Let $\mathcal{L}_A$ be the usual language for arithmetic. Let $\varphi(x)$ be an $\mathcal{L}_A$-formula. $\varphi(x)$ may contain free variables distinct from $x$ as parameters. We consider the following two schemata.

$$(I_{\varphi(x)}) \quad \varphi(0) \land \forall x(\varphi(x) \rightarrow \varphi(x + 1)) \rightarrow \forall x\varphi(x),$$

$$(L_{\varphi(x)}) \quad \exists x\varphi(x) \rightarrow \exists x(\forall y < x \neg \varphi(y) \land \varphi(x)).$$

They are called the induction schema and the least number principle, respectively. $IOpen$, $LOpen$ will denote the theory $PA^- \cup \{I_{\varphi(x)} \mid \varphi(x) : open\}$, $PA^- \cup \{L_{\varphi(x)} \mid \varphi(x) : open\}$, respectively.

In this paper we prove the equivalence of $IOpen$ and $LOpen$. Van den Dries [v.d.D] noted that this can be proven model theoretically by using ideas in the proof of Shepherdson’s theorem in [S1]. Our proof is syntactical and not model theoretical.

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§1. Introduction

The first order language $L_A = \{0, 1, <, +, \cdot\}$, and the axioms of $PA^-$ are the following:

1) $\forall x, y, z((x + y) + z = x + (y + z))$,
2) $\forall x, y(x + y = y + x)$,
3) $\forall x, y, z((x \cdot y) \cdot z = x \cdot (y \cdot z))$,
4) $\forall x, y(x \cdot y = y \cdot x)$,
5) $\forall x, y, z(x \cdot (y + z) = x \cdot y + x \cdot z)$,
6) $\forall x((x + 0 = x) \land (x \cdot 0 = 0))$,
7) $\forall x(x \cdot 1 = x)$,
8) $\forall x, y, z((x < y \land y < z) \rightarrow x < z)$,
9) $\forall x \neg(x < x)$,
10) $\forall x, y(x < y \lor x = y \lor y < x)$,
11) $\forall x, y, z(x < y \rightarrow x + z < y + z)$,
12) $\forall x, y, z((0 < z \land x < y) \rightarrow x \cdot z < y \cdot z)$,
13) $\forall x, y(x < y \rightarrow \exists z(x + z = y))$,
14) $0 < 1 \land \forall x(0 < x \rightarrow 1 \leq x)$,
15) $\forall x(0 \leq x)$.

We employ usual abbreviations such as $x \leq y < z$, $x^n$, $\forall x < y \varphi$ and so on. Note that 1)–13) imply converses to 11), 12) and that $z$ in 13) is unique. We often represent the $z$ by $y - x$.

Note that if we take Robinson Arithmetic $Q$ as a ‘base’ theory, then $IOpen$ and $LOpen$ are equivalent to $Q \cup \{I\varphi(x) \mid \varphi(x) : \text{open}\}$ and $Q \cup \{\forall x(x < x + 1)\} \cup \{L\varphi(x) \mid \varphi(x) : \text{open}\}$, respectively (See [H-P], [K])

It is easy to see that $PA^- \vdash L\varphi(x) \rightarrow I\varphi(x)$ for any formula $\varphi(x)$, hence $LOpen$ proves all axioms of $IOpen$. So we concentrate to see that $IOpen$ proves the least number principle for open formulas.

Now we consider stronger systems than $IOpen$. Let $I\Sigma_n = PA^- \cup \{I\varphi(x) \mid \varphi(x) : \Sigma_n\}$, which is equivalent to the theory $Q \cup \{I\varphi(x) \mid \varphi(x) : \Sigma_n\}$. It is easy to see that $I\Sigma_n$ proves the least number principle for $\Sigma_n$-formulas. Because if we assume $\exists x\varphi(x) \land \forall x(\varphi(x) \rightarrow \exists y < x \varphi(y))$ and apply induction to the formula $\forall y < x \neg\varphi(y)$, then we obtain a contradiction. But we can not apply this argument for $IOpen$.

Shepherdson proved in [S1] that if $M \models PA^-$ and $R(M)$ is the real closure of the ordered ring corresponding to $M$, then $M \models IOpen$ iff $\forall r \in R(M)$ with $r > 0 \exists s \in M$ such that $0 \leq r - s < 1$. One can easily prove that if the latter condition holds then $M$ satisfies the least number principle for open formulas. Therefore we have that if $M \models IOpen$ then $M \models LOpen$, hence $IOpen$ proves all the axioms of $LOpen$.
Our main result is Main Lemma 3.7 which states, roughly speaking, that “any open formula is equivalent to the union of finite intervals” is provable in IOpen. By using the Main Lemma, we will prove proof theoretically that IOpen proves the least number principle for open formulas. We begin investigation into atomic $L_A$-formulas.

§2. Some properties of atomic formulas in $PA^-$

An atomic formula $\varphi$ is of the form $s = t$ or of the form $s < t$, where $s$ and $t$ are terms. We will represent $\varphi(x)$ as an $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0 = b_nx^n + b_{n-1}x^{n-1} + \cdots + b_0$ or $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0 < b_nx^n + b_{n-1}x^{n-1} + \cdots + b_0$ where terms $a_i$ and $b_i$ do not contain the free variable $x$, and either $a_n \neq 0$ or $b_n \neq 0$. We say that $\varphi(x)$ is of degree $n$ (in symbols: $\deg_{x}\varphi = n$) if $\varphi(x)$ is represented as above. We often denote $\deg_{x}\varphi$ simply by $\deg\varphi$.

First we show in $PA^-$ that if $\varphi(x)$ is an equality of degree $n$, then $\varphi(x)$ has at most $n$ ‘solutions’.

