Magic covering of chain of an arbitrary 2-connected simple graph

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Abstract. A simple graph $G = (V, E)$ admits an $H$-covering if every edge in $E$ belongs to a subgraph of $G$ isomorphic to $H$. We say that $G$ is $H$-magic if there is a total labeling $f : V \cup E \rightarrow \{1, 2, 3, \ldots, |V| + |E|\}$ such that for each subgraph $H' = (V', E')$ of $G$ isomorphic to $H$, $\sum_{v \in V'} f(v) + \sum_{e \in E'} f(e)$ is constant. When $f(V) = \{1, 2, \ldots, |V|\}$, then $G$ is said to be $H$-supermagic. In this paper we show that a chain of any 2-connected simple graph $H$ is $H$-supermagic.

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§1. Introduction

The concept of $H$-magic graphs was introduced in [2]. An edge-covering of a graph $G$ is a family of different subgraphs $H_1, H_2, \ldots, H_k$ such that each edge of $E$ belongs to at least one of the subgraphs $H_i$, $1 \leq i \leq k$. Then, it is said that $G$ admits an $(H_1, H_2, \ldots, H_k)$-edge covering. If every $H_i$ is isomorphic to a given graph $H$, then we say that $G$ admits an $H$-covering.

Suppose that $G = (V, E)$ admits an $H$-covering. We say that a bijective function $f : V \cup E \rightarrow \{1, 2, 3, \ldots, |V| + |E|\}$ is an $H$-magic labeling of $G$ if there is a positive integer $m(f)$, which we call magic sum, such that for each subgraph $H' = (V', E')$ of $G$ isomorphic to $H$, we have, $f(H') = \sum_{v \in V'} f(v) + \sum_{e \in E'} f(e) = m(f)$. In this case we say that the graph $G$ is $H$-magic. When $f(V) = \{1, 2, \ldots, |V|\}$, we say that $G$ is $H$-supermagic and we denote its supermagic-sum by $s(f)$.

We use the following notations. For any two integers $n < m$, we denote by $[n, m]$, the set of all consecutive integers from $n$ to $m$. For any set $I \subset \mathbb{N}$ we write $\sum I = \sum_{x \in I} x$ and for any integers $k, I + k = \{x + k : x \in I\}$. Thus
$k + [n, m]$ is the set of consecutive integers from $k + n$ to $k + m$. It can be easily verified that $\sum (\mathbb{I} + k) = \sum \mathbb{I} + k||$. Finally, given a graph $G = (V, E)$ and a total labeling $f$ on it we denote by $f(G) = \sum f(V) + \sum f(E)$.

In [2], A. Gutierrez, and A. Llado studied the families of complete and complete bipartite graphs with respect to the star-magic and star-supermagic properties and proved the following results.

- The star $K_{1,n}$ is $K_{1,h}$-supermagic for any $1 \leq h \leq n$.
- Let $G$ be a $d$-regular graph. Then $G$ is not $K_{1,h}$-magic for any $1 < h < d$.
- (a) The complete graph $K_n$ is not $K_{1,h}$-magic for any $1 < h < n - 1$.
  
  (b) The complete bipartite graph $K_{n,n}$ is not $K_{1,h}$-magic for any $1 < h < n$.
- The complete bipartite graph $K_{n,n}$ is $K_{1,n}$-magic for $n \geq 1$.
- The complete bipartite graph $K_{n,n}$ is not $K_{1,n}$-supermagic for any integer $n > 1$.
- For any pair of integers $1 < r < s$, the complete bipartite graph $K_{r,s}$ is $K_{1,h}$-supermagic if and only if $h = s$.

The following results regarding path-magic and path-supermagic coverings are also proved in [2].

- The path $P_n$ is $P_{h}$-supermagic for any integer $2 \leq h \leq n$.
- Let $G$ be a $P_h$-magic graph, $h > 2$. Then $G$ is $C_h$-free.
- The complete graph $K_n$ is not $P_h$-magic for any $2 < h \leq n$.
- The cycle $C_n$ is $P_h$-supermagic for any integer $2 \leq h < n$ such that $\gcd(n, h(h-1)) = 1$.

