A 2-local characterization of $M(22)$

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Abstract. Let $G$ be a finite group with $O(G) = 1$. We show that if $G$ contains a subgroup $A$ isomorphic to $E_{64}$ such that $C_G(A) = O(N_G(A)) \times A$, $N_G(A)/C_G(A) \cong Sp(6,2)$, and $N_G(A)/O(N_G(A))$ splits over $(O(N_G(A))A)/O(N_G(A))$, then either $G \cong M(22)$ or $G = N_G(A)$.

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§1. Introduction

This paper is concerned with a 2-local characterization of $M(22)$, Fischer’s group of order $2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$.

In the study of finite groups having a standard subgroup $L$ isomorphic to a group of Lie type with rank at least 4 over a field of characteristic 2, the case where $L \cong \Omega^+(8,2)$ is exceptional. In fact, when Seitz [18] tried to classify all such groups, the case where $L \cong \Omega^+(8,2)$ remained unsettled, and it was later settled in [7]. In [7], a 2-local characterization of $M(22)$, which is stated in [4], plays a crucial role. However, the proof given in [4] is erroneous. In fact, the 2-structure described in Sections 2 through 4 of [4] does not coincide with the actual 2-structure of $M(22)$. The purpose of this paper is to remedy the argument in [4], and prove the following theorem.

Main Theorem. Let $G$ be a finite group with $O(G) = 1$, and suppose that $G$ contains a subgroup $A$ isomorphic to $E_{64}$ such that $C_G(A) = O(N_G(A)) \times A$ and $N_G(A)/C_G(A) \cong Sp(6,2)$. Suppose further that $N_G(A)/O(N_G(A))$ splits over $(O(N_G(A))A)/O(N_G(A))$. Then either $G \cong M(22)$ or $G = N_G(A)$.

In passing, we mention that in Section 5 of [4], it is proved that if $N_G(A)/O(N_G(A))$ does not split over $(O(N_G(A))A)/O(N_G(A))$, then $G = N_G(A)$.
Our notation is standard possibly except for the following:

- $E(X)$: the product of the quasisimple subnormal subgroups of $X$,
- $X^\infty$: the final term of the derived series of $X$,
- $X \wr Y$: the wreath product of $X$ by $Y$,
- $X * Y$: a central product of $X$ and $Y$,
- $Z_n$: the cyclic group of order $n$,
- $E_n$: the elementary abelian group of order $n$,
- $D_8$: the dihedral group of order 8,
- $Q_8$: the quaternion group,
- $\Sigma_n$: the symmetric group of degree $n$,
- $\Sigma'_n$: the alternating group of degree $n$,
- $\Gamma L(2, 4) = Z_3 \times SL(2, 4)$ together with an automorphism of order 2 inverting $Z_3$ and inducing $\text{Aut}(SL(2, 4))$ on $SL(2, 4)$,
- $\Gamma U(4, 2) = Z_3 \times SU(4, 2)$ together with an automorphism of order 2 inverting $Z_3$ and inducing $\text{Aut}(SU(4, 2))$ on $SU(4, 2)$,
- $GF(q)$: the field with $q$ elements,
- $M_n(q)$: the set of $n \times n$ matrices with entries in $GF(q)$.

If we write $X = YZ$, it means that $Y \triangleleft X$ and $X = \langle Y, Z \rangle$. If $Y \cap Z = 1$ and if an emphasis is to be placed on that fact, then we write $X = Y \cdot Z$.

If $X$ is a 2-group, then by $J(X)$, we denote the usual Thompson subgroup generated by the abelian subgroups of maximum order.

Let $G$ be a group isomorphic to $\Sigma_5'$ (resp. $\Sigma_5$). Suppose that $G$ acts on a group $V$ isomorphic to $E_{16}$. If the order of the centralizer in $V$ of an element of order 3 of $G$ is 4, then we refer to this action as a “standard action as $\Omega^-(4, 2)$ (resp. $O^-(4, 2)$).” If the action of an element of order 3 of $G$ is fixed-point-free, then we refer to this action as a “standard action as $SL(2, 4)$ (resp. $\text{Aut}(SL(2, 4))$).”

We use the “bar” convention for homomorphic images. Thus if $G$ is a group, $N$ is a normal subgroup and $\bar{G}$ denotes the factor group $G/N$, then, for any subset $X$ of $G$, $\bar{X}$ denotes the image of $X$ under the natural projection $G \to \bar{G}$. Similarly we use the “double bar” convention and the “tilde” convention.

§2. Preliminary Results

In this section, we collect a number of preliminary lemmas to be used in later sections.

The first two lemmas are easy to verify and their proofs are omitted.

**Lemma 2.1.** Let $z$ be an involution actiong on a group $Y$, and let $a$ be an element of $Y$ such that $[a, z]$ is an involution. Then $z$ centralizes $[a, z]$.
Lemma 2.2. Let \( x \) be an involution acting on an elementary abelian 2-group \( C \). Then the following hold.

(1) \(|C_C(x)|^2 \geq |C|\).

(2) If \( A \) is a subgroup of \( C_C(x) \), then \(|C_{C/A}(x)| \geq |[C, x]|\).

(3) If \( A \) is an \( x \)-invariant subgroup of \( C \), then \(|C_C(x)| \geq |C_{C/A}(x)|\).

Lemma 2.3. Let \( R \) be a Sylow 2-subgroup of a group \( G \), and let \( C \) be an abelian subgroup of \( R \) which is weakly closed in \( R \) with respect to \( G \). Let \( \bar{R} = R/C \), \( \Gamma = \{ E \mid E \text{ is a subgroup of } R \text{ not contained in } C \text{ such that } E^g \subseteq C \text{ for some } g \in G \} \), \( \gamma' = \max\{|E| \mid E \in \Gamma\} \).

(1) If \( E \in \Gamma \), then there exists \( g \in G \) such that \( E^g \subseteq C \) and \( N_R(E)^g \subseteq R \).

(2) Suppose that \( x \) is an involution of \( R \) such that \( \langle x \rangle \in \Gamma \). Then \(|C, x| \leq \gamma'^2\)
and, if \( C \) is elementary abelian, then \(|[C, x]| \leq \gamma'\).

Throughout the rest of the statement of this lemma, we assume that \( C \) is elementary abelian, and set \( \gamma = \max\{|E| \mid E \in \Gamma\} \).

(3) If \( E \in \Gamma \) and \(|E| = \gamma \), then \(|C_{C/(C \cap E)}(\bar{x})| \leq |E| \) for every involution \( \bar{x} \) of \( E \).

(4) If \( E \in \Gamma \) and \(|E| = \gamma \), then \(|C/(C \cap E)| \leq |\bar{E}|^2\).

(5) If \( E \in \Gamma \) and \(|E| = \gamma \), then \(|[C, \bar{x}]| \leq |\bar{E}| \) for every involution \( \bar{x} \) of \( \bar{E} \).

(6) If \( E \in \Gamma \) and \(|E| = \gamma \), then \(|C/(C \cap E)|[C, \bar{x}]| \leq |E| \) for every involution \( \bar{x} \) of \( \bar{E} \).

Proof. Statement (1) is (9.3) of Goldschmidt [9], and (2) is Corollary 4 (2) of [9]. Statement (3) follows from (1), and an equivalent statement can be found in the proof of Corollary 4 (1) of [9]. Now (4) follows from (3) and Lemma 2.2(1), and is essentially the same as Corollary 4 (1) of [9]. Similarly (5) follows from (3) and Lemma 2.2(2), and (6) follows from (3) and Lemma 2.2(3). \( \square \)

Lemma 2.4. Let \( F \) be a special 2-group of order \( 2^{2n+1} \) with a subgroup \( B \) such that \( Z(F) = \Phi(F) \cong E_{2^n} \), \( Z(F) \subseteq B \cong E_{2^{n+1}} \) and \([B,F] = Z(F)\). Set \( \bar{F} = F/Z(F) \), and \( \bar{F} = F/B \). Suppose that an involution \( x \) acts on \( F \) and \( B \) is \( x \)-invariant. Then \( C_{\bar{F}}(x) = \bar{C}_F(x) \).

Proof. From the assumption that \([B,F] = Z(F)\), it follows that for each element \( y \) of \( Z(F) \), there exists an element \( \bar{t}(y) \) of \( F \) such that \([B,\bar{t}(y)] = \langle y \rangle\), and such an element is uniquely determined modulo \( B \). Note that the bijection between \( Z(F) \) and \( \bar{F} \) which associates \( \bar{t}(y) \) with \( y \) is an \( \bar{x} \)-isomorphism. Let
Let $C$ be an involution of $B - Z(F)$. Now suppose that the lemma is false. Then there exists an element $y$ of $C_{Z(F)}(x)$ such that $[t(y), x] = \bar{z}$. By Lemma 2.1, $t(y) \notin [F, x]$. Hence $y \notin [Z(F), x]$. By a suitable choice of $z$, we may assume $[t(y), x] = z$. Then again by Lemma 2.1, $[z, x] = 1$. Calculating in the semi-direct product $F \cdot \langle x \rangle$, we get $1 = [(t(y)x)^2, t(y)x] = [t(y)^2z, t(y)x] \in y[Z(F), x]$. Since $y \notin [Z(F), x]$, this is a contradiction. 

Lemma 2.5. Let $P$ be a group. Set $C = O_2(P)$, and suppose that the following three conditions hold.

1. $\Phi(\Phi(C)) = 1$.
2. $C/\Phi(C)$ is isomorphic to $\Phi(C)$ as a $P$-module.
3. Either
   
   (3.1) there exists a $P$-orbit $\{a^P\}$ of $\Phi(C)$ such that $C_{\Phi(C)}(C_P(a)) = \langle a \rangle$ and $\Phi(C) = \langle x \rangle$ both $x$ and $ax$ are in $\{a^P\}$; or
   
   (3.2) $\Phi(C) \cong E_{16}$ and $P/C \cong E_9$, and $P/C$ acts faithfully on $C/\Phi(C)$.

Then $C$ is homocyclic abelian of exponent 4.

Proof. If (3.1) is satisfied, then this is virtually the same as Proposition 1.4 of [6]. Thus assume (3.2) holds, and let $K$ be a complement of $C$ in $P$. Let $\overline{C} = C/\Phi(C)$. Write $K = L \times M$ with $[C, L] \cong [C, M] \cong E_4$. Since $\Phi(C) \subseteq Z(C)$ by (2) and since $||\overline{C}, L|| = 4$, $||[C, L]'|| \leq 2$. Since $[C, L]'$ is $K$-invariant, this means $[C, L]' = 1$. Therefore $[C, L] = C_{[C, L]}(L) \times [C, L]$. Since $[C, L, L] = [C, L]$, this implies $C_{[C, L]}(L) = 1$, and hence $||[C, L]|| = 16$. Similarly $||[C, K]|| = 16$. Consequently $[C, L] \cap [C, K] = 1$. Since $[C, L]$ and $[C, K]$ are both normal in $C$, the desired conclusion immediately follows from this. 

