THE DEGREE THEORY OF A NEW CLASS OF OPERATORS AND ITS APPLICATION

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Abstract. This paper defines a concept of a semi-$k$-set-contraction operator, and establishes a degree theory for it. As its application, we discuss the existence for the solution of two-point boundary value problems for nonlinear second order integro-differential equations in Banach spaces.

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§1. INTRODUCTION

It is well known that the degree theory for the strict-set-contraction operator and the condensing operator has many applications to the existence of the solutions of some equations (see [1], [2], [3], [4]). However, some important operators are not strict-set-contraction operators or condensing operators. Now we give an example.

Let $E$ be a Banach space, $C([0, 1], E) = \{x, \ x \text{ is a mapping from } [0, 1] \text{ into } E \text{ and } x(t) \text{ is continuous at every } t \in [0, 1]\}$. Obviously $C([0, 1], E)$ is a Banach space with norm $\|x\| = \max\{|x(t)|, t \in [0, 1]\}$. For $x \in C([0, 1], E)$, let

$$(Ax)(t) = \int_0^1 G_1(t, s)[x(s) + g(x(s))]ds, (s)$$

where $g \in C(E, E)$, $g(D)$ is relatively compact for any bounded $D \subseteq E$ and $G_1(t, s) = \min\{t, s\}$.

It is difficult to prove that $A$ is a strict-set-contraction operator or a condensing operator from $C([0, 1], E)$ into $C([0, 1], E)$. So it is necessary to establish degree theory for the operators such as $A$ defined by (*).
Now we define a new class of operators.

Let $I$ be a bounded, closed interval of real numbers. Assume that $C^m(I, E) = \{x, x$ is a mapping from $I$ into $E$ and $x(t)$ is $m$-times continuously norm differentiable($m \geq 1$)$\}$. Obviously $C^m(I, E)$ is a Banach space with norm $\|x\| = \max\{\|x\|_0, \|x'\|_0, \ldots, \|x^{(m)}\|_0\}$, here $\|x\|_0 = \max\{\|x(t)\|, t \in I\}$.

Assume that $A$ is an operator from a bounded set $S \subseteq C^m(I, E)$ into $C^m(I, E)$, and $\alpha(S)$ denotes the Kuratowski measure of noncompactness in $C^m(I, E)$.

Now we give a new definition.

**Definition 1.** $A : S \rightarrow C^m(I, E)(S:\text{bounded})$ is called a semi-$k$-set-contraction operator if $A$ is a bounded, continuous operator, $(AS)^{(m)}$ is equicontinuous on $I$, and

$$\alpha(A(D)) \leq k \alpha(D)$$

for any bounded $D \subseteq S$ with equicontinuous $D^{(m)}$, where $0 \leq k < 1$ is a constant, $(AS)^{(m)} = \{y, y(t) = (Ax)^{(m)}(t) \text{ for } t \in I, x \in S\}$. And $A : C^m(I, E) \rightarrow C^m(I, E)$ is called a semi-$k$-set-contraction operator if the restriction $A : S \rightarrow C^m(I, E)$ is a semi-$k$-set-contraction operator for any bounded $S \subseteq C^m(I, E)$.

It is easy to see that this definition is different from that of the $k$-set-contraction operator and that of the condensing operator(see[1], [5]). For example $A$ defined by (*), $A : C(I, E) \rightarrow C(I, E)$ and for any bounded set $S \subseteq C(I, E)$, $AS$ is bounded and equicontinuous. Moreover, by the following lemma 1, for any equicontinuous subset $D \subseteq S$, we have

$$\alpha(AD(t)) = \alpha(\int_0^1 G_1(t, s)[x(s) + g(x(s))]ds, x \in D)$$

$$= \int_0^1 G_1(t, s)[\alpha(D(s)) + \alpha(g(D(s)))]ds$$

$$= \int_0^1 G_1(t, s)\alpha(D(s))ds$$

$$\leq \int_0^1 G_1(t, s)d\alpha(D)$$

$$< \frac{3}{4}\alpha(D).$$

By lemma 2, we have

$$\alpha(AD) \leq \frac{3}{4} \alpha(D).$$

So $A$ is a semi-$\frac{3}{4}$-set-contraction operator. In section 2, we establish the degree theory for the semi-$k$-set-contraction operators and prove some fixed point
theorems. As their application, in section 3 we discuss the existence of the solution of two-point boundary value problems for nonlinear integrodifferential equations in Banach spaces.

The following lemmas are necessary.

**Lemma 1** (see[3]). If \( S \subseteq C(I, E) \) is bounded and equicontinuous, then

\[
\alpha(\{ \int_I x(t) dt, x \in S \}) \leq \int_I \alpha(S(t)) dt.
\]  

(1)

**Lemma 2** (see[2]). If \( S \subseteq C^m(I, E) \) is bounded and \( S^{(m)} \) is equicontinuous on \( I \), then

\[
\alpha(S) = \max \{ \sup \{ \alpha(S(t)), t \in I \}, \sup \{ \alpha(S'(t)), t \in I \}, \ldots, \sup \{ \alpha(S^{(m)}(t)), t \in I \} \}.
\]

§2. ESTABLISHMENT OF THE DEGREE THEORY

Before establishing the degree theory for the class of the semi-\( k \)-set-contraction operator \( A \), we give some lemmas. Let \( \Omega \subseteq C^m(I, E) \) be open and bounded, and \( A : \Omega \to C^m(I, E) \) a semi-\( k \)-set-contraction, \( f = \text{id} - A \), where \( \text{id} \) denotes the identity operator. Then \( f \) is called a semi-\( k \)-set-contraction field.

**Lemma 3.** Assume \( A : \overline{\Omega} \to C^m(I, E) \) is a semi-\( k \)-set-contraction operator, then

1) \( f \) is proper, i.e., \( f^{-1}(D) \) is compact for any compact set \( D \subseteq C^m(I, E) \);

2) \( f \) is a closed mapping, i.e., \( f(S) \) is closed for any closed set \( S \subseteq \overline{\Omega} \).

**Proof.**
1) Let \( D_1 = f^{-1}(D)(D_1 \subseteq \overline{\Omega}) \), then \( D_1 \subseteq A(D_1) + D \). Since \( D^{(m)} \) and \( A(D_1)^{(m)} \) are equicontinuous on \( I \), \( D_1^{(m)} \) is equicontinuous on \( I \). Consequently,

\[
\alpha(D_1) \leq \alpha(A(D_1)) + \alpha(D) = \alpha(AD_1) \leq k\alpha(D_1).
\]

It is easy to see that \( \alpha(D_1) = 0 \). So \( D_1 \) is relatively compact. Consequently, \( D_1 \) is compact.