Proposition 2.1. Let $\varphi(x)$ be an equality with $\deg\varphi = n$ as above. Assume that $0 < n$. Then

$$PA^- \vdash \bigwedge_{i=0}^{n} (a_i \neq b_i) \rightarrow \exists x_0, \cdots, x_n (x_0 < x_1 < \cdots < x_n \land \bigwedge_{i=0}^{n} \varphi(x_i)).$$

Proof. We construct proofs of the formula $\varphi(x)$ by induction on $n(=\deg\varphi)$. For $n = 1$, we work in $PA^-$. Assume that there were $x_0, x_1$ such that $x_0 < x_1$, $\varphi(x_0)$ and $\varphi(x_1)$, that is,

$$a_1x_0 + a_0 = b_1x_0 + b_0,$$
$$a_1x_1 + a_0 = b_1x_1 + b_0.$$

There exists the unique $d$ such that $0 < d$ and $x_1 = x_0 + d$. Substituting $x_0 + d$ for $x_1$ in the second equation and using the first equation, we get $a_1d = b_1d$. Hence $a_1 = b_1$ since $0 < d$. It follows that $a_0 = b_0$, which is a contradiction. Therefore, this proposition holds in case $n = 1$.

Next, assume that the assertion holds for any equation with degree $n - 1$ and consider $\varphi(x)$ with $\deg\varphi = n$. We work in $PA^-$. Suppose that there were $x_0, x_1, \cdots, x_n$ such that $x_0 < x_1 < \cdots < x_n$ and $\varphi(x_i)$ for each $i$, that is

$$\begin{align*}
\left\{ \begin{array}{ll}
a_nx_0^n + a_{n-1}x_0^{n-1} + \cdots + a_0 &= b_nx_0^n + b_{n-1}x_0^{n-1} + \cdots + b_0, \\
a_nx_1^n + a_{n-1}x_1^{n-1} + \cdots + a_0 &= b_nx_1^n + b_{n-1}x_1^{n-1} + \cdots + b_0, \\
& \cdots \\
a_nx_n^n + a_{n-1}x_n^{n-1} + \cdots + a_0 &= b_nx_n^n + b_{n-1}x_n^{n-1} + \cdots + b_0. 
\end{array} \right. 
\tag{*}
\end{align*}$$
There exists the unique \(d_i\) such that \(x_{i+1} = x_0 + d_i\) for each \(0 \leq i \leq n - 1\), and \(0 < d_0 < \cdots < d_{n-1}\). Let \(\psi(y)\) be

\[
a_n \sum_{k=1}^{n} \binom{n}{k} x_0^{n-k} y^{k-1} + a_{n-1} \sum_{k=1}^{n-1} \binom{n-1}{k} x_0^{n-k-1} y^{k-1} + \cdots + a_1 \sum_{k=1}^{1} \binom{1}{k} x_0^{n-k} y^{k-1} = b_n \sum_{k=1}^{n} \binom{n}{k} x_0^{n-k} y^{k-1} + b_{n-1} \sum_{k=1}^{n-1} \binom{n-1}{k} x_0^{n-k-1} y^{k-1} + \cdots + b_1.
\]

Then \(\text{deg}_y \psi = n - 1\) and we have \(\bigwedge_{i=0}^{n-1} \psi(d_i)\) by (\(\ast\)). Let \(\psi(y)\) represent \(a'_{n-1} y^{n-1} + \cdots + a'_0 = b'_{n-1} y^{n-1} + \cdots + b'_0\) where \(a'_i, b'_i\) do not contain \(y\). Then

\[
a'_{n-1} = a_n(n), \\
a'_{n-2} = a_n(n-1)x_0 + a_{n-1}(n-1), \\
a'_{n-3} = a_n(n-2)x_0^2 + a_{n-1}(n-2)x_0 + a_{n-2}(n-2), \\
\vdots \\
a'_0 = a_n(1)x_0^{n-1} + a_{n-1}(n-1)x_0^{n-2} + \cdots + a_1(1).
\]

Similarly for \(b'_i\). Thus, by the induction hypothesis it follows that \(\bigwedge_{i=0}^{n-1} a'_i = b'_i\).

Then \(\bigwedge_{i=1}^{n} a_i = b_i\), hence \(\bigwedge_{i=0}^{n} a_i = b_i\). Thus proposition holds for \(n\). \(\square\)

For any formula \(\varphi(x)\) we define inductively \(\mathcal{J}_n(\varphi(x))\), which means that \(\varphi(x)\) has at most \(n\) ‘solutions’.

**Definition 2.2.** For any formula \(\varphi(x)\), we define \(\mathcal{J}_n(\varphi(x))\) as follows inductively:

\[
\mathcal{J}_0(\varphi(x)) \equiv \forall x \neg \varphi(x), \\
\mathcal{J}_{n+1}(\varphi(x)) \equiv \mathcal{J}_n(\varphi(x)) \lor \exists x_0, x_1, \cdots, x_n(x_0 < x_1 < \cdots < x_n \land \forall y(\varphi(y) \leftrightarrow \bigvee_{i=0}^{n} y = x_i)).
\]

**Lemma 2.3.** Let \(\varphi(x)\) be any formula. Then

\[PA^- \vdash \neg \mathcal{J}_n(\varphi(x)) \rightarrow \exists x_0, x_1, \cdots, x_n(x_0 < x_1 < \cdots < x_n \land \bigwedge_{i=0}^{n} \varphi(x_i)).\]