Also in [2], the authors constructed some families of $H$-magic graphs for a given graph $H$ by proving the following results.

- Let $H$ be any graph with $|V(H)| + |E(H)|$ even. Then the disjoint union $G = kH$ of $k$ copies of $H$ is $H$-magic.

Let $G$ and $H$ be two graphs and $e \in E(H)$ a distinguished edge in $H$. We denote by $G * eH$ the graph obtained from $G$ by gluing a copy of $H$ to each edge of $G$ by the distinguished edge $e \in E(H)$.

- Let $H$ be a 2-connected graph and $G$ an $H$-free supermagic graph. Let $k$ be the size of $G$ and $h = |V(H)| + |E(H)|$. Assume that $h$ and $k$ are not both even. Then, for each edge $e \in E(H)$, the graph $G * eH$ is $H$-magic.
In [3], P. Selvagopal and P. Jeyanthi proved that for any positive integer n, k- polygonal snake of length n is \(C_k\)-supermagic.

In this paper we construct a chain graph \(H_n\) of 2-connected graph \(H\) of length \(n\), and prove that a chain graph \(H_n\) is \(H\)-supermagic.

### §2. Preliminary Results

Let \(P = \{X_1, X_2, \ldots, X_k\}\) be partition set of a set \(X\) of integers. When all sets have the same cardinality we say then \(P\) is a \(k\)-equipartition of \(X\). We denote the set of subsets sums of the parts of \(P\) by \(\sum P = \{\sum X_1, \sum X_2, \ldots, \sum X_k\}\). The following lemmas are proved in [2].

**Lemma 1.** Let \(h\) and \(k\) be two positive integers and let \(n = hk\). For each integer \(0 \leq t \leq \left\lfloor \frac{h}{2} \right\rfloor\) there is a \(k\)-equipartition \(P\) of \([1, n]\) such that \(\sum P\) is an arithmetic progression of difference \(d = h - 2t\).

**Lemma 2.** Let \(h\) and \(k\) be two positive integers and let \(n = hk\). In the two following cases there exists a \(k\)-equipartition \(P\) of a set \(X\) such that \(\sum P\) is a set of consecutive integers.

(i) \(h\) or \(k\) are not both even and \(X = [1, hk]\)

(ii) \(h = 2\) and \(k\) is even and \(X = [1, hk + 1] - \{\frac{k}{2} + 1\}\).

We have the following four results from the above two lemmas.

(a) If \(h\) is odd, then there exists a \(k\)-equipartition \(P = \{X_1, X_2, \ldots, X_k\}\) of \(X = [1, hk]\) such that \(\sum P\) is a set of consecutive integers and \(\sum P = \frac{(h-1)(hk+k+1)}{2} + [1, k]\).

(b) If \(h\) is even, then there exists a \(k\)-equipartition \(P = \{X_1, X_2, \ldots, X_k\}\) of \(X = [1, hk]\) such that subsets sum are equal and is equal to \(\frac{h(hk+1)}{2}\).

(c) If \(h\) is even and \(k\) is odd, then there exists a \(k\)-equipartition \(P = \{X_1, X_2, \ldots, X_k\}\) of \(X = [1, hk]\) such that \(\sum P\) is a set of consecutive integers and \(\sum P = \frac{h(hk+1)}{2} + \left[-\frac{k-1}{2}, \frac{k-1}{2}\right]\).

(d) If \(h = 2\) and \(k\) is even, and \(X = [1, 2k + 1] - \{\frac{k}{2} + 1\}\) then there exists a \(k\)-equipartition \(P = \{X_1, X_2, \ldots, X_k\}\) of \(X\) such that \(\sum P\) is a set of consecutive integers and \(\sum P = \left[\frac{3h}{2} + 3, \frac{5h}{2} + 2\right]\).

We generalise the second part of Lemma 2.