2) Let \( y_n \in f(S), y_n \to y_0 \in C^m(I, E) \). We will prove \( y_0 \in f(S) \). Suppose that \( y_n = f(x_n), x_n \in S \). Let \( S_0 = \{y_0, y_1, y_2, \ldots\} \). Obviously \( S_0 \subseteq C^m(I, E) \) is compact. By the proof of 1), \( f^{-1}(S_0) \subseteq C^m(I, E) \) is compact. So there exists a subsequence \( \{x_{n_i}\}, x_{n_i} \to x_0 \in C^m(I, E) \). Since \( S \) is closed, \( x_0 \in S \). By the continuity of \( f \), \( y_{n_i} = f(x_{n_i}) \to f(x_0) \). Consequently, \( y_0 = f(x_0) \). So \( f(S) \) is closed. The proof is complete. \( \square \)
**Lemma 4.** If $D \subseteq C(I, E)$ is bounded and equicontinuous on $I$, then $\overline{\alpha}(D)$ is bounded and equicontinuous on $I$.

The proof of lemma 4 is routine and may be omitted.

**Lemma 5.** Let $\{S_i\} \subseteq E$ be bounded, closed and $S_1 \supseteq S_2 \supseteq S_3 \supseteq \cdots \supseteq S_n \supseteq \cdots$, $S_n \neq \emptyset$, $n = 1, 2, 3, \cdots$. If $\alpha(S_n) \to 0$, then $S = \bigcap_{i=1}^{\infty} S_i$ is a nonempty compact set.

This Lemma is the exercise 4, page 53, in [1]. In what follows, we give the definition of the degree for a semi-$k$-set-contraction field.

**Definition 2.** Let $\Omega \subseteq C^m(I, E)$ open and bounded, $A : \overline{\Omega} \to C^m(I, E)$ be a semi-$k$-contraction operator, $0 \leq k < 1$, $f = id - A$.

1) Assume that $\theta \notin f(\partial \Omega)$. Let $D_1 = \overline{\alpha}(A(\overline{\Omega}))$ and $D_n = \overline{\alpha}(A(D_{n-1} \cap \overline{\Omega}))$, $n = 2, 3, \cdots$.

2) Now we suppose that $D_n \neq \emptyset$, $n = 1, 2, \cdots$. So $D_n \cap \overline{\Omega}$ is bounded and closed ($n = 1, 2, \cdots$). Let $D = \bigcap_{i=1}^{n} D_i$. Then $D$ is bounded, convex, closed and nonempty as we show below. Obviously $D_1 \supseteq D_2$. If $D_{n-1} \supseteq D_n$, then $D_n = \overline{\alpha}(A(D_{n-1} \cap \overline{\Omega})) \supseteq \overline{\alpha}(A(D_n \cap \overline{\Omega})) = D_{n+1}$. So $D_{n-1} \supseteq D_n$, $n = 2, 3, \cdots$. By lemma 4, $(D_n)^{[m]}$ is equicontinuous on $I$ and

$$\alpha(D_n) = \alpha(A(D_{n-1} \cap \overline{\Omega})) \leq k\alpha(D_{n-1} \cap \overline{\Omega}) \leq k\alpha(D_{n-1}).$$

So $\alpha(D_n) \leq k^{n-1}\alpha(D_1)$. By $k < 1$ and lemma 5, we know $D$ is a nonempty compact set. Because of $D_{n-1} \cap \overline{\Omega} \supseteq D_n \cap \overline{\Omega}$, $D_n \cap \overline{\Omega} \neq \emptyset$ and $\alpha(D_n \cap \overline{\Omega}) \to 0$, we know $D \cap \overline{\Omega} = \bigcap_{n=1}^{\infty} D_n \cap \overline{\Omega}$ is nonempty and compact. On the other hand, from $A(D_n \cap \overline{\Omega}) \subseteq \overline{\alpha}(A(D_{n-1} \cap \overline{\Omega})) = D_n$ we have

$$A(D \cap \overline{\Omega}) \subseteq \bigcap_{n=1}^{\infty} A(D_n \cap \overline{\Omega}) \subseteq \bigcap_{n=1}^{\infty} D_n = D.$$  \hspace{1cm} (2)

Since $D$ is compact, $A : D \cap \overline{\Omega} \to D$ is completely continuous. So by the extension theorem of completely continuous operator (see[1], page 44), there exists a completely continuous operator $A_1 : \overline{\Omega} \to D$ such that $A_1x = Ax$ for every $x \in D \cap \overline{\Omega}$. Let $f_1 = id - A_1$. It is easy to see that $\theta \notin f_1(\partial \Omega)$. So the Leray-Schauder degree $deg_{LS}(f_1, \Omega, \theta)$ can be defined. Let

$$deg(f, \Omega, \theta) = deg_{LS}(f_1, \Omega, \theta),$$  \hspace{1cm} (3)

where $deg_{LS}(f_1, \Omega, \theta)$ denotes the degree of completely continuous operator field $f_1 = id - A_1$. It is easy to find what we defined is independent of
the choice of \( f_1 \). In fact, let \( A_2 : \overline{\Omega} \to D \) be another extension of \( A \), and \( f_2 = id - A_2 \). Let \( H(t, x) = x - tA_1x - (1 - t)A_2x \), \( x \in \overline{\Omega} \), \( 0 \leq t \leq 1 \). We will prove \( H(t, x) \neq \theta \) for \( t \in [0, 1] \) and \( x \in \partial\Omega \). On the contrary, if there exist \( t_0 \), \( 0 \leq t_0 \leq 1 \), and \( x_0 \in \partial\Omega \) such that \( H(t_0, x_0) = \theta \), i.e., \( x_0 = t_0A_1x_0 + (1 - t_0)A_2x_0 \). Since \( A_1x_0 \in D \), \( A_2x_0 \in D \) and \( D \) is convex, we know \( x_0 \in D \). So \( x_0 = t_0A_1x_0 + (1 - t_0)A_2x_0 = Ax_0 \). This contradicts to \( \theta \notin f(\partial\Omega) \). Hence

\[
\deg_{LS}(f_1, \Omega, \theta) = \deg_{LS}(f_2, \Omega, \theta).
\]

(2) Suppose \( p \notin f(\partial\Omega) \). It is easy to see \( \theta \notin (f - p)(\partial\Omega) \) and set

\[
\deg(f, \Omega, p) = \deg(f - p, \Omega, \theta).
\]

Now we have successfully defined the degree \( \deg(f, \Omega, p) \) for a semi-\( k \)-set-contraction operator \( A \).