Lemma 2.6. Let $A$ be a vector space of even dimension over $GF(2)$ with a quadratic form.

1. Assume either that the dimension is greater than or equal to 4 or that the quadratic form is of minus type. Then $A = \langle x \mid x \text{ is non-singular} \rangle$.
2. Assume either that the dimension is greater than or equal to 4 or that the quadratic form is of plus type. Then $A = \langle x \mid x \text{ is singular} \rangle$.

Proof. This is easy to verify. 

Lemma 2.7. Let $G = Z_2^l \Sigma_n^l$, $n \geq 5$. Let $P = (G/Z(G))'$ (thus $P \cong E_{2n-1} \cdot \Sigma_n^l$ or $E_{2n-2} \cdot \Sigma_n^l$ according as $n$ is odd or even). Then there are one or two classes of complements of $O_2(P)$ in $P$ according as $n$ is odd or even (note that in the
Suppose that $M$ that $C$ act on $C$ in a standard way. Then the following hold for every elementary abelian subgroup $E$ of order 8 of $M$.

(1) $|C_C(E)| \leq 4$.

(2) $[C, x] \subseteq C_C(E)$ for every involution $x$ of $E$ such that $|[C, x]| = 2$.

As argued in [1] and [2], we obtain the following two lemmas.

**Lemma 2.9.** Let $M = \text{Aut}(\text{SU}(4, 2))$. Let $a$ be an involution of $M - M'$ such that $C_M(a) \cong \text{Sp}(4, 2)$. Set $D = C_M(a)$. Let $C = \langle e_i, f_i \mid 3 \leq i \leq 6 \rangle \cong E_{256}$. Suppose that $M$ act on $C$, $[f_i, a] = e_i$ for each $i$, $\langle e_i \mid 3 \leq i \leq 6 \rangle$ are $D$-invariant, and $D$ acts on $\langle e_i \mid 3 \leq i \leq 6 \rangle$ as $\text{Sp}(4, 2)$ so that it fixes the alternating form $\theta$ defined by $\theta(e_i, e_j) = \delta_{i,j} - \delta_{i,-j}$, with $\delta_{i,j}$ is Kronecker’s delta. Set $I = C_D((e_3, e_4))$. Let $J$ be an elementary abelian subgroup of order 16 of $M'$ such that $J \supseteq I$, and set $N = N_M(J)$. Then the following hold.

(1) $M$ has four classes of involutions, and $|[C, x]| \geq 4$ for every involution $x$ of $M$.

(2) For every noncentral involution $x$ of $M$, $|[C, x]| = 16$; for every central involution $x$ of $M$, $|[C, x]| = 4$.

(3) If $E$ is an elementary abelian subgroup of order 4 of $M$, then $E$ contains a noncentral involution, and hence $|C_C(E)| \leq 16$ by (2).

(4) $O_2(N) = J$, and $N/O_2(N) \cong \Sigma_5$.

(5) $[C, O_2(N)] = C_C(O_2(N)) = \langle e_3, f_3, e_4, f_4 \rangle$, and $N/O_2(N)$ acts as $\text{Aut}(\text{SL}(2, 4))$ in a standard way on both $\langle e_3, f_3, e_4, f_4 \rangle$ and $C/\langle e_3, f_3, e_4, f_4 \rangle$.

(6) $N/O_2(N)$ acts on $O_2(N)$ as $O^-(4, 2)$ in a standard way. Hence by Lemma 2.6, $O_2(N) = \langle x \in O_2(N) \mid x \text{ is noncentral} \rangle$ and $O_2(N) = \langle x \in O_2(N) \mid x \text{ is central} \rangle$.
(7) \(O_2(N)\) is the unique elementary abelian subgroup of order 16 of \(N'.\)

(8) If \(E\) is an elementary abelian subgroup of order 16 of \(M\) containing \(a\), then \(E \leq \langle a \rangle \times D\) and \(C_C(E) \subseteq C_C(a) = \langle e_i \mid 3 \leq i \leq 6 \rangle\), and hence by Lemma 2.8, \(|C_C(E)| \leq 4\), and \(|\langle C_C(E), [C, x]\rangle| \leq 8\) for every central involution \(x\) contained in \(E\).

(9) If \(E\) is an elementary abelian subgroup of order 16 of \(M\) not contained in \(M'\), then \(E\) contains an involution conjugate to \(a\), and hence by (8), \(|C_C(E)| \leq 4\), and \(|\langle C_C(E), [C, x]\rangle| \leq 8\) for every central involution \(x\) contained in \(E\).

(10) For each element \(v\) of \(C = \langle e_3, f_3, e_4, f_4 \rangle\), \(|v, O_2(N)| = 8\).

(11) There exist elements \(e_i', f_i', e_6', f_6'\) of \(C\) with \(e_i'\langle e_3, f_3, e_4, f_4 \rangle = e_i\langle e_3, f_3, e_4, f_4 \rangle\) and \(f_i'\langle e_3, f_3, e_4, f_4 \rangle = f_i\langle e_3, f_3, e_4, f_4 \rangle\) for each \(i = 5, 6\) such that if we regard \(C\) as a vector space of dimension 4 over GF(4) such that an element \(\alpha(\neq 0, 1)\) of GF(4) acts so that \(e_i'^\alpha = f_i\) for each \(i\), and define a hermitian form \(\varphi\) by \(\varphi(e_i', e_j') = \delta_{i,9-j}\) (here for \(i = 3, 4\), we let \(e_i' = e_i\) and \(f_i' = f_i\)), then \(M\) acts on \(C\) in a standard way.

**Lemma 2.10.** Let \(M = \text{Aut}(\text{GL}(4, 2))\). Let \(a\) be an involution of \(M - M'\) such that \(C_M(a) \cong \text{Sp}(4, 2)\). Set \(D = C_M(a)\). Let \(C = \langle e_i, f_i \mid 3 \leq i \leq 6 \rangle \cong E_{256}\). Suppose that \(M\) act on \(C\), \([f_i, a] = e_i\) for each \(i\), \(\langle e_i \mid 3 \leq i \leq 6 \rangle\) and \(\langle f_i \mid 3 \leq i \leq 6 \rangle\) are \(D\)-invariant, and \(D\) acts on \(\langle e_i \mid 3 \leq i \leq 6 \rangle\) as \(\text{Sp}(4, 2)\) so that it fixes the alternating form \(\theta\) defined by \(\theta(e_i, e_j) = \delta_{i,9-j}\), where \(\delta_{i,9-j}\) is Kronecker’s delta. Set \(I = C_D(\langle e_3, e_4 \rangle)\). Let \(J\) be an elementary abelian subgroup of order 16 of \(M'\) such that \(J \supseteq I\), and set \(N = N_{M,J}\). Then the following hold:

(1) \(M\) has four classes of involutions, and \(|\langle C, x\rangle| \geq 4\) for every involution \(x\) of \(M\).

(2) For every noncentral involution \(x\) of \(M\), \(|\langle C, x\rangle| = 16\); for every central involution \(x\) of \(M\), \(|\langle C, x\rangle| = 4\).

(3) If \(E\) is an elementary abelian subgroup of order 4 of \(M\), then \(|C_C(E)| \leq 32\); if \(E\) is an elementary abelian subgroup of order 8 of \(M\), then \(|C_C(E)| \leq 16\).

(4) \(O_2(N) = J\), and \(N/O_2(N) \cong \text{GL}(2, 2) \times \mathbb{Z}_2 \cong O^+(4, 2)\).

(5) \([C, O_2(N)] = C_C(O_2(N)) = \langle e_3, f_3, e_4, f_4 \rangle\), and \(N/O_2(N)\) acts on \(\langle e_3, e_4, f_3, f_4 \rangle\), \(C/\langle e_3, e_4, f_3, f_4 \rangle\), and \(O_2(N)\) in a standard way (note that the standard action of \(\text{GL}(2, 2) \times \mathbb{Z}_2\) is the same as that of \(O^+(4, 2)\)). Hence by Lemma 2.6, \(O_2(N) = \langle x \in O_2(N) \mid x\) is noncentral \rangle\) and \(O_2(N) = \langle x \in O_2(N) \mid x\) is central \rangle\).
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(6) If $E$ is an elementary abelian subgroup of order 16 of $N$ with $E \neq O_2(N)$, then $|C_C(E)| \leq 8$.

Lemma 2.11. Let $N$ be a group with $N/O(N) \cong \text{PSL}(3,4)$. Suppose that $N$ acts faithfully and irreducibly on an elementary abelian 2-group $J$ with $2^8 \leq |J| \leq 2^{10}$. Let $s,t$ be elements of $N$ such that $sO(N)$ has order 5 in $N/O(N)$ and $tO(N)$ has order 3 in $N/O(N)$, and suppose that $|[J,s]| = 2^8$ and $|[J,t]| = 2^6$. Then $|J| = 512$ and $N \cong \text{PSL}(3,4)$.

Proof. Let

$$O(N) = P_0 \supseteq P_1 \supseteq P_2 \supseteq \cdots \supseteq P_k = 1$$

be a characteristic composition series of $O(N)$ (namely, for each $i$, $P_{i-1}$ is a minimal characteristic subgroup of $O(N)$ properly containing $P_i$). In general, if $p$ is an odd prime, then the 2-rank of $\text{GL}(n,p)$ is $n$ and every elementary abelian 2-subgroup of order $2^n$ contains the involution of the center of $\text{GL}(n,p)$. On the other hand, since $N$ is isomorphic to a subgroup of $\text{GL}(10,2)$, the rank of $P_{i-1}/P_i$ is less than or equal to 4 for each $i$, and hence $N^\infty$ acts on $P_{i-1}/P_i$ trivially. By the three-subgroup lemma, $[O(N), N^\infty] = [O(N), N^\infty, \ldots, N^\infty, N^\infty] = 1$. Thus $N^\infty \cong \text{PSL}(3,4)$ or $\text{SL}(3,4)$. Now we obtain the desired conclusion by arguing as in [1] and [2].

The following lemma also follows from [1] and [2].

Lemma 2.12. Let $N,J$ be as in Lemma 2.11, and let $R \in \text{Syl}_2(N)$. Then the following hold.

(1) $R$ contains precisely two elementary abelian subgroups $A,B$ of order 16.

(2) $N$ acts on $J$ irreducibly.

(3) The order of the centralizer in $J$ of one of $A$ or $B$ is 16, while that of the other is 2. We choose our notation so that $|C_J(A)| = 16$ and $|C_J(B)| = 2$.