**Corollary 1.** Let \(h\) and \(k\) be two even positive integers and \(h \geq 4\). If \(X = [1, hk + 1] - \{\frac{k}{2} + 1\}\), there exists a \(k\)-equipartition \(P\) of \(X\) such that \(\sum P\) is a set of consecutive integers.
Proof. Let $Y = [1, 2k + 1] - \left\{ \frac{k}{2} \right\}$ and $Z = (2k + 1) + [1, (h - 2)k]$. Then $X = Y \cup Z$. By (d), there exists a $k$-equipartition $P_1 = \{Y_1, Y_2, \ldots, Y_k\}$ of $Y$ such that

$$\sum P_1 = \left[ \frac{3k}{2} + 3, \frac{5k}{2} + 2 \right].$$

As $h - 2$ is even, by (b) there exists a $k$-equipartition $P_2' = \{Z_1', Z_2', \ldots, Z_k'\}$ of $[1, (h - 2)k]$ such that

$$\sum P_2' = \left\{ \frac{(h - 2)(hk - 2k + 1)}{2} \right\}.$$ 

Hence, there exists a $k$-equipartition $P_2 = \{Z_1, Z_2, \ldots, Z_k\}$ of $Z$ such that

$$\sum P_2 = \left\{ (h - 2)(2k + 1) + \frac{(h - 2)(hk - 2k + 1)}{2} \right\}.$$ 

Let $X_i = Y_i \cup Z_i$ for $1 \leq i \leq k$. Then $P = \{X_1, X_2, \ldots, X_k\}$ is a $k$-equipartition of $X$ such that $\sum P$ is a set of consecutive integers and

$$\sum P = (h - 2)(2k + 1) + \frac{(h - 2)(hk - 2k + 1)}{2} + \left[ \frac{3k}{2} + 3, \frac{5k}{2} + 2 \right].$$ 

$\square$

§3. Chain of an arbitrary simple connected graph

Let $H_1, H_2, \ldots, H_n$ be copies of a graph $H$. Let $u_i$ and $v_i$ be two distinct vertices of $H_i$ for $i = 1, 2, \ldots, n$. We construct a chain graph $H_n$ of $H$ of length $n$ by identifying two vertices $u_i$ and $v_{i+1}$ for $i = 1, 2, \ldots, n - 1$. See Figures 1 and 2.

§4. Main Result

**Theorem 1.** Let $H$ be a 2-connected $(p, q)$ simple graph. Then $H_n$ is $H$-supermagic if any one of the following conditions is satisfied.

(i) $p + q$ is even

(ii) $p + q + n$ is even

Proof. Let $G = (V, E)$ be a chain of $n$ copies of $H$. Let us denote the $i$th copy of $H$ in $H_n$ by $H_i = (V_i, E_i)$. Note that $|V| = np - n + 1$ and $|E| = nq$. Moreover, we remark that by $H$ is a 2-connected graph, $H_n$ does not contain a subgraph $H$ other than $H_i$. 
Let $v_i$ be the vertex in common with $H_i$ and $H_{i+1}$ for $1 \leq i \leq n-1$. Let $v_0$ and $v_n$ be any two vertices in $H_1$ and $H_n$ respectively so that $v_0 \neq v_1$ and $v_n \neq v_{n-1}$. Let $V'_i = V_i - \{v_{i-1}, v_i\}$ for $1 \leq i \leq n$.

**Case (i):** $p + q$ is even

Suppose $p$ and $q$ are odd. As $p - 2$ is odd, by (a) there exists an $n$-equipartition $P'_1 = \{X'_1, X'_2, \ldots, X'_n\}$ of $[1, n(p-2)]$ such that

$$\sum_{P'_1} = \frac{(p-3)(np-n+1)}{2} + [1, n].$$

Adding $n+1$ to $[1, n(p-2)]$, we get an $n$-equipartition $P_1 = \{X_1, X_2, \ldots, X_n\}$ of $[n+2, np-n+1]$ such that

$$\sum_{P_1} = (p-2)(n+1) + \frac{(p-3)(np-n+1)}{2} + [1, n]$$

Similarly, since $q$ is odd there exists an $n$-equipartition $P_2 = \{Y_1, Y_2, \ldots, Y_n\}$ of $(np-n+1) + [1, nq]$ such that

$$\sum_{P_2} = q(np-n+1) + \frac{(q-1)(nq+n+1)}{2} + [1, n]$$

Define a total labeling $f : V \cup E \rightarrow \{1, 2, 3, \ldots, np + nq - n + 1\}$ as follows:

(i) $f(v_i) = i + 1$ for $0 \leq i \leq n$.

