FRACTIONAL \(3^m\) FACTORIAL DESIGNS WITH SPECIAL REFERENCE TO 18-RUN ORTHOGONAL \(3^4\) FACTORIAL DESIGNS

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Abstract. Loading vectors and loading coefficients of the parameters of a \(3^m\) factorial design and the characteristic vector of its information matrix are introduced. Specific properties of an orthogonal design derived from three-symbol orthogonal array of strength two are discussed. Orthogonal 18-run \(3^4\) factorial designs obtained respectively from the representatives of twelve isomorphic classes are reviewed and two designs among them are recommended for use from the practical point of view.

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§1. \(3^m\) factorial designs

Consider a \(3^m\) factorial experiment with \(m\) factors, \(F(1), F(2), \ldots, F(m)\), each at three levels 0, 1 and 2. Let \((j_1, j_2, \ldots, j_m)\) be an assembly or a treatment combination of \(m\) factors each at three levels \(j_p = 0, 1\) or 2 for every \(p = 1, 2, \ldots, m\). Let \(y(j_1, j_2, \ldots, j_m)\) and \(\eta(j_1, j_2, \ldots, j_m)\) be the corresponding observation and the expectation of the assembly.

Let

\[
\eta(Z) = \begin{bmatrix}
\eta(0, 0, \ldots, 0, 0) \\
\eta(0, 0, \ldots, 0, 1) \\
\eta(0, 0, \ldots, 0, 2) \\
\eta(0, 0, \ldots, 1, 0) \\
\vdots \\
\eta(2, 2, \ldots, 2, 0) \\
\eta(2, 2, \ldots, 2, 1) \\
\eta(2, 2, \ldots, 2, 2)
\end{bmatrix}
\quad \text{and} \quad
\Theta(Z) = \begin{bmatrix}
\theta(0, 0, \ldots, 0, 0) \\
\theta(0, 0, \ldots, 0, 1) \\
\theta(0, 0, \ldots, 0, 2) \\
\theta(0, 0, \ldots, 1, 0) \\
\vdots \\
\theta(2, 2, \ldots, 2, 0) \\
\theta(2, 2, \ldots, 2, 1) \\
\theta(2, 2, \ldots, 2, 2)
\end{bmatrix}
\]

(1.1)
be the vector of the expectation of possible \(3^m\) assemblies and that of factorial effects based on the orthogonal polynomial models. They are linked to each other by
\[
\Theta(Z) = \frac{1}{3^m} D(m) \eta(Z),
\]
where \(D(m) = D \otimes D \otimes \cdots \otimes D\) is the \(m\)-times Kronecker products of the matrix
\[
D = \begin{bmatrix}
d_{00} & d_{01} & d_{02} \\
d_{10} & d_{11} & d_{12} \\
d_{20} & d_{21} & d_{22}
\end{bmatrix} = \begin{bmatrix} d_0, d_1, d_2 \end{bmatrix} = \begin{bmatrix} 1 - \frac{\sqrt{3}}{2} & \frac{1}{\sqrt{2}} \\
1 & 0 & -\frac{1}{\sqrt{2}} \\
1 & \frac{\sqrt{3}}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}.
\]

Of course, \(d_0 = j_3 = (1,1,1)\), and \(d_0, d_1\) and \(d_2\) satisfy \(d_i d_j = 3\delta_{ij}\) for Kronecker \(\delta_{ij}\), \(i, j = 0, 1, 2\).

We may note that the definition of factorial effects here is designed to keep homoscedastic property among the BLUE's obtained under the complete \(3^n\) factorial design in order to compare the effects by their location parameters only.

Solving (1.2), we have
\[
\eta(Z) = D(m) \Theta(Z), \quad \eta(j_1, j_2, \ldots, j_m) = \sum_{p=1,2,\ldots,m} d_{j_1 i_1} d_{j_2 i_2} \cdots d_{j_m i_m} \theta(i_1, i_2, \ldots, i_m).
\]

Let \(U^r = \{p \mid i_p = r\}\) be a subset of \(\Omega = \{1, 2, \ldots, m\}\) with a superscript \(r\) in which the arguments \(i_p\) of \(\theta(i_1, i_2, \ldots, i_m)\) are equal to \(r\) for \(r = 0, 1\) and 2. Then the factorial effect \(\theta(i_1, i_2, \ldots, i_m)\) can be expressed alternatively as \(\theta(U^1 U^2)\), since \(U^0 = \Omega - U^1 - U^2\). If both \(U^1\) and \(U^2\) are null, the parameter or factorial effect \(\theta(0, 0, \ldots, 0, 0)\) is called the general mean and is denoted alternatively by \(\theta(0)\).