(5) $N$ has only one class of involutions, and $|[J,x]| = 16$ for every involution $x$ of $N$.

Lemma 2.13. Let $N$ be a group such that $N/O(N) \cong M_{22}$. Suppose $N$ acts faithfully on an elementary abelian group $J$ of order 1024. Then $N \cong M_{22}$, and the lengths of the $N$-orbits of involutions of $J$ are either

(1) 22,231 and 770; or

(2) 77,330 and 616.
Proof. Arguing as in Lemma 2.12, we obtain \( N \cong M_{22} \) or \( 3M_{22} \). Hence the desired conclusion follows from Hunt [15] and Smith [19]. \( \square \)

Lemma 2.14. Let \( N = M_{22} \), and let \( N \) act faithfully on an elementary abelian group \( J \) of order 1024. Suppose the action is the same as the one described in Lemma 2.13(1). Let \( R \in \text{Syl}_2(N) \). Then

1. \( R \) contains precisely two elementary abelian subgroups \( A, B \) of order 16. We choose our notation so that \( N_N(A)/A \cong \Sigma_6' \) and \( N_N(B)/B \cong \Sigma_5 \). \( N \) has only one class of involutions.

2. Let \( a, b \) and \( c \) be elements of \( J \) such that \( |\{a^N\}| = 22, |\{b^N\}| = 231 \) and \( |\{c^N\}| = 770 \), respectively. Then \( C_N(a) \cong \text{PSL}(3,4) \), \( C_N(b) \cong E_{16} \cdot \Sigma_5 \) and \( C_N(c) \cong E_{16} \cdot (E_9 \cdot Z_4) \). We choose our notation so that \( O_2(C_N(b)) = B \) and \( O_2(C_N(c)) = A \).

3. \( |C_J(x)| = 64 \) for every involution \( x \) of \( N \).

4. \( |C_J(A)| = 32 \), \( ||C_J(A), N_N(A)|| = 16 \), and \( N_N(A) \) acts on \( C_J(A) \) indecomposably. \( ||C_J(A), x|| = 4 \) for every involution \( x \) of \( N_N(A) - A \).

5. All involutions of \( [C_J(A), N_N(A)] \) are conjugate to \( b \) under the action of \( N \). Six of the involutions of \( C_J(A) - [C_J(A), N_N(A)] \) are conjugate to \( a \). The remaining ten involutions of \( C_J(A) - [C_J(A), N_N(A)] \) are conjugate to \( c \).

6. Let \( X \) be the set of pairs \( (x, y) \) of involutions of \( C_J(A) \) such that \( x \in \{b^N\} \) and \( y \in \{c^N\} \). Then \( X \) splits into two \( N(N(A))-\)orbits.


8. Under the action of \( C_N(a) \), the involutions of \( J/A \) split into three orbits.

Proof. Statements(1) and (2) are well-known, and (3) follows from (2) by a counting argument. By (3), \( |C_J(A)| \leq 64 \). By (2), \( C_N(x) \) contains some conjugate of \( A \) for every involution \( x \) of \( J \) (see Lemma 2.12(1)), and hence \( C_J(A) \) contains some involution of each of the three orbits. In view of the action of \( N_N(A)/A \) on \( C_J(A) \), this implies \( |C_J(A)| \geq 32 \). If \( |C_J(A)| = 64 \), then by Lemma 2.7, at least one involution of \( C_J(A) \) is centralized by \( N_N(A) \), which contradicts (2) (note that arguing as in [1], we see that if \( \Sigma_6' \cong \text{Sp}(4,2)' \) acts faithfully and irreducibly on an elementary abelian 2-group \( I_0 \) of order at most 64, then \( |I_0| = 16 \), and the action is the standard action of \( \text{Sp}(4,2)' \), which is the same as the action described in Lemma 2.7). Thus \( |C_J(A)| = 32 \). The other assertions in (4) also follow from the fact that no involution of \( C_J(A) \) is
centralized by $N_N(A)$. Since all involutions of $[C_J(A), N_N(A)]$ are centralized by a Sylow 2-subgroup of $N_N(A)$, they belong to $\{b^N\}$. Since $C_J(A)$ splits into three orbits of involutions under the action of $N_N(A)$, different orbits under this action must correspond to different $N$-orbits. Let $y$ be an involution of $C_J(A) - [C_J(A), N_N(A)]$ such that $|\{y^{N_N(A)}\}| = 6$. Then since $5|C_{N_N(A)}(y)|$ and since $y$ does not belong to $\{b^N\}$, $y$ must belong to $\{a^N\}$. This also shows that an involution $z$ of $C_J(A) - [C_J(A), N_N(A)]$ such that $|\{z^{N_N(A)}\}| = 10$ belongs to $\{c^N\}$. Thus (5) is proved. Since $[C_J(A), N_N(A)]$ splits into two orbits of involutions with lengths 6 and 9 under the action of $C_{N_N(A)}(c) (= C_N(c))$, (6) follows. If we choose the element $a$ so that $C_N(a)$ contains both $A$ and $B$ and consider the action of $C_N(a)$ on $J/C$ then by Lemma 2.12(3), $A$ and $B$ of this lemma correspond to $A$ and $B$ of Lemma 2.12, respectively. Hence $|[J/(a), A]| = 256$. Therefore $|[J/C_J(A), A]| = 16$ and, if we let $dC_J(A)$ denote a fixed element of $J/C_J(A) - [J/C_J(A), A]$, then the bijection from $A$ to $[J/C_J(A), A]$ which associates $[dC_J(A), x] \in [J/C_J(A), A]$ with $x \in A$ gives an $(N_N(A)/A)$-isomorphism. Since the action of $N_N(A)$ on $A$ is irreducible, this also implies $[J/C_J(A), A] = C_{J/C_J(A)}(A)$. Finally (8) follows from (2), (3), (5) and Lemma 2.12.

§3. Notation and Initial Reduction

Throughout the rest of this paper, we let $G$, $A$ be as in the Main Theorem.

Let $Y$ be a complement of $A$ in $N_G(A)$, i.e., $N_G(A) = A \cdot Y$. Then $C_Y(A) = O(Y) = O(N_G(A))$ and $N_G(A)/O(N_G(A)) = (O(N_G(A))/A/O(N_G(A)) \cdot (Y/O(Y)) \cong E_{64} \cdot Sp(6, 2)$. Write $A = \langle e_i \mid 2 \leq i \leq 7 \rangle$ (we here let $i$ range from 2 to 7 so that the notation will be consistent with that in [7]). By a result of Dempwolff [5], the action of $Y/O(Y)$ on $A$ is uniquely determined. In particular, we may assume that $Y$ leaves invariant the alternating form $\theta$ on $A$ defined by $\theta(e_i, e_j) = \delta_{i,9-i}$, where $\delta_{i,9-j}$ is Kronecker’s delta.

We fix the following notation.

**Notation 1.** Let $f_2$ be an involution of $Y$ such that

$$[e_7, f_2] = e_2, \quad [e_k, f_2] = 1 \text{ for } k \neq 7.$$  

For $3 \leq i \leq 6$, let $f_i$ be an involution of $Y$ such that

$$[e_7, f_i] = e_i, \quad [e_9-i, f_i] = e_2,$$

$$[e_k, f_i] = 1 \text{ for } k \neq 7, 9-i.$$  

Let $g_3$ be an involution of $Y$ such that

$$[e_6, g_3] = e_3, \quad [e_k, g_3] = 1 \text{ for } k \neq 6.$$
For \(4 \leq i \leq 5\), let \(g_i\) be an involution of \(Y\) such that

\[
[e_6, g_i] = e_i, \quad [e_{9-i}, g_i] = e_3, \\
[e_k, g_i] = 1 \text{ for } k \neq 6, 9 - i.
\]

Let \(h\) be an involution of \(Y\) such that

\[
[e_5, h] = e_4, \quad [e_k, h] = 1 \text{ for } k \neq 5.
\]

We choose these involutions so that \(S = \langle f_i, g_j, h \mid 2 \leq i \leq 6, 3 \leq j \leq 5 \rangle\) is a Sylow 2-Subgroup of \(Y\).

**Lemma 3.1.** (1) Each involution of the coset \(f_2A\) is conjugate to either \(f_2\) or \(f_2e_3\) in \(N_G(A)\).

(2) Each involution of the coset \(f_3A\) is conjugate to either \(f_3\) or \(f_3e_4\) in \(N_G(A)\).

(3) Each involution of the coset \(f_2g_3A\) is conjugate to either \(f_2g_3\) or \(f_2g_3e_4\) in \(N_G(A)\).

(4) Each involution of the coset \(f_2g_3hA\) is conjugate to \(f_2g_3h\) in \(N_G(A)\).

**Proof.** This immediately follows from Notation 1.

Let \(C_1 = \langle e_i, f_j \mid 2 \leq i \leq 6, 3 \leq j \leq 6 \rangle\) and \(C = \langle C_1, f_2 \rangle\). Then \(C_1 = \langle e_3, f_6 \rangle \langle e_4, f_5 \rangle \langle e_5, f_4 \rangle \langle e_6, f_3 \rangle\), \(C = \langle f_2 \rangle \times C_1\), and \(\langle e_i, f_{9-i} \rangle \cong D_8\) for each \(3 \leq i \leq 6\). In the rest of this section, we determine possible structures for \(N_G(C)/C\) in the following sequence of lemmas.

**Lemma 3.2.** The group generated by all the involutions of \(\langle C, e_7 \rangle - C\) is \(A\).

**Proof.** Let \(e_7x\) be an involution of \(\langle C, e_7 \rangle - C\). Since \(C/e_2\) is elementary abelian, \(x(e_2) \in C_{C/(e_2)}(e_7) = \langle e_i, f_2 \mid 2 \leq i \leq 6 \rangle/e_2\), and hence \(x \in \langle e_i, f_2 \mid 2 \leq i \leq 6 \rangle\). But \(\langle e_i, f_2 \mid 2 \leq i \leq 6 \rangle\) is also elementary abelian, and hence \(x \in C_{\langle e_i, f_2 \mid 2 \leq i \leq 6 \rangle}(e_7) = \langle e_i \mid 2 \leq i \leq 6 \rangle\). Therefore \(x \in A\). On the other hand, \(A = \langle e_7x \mid x \in \langle e_i \mid 2 \leq i \leq 6 \rangle \rangle\). Thus the lemma is proved.

