( - \int_{\vartheta(t)}^{\vartheta} \frac{ds}{\sin(s)} \right) \sin(\vartheta(t)),
\]
where \( w(t) = \int_{0}^{t} \Psi^{-\frac{1}{2}}(s)ds = \int_{0}^{t} ds \exp \left( - \int_{\vartheta(s)}^{\vartheta} \frac{du}{\sin(u)} \right) \) and \( \vartheta(t) \) is the inverse function of \( t(\vartheta) = \frac{1}{2} \eta \int_{0}^{\vartheta} \sin^{-\frac{1}{2}}(s)ds. \)

VI

\[
\ddot{G} = -dt^2 + \eta^2 \sin(\vartheta(t)) \left( 1 + \frac{\eta^2}{4} \right)^{-2} (dx^2 + dy^2 + dz^2),
\]
\[
h(t) = -\frac{\sqrt{3}}{2} \int_{\vartheta(t)}^{\vartheta} \frac{ds}{\sin(s)},
\]
\[
\ddot{G} = -dw^2 + R^2(w) \left( 1 + \frac{\eta^2}{4} \right)^{-2} (dx^2 + dy^2 + dz^2),
\]
\[
R^2(w) = \eta^2 \exp \left( - \int_{\vartheta(t)}^{\vartheta} \frac{ds}{\sin(s)} \right) \sinh(\vartheta(t)),
\]
where \( w(t) = \int_{0}^{t} \Psi^{-\frac{1}{2}}(s)ds = \int_{0}^{t} ds \exp \left( + \int_{\vartheta(s)}^{\vartheta} \frac{du}{\sin(u)} \right) \) and \( \vartheta(t) \) is the inverse function of \( t(\vartheta) = \frac{1}{2} \eta \int_{0}^{\vartheta} \sin^{-\frac{1}{2}}(s)ds. \)

The functions \( R(w) \) in Theorem 2, which are measured by the variables \( w \), have some elementary properties. Especially for type I in Theorem 2, we have

**Lemma 4.** The range of \( w \) becomes \( 0 \leq w < +\infty \) and there exists positive constants \( \alpha, \beta \) such that

1. \[
\lim_{w \to +\infty} R(w) = \alpha.
\]

2. \[
\frac{1}{\beta} \leq \lim_{w \to +\infty} \frac{R(w)}{w^2} \leq \beta.
\]

3. \[
\frac{dR(w)}{dw} > 0
\]

for any \( 0 < w < +\infty. \)
4.

\[
\lim_{w \to 0} \frac{dR(w)}{dw} = 0.
\]

**Proof.** Since

(19) \[ s \leq \sinh(s) \leq (e + e^{-1})s, \]

(20) \[ \left( \frac{1 - e^{-1}}{2} \right) e^s \leq \sinh(s) \leq \frac{1}{2} e^s \]

for any \( 0 \leq s \leq 1, 1 \leq s < +\infty \) respectively, we have

(21) \[ \exp \left( \int_{\vartheta(t)}^{\frac{1}{\vartheta(t)}} \frac{ds}{\sinh(s)} \right) \leq \exp \left( \int_{\vartheta(t)}^{\frac{1}{\vartheta(t)}} \frac{ds}{\sinh(s)} \right) \leq \vartheta(t) \]

for any \( 0 \leq \vartheta(t) \leq 1 \) and

(22) \[ \exp(2e^{-\vartheta(t)}) \leq \exp \left( \int_{\vartheta(t)}^{\frac{1}{\vartheta(t)}} \frac{ds}{\sinh(s)} \right) \leq \exp \left( \left( \frac{2}{1 - e^{-1}} \right) e^{-\vartheta(t)} \right) \]

for any \( 1 \leq \vartheta(t) < +\infty \). By (19), (20) and an explicit form of \( w(t) \):

(23) \[ w(t) = \frac{1}{2} \eta \int_{0}^{\vartheta(t)} d\xi \sinh^{\frac{1}{2}}(\xi) \exp \left( \frac{1}{2} \int_{\xi}^{\frac{1}{\vartheta(t)}} \frac{ds}{\sinh(s)} \right), \]

there exist \( 0 < \lambda_{1} < \delta_{1} \) such that

(24) \[ \lambda_{1} \xi + \left( 1 - \frac{s}{\sinh(s)} \right) \leq \sinh^{\frac{1}{2}}(\xi) \exp \left( \frac{1}{2} \int_{\xi}^{\frac{1}{\vartheta(t)}} \frac{ds}{\sinh(s)} \right) \leq \delta_{1} \]

for any \( 0 \leq \xi \leq 1 \) and by (20) and (22), there exist \( 0 < \lambda_{2} < \delta_{2} \) such that