If \(|U^1 \cup U^2| = 1\) and \(U^1 \cup U^2 = \{p\}\), then the parameters \(\theta(0, 0, \ldots, 1, \ldots, 0)\) and \(\theta(0, 0, \ldots, 2, \ldots, 0)\) both having single nonzero argument in the \(p\)th position are called the linear and the quadratic main effects of the factor \(F(p)\), respectively. They may be denoted alternatively by \(\theta(p^1)\) and \(\theta(p^2)\), respectively. If \(|U^1 \cup U^2| = 2\) and \(U^1 \cup U^2 = \{p, q\}\), then the parameter \(\theta(i_1, i_2, \ldots, i_m)\) having two nonzero arguments \(i_p\) and \(i_q\) is called the linear \(\times\) linear, the linear \(\times\) quadratic, the quadratic \(\times\) linear or the quadratic \(\times\) quadratic two-factor interactions of the factors \(F(p)\) and \(F(q)\) according as \((i_p, i_q)\) is equal to \((1,1), (1,2), (2,1)\) or \((2,2)\), respectively. Those two-factor interactions may be denoted alternatively by \(\theta(p^r q^s)\) for \(i_p, i_q = 1\) or 2, respectively. In general, if \(|U^1 \cup U^2| = k\), then the parameter \(\theta(i_1, i_2, \ldots, i_m)\)
having \( k \) nonzero arguments with respect to \( k \) factors is called the \( k \)-factor interaction and is expressed as \( \theta(U^1U^2) \) by indicating the sets of arguments \( U^1 \) and \( U^2 \).

Let \( T \) be a fraction of \( 3^m \) factorial design with \( m \) factors composed of \( n \) assemblies \((j_1^{(1)}, j_2^{(1)}, \ldots, j_m^{(1)}); j_p^{(a)} = 0, 1 \) or 2, \( p = 1, 2, \ldots, m, \alpha = 1, 2, \ldots, n; \) and suppose \( y(T) \) be the corresponding vector of observations, i.e.,

\[
T = \begin{bmatrix}
  j_1^{(1)}, & j_2^{(1)}, & \ldots, & j_m^{(1)} \\
  \vdots & \vdots & \ddots & \vdots \\
  j_1^{(n)}, & j_2^{(n)}, & \ldots, & j_m^{(n)}
\end{bmatrix}
\quad \text{and} \quad
y(T) = \begin{bmatrix}
  y(j_1^{(1)}, j_2^{(1)}, \ldots, j_m^{(1)}) \\
  \vdots \\
  y(j_1^{(n)}, j_2^{(n)}, \ldots, j_m^{(n)})
\end{bmatrix}.
\]

The vector of observations of the design \( T \) is expressed as

\[
y(T) = E(T)\Theta + e(T)
\]

in terms of \( E(T), \Theta, \) and \( e(T) \), where \( \Theta \) is the parameter vector obtained by rearranging \( \Theta(Z) \) in a natural order of the number of factors and levels concerned, \( E(T) \) is the design matrix whose element in the row and the column correspond respectively to \( \alpha \)th observation and the effect \( \theta(i_1, i_2, \ldots, i_m) \) is given by

\[
e(\alpha; i_1, i_2, \ldots, i_m)) = d_{j_1^{(a)}i_1}d_{j_2^{(a)}i_2}\ldots d_{j_m^{(a)}i_m}
\]

and \( e(T) \) is the error vector with the usual assumption that the components are distributed uncorrelatedly with \((0, \sigma^2)\).

Since \( d_{j0} = 1 \) for every \( j \),

\[
e(\alpha; \phi) = 1, \text{ for every } \alpha,
\]

\[
e(\alpha; p^q) = d_{j_p^{(a)}i_p}, \text{ for } p \in \Omega \text{ and } i_p = 1, 2,
\]

\[
e(\alpha; p^q, q^q) = d_{j_p^{(a)}i_q}d_{j_q^{(a)}i_q}, \text{ for } p \ne q \in \Omega \text{ and } i_p, i_q = 1, 2,
\]

and in general,

\[
e(\alpha; U^1U^2) = \prod_{p \in U^1} d_{j_p^{(a)}i_p} \prod_{q \in U^2} d_{j_q^{(a)}i_q}.
\]

The column vector of the design matrix \( E(T) \) corresponding to the factorial effect \( \theta(i_1, i_2, \ldots, i_m) \) is expressed as:

\[
d(i_1, i_2, \ldots, i_m) = d_{j_1^{(1)}i_1}d_{j_2^{(1)}i_2}\ldots d_{j_m^{(1)}i_m},
\]

\[
d_{j_1^{(a)}i_1}d_{j_2^{(a)}i_2}\ldots d_{j_m^{(a)}i_m}, \ldots, d_{j_1^{(n)}i_1}d_{j_2^{(n)}i_2}\ldots d_{j_m^{(n)}i_m}'.
\]
Since $d_{ij0} = 1$ for every $j$, the above expression may be simplified to a vector of the products of $d_{ji}$’s with nonzero $i$’s only.