Set \(F = C_Y(\langle e_2, e_7 \rangle) \cap N_Y(\langle f_i \mid 2 \leq i \leq 6 \rangle)\). Then \(O(Y)/O(Y) = C_Y(\langle e_2, e_7 \rangle)/O(Y) \cong Sp(4, 2) \cong \Sigma_6\). Set \(M = N_G(C)\) and \(D = M \cap N_G(A)\). Then \(O(D) = O(F)\), and \([C, O(D)] = 1\) and \(D = C\cdot(\langle e_7 \rangle \times F)\). Set \(M = M/C\), then \(\bar{D}/O(\bar{D}) \cong Z_2 \times \Sigma_6\).

**Lemma 3.3.** \(C_{\overline{\Sigma}}(\overline{e_7}) = \overline{D}\)

**Proof.** This follows from Lemma 3.2.

Let \(\overline{H} = C_{\overline{\Sigma}}(C/Z(C))\).
Lemma 3.4. \(|P|\) is odd.

Proof. This is because \(|C_{\overline{P}}(e_7)| = |O(\overline{D})|\) is odd by Lemma 3.3.

Let \(\overline{\text{Aut}(C)} = \text{Aut}(C)/C_{\text{Aut}(C)}(C/Z(C))\). Then \(\overline{\text{Aut}(C)} \cong O^+(8, 2)\). We also let \(\overline{M} = M/H\), and regard \(\overline{M}\) as a subgroup of \(\overline{\text{Aut}(C)}\).

Lemma 3.5. \(\overline{M}/O(\overline{M}) \cong Z_2 \times \Sigma_6, \Sigma_8\) or \(\text{Aut}(SU(4, 2))\).

Proof. Since \(C_{\overline{M}}(\overline{e_7}) = C_{\overline{M}}(\overline{e_7}) = \overline{D} = \langle \overline{e_7} \rangle \times \overline{F}\) by Lemmas 3.4 and 3.3, \(C_{\overline{M}}(\overline{e_7}) \cong Z_2 \times \Sigma_6\). By Harris and Solomon [14], \(E(\overline{M}/O(\overline{M}))\) is isomorphic to one of the following groups:

1. \(\Sigma'_6\) or \(\Sigma'_6 \times \Sigma'_6\);
2. \(\Sigma'_6, SU(4, 2), SL(5, 2), SU(5, 2)\) or \(Sp(4, 4)\);
3. \(PSU(4, 3)\).

By considering the orders of these groups and \(O^+(8, 2)\), we can eliminate \(SL(5, 2), SU(5, 2), Sp(4, 4)\) and \(PSU(4, 3)\). In \(O^+(8, 2)\), no element of order 5 is centralized by a subgroup isomorphic to \(\Sigma'_6\) (see Frame [8]). Hence we can eliminate \(\Sigma'_6 \times \Sigma'_6\). Thus the lemma follows.

We examine the three cases of Lemma 3.5 separately in subsequent sections.

§4. Conjugacy Classes of Involutions

In this section and the next section, we assume that \(\overline{M}/O(\overline{M}) \cong \text{Aut}(SU(4, 2))\) and prove that \(G \cong M(22)\). The principal aim of this section is to show that \(G\) has three classes of involutions.

Lemma 4.1. \(\overline{M}\) is isomorphic to \(\text{Aut}(SU(4, 2))\) or \(\Gamma U(4, 2)\) \((\cong (Z_3 \times SU(4, 2)) \cdot Z_2)\).

Proof. Since 5 divides \(|SU(4, 2)|\) and \(|O(\overline{M})|\) divides \(3 \cdot 5 \cdot 7\), \(|O(\overline{M})C_{\overline{M}}(O(\overline{M}))|/O(\overline{M}) \geq E(\overline{M}/O(\overline{M}))\). Now the lemma follows from the class list of \(O^+(8, 2)\) ([8]).

Lemma 4.2. Every involution of \(\overline{M}\) is conjugate to some involution of \(\langle \overline{e_7} \rangle \times \overline{F}\) in \(\overline{M}\).

Proof. \(\overline{M}\) has 4 classes of involutions, and their representatives are \(\overline{g_3}\) (inner central), \(\overline{g_4}\) (inner non-central), \(\overline{e_7}\) (field) and \(\overline{g_3}(\text{field}) \times \text{inner}\), which are all contained in \(\langle \overline{e_7} \rangle \times \overline{F}\).
Having Lemma 2.9 in mind, we fix the following notation.

**Notation 2.** Let $x_1$ be an element of $M'$ such that $⟨g_3, g_4, h, x_1⟩ \cong E_16$. Then $N_M(⟨g_3, g_4, h, x_1⟩)/⟨g_3, g_4, h, x_1⟩$ is isomorphic to either $\text{Aut}(\text{SL}(2, 4))$ or $\Gamma L(2, 4)$ ($\cong (Z_3 \times \text{SL}(2, 4)) \cdot Z_2$). This factor group acts both on $(Z(C)⟨e_3, f_3, e_4, f_4⟩)/Z(C)$ and on $C/(Z(C)⟨e_3, f_3, e_4, f_4⟩)$ in a standard way as $\text{Aut}(\text{SL}(2, 4))$ or $\Gamma L(2, 4)$, whereas the action on $⟨g_3, g_4, h, x_1⟩$ is the same as that of $O^-(4, 2)$ on a standard module. Thus we can choose $x_1$ so that $g_3$, $g_4$, and $h x_1$ are noncentral involutions. Then $\overline{x_1}$ is conjugate to $g_3$. Since $g_4$ is an involution, we can choose $x_1$ as an involution. Let $x_2$ be an element of $M'$ such that

$$[e_4, x_2] \in f_4 Z(C), \quad [f_4, x_2] \in e_3 f_3 Z(C),$$

$$\langle e_3, f_3 \rangle, x_2 \subseteq Z(C).$$

We choose $x_2$ so that $⟨g_3, x_2⟩ \cong E_1$ and $x_2$ is an involution. Moreover, we choose $x_1$ and $x_2$ so that $⟨AS, x_1, x_2⟩$ is a Sylow 2-subgroup of $M$. Set $R = ⟨AS, x_1, x_2⟩$.

We prove $R \in \text{Syl}_2(G)$ in the following sequence of lemmas.

**Lemma 4.3.** If $x$ is an element of $M$ such that $x^2 \in ⟨e_2⟩$ and $\overline{x}$ is a noncentral involution of $E(\overline{M})$, then $C_C(x)$ contains an abelian subgroup of order 64.

**Proof.** By taking a suitable conjugate of $x$, we may assume $\overline{x} = \overline{g_3}$. Since $x^2 \in ⟨e_2⟩$, $x = g_4 y$, where $y ⟨e_2⟩ \in C_{C/⟨e_2⟩}(g_4) = ⟨e_i, f_i \mid 2 \leq i \leq 4⟩/⟨e_2⟩$. Consequently $C_C(x)$ contains $⟨e_i, f_i \mid 2 \leq i \leq 4⟩$, as desired. \hfill $\square$

**Lemma 4.4.** $C$ is weakly closed in $R$ with respect to $C_{C/⟨e_2⟩}$.

**Proof.** By way of contradiction, let $C_2$ be a subgroup of $R$ such that $C_2 \not\cong C$, $C_2 \cong C$, and $C_2' = ⟨e_2⟩$. Then $C_2/⟨e_2⟩ \cong E_{512}$. If $|C_2| = 2$, then, by Lemma 2.9(1), $|C_{C/⟨e_2⟩}(C_2)| \leq 2 \cdot 64$. This means that $|C_2/⟨e_2⟩| \leq |C_2| \cdot |C_{C/⟨e_2⟩}(C_2)| \leq 256$, which is a contradiction. If $4 \leq |C_2| \leq 8$, then by Lemma 2.9(3), $|C_{C/⟨e_2⟩}(C_2)| \leq 2 \cdot 16$, which leads to a similar contradiction. Thus $|C_2| = 16$. If $C_2 \neq ⟨g_3, g_4, h, x_1⟩$, then by (7) and (9) of Lemma 2.9, $|C_{C/⟨e_2⟩}(C_2)| \leq 2 \cdot 4$, which again leads to the same kind of contradiction. Consequently $C_2 = ⟨g_3, g_4, h, x_1⟩$. We also have $|C_2 \cap C| = 1024/16 = 64$. Since $|C_{C/⟨e_2⟩}(g_4)| \leq 2 \cdot 16 = 32$, this means that $C_2 \ni g_4$ and $C_2/⟨e_2⟩ \supseteq C_{C/⟨e_2⟩}(g_4)$. On the other hand, by Lemma 4.3, the group generated by $g_4$ and the inverse image of $C_{C/⟨e_2⟩}(g_4)$ contains an abelian subgroup of order 128. Since $C_2 \cong C \cong Z_2 \times (D_8 \ast D_8 \ast D_8 \ast D_8)$, this is a contradiction. \hfill $\square$
Lemma 4.5. \( R \in \text{Syl}_2(G) \).

Proof. Since \( Z(R) = \langle e_2 \rangle \), this immediately follows from Lemma 4.4. \( \square \)

We next determine \( J(R) \) and \( N_G(J(R)) \). Set \( J = \langle e_i, f_i, g_3, g_4, h, x_1 \mid 2 \leq i \leq 4 \rangle \). We prove \( J = J(R) \).

Lemma 4.6. \( J \cong E_{1024} \)

Proof. Let \( \tilde{M} = M/Z(C) \). Also set \( J_0 = \langle e_i, f_i \mid 2 \leq i \leq 4 \rangle \), \( J_1 = \langle J_0, g_3, g_4, h \rangle \) and \( J_2 = \langle J_0, g_3, h, x_1 \rangle \). Calculating in \( N_G(A) \), we see that \( J_1 \cong E_{512} \). We show that \( J_2 \cong E_{512} \). By Lemma 2.9(6), there exists \( y \in N_{\tilde{M}}(J_0) \) such that \( \tilde{y}^3 = \tilde{y}, \tilde{h} = h \) and \( \tilde{y}^4 = \tilde{y} \). Note that \( [\tilde{C}, g_3 h] = [\tilde{C}, \tilde{y}] = 0 \). Since both \( \tilde{y} \) and \( \tilde{x} \) are involutions, we get \( g_4^y \in x_1 J_0 \). Similarly \( (g_3 h)^y \in g_3 h J_0 \).

Since \( [\tilde{g}_3 \tilde{h}, \tilde{g}_3 \tilde{y}] = [\tilde{g}_3 \tilde{h}, \tilde{h}^y] = 1 \), we also obtain \( g_3^y \in g_3 J_0 \) and \( h^y \in h J_0 \). Consequently \( J_0^y = J_2 \) and hence \( J_2 \cong J_1 \cong E_{512} \). Note that this implies \( [J_0, g_4] = [J_0, x_1] = 1 \).