**Remark 1:** If \( A \) has a fixed point \( x' \in \overline{\Omega} \), we have \( x' \in D_n \cap \Omega \neq \emptyset \), \( n = 1, 2, \ldots \). So the fixed point set \( F \) is also non-void with \( F \subseteq D \cap \overline{\Omega} \).

**Remark 2:** We can notice the method of establishing \( \{D_n\}_n \) in definition 2 is same as that of \( \{Q_n\}_n \) appearing on page 107 in [5].

**Lemma 6.** Assume that \( A \) is a semi-\( k \)-set-contraction operator as in definition 2, \( f = id - A \), \( \theta \notin f(\partial\Omega) \), and 2) of Definition 2 is satisfied. If \( B : \overline{\Omega} \to S \) is continuous with \( Bx = Ax \) for all \( x \in S \cap \overline{\Omega} \), where \( S \supseteq D(D) \) is the same as in the definition 1) is compact and convex with \( A(S \cap \overline{\Omega}) \subseteq S \). Let \( g = id - B \), then

\[
\deg(f, \Omega, \theta) = \deg_{LS}(g, \Omega, \theta).
\]

**Proof.** Assume that \( A_1 \) and \( f_1 \) are such as those of 2) in definition 2. Let

\[
H(t, x) = x - tA_1x - (1 - t)Bx
\]

for \( x \in \Omega \) and \( 0 \leq t \leq 1 \). Then we have \( H(t, x) \neq \theta \) for \( x \in \partial\Omega \) and \( 0 \leq t \leq 1 \). In fact, suppose that \( H(t_0, x_0) = \theta \) for \( x_0 \in \partial\Omega \), \( 0 \leq t_0 \leq 1 \). Since \( S \supseteq D \) is convex, \( x_0 = t_0A_1x_0 + (1 - t_0)Bx_0 \in S \). So \( Bx_0 = Ax_0 \), \( x_0 = t_0A_1x_0 + (1 - t_0)Ax_0 \). From \( Ax_0 \in D_1 \) and \( A_1x_0 \in D \subseteq D_1 \), we have \( x_0 = t_0A_1x_0 + (1 - t_0)Ax_0 \in D_1 \). So \( Ax_0 \in D_2 \), \( A_1x_0 \in D \subseteq D_2 \). Consequently \( x_0 = t_0A_1x_0 + (1 - t_0)Ax_0 \in D_2 \). Proceeding as before, we have \( x_0 \in D_n(n = 1, 2, 3, \ldots) \). Therefore \( x_0 \in D \). So we have \( A_1x_0 = Ax_0 \), \( x_0 = t_0Ax_0 + (1 - t_0)Ax_0 = Ax_0 \). This contradicts \( \theta \notin f(\partial\Omega) \). So

\[
\deg_{LS}(g, \Omega, \theta) = \deg_{LS}(f_1, \Omega, \theta).
\]

Thus the proof is complete. \( \square \)
\textbf{Theorem 3.} The degree of a semi-\-k-set-contraction field defined in Definition 2 has the following properties:

1) \( \deg(id, \Omega, p) = 1 \) for \( p \in \Omega \);

2) \( \deg(f, \Omega, p) = \deg(f, \Omega_1, p) + \deg(f, \Omega_2, p) \) whenever \( \Omega_1, \Omega_2 \subseteq \Omega \) are open with \( \Omega_1 \cap \Omega_2 = \emptyset \), \( p \not\in \overline{f(\Omega \setminus (\Omega_1 \cup \Omega_2))} \);

3) \( \deg(id - H(t, \cdot), \Omega, p) = \text{const} \) for all \( t \in [0, 1] \) whenever \( H(t, \cdot) \) is a semi-\-k-set-contraction operator for all \( t \in [0, 1] \) and as \( t \to t_0 \) for any \( t_0 \), \( H(t, x) \) converges to \( H(t_0, x) \) in \( C^m(I, E) \) uniformly in \( x \in \overline{\Omega} \), where \( p \not\in h_t(\partial \Omega) \), \( h_t = id - H(t, \cdot) \);

4) if \( \deg(f, \Omega, p) \neq 0 \), then the equation \( f(x) = p \) has a solution in \( \Omega \).

Moreover, set \( g = id - G \), where \( G : \Omega \to C^m(I, E) \) is a semi-\-k-set-contraction operator. Then

5) \( \deg(f, \Omega, p) = \deg(g, \Omega, p) \) whenever \( G|_{\partial \Omega} = A|_{\partial \Omega} \);

6) \( \deg(f, \Omega, p) = \deg(f, \Omega_1, p) \) for every open subset \( \Omega_1 \) of \( \Omega \) such that \( p \not\in \overline{f(\Omega - \Omega_1)} \);

7) \( \deg(f, \Omega, \cdot) \) is constant on every connected subset of \( C^m(I, E) - f(\partial \Omega) \).

\textit{Proof.} We might well suppose that \( p = \theta \). Since 1) is same as the normality of strict-set-contraction field in [5], we can omit the proof. First we prove 2). We discuss three possibilities.

1’ Suppose 2) of (1) in definition 2 for \( \Omega_1 \) and \( \Omega_2 \) is true. Obviously 2) of (1) in definition 2 for \( \Omega \) is true. Now we get

\[
\begin{cases}
D^{(1)} \cap \overline{\Omega_1} \neq \emptyset, & D^{(2)} \cap \overline{\Omega_2} \neq \emptyset, & D \cap \overline{\Omega} \neq \emptyset, & D^{(1)} \subseteq D, & D^{(2)} \subseteq D, \\
A(D^{(1)} \cap \overline{\Omega_1}) \subseteq D^{(1)}, & A(D^{(2)} \cap \overline{\Omega_2}) \subseteq D^{(2)}, & A(D \cap \overline{\Omega}) \subseteq D,
\end{cases}
\]

where \( D^{(1)} \) and \( D^{(2)} \) are obtained as \( D \) in 2) of (1) in definition 2 for \( A|_{\overline{\Omega_1}} \) and \( A|_{\overline{\Omega_2}} \) respectively. And \( D \) is the same as in 2) of (1) in definition 2. Let \( A_1 : \overline{\Omega} \to D \) is the completely continuous operator as in 2) of (1) in definition 2, and \( f_1 = id - A_1 \). According to (3), we get

\[ \deg(f, \Omega, \theta) = \deg_{LS}(f_1, \Omega, \theta). \]