In particular,

\[(1.10) \quad d(\phi) = (1, 1, \ldots, 1)^t, \quad \text{and} \quad d(p^p) = (d_{j_1}^{(1)} p, d_{j_2}^{(2)} p, \ldots, d_{j_{\alpha}}^{(\alpha)} p, \ldots, d_{j_p}^{(n_p)} p)^t\]

for $p \in \Omega$ and $i_p = 1, 2$.

In general,

\[(1.11) \quad d(U_1 U_2^2) = \left( \prod_{p \in U_1} d_{j_p}^{(1)} \prod_{q \in U_2} d_{j_q}^{(1)} d_{j_1}^{(2)} \prod_{p \in U_1} d_{j_p}^{(2)} \prod_{q \in U_2} d_{j_q}^{(2)} \right)^t, \quad \ldots, \quad \prod_{p \in U_1} d_{j_p}^{(n_p)} \prod_{q \in U_2} d_{j_q}^{(n_q)} \prod_{p \in U_1} d_{j_p}^{(n_p)} \prod_{q \in U_2} d_{j_q}^{(n_q)})^t.\]

**Definition 1.** For a fractional $3^m$ factorial design $T$, the vector $d(i_1, i_2, \ldots, i_m)$ or $d(U_1 U_2)$ is called the *loading vector* of a factorial effect $\theta(i_1, i_2, \ldots, i_m)$ or $\theta(U_1 U_2)$.

Using loading vectors of $2m$ main effects given in (1.10), every loading vector can be obtained by the Schur products (*) of related loading vectors for main effects as is given by the formula (1.11). For a simplest example,

$d(p^p q^q) = d(p^p) * d(q^q)$.

Let $S_p[x]$ be the *spur* of a vector $x$ being defined by the sum of its components.

**Definition 2.** The spur $S_p[d(U_1 U_2^2)]$ of the loading vector of a factorial effect $\theta(U_1 U_2^2)$, denoted by $\gamma(U_1 U_2^2)$, is called the *loading coefficient* of $\theta(U_1 U_2^2)$ of the design $T$.

In particular,

\[(1.12) \quad \gamma(\phi) = n, \quad \gamma(p^p) = \sum_{\alpha=1}^n d_{j_p}^{(\alpha)} i_p \quad \text{for} \quad p \in \Omega \quad \text{and} \quad i_p = 1, 2, \]

\[\gamma(p^p q^q) = \sum_{\alpha=1}^n d_{j_p}^{(\alpha)} d_{j_q}^{(\alpha)} i_p i_q \quad \text{for} \quad p \neq q \in \Omega \quad \text{and} \quad i_p, i_q = 1, 2,\]

and, in general,

\[(1.13) \quad \gamma(U_1 U_2^2) = \sum_{\alpha=1}^n \prod_{p \in U_1} d_{j_p}^{(\alpha)} \prod_{q \in U_2} d_{j_q}^{(\alpha)}.\]
The normal equation for estimating $\Theta$ is given by

\begin{equation}
M(T)\Theta = E'(T)y(T)
\end{equation}

where $M(T) = E'(T)E(T)$ is the information matrix of a design $T$.

The element $\varepsilon(i_1i_2 \ldots i_m, k_1k_2 \ldots k_m)$ of the information matrix in the $\theta(i_1, i_2, \ldots, i_m)$ row and the $\theta(k_1, k_2, \ldots, k_m)$ column is given by

\begin{equation}
\varepsilon(i_1i_2 \ldots i_m, k_1k_2 \ldots k_m) = \sum_{\alpha=1}^{n} d_{j_1^{(\alpha)}i_1} d_{j_2^{(\alpha)}i_2} \cdots d_{j_m^{(\alpha)}i_m} d_{j_1^{(\alpha)}k_1} d_{j_2^{(\alpha)}k_2} \cdots d_{j_m^{(\alpha)}k_m}
\end{equation}

for every pair of $x \leq y = 0, 1, 2$. Then, we have:

**Lemma 1.** Every product $d_{jk}d_{jk}$ of the elements of the matrix $D$ satisfies the following irrespective of the value $j = 0, 1, 2$, i.e.,

\begin{equation}
d_{j0}d_{j0} = d_{j0} = 1, \quad d_{j0}d_{j1} = d_{j1}d_{j0} = d_{j1}, \quad d_{j0}d_{j2} = d_{j2}d_{j0} = d_{j2},
\end{equation}

\begin{equation}
d_{j1}d_{j1} = 1 + \frac{1}{2}d_{j2}, \quad d_{j2}d_{j2} = 1 - \frac{1}{2}d_{j2}, \quad d_{j1}d_{j2} = \sqrt{\frac{1}{2}}d_{j1}.
\end{equation}

Let $K^{xy} = (U^x \cap V^y) \cup (U^y \cap V^x)$ with cardinality $|K^{xy}| = k_{xy}$ for every pair of $x \leq y = 0, 1, 2$. Then, we have:

**Theorem 2.** The $(\theta(U^{1}U^{2}), \theta(V^{1}V^{2}))$ element of the information matrix $M(T)$ of a fractional $3^n$ factorial design $T$ is given by

\begin{equation}
\varepsilon(U^{1}U^{2}, V^{1}V^{2}) = \sum_{\alpha=1}^{n} \prod_{p \in K^{01}} d_{j_{p}^{(\alpha)}} \prod_{q \in K^{02}} d_{j_{q}^{(\alpha)}} \prod_{r \in K^{11}} \left(1 + \frac{T}{2}d_{j_{r}^{(\alpha)}}\right) \prod_{s \in K^{22}} \left(1 - \frac{T}{2}d_{j_{s}^{(\alpha)}}\right) \prod_{t \in K^{12}} \left(\sqrt{\frac{1}{2}}d_{j_{t}^{(\alpha)}}\right).
\end{equation}

**Definition 3.** The first row $\Gamma(T)$ of the information matrix $M(T)$ which is composed of all loading coefficients $\gamma(U^{1}U^{2})$’s arranged in a natural order of $\theta(U^{1}U^{2})$ is called the characteristic vector of $M(T)$ or the design $T$.

**Theorem 3.** The information matrix $M(T)$ of the design $T$ is completely determined by its characteristic vector $\Gamma(T)$. 

Proof. The formula (1.17) shows that every component of $M(T)$ is a linear combination of the terms each composed of the sum of the products of at most $m$ $d_{j_p^{(α_i)}}$'s with respect to $α$, i.e., the loading coefficients.

In particular,

$$ε(φ, φ) = n.$$ $$ε(φ, p^i) = γ(p^i)$$ for $p ∈ Ω$ and $i_p = 1, 2.$

$$ε(φ, p^iq^i_q) = γ(p^i_qq^i_q)$$ for $p ≠ q ∈ Ω$ and $i_p, i_q = 1, 2.$

$$ε(φ, U^1U^2) = γ(U^1U^2).$$

$$ε(p^i, p^j) = n + \sqrt{\frac{T}{2}}γ(p^2),$$ $$ε(p^1, p^2) = \sqrt{\frac{T}{2}}γ(p^i),$$ and

$$ε(p^2, p^2) = n - \sqrt{\frac{T}{2}}γ(p^2),$$ for $p ∈ Ω.$

$$ε(p^i, q^i_q) = γ(p^i_qq^i_q)$$ for $p ≠ q ∈ Ω$ and $i_p, i_q = 1, 2.$

$$ε(p^i, p^i_qq^i_q) = γ(q^i) + \sqrt{\frac{T}{2}}γ(p^2q^i),$$ $$ε(p^1, p^2q^i) = \sqrt{\frac{T}{2}}γ(p^i_qq^i_q),$$ and

$$ε(p^2, p^2q^i) = γ(q^i) - \sqrt{\frac{T}{2}}γ(p^2q^i),$$ for $p ≠ q ∈ Ω$ and $i_q = 1, 2.$

$$ε(p^i, q^i_q) = γ(p^i_qq^i_q)$$ for $q ≠ r, p ≠ q, r ∈ Ω$ and $i_p, i_q, i_r = 1, 2.$

$$ε(p^i_q, p^i_q) = n + \sqrt{\frac{T}{2}}γ(p^2) + γ(q^i) + \frac{1}{2}γ(p^2q^i),$$

$$ε(p^i_q, p^i_q) = \sqrt{\frac{T}{2}}γ(p^i) + \frac{1}{2}γ(p^2q^i),$$

$$ε(p^i_q, p^i_q) = \frac{1}{2}γ(p^i_qq^i_q),$$

$$ε(p^i_q, p^i_q) = n + \sqrt{\frac{T}{2}}γ(p^2) - γ(q^i) - \frac{1}{2}γ(p^2q^i),$$

$$ε(p^i_q, p^i_q) = \frac{1}{2}γ(p^i_q),$$

$$ε(p^i_q, p^i_q) = \sqrt{\frac{T}{2}}γ(p^1) - \frac{1}{2}γ(p^i_qq^i_q),$$ and

$$ε(p^i_q, p^i_q) = n - \sqrt{\frac{T}{2}}γ(p^2) + γ(q^i) + \frac{1}{2}γ(p^2q^i),$$ for $p ≠ q ∈ Ω.$

$$ε(p^i_q, p^i_q) = γ(q^i) + \sqrt{\frac{T}{2}}γ(p^2q^i),$$

$$ε(p^i_q, p^i_q) = \sqrt{\frac{T}{2}}γ(p^i_qq^i_q),$$ and

$$ε(p^i_q, p^i_q) = γ(q^i) - \sqrt{\frac{T}{2}}γ(p^2q^i),$$ for $p ≠ q, r ≠ p, q ∈ Ω$ and $i_q, i_r = 1, 2.$
\( \varepsilon(p^jq^r, r^t s^i) = \gamma(p^jq^r r^t s^i) \) for \( p \neq q, \ r \neq p, q, \ s \neq p, q, r \in \Omega \)
and \( i_p, i_q, i_r, i_s = 1, 2. \)