Now in order to prove \( J \cong E_{1024} \), it suffices to show that \( [g_4, x_1] = 1 \). Note that \( g_3 g_4 \) and \( g_4 h \) are noncentral involutions of \( \tilde{M} \), and \( \tilde{y}, \tilde{y}^4 \) are also noncentral involutions of \( \tilde{M} \). Hence by Lemma 2.9(6), there exists \( x \in N_{\tilde{M}}(J_0) \) such that \( \tilde{g}_3 \tilde{g}_4 = \tilde{g}_4 \) and \( \tilde{g}_3 \tilde{h} = \tilde{h} \). As before, we have \( (g_3 g_4)^y \in g_4 J_0 \) and \( (g_3 h)^y \in x_1 J_0 \). On the other hand, \( [g_3 g_4]^2, (g_3 h)^y \) is \( 1 \). Since \( [J_0, g_4] = [J_0, x_1] = 1 \), we now obtain \( [g_4, x_1] = [(g_3 g_4)^2, (g_3 h)^y] = 1 \), as desired. \( \square \)

Lemma 4.7. Let \( \tilde{M} = M/Z(C) \). Then \( \tilde{I} \subseteq \tilde{J} \) for every abelian subgroup \( \tilde{I} \) of \( \tilde{R} \) such that \( \tilde{I} = \tilde{J} \).

Proof. Let \( \tilde{I} \) be an abelian subgroup of \( \tilde{R} \) such that \( \tilde{I} = \tilde{J} \). First note that \( \tilde{I} \cap \tilde{C} \subseteq C_{\tilde{C}}(\langle g_3, g_4, h, x_1 \rangle) = \tilde{C} \cap \tilde{J} \). With each element \( x \) of \( \langle g_3, g_4, h, x_1 \rangle \), we associate an element \( \varphi(x) \) of \( C \) such that \( x \varphi(x) \in \tilde{I} \). Then for any elements \( x, y \) of \( \langle g_3, g_4, h, x_1 \rangle \), we have \( \varphi(x), y = \varphi(y), x \) because \( x \varphi(x), y \varphi(y) = 1 \). Now let \( a \) be an element of \( \langle g_3, g_4, h, x_1 \rangle \) such that \( \tilde{a} \) is a central involution of \( \tilde{E}(\tilde{M}) \).

Suppose that \( \varphi(a) \notin \tilde{C} \cap \tilde{J} \). Then, by Lemma 2.9(10), \( ||\varphi(a), \langle g_3, g_4, h, x_1 \rangle|| = 8 \). Since \( \varphi(y), \tilde{a} = \varphi(a), \tilde{y} \) must hold for every \( y \in \langle g_3, g_4, h, x_1 \rangle \), \( \tilde{C}, \tilde{a} \geq \varphi(a), \langle g_3, g_4, h, x_1 \rangle \). But since \( \tilde{a} \) is central, \( ||\tilde{C}, \tilde{a}|| = 4 \) by Lemma 2.9(2), a contradiction. Thus \( \varphi(a) \in \tilde{J} \). Since \( a \) is arbitrary, the desired conclusion follows from Lemma 2.9(6). \( \square \)

Lemma 4.8. \( J = J(R) \)
Proof. Let $I$ be an abelian subgroup of order at least 1024 of $R$. We have only to show that $I \subseteq J$. Arguing as in Lemma 4.4, we can easily show that $\bar{I} = \bar{J}$. Therefore $I \subseteq J$ by Lemma 4.7.

Lemma 4.9. $N_G(J)/C_G(J) \cong M_{22}$, and the action is the same as the one studied in Lemma 2.14.

Proof. Let $\overline{N_G(J)} = N_G(J)/C_G(J)$. Since $N_M(J)$ contains a Sylow 2-subgroup of $G$, $N_M(J)$ contains a Sylow 2-subgroup of $N_G(J)$. Note that $O_2(\overline{N_M(J)}) = \langle e_5, e_6, f_5, f_6 \rangle$, and $N_M(J)/O_2(\overline{N_M(J)})$ is isomorphic to either $\text{Aut}(\text{SL}(2, 4))$ or $\Gamma L(2, 4)$, where the action on $O_2(\overline{N_M(J)})$ is the same as that on a standard module. Hence a Sylow 2-subgroup of $N_G(J)$ is isomorphic to a Sylow 2-subgroup of $M_{22}$. Note also that $e_2 \sim e_3$ in $G$. Thus in view of Lemmas 4.8, 2.11 and 2.13, it follows from Gorenstein and Harada [11] that $\overline{N_G(J)} \cong M_{22}$. By Lemma 2.14(1), this implies $\overline{N_M(J)}/O_2(\overline{N_M(J)}) \cong \text{Aut}(\text{SL}(2, 4)) \cong \Sigma_5$ (so $\overline{M} \cong \text{Aut}(\text{SU}(4, 2))$). Suppose that the action of $\overline{N_G(J)}$ on $J$ is the same as the one described in Lemma 2.13(2). Then since $C_{\overline{N_G(J)}}(e_2) \supseteq \overline{N_M(J)}$ and $|\overline{N_G(J)} : \overline{N_M(J)}| = |M_{22}|/2^4|\text{Aut}(\text{SL}(2, 4))| = 231$, we have $|\overline{N_G(J)} : C_{\overline{N_G(J)}}(e_2)| = 77$, i.e., $|C_{\overline{N_M(J)}}(e_2) : \overline{N_M(J)}| = 3$. This implies $\overline{N_M(J)} \nsubseteq C_{\overline{N_G(J)}}(e_2)$. Consequently $\langle e_5, e_6, f_5, f_6 \rangle = O_2(\overline{N_M(J)}) \supset C_{\overline{N_G(J)}}(e_2)$, which contradicts the structure of $M_{22}$ described in Lemma 2.14(1). Thus the desired conclusion follows from Lemma 2.13.

Lemma 4.10. $G$ has precisely three classes of involutions.

Proof. In view of Lemmas 4.8 and 4.9, it suffices to show that each involution of $G$ is conjugate to an involution of $C$. Let $x$ be an involution. Since $R$ is a Sylow 2-subgroup of $G$, we may assume $x \in R$. Then $x \in M$. By Lemma 4.2, we may assume $x \in N_G(A)$. If $x \in A$, $x$ is conjugate to $e_2$. Thus we may assume $x \notin A$. Note that $N_G(A)/A$ has four classes of involution and their representatives are $f_2A, f_3A, f_2g_3A$ and $f_2g_3hA$. The desired conclusion now follows from Lemma 3.1.

We proceed to determine the structures of the centralizers of involutions. We include the proof of the following lemma in this section.

Lemma 4.11. $C_G(e_2) = O(C_G(e_2))/N_G(C)$.

Proof. Let $\overline{C_G(e_2)} = C_G(e_2)/\langle e_2 \rangle$. We show $\overline{C}$ is strongly closed in $\overline{R}$ with respect to $C_G(e_2)$. By Lemma 4.4, $\overline{C}$ is weakly closed. Let $\Gamma, \gamma$ be as in Lemma 2.3. Let $E$ be an element of $\Gamma$ such that $|E| = \gamma$. First suppose that $|E| = 2$,
and let \( \varpi \) be the involution of \( E \). Then by Lemma 2.9(1), \( ||\bar{C}, \varpi|| \geq 4 \), which contradicts Lemma 2.3(5). Next suppose that \( 8 \geq |E| \geq 4 \). Then by Lemma 2.9(3), there exists an involution \( \varpi \) of \( E \) such that \( ||\bar{C}, \varpi|| = 16 \). This again contradicts Lemma 2.3(5). Thus \( |E| = 16 \). Suppose that \( E \not\subseteq M \). Then by Lemma 2.9(9), there exists an involution \( \varpi \) of \( E \) such that \( |\langle C, \bar{E}, \varpi \rangle| \leq 2 \cdot 8 = 16 \), which contradicts Lemma 2.3(6). Thus \( E \subseteq M \). By Lemma 2.9(7) and Notation 2, \( E = \langle g_3, g_4, h, x_1 \rangle \). If \( |C \cap E| \leq 2 \), then by Lemma 2.9(2), \( |\langle(C \cap E), \bar{C}, \varpi \rangle| \leq 16 \) for every central involution \( \varpi \) contained in \( E \), which contradicts Lemma 2.3(6). Thus \( |C \cap E| \geq 4 \), which implies \( |E| \geq 128 \), where \( E \) denotes the full inverse image of \( \bar{E} \). But by Lemma 4.7, \( E \subseteq J \), and hence \( E \) is elementary abelian, which is a contradiction because \( C \) does not contain an elementary abelian subgroup of order 128. Consequently \( \bar{C} \) is strongly closed in \( \bar{R} \) with respect to \( C_G(e_2) \). Since \( N_{\bar{R}}(\bar{C}) \) controls the fusion of \( \bar{C} \), we obtain \( (O(C_G(e_2))(e_2))(f_2)/\langle O(C_G(e_2))(e_2) \rangle < C_G(e_2)/(O(C_G(e_2))(e_2)) \) by Glauberman’s \( Z^- \)-Theorem.

Now let \( C_G(e_2) = C_G(e_2)/(O(C_G(e_2))(e_2,f_2)) \). Considering the action of \( \bar{F} \) on \( \bar{C} \), we see that each involution of \( \bar{C} \) is conjugate to \( \bar{e}_3, \bar{f}_3, \bar{e}_3\bar{f}_3, \bar{e}_3\bar{f}_4 \) or \( \bar{e}_3\bar{f}_6 \). In view of the action of \( N_{\bar{M}}(\bar{F})/\bar{F} \), it follows that each involution of \( \bar{C} \) is conjugate to \( \bar{e}_3 \) or \( \bar{e}_3\bar{f}_6 \). Therefore by the main theorem of Goldschmidt [9], \( \bar{C} < C_G(e_2) \), as desired.

\[ \square \]

§5. Centralizers of Involutions

We continue with the notation of the preceding section, and complete the proof for the case where \( \bar{M}/O(\bar{M}) \cong Aut(SU(4,2)) \).

Our aim is to show that \( O(C_G(e_2)) = 1 \). For this purpose, we need to determine the structure of \( C_G(e_3f_2) \) and \( C_G(f_2) \). Before this is done, we make some more preparations. Let \( I = \langle e_2, f_2, e_3, f_3, g_3 \rangle \).