By virtue of lemma 6, we have

\[
\begin{cases}
\deg(f, \Omega_1, \theta) = \deg_{LS}(f_1, \Omega_1, \theta), \\
\deg(f, \Omega_2, \theta) = \deg_{LS}(f_1, \Omega_2, \theta).
\end{cases}
\]

By virtue of the degree theory of Leray-Schauder, we get

\[ \deg_{LS}(f_1, \Omega, \theta) = \deg_{LS}(f_1, \Omega_1, \theta) + \deg_{LS}(f_1, \Omega_2, \theta). \]

According to above conclusion, we get

\[ \deg(f, \Omega, \theta) = \deg(f, \Omega_1, \theta) + \deg(f, \Omega_2, \theta). \]
2’ Suppose that one of $\Omega_1$ and $\Omega_2$ satisfies 2) of (1) in definition 2 (for example, $\Omega_1$), one of $\Omega_1$ and $\Omega_2$ satisfies 1) of (1) in definition 2 (for example, $\Omega_2$). Obviously $\Omega$ satisfies (1) 2) in definition 2. Therefore

$$\deg(f, \Omega_2, \theta) = 0.$$ 

By virtue of lemma 6, we have

$$\deg(f, \Omega_1, \theta) = \deg_{LS}(f_1, \Omega_1, \theta),$$

where $f_1$ is as in 1’. Now we will prove

$$\deg_{LS}(f_1, \Omega_2, \theta) = 0.$$ 

In fact, if $\deg_{LS}(f_1, \Omega_2, \theta) \neq 0$, then there exists an $x_0 \in \Omega$ such that $f_1(x_0) = 0$, i.e. $x_0 = A_1x_0 \in D$. So $A_1x_0 = A_2x_0$, $x_0 = A_2x_0$. By the Remark 1, $\Omega_2$ satisfies 2) of (1) in definition 2. This is a contradiction. Now by (**), we have

$$\deg(f, \Omega, \theta) = \deg(f, \Omega_1, \theta) + \deg(f, \Omega_2, \theta).$$

3’ Suppose $\Omega_1$ and $\Omega_2$ satisfy 1) of (1) in definition 2. Now we have

$$\deg(f, \Omega_1, \theta) = 0, \quad \deg(f, \Omega_2, \theta) = 0.$$ 

By the Remark 1, $\theta \not\in f(\Omega_1 \cup \Omega_2)$. Hence, $\theta \not\in f(\overline{\Omega})$. Then we have

$$\deg(f, \Omega, \theta) = 0.$$

So

$$\deg(f, \Omega, \theta) = \deg(f_1, \Omega_1, \theta) + \deg(f_1, \Omega_2, \theta).$$

And Since the proof of (2) includes that of (4), we can omit the proof of (4).

Next we prove 3). First we need to prove $H([0,1] \times \overline{\Omega})$ is bounded. In fact, assume that there exists a sequence $\{t_n\} \subseteq [0,1]$ and a $\{x_n\} \subseteq \overline{\Omega}$ such that

$$\|H(t_n, x_n)\| \rightarrow \infty, \quad n \rightarrow \infty. \quad (8)$$

We might as well suppose that $t_n \rightarrow t_0$. We have

$$\|H(t_n, x_n)\| \leq \|H(t_n, x_n) - H(t_0, x_n)\| + \|H(t_0, x_n)\|. \quad (9)$$

Since $H(t_0, \cdot)$ is a semi-$k$-set-contraction operator, $\|H(t_0, x_n)\|$ is bounded. And because $\|H(t_n, x) - H(t_0, x)\| \rightarrow 0 (n \rightarrow +\infty)$ uniformly in $x \in \overline{\Omega}$, $\|H(t_n, x_n) - H(t_0, x_n)\|$ is bounded. So $\|H(t_n, x_n)\|$ is bounded. This contradicts (9). Consequently, $H([0,1] \times \overline{\Omega})$ is bounded. Let $D^*_1 = \overline{cO}(H([0,1] \times \overline{\Omega}))$, and

$$D^*_n = \overline{cO}(H([0,1] \times (\overline{\Omega} \cap D^*_{n-1}))), \quad n = 2, 3, \cdots.$$ 

Obviously $D^*_1 \supseteq D^*_2$. If $D^*_{n-1} \supseteq$
$D_n^*$, then $D_n^* = \overline{\text{co}}(H([0, 1] \times (D_{n-1}^* \cap \overline{\Omega}))) \supseteq \overline{\text{co}}(H([0, 1] \times (D_n^* \cap \overline{\Omega}))) = D_{n+1}^*$.

So $D_{n+1}^* \supseteq D_n^*$, $n = 2, 3, \ldots$. We need to prove $D_n^*[m]$ is equicontinuous on $I$.

First we will prove that $D_1^*[m]$ is equicontinuous. From lemma 4, we have only to prove that $H([0, 1] \times \overline{\Omega})^*[m]$ is equicontinuous. Assume that $H([0, 1] \times \overline{\Omega})^*[m]$ is not equicontinuous. Then there exists an $\varepsilon > 0$, a subsequence $\{x_n\} \subseteq H([0, 1] \times \overline{\Omega})$ with $x_n = H(t_n, y_n)$, and $|t_{1,n} - t_{2,n}| < \frac{1}{n}$ such that

$$\|x_n^*[m](t_{1,n}) - x_n^*[m](t_{2,n})\| \geq \varepsilon. \quad (10)$$

We might as well suppose $t_n \to t_0$, then we have

$$\|H(t_n, y_n)^*[m](t_{1,n}) - H(t_n, y_n)^*[m](t_{2,n})\| \\
\leq \|H(t_n, y_n)^*[m](t_{1,n}) - H(t_0, y_n)^*[m](t_{1,n})\| \\
+ \|H(t_0, y_n)^*[m](t_{2,n}) - H(t_n, y_n)^*[m](t_{2,n})\| \\
+ \|H(t_0, y_n)^*[m](t_{1,n}) - H(t_0, y_n)^*[m](t_{2,n})\| \\
= I_{1,n} + I_{2,n} + I_{3,n}.$$

And because $\|H(t_n, x) - H(t_0, x)\| \to 0(n \to +\infty)$ uniformly in $x \in \overline{\Omega}$, we have $I_{1,n} + I_{2,n} \to 0, n \to +\infty$. Since $H(t_0, \cdot)$ is a semi-$k$-set-contraction operator, we have $I_{3,n} \to 0, n \to +\infty$. Then $I_{1,n} + I_{2,n} + I_{3,n} \to 0, n \to +\infty$. This contradicts (10). By lemma 4, $D_1^*[m]$ is equicontinuous on $I$. By the monotonicity of $\{D_n^*[m]\}$, $D_n^*[m]$ is equicontinuous.