**Proof.** Induction on \(n\). We work in \(PA^-\). The assertion is trivial for \(n = 0\). For induction step, assume that the assertion holds for \(n\). Suppose
that \( \neg J_{n+1}(\varphi(x)) \), that is \( \neg J_n(\varphi(x)) \) and

\[
\forall x_0, x_1, \ldots, x_n (x_0 < x_1 < \cdots < x_n \rightarrow \exists y((\varphi(y) \land \bigwedge_{i=0}^{n} y \neq x_i) \lor (\neg \varphi(y) \land \bigvee_{i=0}^{n} y = x_i))).
\]

By \( \neg J_n(\varphi(x)) \) and the induction hypothesis, we have \( a_0, a_1, \ldots, a_n \) such that

\[
a_0 < a_1 < \cdots < a_n \land \bigwedge_{i=0}^{n} \varphi(a_i).
\]

Then there exists a \( b \) such that

\[
(\varphi(b) \land \bigwedge_{i=0}^{n} b \neq a_i) \lor (\neg \varphi(b) \land \bigvee_{i=0}^{n} b = a_i).
\]

Since \( \bigwedge_{i=0}^{n} \varphi(a_i), \varphi(b) \land \bigwedge_{i=0}^{n} b \neq a_i \) holds. It is easy to see that

\[
a_0 < a_1 < \cdots < a_n \land \bigwedge_{i=0}^{n} b \neq a_i \rightarrow (b < a_0 < a_1 < \cdots < a_n \lor
a_0 < b < a_1 < \cdots < a_n \lor
\cdots
a_0 < a_1 < \cdots < b < a_n \lor
a_0 < a_1 < \cdots < a_n < b),
\]

then we have \( \exists x_0, x_1, \ldots, x_{n+1} (x_0 < x_1 < \cdots < x_{n+1} \land \bigwedge_{i=0}^{n+1} \varphi(x_i)). \square \)

**Corollary 2.4.** Let \( \varphi(x) \) be an equality with \( \deg \varphi = n \) and \( 0 < n \). Then

\[
PA^- \vdash \bigvee_{i=0}^{n} (a_i \neq b_i) \rightarrow J_n(\varphi(x))
\]

**Proof.** Immediate from Proposition 2.1 and Lemma 2.3. \( \square \)

Let \( \varphi(x) \) be an inequality. Consider a sequence \( x_0 < x_1 < \cdots < x_k \) with

\[
\varphi(x_0) \land \neg \varphi(x_1) \land \cdots \land \neg \varphi(x_i) \land \cdots \land \neg \varphi(x_k) \text{ or } \neg \varphi(x_0) \land \neg \varphi(x_1) \land
\cdots \land \neg \varphi(x_i) \land \cdots \land \neg \varphi(x_k),
\]

in which affirmation alternates with negation since \( \vdash \neg \neg \psi \leftrightarrow \psi \) for any formula \( \psi \). We shall show that the number of such alternation is bounded by \( \deg \varphi \). For this purpose we introduce the following notation.
Definition 2.5. For terms $t_0, t_1, \cdots, t_k$, and $i + j \leq k$, 

$$S_{i,j}^n(t_0, t_1, \cdots, t_k) = \sum_{n_0+n_1+\cdots+n_j=n} t_i^{n_0} \cdot t_{i+1}^{n_1} \cdot \cdots \cdot t_{i+j}^{n_j}.$$ 

For example, $S_{0,0}^n(x, y) = x^n$, $S_{0,1}^n(x, y, z) = x^n + x^{n-1}y + \cdots + y^n$, $S_{1,1}^n(x, y, z) = y^n + y^{n-1}z + \cdots + z^n$, and so on. Especially $S_{i,j}^0(t_0, t_1, \cdots, t_k) = 1$.

Lemma 2.6. Let $t_0, t_1, \cdots, t_k$ be terms and $i + j + 1 \leq k$. Then

$$PA^- \vdash t_0 < t_1 < \cdots < t_k \rightarrow S_{i,j}^n(t_0, t_1, \cdots, t_k) < S_{i+1,j}^n(t_0, t_1, \cdots, t_k),$$

$$PA^- \vdash t_0 < t_1 < \cdots < t_k \rightarrow S_{i+1,j}^n(t_0, t_1, \cdots, t_k) - S_{i,j}^n(t_0, t_1, \cdots, t_k) = (t_{i+j+1} - t_i) \cdot S_{i,j+1}^{n-1}(t_0, t_1, \cdots, t_k).$$

More precisely,

$$PA^- \vdash t_0 < t_1 < \cdots < t_k \rightarrow \exists d(t_{i+j+1} = t_i + d \land S_{i+1,j}^n(t_0, \cdots, t_k) = S_{i,j}^n(t_0, \cdots, t_k) + d \cdot S_{i,j+1}^{n-1}(t_0, t_1, \cdots, t_k)).$$

Proof. The first assertion is clear. The second assertion is proved in $PA^-$. Assume $t_0 < t_1 < \cdots < t_k$.

$$S_{i+1,j}^n(t_0, \cdots, t_k) - S_{i,j}^n(t_0, \cdots, t_k)$$

$$= \sum_{n_0+\cdots+n_j=n} t_i^{n_0} \cdot t_{i+1}^{n_1} \cdot \cdots \cdot t_{i+j}^{n_j} - \sum_{n_0+\cdots+n_j=n} t_i^{n_0} \cdot t_{i+1}^{n_1} \cdot \cdots \cdot t_{i+j}^{n_j}$$

$$= \sum_{n_0+\cdots+n_j=n} t_i^{n_0} \cdot t_{i+1}^{n_1} \cdot \cdots \cdot t_{i+j}^{n_j} - \sum_{n_0+\cdots+n_j=n} t_i^{n_0} \cdot t_{i+1}^{n_1} \cdot \cdots \cdot t_{i+j}^{n_j} - t_i^{n_0} \cdot t_{i+1}^{n_1} \cdot \cdots \cdot t_{i+j}^{n_j}$$