**Lemma 5.1.** Let \( N_{\bar{G}}(J) = N_G(J)/C_G(J) \). Then the following hold:

1. \( \bar{R} \) contains exactly two elementary abelian subgroups \( \langle e_5, f_5, e_6, f_6 \rangle \), \( \langle e_5, f_5, g_5, x_2 \rangle \) of order 16.
2. \( N_{\bar{G}(J)}(\langle e_5, f_5, e_6, f_6 \rangle) \cong E_{16} \cdot Aut(SL(2,4)) \), \( N_{\bar{G}(J)}(\langle e_5, f_5, g_5, x_2 \rangle) \cong E_{16} \cdot Sp(4,2)' \), where the actions are standard.
3. \( C_J(\langle e_5, f_5, g_5, x_2 \rangle) = I \).
4. \( e_2, f_2 \) and \( e_3f_2 \) are the representatives of the three conjugacy classes of involutions of \( G \).
(5) If \( x, y \) are elements of \( I \) such that \( x \in \{ e_2^2 \} \) and \( y \in \{ (e_3 f_2) G \} \), then there exists an element \( \tilde{g} \in N_{N_G(J)}(e_5 f_5, g_5, x_2) \) such that \( x \tilde{g} = e_2 \) and \( y \tilde{g} = e_3 f_2 \) or such that \( x \tilde{g} = e_2 \) and \( y \tilde{g} = e_2 g_3 \).

(6) \( O_2(C_{N_G(J)}(e_3 f_2)) = \langle e_5 f_5, g_5, x_2 \rangle \).

(7) \( I < C_{N_G(J)}(e_3 f_2) \).

Proof. Statement (1) follows from Lemma 2.14(1) and Notation 2. As noted in the proof of Lemma 4.9, \( N_M(J)/\langle e_5, f_5, e_6, f_6 \rangle \cong \Sigma_5 \). Hence (2) follows from Lemma 2.14(1). We now prove (3). By Lemma 2.9(5), \( [C/Z(C), \tilde{J}] = C_{C/Z(C)}(\tilde{J}) = \langle e_i, f_i \mid 2 \leq i \leq 4 \rangle / Z(C) \). Hence it follows from Lemma 2.9(2) that if \( \pi \) is an involution of \( \tilde{J} \) which centralizes \( Z(C)e_5 f_5/Z(C) \), then \( \pi \) is central. In view of Lemma 2.9(6), it follows form the choice of \( x_1 \) that every involution in \( \langle g_i, h, x_1 \rangle - \langle h \rangle \) is noncentral. Since \( \tilde{K} \) does not centralize \( Z(C)e_5 f_5/Z(C) \) and \( g_3 \) centralizes \( Z(C)e_5 f_5/Z(C) \), we get \( C_{\tilde{J}}(Z(C)e_5 f_5/Z(C)) = \langle g_3 \rangle \). Consequently, calculating in \( M \), we obtain \( C_{\tilde{J}}(e_5 f_5) = \langle e_2, f_2, e_3, f_3, g_3 \rangle \). On the other hand, by Lemma 2.14(4), \( |C_{\tilde{J}}(e_5 f_5, g_5, x_2)\rangle = 32 \). Therefore (3) holds. We proceed to the proof of (4). In view of Lemmas 4.8 and 4.10, it suffices to show that \( e_2 \not\sim f_2 \not\sim e_3 f_2 \not\sim e_2 \) in \( N_G(J) \). Set \( \tilde{X} = N_{N_G(J)}(e_5 f_5, g_5, x_2) \). We first prove \( [I, \tilde{X}] = \langle e_2, f_3, f_2 f_3, f_2 g_3 \rangle \). Clearly \( e_2 = [f_2, e_7] \in [I, \tilde{R}] \subseteq [I, \tilde{X}] \). Since \( e_2 \sim e_3 \) in \( N_G(A) \), \( e_2 \sim e_3 \) in \( N_G(J) \) by Lemma 4.8. Hence \( e_3 \in [I, \tilde{X}] \) by Lemma 2.14(5). Suppose that \( f_2 f_3 \not\in [I, \tilde{X}] \). Then since \( f_2 f_3 \sim f_2 g_3 \) in \( Y \) (recall that \( Y \) is a complement of \( A \) in \( N_G(A) \)), it follows from Lemmas 4.8 and 2.14(5) that \( f_2 g_3 \not\in [I, \tilde{X}] \). Since \( |I : [I, \tilde{X}]| = 2 \) by Lemma 2.14(4), this implies \( f_2 g_3 = (f_2 f_3)(f_2 g_3) \in [I, \tilde{X}] \). But since \( f_2 g_3 \sim f_2 f_3 \), this is a contradiction. Thus \( f_2 f_3 \in [I, \tilde{X}] \), and hence \( f_2 g_3 \in [I, \tilde{X}] \). Therefore \( [I, \tilde{X}] = \langle e_2, e_3, f_2, f_3, f_2 g_3 \rangle \). By Lemma 2.14(5), this implies \( e_2 \not\sim f_2 \) and \( e_2 \not\sim e_3 f_2 \) in \( N_G(J) \). We have \( f_2 \sim e_2 f_2 \sim g_3 \sim e_3 g_3 \sim f_2 f_3 g_3 \sim e_2 e_3 f_2 f_3 g_3 \), \( e_3 f_2 \sim e_2 e_3 f_2 \sim e_2 g_3 \sim e_2 e_3 g_3 \sim e_2 f_2 f_3 g_2 \sim e_3 f_2 f_3 g_3 \) and \( f_3 \sim e_2 f_3 \sim e_3 f_3 \sim e_2 e_3 f_3 \) in \( N_G(A) \). In view of Lemma 2.14(5), this implies \( f_2 \not\sim e_3 f_2 \). Note that \( C_{N_G(J)}(e_2) \) contains \( N_M(J) \). By (2) and (5) of Lemma 2.14, this means that \( f_2 \) corresponds to the element \( a \) in Lemma 2.14 and \( e_3 f_2 \) corresponds to \( c \) (so \( f_2 \not\sim f_3 \) and \( e_3 f_2 \not\sim f_3 \)). Recall that \( e_3 f_2 \sim e_2 g_3 \). On the other hand, it follows from Lemma 4.11 that \( e_3 f_2 \) and \( e_2 g_3 \) are not conjugate in \( C_G(e_2) \). Consequently (5) follows from Lemma 2.14(6). Now (6) follows from Lemma 2.14(2), and (7) follows from (3) and (6). \( \square \)

Lemma 5.2. \( O(C_G((e_2, e_3 f_2)))I < C_G((e_2, e_3 f_2)) \)
Proof. Let \( \overline{C_G(e_2)} = C_G(e_2)/(O(C_G(e_2))Z(C)) \). Clearly \( \overline{C_G((e_2,e_3f_2)} \subseteq C_{\overline{C_G(e_2)}}(\bar{e_3}) \). In the sense of Lemma 2.9(11), \( \langle e_3,f_3 \rangle \) is the “1-dimensional subspace of \( \overline{C} \) over \( GF(4) \) spanned by \( \bar{e_3} \).” Hence \( \langle e_3,f_3 \rangle \trianglelefteq C_{\overline{C_G(e_2)}}(\bar{e_3}) \).

By way of contradiction, suppose that \( \bar{I} \not\trianglelefteq C_G(\langle e_2,e_3f_2 \rangle) \). Then there exists an element \( x \) of \( C_G(\langle e_2,e_3f_2 \rangle) \) such that \( g_3x = g_3a \) where \( a \in C \), and \( \bar{a} \in C_{\overline{C_G(g_3)}}(\bar{C},\bar{g_3}) \). Since the full inverse image of \( C_{\overline{C_G(g_3)}}(\overline{\bar{C}}) \) is centralized by \( g_3O(C_{E_G}(\bar{e_2})) \) and since \( g_3 \) and \( g_3a \) are both involutions, \( aO(C_G(\langle e_2 \rangle)) \) is an involution. Hence there exists an element \( g \) of \( \langle e_2,e_3,f_2,g_3 \rangle \) \( \subseteq C_G(\langle e_2,e_3f_2 \rangle) \cap G \) such that \( g^o \in \langle e_1,f_i \mid 2 \leq i \leq 4 \rangle \). Thus this fusion must occur in \( NC_G(\langle e_2,e_3f_2 \rangle)(J) \).

The proof of Lemma 5.3 is similar to and easier than that of Lemma 5.2, and so it is omitted.

Lemma 5.3. \( O(C_G(\langle e_2,e_3f_2 \rangle)) \trianglelefteq C_G(\langle e_2,e_3f_2 \rangle)). \)

Lemma 5.4. If \( x,y \) are elements of \( I \) such that \( x \in \{ e_2^G \} \) and \( y \in \{ (e_3f_2)^G \} \), then \( O(C_G((x,y))) \trianglelefteq C_G((x,y)). \)

Proof. This follows from (3) and (5) of Lemma 5.1 and Lemmas 5.2 and 5.3.

We now determine the structure of \( C_G(e_3f_2) \).

Lemma 5.5. \( O(C_G(e_3f_2)) \trianglelefteq C_G(e_3f_2). \)

Proof. By Lemma 5.1(6), \( J(e_5,f_5,g_5,x_2) \subseteq C_R(e_3f_2) \). Since \( e_6, e_7f_6 \in C_R(e_3f_2) \), this implies \( |C_R(e_3f_2)| = 2^{16} \). Since \( Z(R) = \langle e_2 \rangle \), it follows from Lemma 5.1(4) that \( C_R(e_3f_2) \subseteq Syl_2(C_G(e_3f_2)) \). Note also that \( (e_7f_6)^2 = e_6, C_{J(e_5,f_5,g_5,x_2)} / J(e_6) = J(e_5,f_5)/J \) and \( C_G(e_6) = \langle e_2,f_2,e_3 \rangle \).

We show that \( I \) is strongly closed in \( C_R(e_3f_2) \) with respect to \( C_G(e_3f_2) \). By way of contradiction, let \( x \) be an element of \( C_R(e_3f_2) \) such that \( x \notin I \) and \( x^g \in I \) for some \( g \in C_G(e_3f_2) \). Since \( C_{NC_G(J)(e_3f_2)}(J) \) controls the fusion of \( J \) in \( C_G(e_3f_2) \), \( x \notin J \) (see Lemma 5.1(7)). Hence by Lemma 2.3(2), we may assume \( C_{NC_G(J)(e_3f_2)}(x)^g \subseteq C_R(e_3f_2) \). First suppose that \( x \in J(e_5,f_5,g_5,x_2) \). By Lemma 2.14(5), \( J \) contains a subgroup \( I_1 \) of order 16 all of whose involutions are conjugate to \( e_2 \) in \( G \). Let \( y \) be an involution of \( I_1 \). Suppose that \( y^g \in J \). Then since \( C_{NC_G(J)(e_3f_2)}(e_3f_2) \) controls the fusion, there exists an element \( g_1 \) of \( C_{NC_G(J)(e_3f_2)}(e_3f_2) \) such that \( (y^g)^{g_1} = y \). But then
by Lemma 5.1(7), \( x^{g_1} \in I \), which contradicts Lemma 5.4. Thus \( y^g \notin J \).