For given $t \in [0, 1]$, let $D_1(t) = \overline{\text{co}}(H(t, \overline{\Omega}))$,

$$D_n(t) = \overline{\text{co}}(H(t, D_{n-1}(t) \cap \overline{\Omega})), \quad n = 2, 3, \ldots \quad (11)$$

Obviously $D_n(t) \subseteq D_{n-1}(t), n = 2, 3, \ldots$. If there exists an $n_0$ with $D_{n_0}^* \cap \overline{\Omega} = \emptyset$ for every, then $D_{n_0}(t) \cap \overline{\Omega} = \emptyset, t \in [0, 1]$. Then we have

$$\text{deg}(h_t, \Omega, \theta) \equiv 0, \quad t \in [0, 1].$$

Now suppose $D_n^* \cap \overline{\Omega} \neq \emptyset(n = 1, 2, \ldots)$.

Take any $\varepsilon > 0$ and $t_0 \in [0, 1]$. Then for each $n \geq 2$ there exist a finite covering $\{S_i\}_{i=1}^r$ such that $H(t_0, D_{n-1}^* \cap \overline{\Omega}) \subseteq \bigcup_{i=1}^r S_i$ with $d(S_i) \leq k\alpha(D_{n-1}^* + \varepsilon, i = 1, 2, \ldots, r$ since $\alpha(H(t_0, D_{n-1}^* \cap \overline{\Omega})) \leq k\alpha(D_{n-1}^* \cap \overline{\Omega}) \leq k\alpha(D_{n-1}^*).$ On the other hand, from the assumption, there is a $\delta > 0$ such that $\|H(t, x) - H(t_0, x)\| < \varepsilon$ for all $x \in \overline{\Omega}$ when $|t - t_0| < \delta$. Let $S_i^\varepsilon = \{x, d(x, S_i) < \varepsilon\}, I(t_0, \delta) = (t_0 - \delta, t_0 + \delta) \cap [0, 1]$. So $H(I(t_0, \delta) \times (D_{n-1}^* \cap \overline{\Omega})) \subseteq \bigcup_{i=1}^r S_i^\varepsilon$.

$$d(S_i^\varepsilon) \leq d(S_i) + 2\varepsilon \leq k\alpha(D_{n-1}^*) + 3\varepsilon.$$
We have $\alpha(H(I(t_0, \delta) \times (D_{n-1}^* \cap \overline{\Omega}))) \leq \kappa \alpha(D_{n-1}^*) + 3\varepsilon$. By the compactness of the interval $[0, 1]$, there exist $t_i \in [0, 1]$, $\delta_i > 0$, $i = 1, 2, \cdots, s$ such that $[0, 1] = \bigcup_{i=1}^{s} I(t_i, \delta_i)$, and

$$
\alpha(H(I(t_i, \delta_i) \times (D_{n-1}^* \cap \overline{\Omega}))) \leq \kappa \alpha(D_{n-1}^*) + 3\varepsilon, \quad i = 1, 2, \cdots, s.
$$

So

$$
\alpha(D_n^*) = \alpha(H([0, 1] \times (D_{n-1}^* \cap \overline{\Omega})))
= \alpha(\bigcup_{i=1}^{s} H(I(t_i, \delta_i) \times (D_{n-1}^* \cap \overline{\Omega})))
= \max\{\alpha(H(I(t_i, \delta_i) \times (D_{n-1}^* \cap \overline{\Omega}))), i = 1, 2, \cdots, s\}
\leq \kappa \alpha(D_{n-1}^*) + 3\varepsilon.
$$

By the arbitrariness of $\varepsilon$, we have $\alpha(D_n^*) \leq \kappa \alpha(D_{n-1}^*)$, $n = 2, 3, \cdots$. Consequently, $\alpha(D_n^*) \leq \kappa^{n-1} \alpha(D_1^*)$. This implies $\alpha(D_n^*) \rightarrow 0$. By lemma 5, $D^* = \bigcap_{n=1}^{\infty} D_n^*$ is nonempty, convex and compact (recall that we are now assuming $D_n^* \cap \overline{\Omega} \neq \emptyset$ for $n = 1, 2, \cdots$). By the same proof, $D^* \cap \overline{\Omega}$ also is shown to be nonempty and compact. Since $H([0, 1] \times (D_n^* \cap \overline{\Omega})) \subseteq \overline{\alpha(H([0, 1] \times (D_n^* \cap \overline{\Omega})) = D_{n+1}^* \subseteq D_n^*$. So $H([0, 1] \times (D^* \cap \overline{\Omega})) \subseteq \bigcap_{n=1}^{\infty} H([0, 1] \times (D_n^* \cap \overline{\Omega})) \subseteq \bigcap_{n=1}^{\infty} D_n^* = D^*$.

By the extention theorem of completely continuous function, there exists a $G : [0, 1] \times \overline{\Omega} \rightarrow D^*$ such that $G(t, x) = H(t, x)$ when $(t, x) \in [0, 1] \times (D^* \cap \overline{\Omega})$. Let $g_t = x - G(t, x)$. We will prove $\deg(h_t, \Omega, \theta) = \deg_{LS}(g_t, \Omega, \theta)$. It is easy to see that $\theta \not\in g_t(\partial \Omega)$. In fact, if there exist $t_0$ with $0 \leq t_0 \leq 1$, and $x_0 \in \partial \Omega$ such that $g_{t_0}(x_0) = 0$. Then $x_0 = G(t_0, x_0) \in D^*$. So $G(t_0, x_0) = H(t_0, x_0)$. This contradicts $\theta \not\in h_t(\partial \Omega)$. So $\theta \not\in g_t(\partial \Omega)$.