### §2. Normal equation of a fractional \( 3^m \) factorial design

The first member of the normal equation (1.14) is given by

\[
(2.1) \quad n\theta(\phi) + \sum_{k=1}^{m} \sum_{|V^1 \cup V^2| = k} \gamma(V^1 V^2)\theta(V^1 V^2) = d'(\phi)y(T),
\]

or
\[
= \sum_{\alpha=1}^{n} y(j_1^{(\alpha)}, j_2^{(\alpha)}, \ldots, j_m^{(\alpha)}).
\]

In some sense, the left hand member of equation (2.1) may be called the
**defining formula.** This is an extension of the so-called **defining relation** introduced by Box and Hunter [1,2] in the case of \( 2^n \) factorial designs.

Those members corresponding to the main effects \( \theta(p^1) \) and \( \theta(p^2) \), for \( p \in \Omega \), are given by

\[
(2.2) \quad \gamma(p^1)\theta(\phi) + \left( n + \sqrt{\frac{T}{2}} \gamma(p^2) \right)\theta(p^1) + \sqrt{\frac{T}{2}} \gamma(p^1)\theta(p^2)
\]
\[+ \sum_{q \neq p} \left( \gamma(p^1 q^1)\theta(q^1) + \gamma(p^1 q^2)\theta(q^2) \right) \]
\[+ \sum_{k=2}^{m} \sum_{|V^1 \cup V^2| = k} \varepsilon(p^1, V^1 V^2)\theta(V^1 V^2) = d'(p^1)y(T),
\]

or
\[
= \sum_{\alpha=1}^{n} d_{j_p^{(\alpha)}} y(j_1^{(\alpha)}, j_2^{(\alpha)}, \ldots, j_m^{(\alpha)}) \text{ for } \theta(p^1), \text{ and}
\]

\[
(2.3) \quad \gamma(p^2)\theta(\phi) + \left( n - \sqrt{\frac{T}{2}} \gamma(p^2) \right)\theta(p^1) + \sqrt{\frac{T}{2}} \gamma(p^2)\theta(p^2)
\]
\[+ \sum_{q \neq p} \left( \gamma(p^2 q^1)\theta(q^1) + \gamma(p^2 q^2)\theta(q^2) \right) \]
\[+ \sum_{k=2}^{m} \sum_{|V^1 \cup V^2| = k} \varepsilon(p^2, V^1 V^2)\theta(V^1 V^2) = d'(p^2)y(T),
\]

or
\[
= \sum_{\alpha=1}^{n} d_{j_p^{(\alpha)}} y(j_1^{(\alpha)}, j_2^{(\alpha)}, \ldots, j_m^{(\alpha)}) \text{ for } \theta(p^2).
\]

The member corresponding to \( \theta(p^q q^r) \), for \( p \neq q \in \Omega \), is given by

\[
(2.4) \quad \gamma(p^q q^r)\theta(\phi) + \sum_{r} \sum_{i_r=1,2} \varepsilon(p^q q^r, r^{i_r})\theta(r^{i_r})
\]
principal equation

main effects and those
hold true for all
simultaneously.

\[
\varepsilon = \varepsilon \theta(\epsilon s^t)
\]

\[
\sum_{p \neq s, r, i_s=1,2} \varepsilon(p^i q^\alpha, r^i s^t) \theta(\epsilon s^t)
\]

\[
\sum_{k=3}^{m} \sum_{|V^1 \cup V^2|=k} \varepsilon(p^i q^\alpha, V^1 V^2) \theta(V^1 V^2) = d'(p^i q^\alpha) y(T).
\]

In general, the member corresponding to \(\theta(U^1 U^2)\), which may be called principal equation of estimating the parameter, is given by

\[
\sum_{k=0}^{m} \sum_{|V^1 \cup V^2|=k} \varepsilon(U^1 U^2, V^1 V^2) \theta(V^1 V^2) = d'(U^1 U^2) y(T),
\]

or

\[
\sum_{\alpha=1}^{n} \prod_{p \in U^1} d(j_1, j_2, \ldots, j_n).
\]

Those left hand members of (2.2), (2.3), (2.4) and, in general, (2.5) may be called the derived formulas of the design.

§3. Designs derived from three-symbol orthogonal arrays of strength 2

Let \(T\) in (1.5) be a design derived from a three-symbol orthogonal array of strength 2, \(m\) constraints and index \(\lambda\), denoted by 3-OA(2, \(m\), \(\lambda\)), having \(n = 9\lambda\) runs. Then, since \(\sum_{\alpha=1}^{n} d(j_1, j_2, \ldots, j_n) = 0\) and also \(\sum_{\alpha=1}^{n} d(j_1, j_2, \ldots, j_n) = 0\) hold true for all \(p \neq q \in \Omega\) and \(i_p, i_q = 1, 2\), all loading coefficients \(\gamma(p^i q^\alpha)\) of 2m main effects and those \(\gamma(p^i q^\alpha)\) of 2m(m - 1) two-factor interactions vanish simultaneously.