(25) \[ \lambda_{2} e^{\frac{s}{2}} \leq \sinh^{\frac{1}{2}}(\xi) \exp \left( \frac{1}{2} \int_{\xi}^{\frac{1}{\vartheta(t)}} \frac{ds}{\sinh(s)} \right) \leq \delta_{2} e^{\frac{s}{2}} \]

for any \( 1 \leq \xi < +\infty \). Using (23) \( \sim (25) \) and

\[
\frac{dw}{dt} = \Psi^{-\frac{1}{2}}(t) = \exp \left( -\frac{1}{2} \int_{\vartheta(t)}^{\frac{1}{\vartheta(t)}} \frac{ds}{\sinh(s)} \right),
\]

we have \( \frac{dw}{dt} > 0 \) for any \( 0 \leq t < +\infty, \lim_{t \to +\infty} w(t) = 0 \) and \( \lim_{t \to +\infty} w(t) = +\infty \). Under these preparations, we will prove \( 1 \sim 4 \). By (24), (25) and the explicit form such that

(26) \[ R^{2}(w) = \eta^{2} \sinh(\vartheta(t)) \exp \left( +\int_{\vartheta(t)}^{\frac{1}{\vartheta(t)}} \frac{ds}{\sinh(s)} \right), \]
there exist $\lambda_3, \delta_3 > 0$ such that
\begin{equation}
\eta^2((\lambda_2)^2 e^\vartheta + \lambda_3) \leq R^2(w) \leq \eta^2((\delta_2)^2 e^\vartheta + \delta_3)
\end{equation}
for any $1 \leq \vartheta < +\infty$. On the other hand by (23) $\sim$ (25), there exist $\lambda_4, \delta_4 > 0$ such that
\begin{equation}
\eta \left( \lambda_2 e^{\frac{\vartheta}{\varphi}} + \lambda_4 \right) \leq u(t) \leq \eta \left( \delta_2 e^{\frac{\vartheta}{\varphi}} + \delta_4 \right)
\end{equation}
for any $0 \leq t < +\infty$. By (26) $\sim$ (28), we have
\[
\lim_{w \to +\infty} R(w) = \eta \sqrt{\mu_1}, \quad \frac{\lambda_2}{\delta_2} \leq \lim_{w \to +\infty} \frac{R(w)}{w^2} \leq \frac{\delta_2}{\lambda_2},
\]
where $\mu_1 = \lim_{\vartheta \to +\infty} \sinh(\vartheta) \exp \left( \int_{\vartheta}^{\infty} \frac{ds}{\sinh(s)} \right)$. Using an equality:
\[
\frac{dR(w)}{dw} = \frac{1}{\sinh(\vartheta)} \left\{ \cosh(\vartheta) - 1 \right\} \exp \left( \frac{1}{2} \int_{\vartheta}^{\infty} \frac{ds}{\sinh(s)} \right),
\]
we have $\frac{dR(w)}{dw} > 0$ for any $0 < w < +\infty$ and $\lim_{w \to +\infty} \frac{dR(w)}{dw} = 0$.

Next we discuss the functions $R(w)$ of type V in Theorem 2.

**Lemma 5.** For the type V in Theorem 2, there exist $\delta, \nu > 0$ such that

1. $\frac{dR(w)}{dw} < 0$ for any $0 < w < \nu$,
2. $\lim_{w \to 0} \frac{dR(w)}{dw} = 0$,
3. $\lim_{w \to \nu-0} \frac{dR(w)}{dw} = 0$,
4. $\lim_{w \to +\infty} R(w) = \delta$,
5. $\lim_{w \to +\infty} R(w) = 0$.