(a) If the condition 1) of definition 2 is satisfied for $h_t$, we have $\deg(h_t, \Omega, \theta) = 0$. In this case, since $H(t, x)$ has not fixed points in $\overline{\Omega}$, $G(t, x)$ also has not fixed points in $\overline{\Omega}$. By the theory of Leray-Schauder degree, we have

$$
\deg_{LS}(g_t, \Omega, \theta) = 0.
$$

(b) If $h_t$ satisfies the condition 2) in definition 2, by lemma 6, we have

$$
\deg(h_t, \Omega, \theta) = \deg_{LS}(g_t, \Omega, \theta).
$$

Therefore we have

$$
\deg_{LS}(g_t, \Omega, \theta) = \text{const}, \quad 0 \leq t \leq 1.
$$

Hence

$$
\deg(h_t, \Omega, \theta) = \text{const}, \quad 0 \leq t \leq 1.
$$
If \( p \neq \theta \), let \( \overline{h}_t = id - H(t, \cdot) - p \). Then by the result proved above, we have
\[
\deg(\overline{h}_t, \Omega, \theta) = \text{const}.
\]

Finally since the proofs of 5), 6), 7) are similar to the proofs of the relative properties of degree theory of strict-set-contraction field in [1], we omit the proofs. Thus proof is complete. \( \square \)

**Theorem 4.** Let \( \Omega \) be a bounded, convex open set in \( C^m(I, E), A : \overline{\Omega} \to C^m(I, E) \) be a semi-\( k \)-set-contraction operator, \( 0 \leq k < 1 \), \( A(\partial \Omega) \subseteq \overline{\Omega} \) without fixed point in \( \partial \Omega \), then \( \deg(id - A, \Omega, \theta) = 1 \).

**Proof.** Choose an \( x_0 \in \Omega \) arbitrarily. Let \( h_t = t(x - Ax) + (1 - t)(x - x_0) = x - H(t, x), \) here \( H(t, x) = tAx + (1 - t)x_0 \). Obviously \( \|H(t_n, x) - H(t_0, x)\| \to 0(n \to +\infty) \) uniformly in \( x \in \overline{\Omega} \). And \( H(t, \cdot) \) is a semi-\( k \)-set-contraction operator for all \( t \in [0, 1] \). In virtue of the fact: let \( A \) be a convex set in a topological vector space \( E \) with a interior point \( x_0 \), then for any \( x_1 \in \mathcal{A} \), the open segment with end points \( x_0 \) and \( x_1 \) is contained in \( \mathcal{A} \) (cf. N.Bourbaki, "Espace Vectoriels Topologiques", Prop.16 in Chap.2, \( \S \)2, n°6), it is easy to see that \( \theta \notin h_t(\partial \Omega), 0 \leq t \leq 1 \). By Theorem 3, \( \deg(id - A, \Omega, \theta) = \deg(id, \Omega, \theta) = 1 \). The proof is complete. \( \square \)

§3. EXISTENCE OF THE SOLUTION FOR TWO-POINT BOUNDARY VALUE PROBLEMS IN BANACH SPACES

Now we consider the following boundary value problem
\[
\begin{align*}
-x''(t) &= f(t, x(t), x'(t), (Tx)(t), (Sx)(t)), 0 \leq t \leq 1; \\
ax(0) - bx'(0) &= x_0, \\
ax(1) + dx'(1) &= x_1,
\end{align*}
\]
(12)
where
\[
(Tx)(t) = \int_0^t k(t, s)x(s)ds, \quad (Sx)(t) = \int_0^1 h(t, s)x(s)ds.
\]
(13)

Here \( k \in C(D, R^+) \), \( D = \{(t, s) \in R^2; 0 \leq s \leq t \leq 1 \} \) and \( h \in C(D_0, R^+) \), \( D_0 = \{(t, s) \in R^2; 0 \leq t, s \leq 1 \} \). \( E \) is Banach space. And assume \( a \geq 0, b \geq 0, c \geq 0, d \geq 0 \) and \( J = ac + ad + bc > 0 \) throughout this section.

In order to investigate BVP (12), we first consider the integral operator
\[
(Ax)(t) = \int_0^1 G(t, s)f(s, x(s), x'(s), (Tx)(s), (Sx)(s))ds + y(t),
\]
(14)
where \( f \in C(I \times E \times E \times E \times E, P) \), \( y \in C^2(I, E) \) and \( y(t) \geq \theta \) for \( t \in I \) and \( P \subseteq E \) is a normal solid cone of \( E \) with normal constant \( N \geq 1 \)(i.e. if
we define the relation \( x \leq y \) by \( y - x \in P \), then ‘\( \leq \)’ is an order relation in \( E \). Moreover, \( \theta \leq x \leq y \) implies \( \|x\| \leq N\|y\| \). We denote the relation \( y - x \in \tilde{P} \) by \( x \ll y \).

Let
\[
G(t, s) = \begin{cases} 
J^{-1}(at + b)(c(1 - s) + d), & t \leq s; \\
J^{-1}(as + b)(c(1 - t) + d), & t > s,
\end{cases}
\]  
(15)

here \( a \geq 0, b \geq 0, c \geq 0, d \geq 0 \) and \( J = ac + ad + bc > 0 \). Moreover, \( T \) and \( S \) are defined by (13). In the following, let \( B_R = \{x \in E : \|x\| \leq R\} \) (\( R > 0 \)) and
\[
k_0 = \max\left\{\int_0^t k(t, s)ds, t \in I\right\}, \quad h_0 = \max\left\{\int_0^1 h(t, s)ds, t \in I\right\},
\]  
(16)

Furthermore, let \( P(I) = \{x \in C^1(I, E) : x(t) \geq \theta \text{ for } t \in I\} \). Then \( P(I) \) is a cone in \( C^1(I, E) \). Usually, \( P(I) \) is not normal in \( C^1(I, E) \) even if \( P \) is a normal cone in \( E \). Let
\[
q_1 = \sup_{t \in [0,1]} \int_0^1 G(t, s)ds, \quad q_2 = \sup_{t \in [0,1]} \int_0^1 |G'_t(t, s)ds,
\]

and
\[
q = \max\{q_1, q_2\}
\]  
(17)

Then we have the following lemma 7.

**Lemma 7.** Let \( f \) be uniformly continuous on \( I \times B_R \times B_R \times B_R \times B_R \) for any \( R > 0 \). Suppose that there exist constants \( L_i \geq 0 \) (\( i = 1, 2, 3, 4 \)) such that
\[
\alpha(f(t, X, Y, Z, W)) \leq L_1 \alpha(X) + L_2 \alpha(Y) + L_3 \alpha(Z) + L_4 \alpha(W)
\]  
(18)

for any bounded \( X, Y, Z, W \subseteq E \), \( t \in I \) and
\[
\mathcal{K} = q(L_1 + L_2 + k_0L_3 + h_0L_4) < 1.
\]  
(19)

Then the operator \( A \) defined by (14) is a semi-\( \mathcal{K} \)-set-contraction operator from \( C^1(I, E) \) into \( P(I) \).