Since \( y \) is arbitrary, \( I_1 \cap J = 1 \). Since \( I_1 \subseteq C_{C_R(x_3f_2)}(x) \), \( I_1^g \subseteq C_{R(x_3f_2)} \).

Since \( (e_7f_6)^2 = e_6 \) and \( C_{J(e_5f_5,g_5,x_2)}(e_6) = J(e_5,f_5)/J \), \( J(e_5,f_5,g_5,x_2)/J \) is the only elementary abelian subgroup of order 16 of \( C_{R(x_3f_2)}/J \).

Consequently \( JI_1^g = J(e_5,f_5,g_5,x_2) \), and hence \( C_{C_R(x_3f_2)}(I_1^g) = C_{C_R(x_3f_2)}(I_1) \cdot I_1^g = C_{J(g_5,x_2,f_5,e_5)}(I_1^g) \). Since \( C_J(x)^g = C_{C_{R(x_3f_2)}(x)} \cap C_{C_{G(x_3f_2)}/I_1^g)(I_1)^g = C_{C_R(x_3f_2)}(I_1) \), we get \( |C_J(x)| \subseteq I \cdot I_1^g \). Since \( |C_J(x)| = 64 \) by Lemma 2.14(3), it follows that there exists an element \( z \) of \( C_J(x) - I \) such that \( z^g \in I \). But this contradicts the fact that \( C_{N_G(J_2)}(x_3f_2) \) controls the fusion of \( J \).

Consequently no element of \( J(e_5,f_5,g_5,x_2) - I \) is fused into \( I \).

Recall that \( (e_7f_6)^2 = e_6 \) and \( C_I(e_6) = (e_2,f_2,e_3) \), and \( J(e_5,f_5,g_5,x_2) \subseteq C_R(I) \) by Lemma 5.1(3). Thus \( I \) is weakly closed in \( C_R(x_3f_2) \). Now if we define \( \gamma' \) as in Lemma 2.3, then \( \gamma' = 2 \), which contradicts Lemma 2.3(2). Therefore \( I \) is strongly closed.

Now let \( C_{G(x_3f_2)} = C_{G(x_3f_2)}/(O(G)(x_3f_2)) \). Suppose that \( \tilde{I} \not\triangleleft C_{G(x_3f_2)} \), and set \( \tilde{X} = \langle \tilde{I},C_{G(x_3f_2)} \rangle \). By Lemma 4.8, \( N_{C_{G(x_3f_2)}}(J) \) controls the fusion of \( J \), and hence it follows from the proof of Lemma 2.14(6) that the involutions of \( \tilde{I} \) split into two classes of sizes 6 and 9. Therefore by the main theorem of [9], \( \tilde{X} \cong \text{PSL}(2,q) \times \text{PSL}(2,q) \) for some \( q \) with \( q \equiv 3,5 \) (mod 8). Set \( Q = J(e_5,f_5,g_5,x_2) \). Then by Lemma 5.1(3), \( Q \subseteq C_{C_{G(x_3f_2)}}(\tilde{I}) \).

This implies \( \tilde{X}Q = \tilde{X} \times C_Q(\tilde{X}) \), and hence \( \tilde{Q} = \tilde{I} \times C_Q(\tilde{X}) \). But then \( \tilde{I} \cap \tilde{Q} = 1 \), which contradicts Notation 1. Consequently \( \tilde{I} \triangleleft C_{G(x_3f_2)} \), as desired.

\[ \square \]

**Lemma 5.6.** \( C_{G(x_3f_2)} \) is solvable.

\[ \text{Proof.} \text{ Set } B = N_{C_{G(x_3f_2)}}(I) \text{ and } \tilde{B} = B/I \text{. In view of Lemma 5.5, the lemma is equivalent to the assertion that } \tilde{B} \text{ is solvable. Note that } J(e_5,f_5,g_5,x_2) = C_{N_{B}(J)} \subseteq \text{Syl}_2(C_{B}(J)) \text{ and } C_{R(x_3f_2)} = C_{R(J)}(e_7f_6) \subseteq \text{Syl}_2(B) \text{ (see the first paragraph of the proof of Lemma 5.5). In particular, a Sylow-2 subgroup of } B/C_{B}(I) \text{ is a cyclic group of order 4, and hence } B/C_{B}(I) \text{ is solvable. Thus it suffices to show that } C_{B}(I) \text{ is solvable. Set } W = N_{N_{B}(J)}(C_{R}(I)) \text{ and } Z/J = C_{W/J}(C_{R}(I)/J) \text{. By Lemma 2.14(7), } |C_{R}(I)/[J,C_{R}(I)]| = 32 \text{. Since } W/Z \cong E_{32} \times Z_4, C_{R}(I)/[J,C_{R}(I)] \cong E_{32} \text{ or } D_{8} \times D_{8} \text{. Since } (e_5,f_5,g_5) \cong E_{8}, \text{this implies } C_{R}(I)/[J,C_{R}(I)] \cong E_{32} \text{. Hence if we write } C_{R}(I)/[J,C_{R}(I)], O(W/J) = E/[J,C_{R}(I)], \text{then } |E/[J,C_{R}(I)]| = 16, \text{i.e., } |E| = 256 \text{. Note that } E/[J,C_{R}(I)] \cong C_{R}(I)/J \text{ and } E \cap J = [J,C_{R}(I)] = C_{R}(I)/J, \text{and } C_{R}(I)/J \text{ and } [J,C_{R}(I)] \text{ are isomorphic as a } (W/Z)-\text{module by Lemma 2.14(7). Consequently } E \text{ is abelian by Lemma 2.5. By Lemma 2.3(2), } E \text{ is strongly closed in } C_{R}(I) \text{ with respect to } C_{B}(I). \]
Suppose that $O(C_B(I))\cong E \not\cong C_B(I)$, and set $X = \langle E^{C_B(I)} \rangle$. In view of the action of $W/Z$ and $f/\langle f \rangle$ on $E$, $\langle f \rangle$ is the only nontrivial proper $N_B(E)$-invariant subgroup of $E$. Hence $O_2((O(C_B(I))\langle f \rangle)/O(C_B(I))) = 1$ by the main theorem of [9]. Considering the action of $O(W/Z)$ on $E$, it also follows from the main theorem of [9] that $(O(C_B(I))\langle f \rangle)/O(C_B(I))$ is the direct product of groups isomorphic to $\text{SL}(2,2^n)$ ($n = 2, 4, 8$). By Lemma 2.12(4), $E'$ is a subgroup of $I$ with $|E'| \geq 16$. Hence $|X^{\infty} \cap I| \geq 16$. On the other hand, $I \subseteq Z(E)$ by Lemma 5.1(3), and hence $X^{\infty} \cap I \subseteq Z(X^{\infty})$. Note that Sylow 2-subgroups of $2\text{SL}(2, 4)$ are not abelian. Thus we get a contradiction to the structure of the Schur multiplier of $\text{SL}(2, 2^n)$ (see the argument used in the proof of Lemma 2.11). Consequently $O(C_B(I)) \cong E \not\cong C_B(I)$. Since $|C_R(I) : E| = 2$, this implies that $C_B(I)$ is solvable, as desired.

\[\text{Lemma 5.7.}\] Let $C_G(f_2) = C_G(f_2)/\langle f_2 \rangle$. Then $C_G(f_2)$ has three classes of involutions with representatives $e_2$, $e_3$ and $f_4g_3$.

Proof. By Lemma 2.14(2), $N_{C_G(f_2)}(\langle J \rangle)/C_{C_G(f_2)}(\langle J \rangle) \cong \text{PSL}(4, 3)$ (see the proof of (4) and (5) of Lemma 5.1). Hence by Lemma 2.14(8), the involutions of $\langle J \rangle$ split into three classes under the action of $N_{C_G(f_2)}(\langle J \rangle)$. Let $\bar{e}$ be an involution of $N_{C_G(f_2)}(\langle J \rangle)$. We prove that $\bar{e}$ is fused into $\langle J \rangle$. By Lemma 2.12(5), we may assume $\bar{e} = e_5\langle J \rangle$. By Notation 1, $[g_3, e_5] = 1$. Since $J \subseteq N_G(C)$ and $e_5 \in C$ and $f_4g_3, g_3 \not\in C$, we have $f_4g_3, g_3 \not\in [J, e_5]$; that is, $g_3 \in C_\langle e_5 \rangle - [\langle J \rangle, e_5]$. Hence by Lemma 2.12(6), $\bar{e}$ is conjugate to $e_5$ or $e_3g_3$. Recall that $N_G(A) = A \cdot Y$. Thus there exists $v \in Y$ such that $f_2^v = f_2$, $e_3^v = e_4$, $g_3^v = g_3$. Hence $e_5$ and $g_3$ are conjugate to $e_4$ and $e_4g_3$, respectively, in $C_{N_G(A)}(f_2)$. Therefore every involution of $N_{C_G(f_2)}(\langle J \rangle)$ is fused into $\langle J \rangle$. Since $N_{C_G(f_2)}(\langle J \rangle)$ controls the fusion of $\langle J \rangle$, this means that $C_G(f_2)$ has precisely three classes of involutions.

We now show that $f_4g_3$ is conjugate to $e_3f_2$ in $G$. In $Y$, 28 of the involutions of $\langle f_2, f_3, f_4, g_3, h \rangle$ are conjugate to $f_4g_3$. For each such involution $x$, every element of the coset $\langle e_2, e_3, e_4 \rangle$ is conjugate to $x$ in $N_G(A)$ by Lemma 3.1(4). Hence $J$ contains 224 involutions conjugate to $f_4g_3$ in $N_G(A)$, and none of them is contained in $I$. By Lemma 2.14(5), 15 of the involutions of $I$ are conjugate to $e_2$. Since $224 + 15 > 231$, it follows from Lemma 2.14(2) that $f_4g_3$ cannot be conjugate to $e_2$ or $f_2$. Consequently $f_4g_3$ is conjugate to $e_3f_2$. Note that $f_4g_3$ and $f_4g_3f_2$ are conjugate in $Y$. Thus both elements of the coset $f_4g_3\langle f_2 \rangle$ are conjugate to $e_3f_2$ in $G$. On the other hand, one element of the coset $e_3\langle f_2 \rangle$ is conjugate to $e_2$ and the other is (conjugate to) $e_3f_2$ in $G$, and one element of the coset $e_2\langle f_2 \rangle$ is $e_3$ and the other is conjugate to $f_2$ in $G$. Therefore $f_4g_3 \not\in e_3 \not\in e_2 \not\in f_4g_3$ in $C_G(f_2)$, and the lemma is proved.
Lemma 5.8. $C_G(f_2)/\langle (f_2) O(C_G(f_2)) \rangle \cong PSU(6,2)$.