$$= \sum_{n_0+\cdots+n_j=n} t_i^{n_0} \cdot t_{i+1}^{n_1} \cdot \cdots \cdot t_{i+j}^{n_j} - t_i^{n_0} \cdot t_{i+1}^{n_1} \cdot \cdots \cdot t_{i+j}^{n_j}$$

$$= (t_{i+j+1} - t_i) \sum_{n_0+\cdots+n_j=n} t_i^{n_0} \cdot t_{i+1}^{n_1} \cdot \cdots \cdot t_{i+j}^{n_j-1} \sum_{n_0+m_1=n-j-1} t_i^{n_0} \cdot t_{i+1}^{n_1} \cdot t_{i+j+1}^{m_1}$$

$$= (t_{i+j+1} - t_i) \sum_{n_0+\cdots+n_j+n_0+m_1=n-1} t_i^{n_0} \cdot t_{i+1}^{n_1} \cdot t_{i+j+1}^{m_1}$$

$$= (t_{i+j+1} - t_i) \cdot S_{i,j+1}^{n-1}(t_0, \cdots, t_k).$$

$\square$
Proposition 2.7. Let \( \varphi(x) \) be an inequality with \( \deg \varphi = n \) and \( 0 < n \). Then

\[
PA^- \vdash \neg \exists x_0, \ldots, x_{n+1}(x_0 < x_1 < \cdots < x_{n+1} \wedge \varphi(x_0) \wedge \varphi(x_1) \wedge \cdots \wedge \varphi(x_{n+1})),
\]

where \( \varphi \) is \( x_0, \ldots, x_{n+1} \), and \( \rho \) is \( \leq \) or \( \geq \) if \( n+1 \) is even and is odd. The system \( \Lambda_0 \) consists of \( n+2 \) inequalities. By the first and the second inequalities, we have

\[
\begin{align*}
a_n(S_{1,0}^n(\bar{x}) - S_{0,0}^n(\bar{x})) + a_{n-1}(S_{1,0}^{n-1}(\bar{x}) - S_{0,0}^{n-1}(\bar{x})) + \cdots + a_1(S_{1,0}^1(\bar{x}) - S_{0,0}^1(\bar{x})) > b_n(S_{1,0}^n(\bar{x}) - S_{0,0}^n(\bar{x})) + b_{n-1}(S_{1,0}^{n-1}(\bar{x}) - S_{0,0}^{n-1}(\bar{x})) + \cdots + b_1(S_{1,0}^1(\bar{x}) - S_{0,0}^1(\bar{x})),
\end{align*}
\]

hence by using Lemma 2.6,

\[
a_nS_{0,1}^{n-1}(\bar{x}) + a_{n-1}S_{0,1}^{n-2}(\bar{x}) + \cdots + a_1 > b_nS_{0,1}^{n-1}(\bar{x}) + b_{n-1}S_{0,1}^{n-2}(\bar{x}) + \cdots + b_1.
\]

In the same way, we have

\[
\begin{align*}
a_nS_{0,1}^{n-1}(\bar{x}) + a_{n-1}S_{0,1}^{n-2}(\bar{x}) + \cdots + a_1 > b_nS_{0,1}^{n-1}(\bar{x}) + b_{n-1}S_{0,1}^{n-2}(\bar{x}) + \cdots + b_1,
\end{align*}
\]

where \( \varphi \) is \( x_0, \ldots, x_{n+1} \) and \( \rho \) is \( \leq \) if \( n+1 \) is even and is odd. The system \( \Lambda_1 \) consists of \( n+1 \) inequalities. We continue further this reduction \( n-2 \) times.

Then we get that, in the case that \( n \) is even

\[
\begin{align*}
\Lambda_{n-1} \begin{cases}
a_nS_{0,n-1}^1(\bar{x}) + a_{n-1} > b_nS_{0,0}^{1}(\bar{x}) + b_{n-1}, \\
a_nS_{1,n-1}^1(\bar{x}) + a_{n-1} < b_nS_{1,0}^{1}(\bar{x}) + b_{n-1}, \\
a_nS_{2,n-1}^1(\bar{x}) + a_{n-1} > b_nS_{2,0}^{1}(\bar{x}) + b_{n-1},
\end{cases}
\end{align*}
\]
and in the case that $n$ is odd,

$$
\Lambda_{n-1}' \begin{cases} 
  a_nS_{0,n-1}^1(\bar{x}) + a_{n-1} < b_nS_{0,n-1}^1(\bar{x}) + b_{n-1}, \\
  a_nS_{1,n-1}^1(\bar{x}) + a_{n-1} > b_nS_{1,n-1}^1(\bar{x}) + b_{n-1}, \\
  a_nS_{2,n-1}^1(\bar{x}) + a_{n-1} < b_nS_{2,n-1}^1(\bar{x}) + b_{n-1}.
\end{cases}
$$

We proceed the same reduction once again. We have

$$
\Lambda_n \begin{cases} 
  a_nS_{0,n}^0(\bar{x}) < b_nS_{0,n}^0(\bar{x}), \\
  a_nS_{1,n}^0(\bar{x}) > b_nS_{1,n}^0(\bar{x}),
\end{cases} \quad \text{or} \quad \Lambda_n' \begin{cases} 
  a_nS_{0,n}^1(\bar{x}) > b_nS_{0,n}^1(\bar{x}), \\
  a_nS_{1,n}^1(\bar{x}) < b_nS_{1,n}^1(\bar{x}).
\end{cases}
$$

In any case, we have $a_n < b_n \land b_n < a_n$, which is a contradiction.