In such a circumstance, we have the following:

\[
\varepsilon(\phi, \phi) = n, \ \varepsilon(\phi, p^i) = \varepsilon(\phi, p^i q^\alpha) = 0, \ \text{for} \ p \neq q \in \Omega \text{ and } i_p, i_q = 1, 2.
\]

\[
\varepsilon(p^1, p^1) = \varepsilon(p^2, p^2) = n \text{ and } \varepsilon(p^1, p^2) = \varepsilon(p^2, p^1) = 0, \ \text{for} \ p \in \Omega.
\]

\[
\varepsilon(p^i, q^\alpha) = 0 \ \text{for} \ p \neq q \in \Omega \text{ and } i_p, i_q = 1, 2.
\]

\[
\varepsilon(p^1, p^1 q^\alpha) = \varepsilon(p^2, p^2 q^\alpha) = \varepsilon(p^2, q^\alpha) = 0, \ \text{for} \ p \neq q \in \Omega \text{ and } i_q = 1, 2.
\]

\[
\varepsilon(p^1, q^\alpha r^i) = \gamma(p^1 q^\alpha r^i) \ \text{for} \ q \neq r, p \neq q, r \in \Omega \text{ and } i_p, i_q, i_r = 1, 2.
\]

\[
\varepsilon(p^1 q^1, p^1 q^1) = \gamma(p^1 q^1, p^1 q^1) = \gamma(p^2 q^1, p^2 q^1) = n, \ \text{and}
\]

\[
\varepsilon(p^1 q^1, p^1 q^2) = \gamma(p^1 q^1, p^1 q^2) = \gamma(p^2 q^1, p^2 q^2) = \gamma(p^1 q^2, p^1 q^2) = \gamma(p^1 q^2, p^2 q^2) = 0,
\]

\[
\text{for} \ p \neq q \in \Omega.
\]

\[
\varepsilon(p^1 q^1, p^1 r^i) = \sqrt{\frac{T}{2}} \gamma(p^1 q^1 r^i),
\]
\[ \varepsilon(p^2q^q, p^2r^r) = -\sqrt{\frac{T}{2}} \gamma(p^2q^q r^r), \text{ and} \]
\[ \varepsilon(p^1q^q, p^2r^r) = \sqrt{\frac{T}{2}} \gamma(p^1q^q r^r), \]
for \( p \neq q, r \neq p, q \in \Omega \) and \( i_q, i_r = 1, 2 \).
\[ \varepsilon(p^q q^q, r^r s^s) = \gamma(p^q q^q r^r s^s) \]
for \( p \neq q, r \neq p, q \neq r, p, q \in \Omega \) and \( i_p, i_q, i_r, i_s = 1, 2 \).

In this orthogonal case, the principal member of the normal equation for the general mean \( \theta(\phi) \) is given by

\[ n\theta(\phi) + \sum_{k=3}^{m} \sum_{V^1 \cup V^2 = k} \gamma(V^1 V^2) \theta(V^1 V^2) = d'(\phi)y(T), \]
\[ \text{or } = \sum_{\alpha=1}^{n} y(j_1^{(\alpha)}, j_2^{(\alpha)}, \ldots, j_m^{(\alpha)}). \]

The principal member of the normal equation for the main effect \( \theta(p^r) \) is given by

\[ n\theta(p^r) + \sum_{\{q,r\}(q,r \neq p) \ i_q, i_r = 1,2} \gamma(p^r q^q r^r) \theta(q^q r^r) \]
\[ + \sum_{k=3}^{m} \sum_{V^1 \cup V^2 = k} \varepsilon(p^r q^q, V^1 V^2) \theta(V^1 V^2) = d'(p^r) y(T), \]
\[ \text{or } = \sum_{\alpha=1}^{n} d_{j_p^{(\alpha)}} y(j_1^{(\alpha)}, j_2^{(\alpha)}, \ldots, j_m^{(\alpha)}), \]
for \( p \in \Omega, i_p = 1, 2 \).