**Proof.** By direct differentiation of (14), we have for \( x \in C^1(I, E) \),
\[
(Ax(t))' = \int_0^1 G'_t(t, s)f(s, x(s), x'(s), (Tx)(s), (Sx)(s))ds + y'(t),
\]  
(20)

where
\[
G'_t(t, s) = \begin{cases} 
J^{-1}a(c(1 - s) + d), & t < s; \\
J^{-1}(-c)(as + b), & t > s,
\end{cases}
\]  
(21)
and

\[
((Ax)(t))'' = -f(t, x(t), x'(t), (Tx)(t), (Sx)(t)) + g'(t).
\]  (22)

It is easy to see that the uniform continuity of \( f \) on \( I \times B_R \times B_R \times B_R \times B_R \) implies the boundedness of \( f \) on \( I \times B_R \times B_R \times B_R \times B_R \). So \( A \) is bounded and continuous from \( C^1(I, E) \) into \( P(I) \). Now, let \( Q \subseteq C^1(I, E) \) be bounded. By virtue of (22), \( \| (Ax(t))'' \| : x \in Q, t \in I \) is a bounded set of \( E \). So \( A(Q)' \) is equicontinuous, and hence lemma 2 implies that

\[
\alpha(A(Q)) = \max \{ \sup \{ \alpha(AQ(t)), t \in I \}, \sup \{ \alpha((AQ)'(t)), t \in I \} \}. \]  (23)

On the other hand, it is easy to see that for any bounded \( Q \subseteq C^1(I, E) \) with equicontinuous \( Q' \), \( \{ f(s, x(s), x'(s), (Tx)(s), (Sx)(s)), x \in Q \} \) is equicontinuous because of the uniform continuity of \( f \). By lemma 1, lemma 2 and (18) we have

\[
\alpha(AQ(t)) \leq \int_0^1 G(t, s) f(s, x(s), x'(s), (Tx)(s), (Sx)(s)) ds + g(t) + y(t), x \in Q \}
\]

\[
\leq \int_0^1 G(t, s) \alpha \{ f(s, x(s), x'(s), (Tx)(s), (Sx)(s)), x \in Q \} ds \]

\[
\leq \int_0^1 G(t, s)|L_1(\alpha(Q(s)) + L_2(\alpha(Q'(s)) + L_3(\alpha((TQ)(s)) + L_4(\alpha((SQ)(s)))] ds \]

\[
\leq \int_0^1 G(t, s)[L_1(\alpha(Q(s)) + L_2(\alpha(Q'(s)) + L_3 \int_0^s k(s, r) \alpha(Q(r)) dr]
\]

\[+ L_4 \int_0^1 h(s, r) \alpha(Q(r)) dr] ds \]

\[
\leq \int_0^1 G(t, s) ds [L_1 + L_2 + L_3k_0 + L_4h_0] \alpha(Q) \]

\[
\leq q_1[L_1 + L_2 + L_3k_0 + L_4h_0] \alpha(Q) \]

\[
\leq q[L_1 + L_2 + L_3k_0 + L_4h_0] \alpha(Q). \]  (24)

Similarly, we have

\[
\alpha((AQ)'(t)) \leq \int_0^1 G'(t, s) f(s, x(s), x'(s), (Tx)(s), (Sx)(s)) ds + g(t), x \in Q \}
\]

\[
\leq \int_0^1 G'(t, s) \alpha \{ f(s, x(s), x'(s), (Tx)(s), (Sx)(s)), x \in Q \} ds \]

\[
\leq \int_0^1 G'(t, s)[L_1(\alpha(Q(s)) + L_2(\alpha(Q'(s)) + L_3k_0(\alpha(Q) + L_4h_0(\alpha(Q)) ds \]

\[
\leq \int_0^1 G'(t, s) ds [L_1 + L_2 + L_3k_0 + L_4h_0] \alpha(Q). \]
\[ q_2[L_1 + L_2 + L_3k_0 + L_4h_0] \alpha(Q) \leq q[L_1 + L_2 + L_3k_0 + L_4h_0] \alpha(Q). \] 

From (23), we have

\[
\alpha(A(Q)) = \max \{ \sup \{ \alpha(A(Q(t)), t \in I) \}, \ \sup \{ \alpha((AQ)'(t)), t \in I) \} \leq R \alpha(Q)
\]

(26)

So \( A \) is a semi-\( k \)-set-contraction operator. The proof is complete. □

Let us list some conditions for convenience:

(H1) \( x_0 \geq \theta, x_1 \geq \theta, f \in C(I \times E \times E \times E \times E, P) \) is uniformly continuous on \( I \times B_R \times B_R \times B_R \times B_R \) for any \( R > 0 \) and there exists \( L_i \geq 0 (i = 1, 2, 3, 4) \) such that (18) and (19) hold;

(H2) \( \lim_{R \to +\infty} \sup \frac{M(R)}{R} < \frac{1}{q \alpha_n} \), where \( M(R) = \sup \{ \| f(t, x, y, z, w) \| : (t, x, y, z, w) \in I \times B_R \times B_R \times B_R \times B_R \}, m = \max \{ 1, k_0, h_0 \} \) and \( q \) is defined by (17);

**Theorem 5.** Let (H1), (H2), (H3) be satisfied. Then BVP (12) has at least one nonnegative solution in \( C^2(I, E) \).

**Proof.** It is well known that the \( C^2(I, E) \) solution of (12) is equivalent to \( C^1(I, E) \) solution of the following integral equation

\[
x(t) = \int_0^1 G(t, s)f(s, x(s), x'(s), (Tx)(s), (Sx)(s)) ds + y(t),
\]

where \( G(t, s) \) is the Green function given by (15) and \( y(t) \) denotes the unique solution of BVP

\[
\begin{align*}
x'' &= 0, \quad 0 \leq t \leq 1; \\
x(x(0) - bx'(0) = x_0, \quad cx(1) + dx'(1) = x_1,
\end{align*}
\]

which is given by

\[
y(t) = J^{-1}\{ (e(1-t) + d)x_0 + (at + b)x_1 \}.
\]

Evidently, \( y \in C^2(I, E) \cap P(I) \). Let \( A \) be defined by (14). Then condition (H1) and lemma 7 imply that \( A \) is a semi-\( k \)-set-contraction operator from \( C^1(I, E) \) to \( P(I) \). By (H2), there exist \( \delta > 0 \) and \( R > 2\| u_0 \| \) such that for any \( R' \geq R \)

\[
\frac{M(R')}{R'} < \frac{1}{q(m + \delta)},
\]

and

\[
\frac{m}{m + \delta} + \frac{\| y \|_1}{R} < 1
\]