Since we get a contradiction in the same way when we assume that $x_0 < x_1 < \cdots < x_{n+1}$ and $\neg \varphi(x_0) \land \neg \varphi(x_1) \land \cdots \neg \varphi(x_{n+1})$, we have proved the proposition. \qed

§3. \textbf{IOpen is equivalent to IOpen}

Let $\varphi(x)$ be an open formula. Suppose $\varphi(x_0)$ and $\neg \varphi(x_1)$ with $x_0 < x_1$. Intuitively it is clear that we can find a $x$ such that $\varphi(x)$ and $\neg \varphi(x+1)$ with $x_0 \leq x < x_1$. In fact this can be proved in IOpen.

\textbf{Lemma 3.1.} Let $\varphi(x)$ be an open formula. Then

$I_{\text{Open}} \vdash \forall x_0, x_1(x_0 < x_1 \land \varphi(x_0) \land \neg \varphi(x_1) \rightarrow \exists x(x_0 \leq x < x_1 \land \varphi(x) \land \neg \varphi(x+1))).$

\textbf{Proof.} Assume that $x_0 < x_1 \land \varphi(x_0) \land \neg \varphi(x_1)$. There exists the unique $d$ such that $x_1 = x_0 + d$ since $x_0 < x_1$. We represent the $d$ by $x_1 - x_0$ and $\psi(z)$ as

$$
\varphi(x_0 + z) \land 0 \leq z \leq x_1 - x_0.
$$

Then $\psi(0)$ and $\neg \psi(x_1 - x_0)$, hence $\psi(0) \land \exists z \neg \psi(z)$. By using $I_{\psi(z)}$ we have $\exists z(\psi(z) \land \neg \psi(z + 1))$, consequently $\exists x(\varphi(x) \land \neg \varphi(x+1) \land x_0 \leq x \leq x_1)$. Finally $x \neq x_1$ follows from $\neg \varphi(x_1)$. \qed

Note that the $L_A$-sentence in Lemma 3.1 implies $I_{\varphi(x)}$ in $PA^-$. Let us say that a formula of the form $x < a$, $a \leq x < b$, or $b \leq x$ is an interval. Let $\varphi(x)$ be an inequality of degree $n$. We show in IOpen that there exist $x_0, x_1, \ldots, x_{n+1}$ such that $\varphi(x)$ is equivalent to the disjunction of intervals whose endpoints are $x_0, x_1, \ldots, x_{n+1}$. Note that the number of these intervals is at most $\lfloor \frac{n}{2} \rfloor + 1$ since it may be the case that $x_{i+1} = x_i$ for some $i$. To be precise, we define inductively the formula $I_n(\varphi(x))$ as follows.
Lemma 3.3. Let \( \varphi \) that is, 1). By induction on \( n \), we conclude that hypothesis, \( \varphi(0) \land \neg \mathcal{I}_n(\varphi(x)) \), that is, \( \varphi(0) \land \neg \mathcal{I}_{2m}(\varphi(x)) \land \neg \theta_{2m+1}(\varphi(x)) \land \neg \theta'_{2m+1}(\varphi(x)) \). By the induction hypothesis, \( \varphi(0) \) and \( \neg \mathcal{I}_{2m}(\varphi(x)) \), there exist \( a_0, a_1, \ldots, a_{2m+1} \) such that

\[
\neg \varphi(0) \land \neg \mathcal{I}_n(\varphi(x)) \rightarrow \exists x_0, x_1, \ldots, x_{n+1}(x_0 < x_1 < \cdots < x_{n+1} \land \neg \mathcal{I}_{n+1}(\varphi(x)))
\]

\[
\neg \varphi(0) \land \neg \mathcal{I}_n(\varphi(x)) \rightarrow \exists x_0, x_1, \ldots, x_{n+1}(x_0 < x_1 < \cdots < x_{n+1} \land \neg \mathcal{I}_{n+2}(\varphi(x)))
\]

\[
\neg \varphi(0) \land \neg \mathcal{I}_n(\varphi(x)) \rightarrow \exists x_0, x_1, \ldots, x_{n+1}(x_0 < x_1 < \cdots < x_{n+1} \land \neg \mathcal{I}_{n+2}(\varphi(x)))
\]

\[
\neg \varphi(0) \land \neg \mathcal{I}_n(\varphi(x)) \rightarrow \exists x_0, x_1, \ldots, x_{n+1}(x_0 < x_1 < \cdots < x_{n+1} \land \neg \mathcal{I}_{n+2}(\varphi(x)))
\]
\[ a_0 < a_1 < \cdots < a_{2m+1} \text{ and } \varphi(a_0), \neg\varphi(a_1), \cdots, \neg\cdots\neg\varphi(a_{2m+1}). \] By Lemma 3.1 there exist \( b_0, b_1, \cdots, b_{2m} \) such that \( a_0 \leq b_0 < a_1 < b_1 < a_2 \leq \cdots < a_{2m} \leq b_{2m} < a_{2m+1} \) and

\[
\begin{align*}
\varphi(b_0) & , & \neg\varphi(b_0 + 1) \\
\neg\varphi(b_1) & , & \neg\neg\varphi(b_1 + 1) \\
& \vdots \\
\neg\cdots\neg\varphi(b_{2m}) & , & \neg\cdots\neg\neg\varphi(b_{2m+1}).
\end{align*}
\]

Since \( 0 < b_0 + 1 < b_1 + 1 < \cdots < b_{2m} + 1 \), we apply \( b_0 + 1, b_1 + 1, \cdots, b_{2m} + 1 \) to \( \neg\varphi_{2m+1}(\varphi(x)) \). Then we have

\[
\begin{align*}
\exists y(\varphi(y) \land (\bigvee_{i=0}^{m-1} b_{2i} + 1 \leq y < b_{2i+1} + 1) \lor b_{2m} + 1 \leq y)) & \text{ or } \\
\exists y(\neg\varphi(y) \land (y < b_0 + 1 \lor (\bigvee_{i=1}^m b_{2i-1} + 1 \leq y < b_{2i} + 1))).
\end{align*}
\]

Case 1.