The principal member corresponding to the two-factor interaction \( \theta(p^r q^q) \) is given by

\[ \sum_{r \neq p, q} \sum_{i_r = 1,2} \gamma(p^r q^q r^r) \theta(r^r) + n\theta(p^r q^q) \]
\[ + \sum_{\{r,s\} \neq \{p,q\} \ i_r, i_s = 1,2} \varepsilon(p^r q^q, r^r s^s) \theta(r^r s^s) \]
\[ + \sum_{k=3}^{m} \sum_{V^1 \cup V^2 = k} \varepsilon(p^r q^q, V^1 V^2) \theta(V^1 V^2) = d'(p^r q^q) y(T), \]
\[ \text{or } = \sum_{\alpha=1}^{n} d_{j_p^{(\alpha)}} d_{j_q^{(\alpha)}} y(j_1^{(\alpha)}, j_2^{(\alpha)}, \ldots, j_m^{(\alpha)}), \]
for \( p \neq q \in \Omega, i_p, i_q = 1, 2 \).
If three-factor or more interactions are assumed to be negligible, those equations (3.1), (3.2) and (3.3) may be simplified as follows:

\[
\begin{align*}
\text{(3.4)} & \quad n\theta(\phi) = d'(\phi)y(T), \\
\text{(3.5)} & \quad n\theta(p^\gamma) + \sum_{\{q,r\}(q,r\neq p)} \sum_{i_q,i_r=1,2} \gamma(p^\theta q^\gamma r^r)\theta(q^\gamma r^r) = d'(p^\gamma)y(T), \\
\text{for } \theta(p^\gamma), \ p \in \Omega, \ i_p = 1, 2, \text{ and} \\
\text{(3.6)} & \quad \sum_{\{r,s\}(r\neq p,q)} \sum_{i_r,i_s=1,2} \varepsilon(p^\theta q^\gamma r^r)\theta(r^r s^i) + n\theta(p^\theta q^\gamma) \\
& \quad + \sum_{\{r,s\}(r\neq p,q)} \sum_{i_r,i_s=1,2} \varepsilon(p^\theta q^s, r^i r^s)\theta(r^i s^i) = d'(p^\theta q^\gamma)y(T), \\
\text{for } \theta(p^\theta q^\gamma), \ p \neq q \in \Omega, \ i_p, i_q = 1, 2.
\end{align*}
\]

It can be seen that in estimating the main effects \(\theta(p^\gamma)\)'s and the two-factor interactions \(\theta(p^\theta q^\gamma)\)'s using principal equations (3.5) and (3.6), the estimates may be more or less confounded by several effects, i.e., the estimate of a main effect may be partially confounded by at most \(4 \times (m-1)\) two-factor interactions and that of a two-factor interaction may be partially confounded by \(2 \times (m-2)\) main effects and \(2m(m-1)-4\) two-factor interactions.

With respect to the confounding coefficient of a two-factor interaction to the main effect to be estimated and that to the two-factor interaction to be estimated, the following theorem shows that,

\[
|\gamma(p^\theta q^\gamma r^r)|/n \leq 1, \text{ and } |\varepsilon(p^\theta q^s, r^i r^s)|/n \leq 1,
\]

hold true, respectively.

**Theorem 4.** The absolute value of the coefficient \(\gamma(p^\theta q^\gamma r^r)\) of \(\theta(q^\gamma r^r)\) in equation (3.5) is bounded by \(n\). The absolute values of the coefficients \(\gamma(p^\theta q^s, r^i r^s)\) of \(\theta(r^i s^i)\) and \(\varepsilon(p^\theta q^s, r^i r^s)\) of \(\theta(r^i s^i)\) in (3.6) are also bounded by \(n\), respectively.

**Proof.** In proving the theorem, it is sufficient to show the following:

\[
\begin{align*}
(\gamma(p^\theta q^\gamma r^r))^2 &= \left(\sum_{\alpha=1}^{n} d_{j_p}^{(\alpha)_{i_p}} d_{j_q}^{(\alpha)_{i_q}} d_{j_r}^{(\alpha)_{i_r}}\right)^2 \\
&\leq \left(\sum_{\alpha=1}^{n} d_{j_p}^{2(\alpha)_{i_p}}\right)\left(\sum_{\alpha=1}^{n} d_{j_q}^{2(\alpha)_{i_q}} d_{j_r}^{2(\alpha)_{i_r}}\right) = n^2, \text{ and} \\
(\varepsilon(p^\theta q^s, r^i r^s))^2 &= \left(\sum_{\alpha=1}^{n} d_{j_p}^{(\alpha)_{i_p}} d_{j_q}^{(\alpha)_{i_q}} d_{j_r}^{(\alpha)_{i_r}} d_{j_s}^{(\alpha)_{i_s}}\right)^2 \\
&\leq \left(\sum_{\alpha=1}^{n} d_{j_p}^{2(\alpha)_{i_p}} d_{j_q}^{2(\alpha)_{i_q}}\right)\left(\sum_{\alpha=1}^{n} d_{j_r}^{2(\alpha)_{i_r}} d_{j_s}^{2(\alpha)_{i_s}}\right) = n^2.
\end{align*}
\]
§4. 18-run orthogonal 3⁴ factorial designs

An orthogonal $n$-run 3⁴ factorial design can be provided by a three-symbol orthogonal array, 3-OA($t, m, \lambda$), of size $n$, $m$ constraints, strength $t = 2$ and index $\lambda$, where $n = 9\lambda$.