Proof. Let $\widetilde{C_G}(f_2) = C_G(f_2)/\langle (f_2) O(C_G(f_2)) \rangle$. By Lemma 5.7, $\widetilde{C_G}(f_2)$ contains three classes of involutions. Let $\tilde{x}$ be an involution which is conjugate to either $e_3 f_2$ or $f_2 g_3$. We may assume $x$ is conjugate to $e_3 f_2$ in $G$ (see the second paragraph of the proof of Lemma 5.7). Thus $C_G(x)$ is solvable by Lemma 5.6, and hence $C_{CG}(x)(f_2)$ is solvable. Since $|C_{CG}(f_2)(\tilde{x}) : C_{CG}(x)(f_2)| \leq 2$, this means that $\widetilde{C_{CG}(f_2)}(\tilde{x})$ is solvable. Now since $e_2 \not\sim f_2 \sim e_2 f_2$ in $G$,

$\widetilde{C_{CG}(f_2)}(\tilde{e}_2) = \widetilde{C_{CG}(f_2)}(e_2)$. By Lemma 4.11, $C_{CG}(f_2)(e_2)/O(C_{CG}(e_2))$ is an extension of $D_8 * D_8 * D_8 * D_8$ by $SU(4,2)$ (see the parenthetic remark about $\overline{M}$ in the proof of Lemma 4.9). Consequently the centralizer of each involution of $C_{E(G)}(f_2)$ is 2-constrained. Since $C_{G}(f_2)$ is connected in the sense of Gorenstein and Walter [13], $O(C_{CG}(f_2)(e_2)) = 1$ by Theorem B of [13]. Therefore $\widetilde{C_{E(G)}}(f_2) \cong PSU(6,2)$ by a result of Parrot [17].

We are now in a position to complete the proof for the case where $\overline{M}/O(\overline{M}) \cong \text{Aut}(SU(4,2))$. Note that PSU(6,2) is 2-generated and 2-balanced. Hence in view of Lemmas 5.1(4), 4.11, 5.6 and 5.8, it follows from Theorem A of [13] that $O(C_G(x)) = 1$ for every involution $x$ of $G$. Therefore we obtain $E(G) \cong M(22)$ by a result of Hunt [15] or Parrot [17].

§6. Contradiction

In this section, we assume that $\overline{M}/O(\overline{M}) \cong \Sigma_8$, and derive a contradiction. Arguing as in Section 4, we obtain the following lemmas.

Lemma 6.1. $\overline{M} \cong \Sigma_8(\cong \text{Aut}(GL(4,2)))$.

Lemma 6.2. Every involution of $\overline{M}$ is conjugate to some involution of $\langle (\overline{e}_7) \rangle$ in $\overline{M}$.

Having Lemma 2.10 in mind, we fix the following notation.

Notation 3. Let $x_1$ be an element of $M'$ such that $(\overline{g_3}, \overline{g_4}, \overline{h}, \overline{x_1}) \cong E_{16}$. Then $\mathcal{N}_{\overline{M}}((\overline{g_3}, \overline{g_4}, \overline{h}, \overline{x_1}))/((\overline{g_3}, \overline{g_4}, \overline{h}, \overline{x_1}) \cong GL(2,2) \wr Z_2 \cong O^+(4,2)$, and $\mathcal{N}_{\overline{M}}((\overline{g_3}, \overline{g_4}, \overline{h}, \overline{x_1}))/((\overline{g_3}, \overline{g_4}, \overline{h}, \overline{x_1})$ acts on $(Z(C)(e_3, e_4, f_3, f_4))/Z(C)$.

Let $x_2$ be an element of $M'$ such that

$[e_4 f_4, x_2] \in e_3 f_3 Z(C), [e_3, f_3, f_4, x_2] \subseteq Z(C)$.
We choose $x_1$ and $x_2$ as involutions. Moreover we choose them so that $(\overline{g_3h}, \overline{x_2}) \cong E_4$ and $(\overline{AS}, x_1, x_2)$ is a Sylow 2-subgroup of $M$. Set $R = \langle AS, x_1, x_2 \rangle$.

We now argue as in Section 4, using Lemma 2.10 in place of Lemma 2.9. Then we obtain the following lemma.

**Lemma 6.3.** $R \in \text{Syl}_2(G)$.

**Lemma 6.4.** There exists $x_i' \in x_iC$ such that $\langle e_i, f_i, g_3h, g_4, x_i' \mid 2 \leq i \leq 4 \rangle \cong E_{512}$.

**Proof.** By Lemma 2.10(5), there exists $\overline{y} \in N_{\overline{M}}(\langle e_i, f_i \mid 2 \leq i \leq 4 \rangle)$ such that $\overline{g_3h} = \overline{g_i}, \overline{g_3h} = \overline{g_3h}$ and $\overline{h} = \overline{e_i}$. Set $x_i' = h^y$. Then arguing as in the first paragraph of the proof of 4.6, we obtain $\langle e_i, f_i, g_3h, g_4, x_i' \mid 2 \leq i \leq 4 \rangle = (e_i, f_i, g_3h, g_4, h \mid 2 \leq i \leq 4)^y \cong E_{512}$.

Let $x_i'$ be as in Lemma 6.4, and set $J = \langle e_i, f_i, g_3, g_4, h, x_i' \mid 2 \leq i \leq 4 \rangle$.

**Lemma 6.5.** $J \cong E_{1024}$.

**Proof.** In view of Lemma 6.4, it suffices to show that $[g_4h, g_3hx_i'] = 1$. Note that $g_4$ and $g_3$ are noncentral involutions and $g_3g_4h$ is a central involution, and that $g_4h$ and $g_3hx_i'$ are noncentral involutions and $g_3g_4x_i'$ is a central involution. Hence by Lemma 2.10(5), there exists $\overline{z} \in N_{\overline{M}}(\langle e_i, f_i \mid 2 \leq i \leq 4 \rangle)$ such that $\overline{g_4h} = \overline{g_3h}$ and $\overline{g_3h} = \overline{g_3hx_i'}$. Therefore arguing as in the second paragraph of the proof of Lemma 4.6, we obtain $[g_4h, g_3hx_i'] = [g_4, (g_3h)^2] = 1$.

Arguing as in Section 4, we also obtain the following lemma.

**Lemma 6.6.** $J = J(R) \cong E_{1024}$.

We are now in a position to derive a contradiction. Let $\widehat{N_G(J)} = N_G(J)/C_G(J)$. Since $N_M(J)$ contains a Sylow 2-subgroup of $G$, $N_M(J)$ contains a Sylow 2-subgroup of $N_G(J)$. Note that $O_2(N_M(J)) = \langle e_5, e_6, f_5, f_6 \rangle$, $\widehat{N_M(J)}/O_2(N_M(J)) \cong \text{GL}(2,2) \rtimes Z_2$ and $N_M(J)/O_2(N_M(J))$ acts on $O_2(N_M(J))$ in a standard way. Hence a Sylow 2-subgroup of $N_M(J)$ is isomorphic to a Sylow 2-subgroup of $\Sigma_8$. Therefore we see from Gorenstein and Harada [12] that the action of $\widehat{N_M(J)}$ on $J$ cannot be consistent with the fusion of $J$, which is a contradiction. This concludes the discussion for the case where $\overline{M}/O(\overline{M}) \cong \Sigma_8$. 
§7. Normal Case

In this section, we assume that $\overline{M}/O(M) \cong \mathbb{Z}_2 \times \Sigma_6$, and show that $A$ is normal in $G$.

Arguing as in Section 4, we obtain the following two lemmas.

Lemma 7.1. $AS \in \text{Syl}_2(G)$

Lemma 7.2. $J(AS) = \langle e_2, e_3, e_4, e_5, e_6, e_7 \rangle$.

Proof. Set $W = N_G(J(AS))$, $\overline{W} = W/C_W(J(AS))$, $B = \langle e_5, e_6, e_7 \rangle$, $W_1 = N_Y((f_1, g_3, g_4, h \mid 2 \leq i \leq 4))$. Then $N_A(J(AS)) = \overline{B} \cdot W_1 \cong E_8 \cdot \text{SL}(3, 2)$ and $AS \in \text{Syl}_2(BW_1)$. We prove that $\overline{B}$ is strongly closed in $AS$ with respect to $\overline{W}$. Define $\Gamma$ and $\gamma$ as in Lemma 2.3 (note that we have not yet proved that $\overline{B}$ is weakly closed) and, by way of contradiction, suppose that $\Gamma \neq \emptyset$. Let $\overline{E}$ be a member of $\Gamma$. Then $[J(AS), \overline{y}'] = \langle e_2, e_3, e_4 \rangle$ for every involution $\overline{y}$ of $\overline{B}$, $[J(AS), \overline{y}_1] = [J(AS), \overline{y}_2]$ for any involutions $\overline{y}_1, \overline{y}_2$ of $\overline{E}$. But $|[J(AS)/\langle e_2, e_3, e_4 \rangle, \overline{y}]| = 4$ for each involution $\overline{y}$ of $\overline{A}S - \overline{B}$, and $[J(AS)/\langle e_2, e_3, e_4 \rangle, \overline{y}_1] \neq [J(AS)/\langle e_2, e_3, e_4 \rangle, \overline{y}_2]$ if $\overline{y}_1 \overline{B} \neq \overline{y}_2 \overline{B}$. Hence $|E| = 2$. Since $\overline{E}$ is arbitrary, this means $\gamma = 2$. In particular, $\overline{B}$ is weakly closed. These contradict Lemma 2.3(6). Thus $\overline{B}$ is strongly closed. Consequently $\overline{W} = O(\overline{W})(BW_1)$ by Goldschmidt [9]. We next prove $O(\overline{W}) = 1$. Arguing as in Lemma 2.11, we can easily show that $O(\overline{W})$ centralizes $BW_1$. On the other hand, $\langle e_2, e_3, e_4 \rangle$ is the unique minimal $BW_1$-invariant subgroup of $J(AS)$, and $\langle f_3, f_4, g_3, g_4, e_2, e_3, e_4 \rangle/\langle e_2, e_3, e_4 \rangle$ is the unique minimal $BW_1$-invariant subgroup of $J(AS)/\langle e_2, e_3, e_4 \rangle$. Therefore $O(\overline{W})$ centralizes $J(AS)$, and hence $O(\overline{W}) = 1$, as desired.

We can now easily show that $A$ is strongly closed in $AS$. Note that each involution of $AS$ is conjugate to an involution of $J$ (see the last few sentences of the proof of Lemma 4.10). Since $N_G(J)$ controls the fusion of $J$ by Lemma 7.2, it follow from Lemma 7.3 that no involution of $J - A$ is conjugate to an involution of $J \cap A$. Consequently $A$ is strongly closed. Therefore $A \triangleleft G$ by Goldschmidt [9], as desired.

References


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