Let \( b_{2i} + 1 \leq c < b_{2i+1} + 1 \) and \( \varphi(c) \) for some \( i \) with \( 0 \leq i < m \). Since \( \neg\varphi(b_{2i} + 1) \) and \( \neg\varphi(b_{2i+1}) \), we have \( b_{2i} + 1 < c < b_{2i+1} \). Then we find \( 2m + 3 \) terms, that is, \( b_0 < \cdots < b_{2i} < b_{2i} + 1 < c < b_{2i+1} < \cdots < b_{2m} \) such that

\[
\begin{align*}
\varphi(b_0), \cdots, \varphi(b_{2i}), \neg\varphi(b_{2i} + 1), \varphi(c), \neg\varphi(b_{2i+1}), \cdots, \varphi(b_{2m}).
\end{align*}
\]

Let \( b_{2m} + 1 \leq c \) and \( \varphi(c) \). Since \( \neg\varphi(b_{2m} + 1) \), we find \( 2m + 3 \) terms, that is, \( b_0 < b_1 < \cdots < b_{2m} < b_{2m} + 1 < c \) such that

\[
\begin{align*}
\varphi(b_0), \neg\varphi(b_1), \cdots, \varphi(b_{2m}), \neg\varphi(b_{2m} + 1), \varphi(c).
\end{align*}
\]

Case 2.

Let \( c < b_0 + 1 \) and \( \neg\varphi(c) \). Since \( \varphi(0) \) and \( \varphi(b_0) \), we have \( 0 < c < b_0 \). Then we find \( 2m + 3 \) terms, that is, \( 0 < c < b_0 < b_1 < \cdots < b_{2m} \) such that

\[
\begin{align*}
\varphi(0), \neg\varphi(c), \varphi(b_0), \cdots, \varphi(b_{2m}).
\end{align*}
\]

Let \( b_{2i-1} + 1 \leq c < b_{2i} + 1 \) and \( \neg\varphi(c) \) for some \( i \) with \( 1 \leq c < m \). Since \( \varphi(b_{2i-1} + 1) \) and \( \varphi(b_{2i}) \), we have \( b_{2i-1} + 1 < c < b_{2i} \). Then we find \( 2m + 3 \) terms, that is, \( b_0 < \cdots < b_{2i-1} < b_{2i-1} + 1 < c < b_{2i} < \cdots < b_{2m} \) such that

\[
\begin{align*}
\varphi(b_0), \cdots, \neg\varphi(b_{2i-1}), \varphi(b_{2i-1} + 1), \neg\varphi(c), \varphi(b_{2i}), \cdots, \varphi(b_{2m}).
\end{align*}
\]

Thus we have proved 1) for \( n = 2m + 1 \). We can prove similarly that 1) for \( n = 2m + 1 \) implies 1) for \( n = 2m + 2 \).

2). It is easy to see that \( PA^- \vdash \mathcal{I}_n(\neg\psi(x)) \leftrightarrow \mathcal{I}_n(\psi(x)) \) for any formula \( \psi(x) \) by induction on \( n \). Then we have the assertion 2) by 1). \( \square \)
Corollary 3.4. Let $\varphi(x)$ be an inequality with $\deg \varphi = n$ and $0 < n$. Then

$$\text{IOpen} \vdash I_n(\varphi(x)).$$

Proof. Immediate from Proposition 2.7 and Lemma 3.3. \qed

In order to deal with an arbitrary open formula, we need some technical lemmas. Now recall Corollary 2.4, which says that any equality $\varphi(x)$ with $\deg_x \varphi = n$ has at most $n$ ‘solutions’. We rewrite Corollary 2.4 in terms of $I_n(\varphi(x))$.

Lemma 3.5. Let $\varphi(x)$ be an equality with $\deg \varphi = n$ and $0 < n$. Then

$$\text{PA}^- \vdash I_{2n+2}(\varphi(x)).$$

Proof. We work in $\text{PA}^-$. If $a_i = b_i$ for all $i$ with $0 \leq i \leq n$, we have $\forall x \varphi(x)$, then $I_0(\varphi(x))$. By the definition for $I$, we have $I_{2n+2}(\varphi(x))$. Otherwise, since $y = a \leftrightarrow a \leq y < a + 1$ we have the assertion by Corollary 2.4. \qed

Next we show that the set of formulas $\varphi(x)$ for which $I_n(\varphi(x))$ are provable in $\text{PA}^-$ for some $n$ is closed under conjunction and disjunction.

Lemma 3.6. For any formulas $\varphi(x)$ and $\psi(x)$

$$\text{PA}^- \vdash I_n(\varphi(x)) \land I_m(\psi(x)) \rightarrow I_{n+m}(\varphi(x) \land \psi(x)),$$
$$\text{PA}^- \vdash I_n(\varphi(x)) \land I_m(\psi(x)) \rightarrow I_{n+m}(\varphi(x) \lor \psi(x)).$$