The class of three-symbol orthogonal arrays of strength $t$ having $m = t + 2$ and index $\lambda = 2$ (3-OA($t, m = t + 2, \lambda = 2$)) has been investigated by Yamamoto, Fujii and Mitsuoka [4]. They have shown that the number of all possible 3-OA(2,4,2)’s amounts to 31,356 and these arrays are classified into 12 cosets with respect to the group of the symbol (level) and column (factor) permutations. In the case of 3-OA(3,5,2)’s the number amounts to 62,944 and these arrays are classified into 4 cosets. In their subsequent paper [5], the class of three-symbol orthogonal arrays of strength 2 and index 2 having maximal or saturated ($m = 7$) constraints have been investigated and it has been shown that there are three nonisomorphic classes of 3-OA($t=2, m=7, \lambda=2$).

Representative arrays of the 3 cosets of 3-OA($t=2, m=7, \lambda=2$) (labeled as [A], [B], [C]) and those of 12 cosets of 3-OA($t=2, m=4, \lambda=2$) (labeled as (1), (2), . . . , (12)) will be referred to here in Table 1.
Table 1. Representatives of the three isomorphic classes of saturated 3-OA($t=2$, $m=7$, $\lambda=2$) and twelve classes of 3-OA($t=2$, $m=4$, $\lambda=2$)

<table>
<thead>
<tr>
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<th>[B]</th>
<th>[C]</th>
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<td>0200112</td>
</tr>
<tr>
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<td>1010120</td>
<td>1010120</td>
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</tr>
<tr>
<td>2212010</td>
<td>2212010</td>
<td>2212010</td>
</tr>
</tbody>
</table>

(1) (2) (3) (4) (5) (6) (7) (8) (9) (10) (11) (12)
0022 0022 0022 0022 0022 0022 0022 0022 0022 0022 0022 0022
0111 0111 0110 0110 0110 0110 0110 0110 0110 0110 0110 0110
0111 0111 0111 0111 0112 0112 0112 0112 0112 0112 0112 0112
0200 0200 0200 0200 0200 0200 0200 0200 0200 0200 0200 0200
0200 0200 0201 0201 0211 0201 0201 0201 0201 0201 0201 0201
1010 1001 1001 1000 1001 1010 1000 1000 1000 1000 1000 1000
1010 1010 1011 1011 1010 1010 1010 1010 1010 1010 1010 1010
1102 1102 1102 1102 1102 1102 1102 1102 1102 1102 1102 1102
1102 1120 1120 1121 1121 1121 1121 1121 1121 1121 1121 1121
1221 1212 1212 1212 1212 1212 1212 1212 1212 1212 1212 1212
1212 1220 1220 1220 1220 1220 1220 1220 1220 1220 1220 1220
2120 2102 2102 2102 2102 2102 2102 2102 2102 2102 2102 2102
2120 2120 2120 2120 2120 2120 2120 2120 2120 2120 2120 2120
2212 2212 2212 2212 2212 2212 2212 2212 2212 2212 2212 2212
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After editing the results appearing in Yamamoto, Fujii and Mitsuoka [5], the possibility of embedding those 12 classes of 3-OA($t=2$, $m=4$, $\lambda=2$) into those 3 classes of saturated 3-OA($t=2$, $m=7$, $\lambda=2$) having maximal constraints can be summarized in the following Table 2. This table, of course, shows the possibility of deriving the former from the latter.
Table 2. Possibility of deriving 3-OA($t=2, m=4, \lambda=2$)
from saturated 3-OA($t=2, m=7, \lambda=2$)

<table>
<thead>
<tr>
<th>3-OA($t=2, m=4, \lambda=2$)</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
<th>(6)</th>
<th>(7)</th>
<th>(8)</th>
<th>(9)</th>
<th>(10)</th>
<th>(11)</th>
<th>(12)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3-OA($t=2, m=7, \lambda=2$)</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
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<td>$\circ$</td>
<td>$\circ$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$\lambda=2$ [C]</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\circ$</td>
<td>$\circ$</td>
<td>$\circ$</td>
<td>$\circ$</td>
<td>$\circ$</td>
<td>$\circ$</td>
<td></td>
</tr>
</tbody>
</table>

(\circ: possible, $\times$: impossible)

With respect to each of the 18-run orthogonal 3$^4$ factorial designs provided by the 12 representative arrays (1), (2), ..., (12), the loading vectors, the characteristic vector and the information matrix are calculated. Those 12 information matrices $M(2, T)$, under the assumption that three or more factor interactions are negligible, will be given in Table 3.

The number of two-factor interactions actually confounded with (though partially) the main effect to be estimated by the principal equation is at most 12 in this case. The number, however, varies from a main effect to another and from a design to another. The largest is 12 which can be seen in the design (1) and some of others and the smallest is 4 which can be seen in the design (7).

The average of the confounding coefficients among 12 two-factor interactions having the possibility of confounding also varies from a main effect to another and a design to another (see Table 3).

These considerations along Table 3 show that the design (11) and also the design (7) seem to be recommendable for the practical use.
References


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