(27)
Let \( U = \{ x \in C^1(I, E), \| x \|_1 < R \} \). So \( U \) is bounded convex open set. For \( x \in \overline{U} \), we have \( \| x \|_1 \leq R \) and
\[
\| Ax \|_0 = \max \{ \| \int_0^1 G(t, s) f(s, x(s), x'(s), (Tx)(s), (Sx)(s)) ds + y(t) \|, t \in I \}
\leq \max \{ \int_0^1 G(t, s) \| f(s, x(s), x'(s), (Tx)(s), (Sx)(s)) \| ds + \| y(t) \|, t \in I \}
\leq M(mR) \max \{ \int_0^1 G(t, s) ds, t \in I \} + \| y \|_1
\leq mR \frac{1}{q(m+\delta)} q_1 + \| y \|_1
< R \left( \frac{m}{m+\delta} + \frac{\| y \|_1}{R} \right)
< R
\]
and
\[
\| (Ax)' \|_0 = \max \{ \| \int_0^1 G'_t(t, s) f(s, x(s), x'(s), (Tx)(s), (Sx)(s)) ds + y'(t) \|, t \in I \}
\leq \max \{ \int_0^1 |G'_t(t, s)| \| f(s, x(s), x'(s), (Tx)(s), (Sx)(s)) \| ds + \| y'(t) \|, t \in I \}
\leq M(mR) \max \{ \int_0^1 |G'_t(t, s)| ds, t \in I \} + \| y \|_1
\leq mR \frac{1}{q(m+\delta)} q_2 + \| y \|_1
< R \left( \frac{m}{m+\delta} + \frac{\| y \|_1}{R} \right)
< R
\]
hence \( \| Ax \|_1 < R \).

In virtue of (29), (30), \( A \overline{U} \subseteq U \). Then by theorem 4 we get
\[
\deg(id - A, U, \emptyset) = 1,
\]
i.e., there is a fixed point \( x \in U \). The proof is complete.\( \square \)

**Example 1.**

We consider the following system of scalar valued differential equations
\[
\begin{align*}
-x''_n &= 3[x_n + 1]^\frac{1}{n} + \frac{1}{n+1}(x_{n+1}^n)^\frac{1}{n} + \frac{1}{2n} \int_0^t \frac{1}{1+t+s} x_{2n}(s) ds |^\frac{1}{n} \\
+ \frac{1}{4n} \int_0^t \cos(t-s) x_{3n}(s) ds |^\frac{1}{2} + 17, \\
x_n(0) &= x_n(1) = 0, \quad n = 1, 2, \ldots.
\end{align*}
\] (31)
Conclusion: equation (31) has at least one positive solution.

Proof. Let $E = \{ x = (x_1, x_2, \cdots, x_n, \cdots), \sup_{n \in \mathbb{N}} |x_n| < +\infty \}$ with norm $\|x\| = \sup_{n \in \mathbb{N}} |x_n|$, and $P = \{ x = (x_1, x_2, \cdots) \in E, x_n \geq 0, n = 1, 2, \cdots \}$. Then $P$ is a normal solid cone of $E$ and (31) can be regarded as a BVP of the form (12), where $a = c = 1$, $b = d = 0$, $x_0 = x_1 = \theta$, $k(t,s) = \frac{1}{1 + t + s}$, $h(t,s) = \cos(t - s)$, $x = (x_1, x_2, \cdots)$, $y = (y_1, y_2, \cdots)$, $z = (z_1, z_2, \cdots)$, $w = (w_1, w_2, \cdots)$, and $f = g + h = (g_1, g_2, \cdots) + (h_1, h_2, \cdots)$ in which

$$\begin{align*}
g_n(t,x,y,z,w) &= 3(|x_n| + 1)^{\frac{1}{3}} + 17, \tag{32}
\end{align*}$$

and

$$\begin{align*}
h_n(y,z,w) &= \frac{1}{n + 1} (y_{n+1})^{\frac{1}{3}} + \frac{1}{2n} z_{2n}^{\frac{1}{3}} + \frac{1}{3n} w_{3n}^{\frac{1}{3}}. \tag{33}
\end{align*}$$

Then

$$\begin{align*}
\|f\| &\leq 3(\|x\| + 1)^{\frac{1}{3}} + \frac{1}{2}(\|y\|)^{\frac{1}{3}} + \frac{1}{2}\|z\|^{\frac{1}{3}} + \frac{1}{3}\|w\|^{\frac{1}{3}} + 17. \tag{34}
\end{align*}$$

which implies

$$\begin{align*}
M(R) &\leq 3(R + 1)^{\frac{1}{3}} + \frac{1}{2} R^{\frac{2}{3}} + \frac{1}{2} R^{\frac{1}{3}} + \frac{1}{3} R^{\frac{2}{3}} + 17
\end{align*}$$

and consequently

$$\begin{align*}
\lim_{R \to +\infty} \frac{M(R)}{R} = 0.
\end{align*}$$

This shows that condition (H2) is satisfied.

Obviously, $f \in C(I \times E \times E \times E \times E, P)$ and $f$ is uniformly continuous on $I \times B_R \times B_R \times B_R \times B_R$ for any $R > 0$. Now for any bounded $D \subseteq E$, it is easy to see that $\alpha(g(D)) \leq \frac{2}{3} \alpha(D)$. And for any bounded $Y \subseteq E$, $Z \subseteq P$, $W \subseteq P$, we have $\alpha(h(Y, Z, W)) = 0$. In fact, let $(y^{(m)}) \subseteq Y$, $(z^{(m)}) \subseteq Z$, $(w^{(m)}) \subseteq W$, and $v^{(m)}_n = h_n(y^{(m)}, z^{(m)}, w^{(m)})$. By (33), we get

$$\begin{align*}
|v^{(m)}_n| &\leq \frac{1}{n + 1} \|y^{(m)}\|^{\frac{1}{3}} + \frac{1}{2n} \|z^{(m)}\|^{\frac{1}{3}} + \frac{1}{3n} \|w^{(m)}\|^{\frac{1}{3}}.
\end{align*}$$

Now by the diagonal method, we can select a subsequence $\{v^{(m_i)}\} \subseteq \{v^{(m)}\}$ such that

$$\begin{align*}
v^{(m_i)} \to v^0 \in P.
\end{align*}$$

So $\alpha(h(Y, Z, W)) = 0$. On the other hand, it is easy to see that in this case

$$\begin{align*}
q = \frac{1}{2}, \quad m = 1.
\end{align*}$$
So the condition $(H_1)$ is satisfied. Consequently, our conclusion follows from theorem 5. □

The operator $A$ defined by (31) is not a strict-set-contraction operator or a condensing operator. So the degree theory of the condensing operator or the strict-set-contraction operator is not suitable.

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