(ii) $f(V'_i) = X_{n-i+1}$ for $1 \leq i \leq n$.

(iii) $f(E_i) = Y_{n-i+1}$ for $1 \leq i \leq n$.

Then for $1 \leq i \leq n$,

$$f(H_i) = f(v_{i-1}) + f(v_i) + \sum_{V'_i} f(V'_i) + \sum_{E_i} f(E_i)$$

$$= f(v_{i-1}) + f(v_i) + \sum X_{n-i+1} + \sum Y_{n-i+1}$$

$$= \frac{n(p+q)^2 + 3(p+q) - 2n(p+q) + 2n - 2}{2}$$

As $H_i \cong H$ for $1 \leq i \leq n$, $Hn$ is $H$-supermagic.

Suppose both $p$ and $q$ are even. As $p$ is even, by Lemma 1, there exists an $n$-equipartition $P'_1 = \{X'_1, X'_2, \ldots, X'_n\}$ of $[1, n(p-2)]$ such that $\sum P'_1$ is arithmetic progression of difference 2 and

$$\sum P'_1 = \left\{ \frac{n[(p-2)^2 - 2]}{2} + \frac{p-4}{2} + 2r : 1 \leq r \leq n \right\}.$$
Adding $n+1$ to $[1, n(p-2)]$, we get an $n$-equipartition $P_1 = \{X_1, X_2, \ldots, X_n\}$ of $[n+2, np-n+1]$ such that

$$\sum P_1 = \{(p-2)(n+1) + \frac{n[(p-2)^2 - 2] + p - 4}{2} + 2i : 1 \leq i \leq n\}$$

As $q$ is even, by (b), there exists an $n$-equipartition $P_2' = \{Y'_1, Y'_2, \ldots, Y'_n\}$ of $[1, nq]$ such that $\sum P_2' = \{\frac{q(nq+1)}{2}\}$.

Adding $np-n+1$ to $[1, nq]$ there exists an $n$-equipartition $P_2 = \{Y_1, Y_2, \ldots, Y_n\}$ of $(np-n+1) + [1, nq]$ such that

$$\sum P_2 = \{q(np-n+1) + \frac{q(nq+1)}{2}\}$$

Define a total labeling $f : V \cup E \rightarrow \{1, 2, 3, \ldots, np+nq-n+1\}$ as follows:

(i) $f(v_i) = i + 1$ for $0 \leq i \leq n$.

(ii) $f(V'_i) = X_{n-i+1}$ for $1 \leq i \leq n$.

(iii) $f(E_i) = Y_{n-i+1}$ for $1 \leq i \leq n$.

Then for $1 \leq i \leq n$,

$$f(H_i) = f(v_{i-1}) + f(v_i) + \sum f(V'_i) + \sum f(E_i)$$

$$= f(v_{i-1}) + f(v_i) + X_{n-i+1} + Y_{n-i+1}$$

$$= \frac{n(p+q)^2 + 3(p+q) - 2n(p+q) + 2n - 2}{2}$$

As $H_i \cong H$ for $1 \leq i \leq n$, $H_n$ is $H$-supermagic.

**Case (ii):** $p + q + n$ is even: Suppose $p$ is odd, $q$ is even and $n$ is odd. Since $p$ is odd as in proof of Case (i), there exists an $n$-equipartition $P_1 = \{X_1, X_2, \ldots, X_n\}$ of $[n+2, np-n+1]$ such that

$$\sum P_1 = (p-2)(n+1) + \frac{(p-3)(np-n+1)}{2} + [1, n]$$

Since $q$ is even and $n$ is odd, by (c) there exists an $n$-equipartition $P_2' = \{Y'_1, Y'_2, \ldots, Y'_n\}$ of $[1, nq]$ such that

$$\sum P_2' = \frac{q(nq+1)}{2} + \left[\frac{n-1}{2}, \frac{n-1}{2}\right].$$

Adding $np-n+1$ to $[1, nq]$ there exists an $n$-equipartition $P_2 = \{Y_1, Y_2, \ldots, Y_n\}$ of $(np-n+1) + [1, nq]$ such that

$$\sum P_2 = q(np-n+1) + \frac{q(nq+1)}{2} + \left[\frac{n-1}{2}, \frac{n-1}{2}\right]$$