Proof. We first show the following results: For $0 < n$ and $0 < m$

$$\text{PA}^- \vdash \Theta_n(\varphi(x)) \land \Theta_m(\psi(x)) \rightarrow I_{n+m}(\varphi(x) \land \psi(x)),$$
$$\text{PA}^- \vdash \Theta_n(\varphi(x)) \land \Theta_m(\psi(x)) \rightarrow I_{n+m}(\varphi(x) \lor \psi(x)),$$

where $\Theta_k$ is $\theta_k$ or $\theta'_k$. We work in $\text{PA}^-$. Assume that $\theta_{2n+1}(\varphi(x))$ and $\theta_{2m+2}(\psi(x))$. There exist $a_0, a_1, \ldots, a_{2n}$, and $b_0, b_1, \ldots, b_{2m+1}$ such that $0 < a_0 < a_1 < \cdots < a_{2n}$ and $0 < b_0 < b_1 < \cdots < b_{2m+1}$ and

$$\forall y(\varphi(y) \leftrightarrow y < a_0 \lor (\bigvee_{i=1}^{n} a_{2i-1} \leq y < a_{2i})), $$
$$\forall y(\varphi(y) \leftrightarrow \bigvee_{i=0}^{m} b_{2i} \leq y < b_{2i+1}).$$
Consider all the possibilities of ordering between $a_0, a_1, \ldots, a_{2n}$ and $b_0, b_1, \ldots, b_{2m+1}$. For example; if $2n \leq 2m+1$ and $a_0 > b_0 < b_1 < a_1 < \cdots < a_{2i} < b_{2i} < b_{2i+1} < a_{2i+1} < \cdots < a_{2n} < b_{2n} < b_{2n+1} < \cdots < b_{2m+1}$ then we have

$$\forall x \neg (\varphi(x) \land \psi(x)),$$

$$\forall y((\varphi(y) \lor \psi(y)) \leftrightarrow (y < a_0 \lor (\bigvee_{i=1}^{n} a_{2i-1} \leq y < a_{2i}) \lor (\bigvee_{i=0}^{m} b_{2i} \leq y < b_{2i+1}))),$$

thus $I_0(\varphi(x) \land \psi(x))$ and $I_{2n+1+2m+2}(\varphi(x) \lor \psi(x))$. If $2m+1 < 2n$ and $b_0 < a_0 < a_1 < b_1 < \cdots < b_{2i} < a_{2i} < a_{2i+1} < b_{2i+1} < \cdots < a_{2m+1} < b_{2m+1} < a_{2m+2} < a_{2m+3} < \cdots < a_{2n}$ then we have

$$\forall y((\varphi(y) \land \psi(y)) \leftrightarrow (\bigvee_{i=0}^{m} b_{2i} \leq y < a_{2i}) \lor (\bigvee_{i=0}^{m} a_{2i+1} \leq y < b_{2i+1}))),$$

$$\forall y((\varphi(y) \lor \psi(y)) \leftrightarrow (y < a_{2m+2} \lor (\bigvee_{i=m+2}^{n} a_{2i-1} \leq y < a_{2i}))),$$

thus $I_{4m+4}(\varphi(x) \land \psi(x))$ and $I_{2n-2m-1}(\varphi(x) \lor \psi(x))$. Similarly, for the other orderings we have $I_k(\varphi(x) \land \psi(x))$ and $I_l(\varphi(x) \lor \psi(x))$ for some $k, l \leq 2n+1+2m+2$. So we have $I_{2n+1+2m+2}(\varphi(x) \land \psi(x))$ and $I_{2n+1+2m+2}(\varphi(x) \lor \psi(x))$. Therefore we get the result in this case. Similarly for the other cases on $\Theta_n$ and $\Theta_m$, and hence we can show the first results.

Secondly we show the following results: For $0 < m$

$$PA^- \vdash I_n(\varphi(x)) \land \Theta_m(\psi(x)) \rightarrow I_{n+m}(\varphi(x) \land \psi(x)),$$

$$PA^- \vdash I_n(\varphi(x)) \land \Theta_m(\psi(x)) \rightarrow I_{n+m}(\varphi(x) \lor \psi(x)),$$

where $\Theta_m$ is $\theta_m$ or $\theta'_m$. We prove these results by induction on $n$. Let $n = 0$, and we work in $PA^-$. Recall that $I_0(\varphi(x))$ is $\forall x \varphi(x) \lor \forall x \neg \varphi(x)$. If $\forall x \varphi(x)$, we have $\forall x (\varphi(x) \land \psi(x) \rightarrow \psi(x))$ and $\forall x (\varphi(x) \lor \psi(x))$. Then $\Theta_m(\psi(x))$ implies $I_m(\varphi(x) \land \psi(x))$ and $I_0(\varphi(x) \lor \psi(x))$. If $\forall x \neg \varphi(x)$, we have $\forall x (\varphi(x) \land \psi(x))$ and $\forall x (\varphi(x) \lor \psi(x))$. Then $\Theta_m(\psi(x))$ implies $I_0(\varphi(x) \land \psi(x))$ and $I_m(\varphi(x) \lor \psi(x))$. In both cases, we have the second results for $n = 0$. Assume the results for $n$ and $I_{n+1}(\varphi(x)) \land \Theta_m(\psi(x))$, that is, $(I_n(\varphi(x)) \land \Theta_m(\psi(x))) \lor ((\theta_{n+1}(\varphi(x)) \land \Theta_m(\psi(x))) \land \theta'_m(\varphi(x)) \land \Theta_m(\psi(x)))$. If $I_n(\varphi(x)) \land \Theta_m(\psi(x))$, we have $I_{n+m}(\varphi(x) \land \psi(x))$ and $I_{n+m}(\varphi(x) \lor \psi(x))$ by the induction hypothesis. If $\theta_{n+1}(\varphi(x)) \land \Theta_m(\psi(x))$ or $\theta'_m(\varphi(x)) \land \Theta_m(\psi(x))$, we have $I_{n+1+m}(\varphi(x) \land \psi(x))$ and $I_{n+1+m}(\varphi(x) \lor \psi(x))$ by the first result. Thus the second results are proved.