**Proof.** Since
\begin{equation}
\frac{2}{\pi} s \leq \sin(s) \leq s,
\end{equation}
\begin{equation}
\frac{2}{\pi} (\pi - s) \leq \sinh(s) \leq \pi - s
\end{equation}
for any $0 \leq s \leq \frac{\vartheta}{2}, \frac{\vartheta}{2} \leq s \leq \pi$ respectively, we have
\begin{equation}
\frac{\pi}{2\vartheta} \leq \exp \left( \int_{0}^{\frac{\vartheta}{2}} \frac{ds}{\sin(s)} \right) \leq \left( \frac{\pi}{2\vartheta} \right)^{\frac{\vartheta}{2}}
\end{equation}
for any $0 \leq \vartheta \leq \frac{\pi}{2}$ and
\[
\left\{ \frac{2}{\pi}(\pi - \vartheta) \right\}^{\frac{\pi}{2}} \leq \exp \left( \frac{1}{\eta} \int_{\vartheta}^{\pi} \frac{ds}{\sin(s)} \right) \leq \frac{2}{\pi}(\pi - \vartheta)
\]
for any $\frac{\pi}{2} \leq \vartheta \leq \pi$. By (29), (31) and an explicit form of $w(t)$ such that
\[
w(t) = \frac{1}{2\eta} \int_{0}^{\vartheta} d\xi \sin^{\frac{1}{2}}(\xi) \exp \left( \frac{1}{2} \int_{\vartheta}^{\pi} \frac{ds}{\sin(s)} \right),
\]
we have
\[
1 \leq \sin^{\frac{1}{2}}(\xi) \exp \left( \frac{1}{2} \int_{\vartheta}^{\pi} \frac{ds}{\sin(s)} \right) \leq \left( \frac{\pi}{2} \right) ^{\frac{\pi}{2}} \left( \frac{1}{\xi} \right) ^{\frac{\pi}{2} - 1}
\]
for any $0 < \xi \leq \frac{\pi}{2}$ and by (30) and (32), we have
\[
\left\{ \frac{\pi}{2}(\pi - \xi) \right\} ^{\frac{\pi}{2} + 1} \leq \sin^{\frac{1}{2}}(\xi) \exp \left( \frac{1}{2} \int_{\vartheta}^{\pi} \frac{ds}{\sin(s)} \right) \leq \sqrt{\frac{\pi}{2}}(\pi - \xi)
\]
for any $\frac{\pi}{2} \leq \xi \leq \pi$. Using the explicit form of $\vartheta(t)$, (33) $\sim$ (35) and
\[
\frac{dw}{dt} = \Psi^{-\frac{1}{2}}(t) = \exp \left( \frac{1}{2} \int_{0}^{\pi} \frac{ds}{\sin(s)} \right),
\]
we have $\frac{dw}{dt} > 0$ for any $0 < \vartheta < \pi$, $\lim_{\vartheta \to +0} w(t) = 0$ and $\lim_{\vartheta \to \pi - 0} w(t) = \nu < +\infty$, where
\[
\nu = \frac{1}{2\eta} \int_{0}^{\pi} d\xi \sin^{\frac{1}{2}}(\xi) \exp \left( \int_{0}^{\pi} \frac{ds}{\sin(s)} \right).
\]
Under these preparations we will prove 1 $\sim$ 5. By (33) $\sim$ (35) and the formula:
\[
R^2(w) = \eta^2 \sin(\vartheta) \exp \left( \int_{0}^{\pi} \frac{ds}{\sin(s)} \right),
\]
we have
\[
\frac{dR(w)}{dw} = \left( \cos(\vartheta) - \frac{1}{\sin(\vartheta)} \right) \exp \left( \frac{1}{2} \int_{0}^{\pi} \frac{ds}{\sin(s)} \right)
\]
The right side of (36) is negative on $0 < w < \nu$ and tends to zero when $\vartheta \to +0$ i.e. $w \to +0$, so is a bounded function on $0 \leq w \leq \nu_1$, where $\nu_1$ is a small positive number. Thus $R(w)$ is a bounded function on $0 \leq w \leq \nu$ and has the following properties:
\[
\frac{dR[w]}{dw} < 0 \quad \text{for any } 0 < w < \nu, \\
\lim_{w \to +0} R(w) = \eta \sqrt{\lambda_1} = \delta < +\infty, \\
\lim_{w \to -\nu} R(w) = 0, \quad \lim_{w \to +0} \frac{dR[w]}{dw} = 0, \\
\lim_{w \to -\nu} \frac{dR[w]}{dw} = -\infty,
\]

where

\[
\lambda_1 = \lim_{\theta \to +0} \sin(\theta) \exp\left( \int_0^\theta \frac{d\theta}{\sin(s)} \right) < +\infty.
\]

Regarding properties of the functions \( R(w) \), the other cases in Theorem 2 are trivial or almost the same as Lemma 4 or Lemma 5. Now we have the following

**Theorem 3.** The functions \( R(w) \) in Theorem 2, which are measured by the variables \( w \), have the following properties:

**I** The range of \( w \) becomes \( 0 \leq w < +\infty \) and there exist \( \alpha, \beta > 0 \) such that

1. \( \lim_{w \to +0} R(w) = \alpha, \)
2. \( \frac{1}{\beta} \leq \lim_{w \to +\infty} \frac{R(w)}{w} \leq \beta, \)
3. \( \frac{dR[w]}{dw} > 0 \quad \text{for any } 0 < w < +\infty, \)
4. \( \lim_{w \to +0} \frac{dR[w]}{dw} = 0. \)

**II** The range of \( w \) becomes \( 0 \leq w < +\infty \) and there exists \( \delta > 0 \) such that

1. \( \lim_{w \to +0} R(w) = 0, \)
2. \( \frac{1}{\delta} \leq \lim_{w \to +\infty} \frac{R(w)}{w} \leq \delta, \)
3. \( \frac{dR[w]}{dw} > 0 \quad \text{for any } 0 < w < +\infty, \)
4. \( \lim_{w \to +0} \frac{dR[w]}{dw} = +\infty. \)

**III** \( R(w) = \text{const.} \). Thus, this case is trivial.

**IV** \( R(w) = \text{const.} \sqrt{w} \). Thus, this case is trivial.

**V** There exist \( \delta, \nu > 0 \) such that

1. \( R(w) \) is defined on \( 0 \leq w \leq \nu, \)
2. \( \frac{dR[w]}{dw} < 0 \quad \text{for any } 0 < w < \nu, \)
3. \( \lim_{w \to +0} \frac{dR[w]}{dw} = 0, \)
4. \( \lim_{w \to -0} \frac{dR(w)}{dw} = -\infty \),
5. \( \lim_{w \to +0} R(w) = \delta \),
6. \( \lim_{w \to +0} R(w) = 0 \).

VI The functions \( R(w) \) of these models are the same as \( R(\nu - w) \) in \( V \).

References


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