Define a total labeling $f : V \cup E \rightarrow \{1, 2, 3, \ldots, np+nq-n+1\}$ as follows:
(i) \( f(v_i) = i + 1 \) for \( 0 \leq i \leq n \).

(ii) \( f(V'_i) = X_{n-i+1} \) for \( 1 \leq i \leq n \).

(iii) \( f(E_i) = Y_{n-i+1} \) for \( 1 \leq i \leq n \).

Then for \( 1 \leq i \leq n \),

\[
f(H_i) = f(v_{i-1}) + f(v_i) + \sum f(V'_i) + \sum f(E_i) \\
= f(v_{i-1}) + f(v_i) + \sum X_{n-i+1} + \sum Y_{n-i+1} \\
= \frac{n(p+q)^2 + 3(p+q) - 2n(p+q) + 2n - 2}{2}
\]

As \( H_i \cong H \) for \( 1 \leq i \leq n \), \( Hn \) is \( H \)-supermagic.

Suppose \( p \) is even, \( q \) is odd and \( n \) is odd. Since \( p - 2 \) is even and \( n \) is odd, by (c) there exists an \( n \)-equipartition \( P'_1 = \{ X'_1, X'_2, \ldots, X'_n \} \) of \([1, n(p-2)]\) such that

\[
\sum P'_1 = \frac{(p-2)[n(p-2) + 1]}{2} + \left[ -\frac{n-1}{2}, \frac{n-1}{2} \right].
\]

Adding \( n+1 \) to \([1, n(p-2)]\), we get an \( n \)-equipartition \( P_1 = \{ X_1, X_2, \ldots, X_n \} \) of \([n+2, np-n+1]\) such that

\[
\sum P_1 = (p-2)(n+1) + \frac{(p-2)[n(p-2) + 1]}{2} + \left[ -\frac{n-1}{2}, \frac{n-1}{2} \right]
\]

Since \( q \) is odd, as in Case (i) there exists an \( n \)-equipartition \( P_2 = \{ Y_1, Y_2, \ldots, Y_n \} \) of \((np-n+1) + [1, nq]\) such that

\[
\sum P_2 = q(np-n+1) + \frac{(q-1)(nq+n+1)}{2} + [1, n]
\]

Define a total labeling \( f : V \cup E \to \{1, 2, 3, \ldots, np+q-n+1\} \) as follows:

(i) \( f(v_i) = i + 1 \) for \( 0 \leq i \leq n \).

(ii) \( f(V'_i) = X_{n-i+1} \) for \( 1 \leq i \leq n \).

(iii) \( f(E_i) = Y_{n-i+1} \) for \( 1 \leq i \leq n \).

Then for \( 1 \leq i \leq n \),

\[
f(H_i) = f(v_{i-1}) + f(v_i) + \sum f(V'_i) + \sum f(E_i) \\
= f(v_{i-1}) + f(v_i) + \sum X_{n-i+1} + \sum Y_{n-i+1} \\
= \frac{n(p+q)^2 + 3(p+q) - 2n(p+q) + 2n - 2}{2}
\]

As \( H_i \cong H \) for \( 1 \leq i \leq n \), \( Hn \) is \( H \)-supermagic. \( \square \)
§5. Illustrations

A chain of a 2-connected \((5, 7)\) simple graph \(H\) of length 5 is shown in Figure 1 and a chain of a 2-connected \((6, 9)\) simple graph \(H\) of length 3 is shown in Figure 2.

Figure 1. \(p = 5, q = 7, s(f) = 322\).

Figure 2. \(p = 6, q = 9, s(f) = 317\).
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