Finally the assertion in this lemma is proved by induction on $m$, by using the second results. $\Box$
Now let $\varphi(x)$ be an open formula. Let a disjunctive normal form of $\varphi(x)$ be $\bigvee \bigwedge_{j=0}^{q} \varphi_{ij}(x)$, where $\varphi_{ij}(x)$ is atomic or negated atomic. Since $\neg s = t$ and $\neg s < t$ are equivalent to $s < t \lor t < s$ and $t < s \lor t = s$ in $PA^-$ respectively, we may suppose, by using the distributive law, $\varphi(x) \equiv \bigvee \bigwedge_{j=0}^{q} \varphi_{ij}(x)$ where $\varphi_{ij}(x)$ is of the form $s < t$ or $s = t$.

**Main Lemma 3.7.** For any open formula $\varphi(x)$ there exists an $n$ such that

$$I\text{Open} \vdash \mathcal{I}_n(\varphi(x)),$$

that is,

1) for any atomic formula $\varphi(x)$ there exists an $n$ such that

$$I\text{Open} \vdash \mathcal{I}_n(\varphi(x)),$$

2) $I\text{Open} \vdash \mathcal{I}_n(\varphi(x))$ and $I\text{Open} \vdash \mathcal{I}_m(\psi(x))$ then,

2.1) $I\text{Open} \vdash \mathcal{I}_{n+m}(\varphi(x) \land \psi(x))$,

2.2) $I\text{Open} \vdash \mathcal{I}_{n+m}(\varphi(x) \lor \psi(x))$.

**Proof.** 1) is immediately obtained from Corollary 3.4 and Lemma 3.5. 2) is from Lemma 3.6. □

**Theorem 3.8.** $I\text{Open}$ proves $L_\varphi(x)$ for any open formula $\varphi(x)$.

**Proof.** By Main Lemma 3.7, there exists an $n$ such that $I\text{Open} \vdash \mathcal{I}_n(\varphi(x))$. By induction on $n$. For $n = 0$, we work in $I\text{Open}$. Assume that $\exists x \varphi(x)$. Since $\mathcal{I}_0(\varphi(x))$ is $\forall x \varphi(x) \lor \forall x \neg \varphi(x)$, we obtain $\forall x \varphi(x)$. We have $\forall y < 0 \neg \varphi(y) \land \varphi(0)$. Hence $\exists x (\forall y < x \neg \varphi(y) \land \varphi(x))$. Next, assume that the assertion holds for $n = 2m$. Recall that $\mathcal{I}_{2m+1}(\varphi(x)) \equiv \mathcal{I}_{2m}(\varphi(x)) \lor \theta_{2m+1}(\varphi(x)) \lor \theta'_{2m+1}(\varphi(x))$. If $\mathcal{I}_{2m}(\varphi(x))$ holds, we have the assertion by the induction hypothesis. In the other case, there exist $a_0, a_1, \ldots, a_{2m}$ such that $a_0 < a_1 < \cdots < a_{2m}$, and $\forall y (\varphi(y) \rightarrow y < a_0 \lor (\bigvee_{i=0}^{m-1} a_{2i+1} \leq y < a_{2i+2}))$ or $\forall y (\varphi(y) \rightarrow (\bigwedge_{i=0}^{m} a_{2i} \leq y < a_{2i+1}) \lor a_{2m} \leq y)$. In the first case $\forall y < 0 \neg \varphi(y) \land \varphi(0)$, and in the second case $\forall y < a_0 \neg \varphi(y) \land \varphi(a_0)$. In either case, we have $\exists x (\forall y < x \neg \varphi(y) \land \varphi(x))$.

Assume that the assertion holds for $n = 2m + 1$. We can show similarly that the assertion holds for $n = 2m + 2$. □
Corollary 3.9.  \( I_{\text{Open}} \) is equivalent to \( L_{\text{Open}} \).

Proof.  It is easy to see that \( PA^- \vdash L_{\varphi(x)} \rightarrow I_{\varphi(x)} \) for any open formula \( \varphi(x) \), hence \( L_{\text{Open}} \) proves all axioms of \( I_{\text{Open}} \). Thus the assertion is immediately obtained from Theorem 3.8.  \( \square \)

We have used open induction only in showing that \( I_n(\varphi(x)) \) for an inequality \( \varphi(x) \) of degree \( n \). For any \( n \), \( I_n(\varphi(x)) \) can not be proved in \( PA^- \). To see this, consider \( M = \{ f \in Z[t] \mid 0 \leq LC(f) \} \), where \( Z[t] \) is the polynomial ring over the integers and \( LC(f) \) is the leading coefficient of \( f \). \( M \) is made into an \( LA \)-structure by defining an order \( < \) making \( t \) ‘infinitely large’. More specifically, if \( f, g \in Z[t] \), then we define \( g < f \leftrightarrow LC(f-g) > 0 \). \( M \) is a model of \( PA^- \), but not of \( I_{\text{Open}} \); let \( \varphi(x,y) \equiv x^2 < y \) and consider \( \varphi(x,t) \equiv x^2 < t \), then \( \varphi(0,t) \land \exists x \varphi(x,t) \) but there does not exist an element \( x \) in \( M \) such that \( \varphi(x,t) \) and \( \neg \varphi(x+1,t) \). Now assume that \( I_2(\varphi(x,t)) \). Since \( \forall n \in N \varphi(n,t) \) and \( \neg \varphi(t,t), \varphi(x,t) \) must be equivalent to \( x < a \) or \( x < a \lor b \leq x \) for some \( a, b \in M \) with \( 0 < a < b \) by Corollary 3.4. In either case we have \( \varphi(a-1,t) \land \neg \varphi(a,t) \), which is a contradiction.

References


